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Well-Posedness of the Keller–Segel System in Fourier–Besov–Morrey Spaces

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Abstract. In this note, we investigate the Cauchy problem for Keller–Segel system with fractional diffusion for the initial data (u_0, v_0) in the critical Fourier–Besov–Morrey spaces $\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}(\mathbb{R}^d) \times \mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}(\mathbb{R}^d)$ with $1 < \alpha \leq 2$. The global well-posedness with a small initial data of the solution to Keller–Segel system of double-parabolic type is established.

Keywords. Keller–Segel system, fractional diffusions, Littlewood–Paley theory, well-posedness, Fourier–Besov space

Mathematics Subject Classification (2010). Primary 35K55, secondary 47J35

1. Introduction

In this paper, we consider the following Cauchy problem for the parabolic-parabolic Keller–Segel system in $\mathbb{R}^d \times \mathbb{R}^+$ with fractional Laplacian

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = \pm \nabla \cdot (u \nabla v), & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \varepsilon \partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ (u,v)|_{t=0} = (u_0, v_0), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where u is the density of cells, v is the concentration of the chemoattractant and $1 < \alpha \leq 2$. In the case $\alpha = 2$, (1) has been used to describe some collective motions of cells attracted by a chemical secreted by themselves. In fact, over the years, Keller–Segel type models have attracted the attention of several researchers, e.g. mathematicians, applied mathematicians, physcists, biologists etc.

The case $\varepsilon = 0$ and $\alpha = 2$ corresponds to the so-called Patlak–Keller–Segel. In this case, it is known there exists a threshold value for the mass

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$m_0 = \int u(x,0)dx = \int u(x,t)dx$ which decides whether solutions blow up or not at finite time. More precisely, if $m_0 < 8\pi$, there is global solution, while the solution blows up at a finite time T when $m_0 > 8\pi$. The value 8π is called the critical mass. In fact, for $m_0 > 8\pi$ solutions can blow up as measures (Dirac masses)(see e.g. Seki–Sugiyama–Velázquez 2013). In the case $\varepsilon = 1$, it is an open problem to know whether smooth solutions for (1) blow up(or not) at a finite time T.

In view of the above comments, it is natural to study system (1) in frameworks containing singular data or measures. There is a rich literature about global well-posedness for (1) with singular data in different critical spaces, such as weak-L^p spaces [11], Morrey spaces [3], Besov spaces[4, 15, 16, 18], Besov-Morrey space [8], Fourier–Besov spaces [9], and Fourier–Herz spaces [17]. Of course, there are also other famous works on this topic (see [12, 14]).

The purpose of this paper is to establish the existence of global solution to (1) with $\varepsilon = 1$ in the critical Fourier–Besov–Morrey spaces $\mathcal{FN}^{s_1}_{q,\mu,r} \times \mathcal{FN}^{s_2}_{q,\mu,r}$ with $0 \le \mu < d, 1 \le q < \infty$. From the definition below, it is easy to know that Fourier–Besov–Morrey space $\mathcal{FN}_{q,\mu,r}$ is larger than Fourier–Besov space $\mathcal{FN}_{q,0,r}$. Before stating our result, we first introduce some notations.

Denote the set of all polynomials by $\mathcal{P}$ and the Morrey spaces by $\mathcal{M}_{q,\mu}$, with norm

$$\|f\|_{q,\mu} = \sup_{x_0 \in \mathbb{R}^n, R>0} R^{-\frac{\mu}{q}} \|f\|_{L^q(B_R(x_0))} < \infty.$$

We define

$$\mathcal{FN}^s_{q,\mu,r} = \left\{ f \in S' \backslash \mathcal{P} \big| \|f\|_{\mathcal{FN}^s_{q,\mu,r}} = \left\| 2^{js} \|\varphi_j \hat{f}\|_{q,\mu} \right\|_{\ell^r(\mathbb{Z})} < \infty \right\}$$

where $\{\varphi_j\}$ is the Littlewood–Paley decomposition (see Section 2 for details). This space has been introduced to study self-similar solutions for a family of nonlinear active scalar equations in [7] and Ferreira et al. also used it to investigate the global well-posedness of the Navier–Stokes–Coriolis system in [6].

Let us firstly recall the scaling property of the systems. We can easily get that if the solutions $(u(x,t), v(x,t))$ solves (1), then $(u_\lambda(x,t), v_\lambda(x,t))$ with

$$u_\lambda(x,t) = \lambda^{2\alpha-2} u(\lambda x, \lambda^\alpha t), v_\lambda(x,t) = \lambda^{\alpha-2} v(\lambda x, \lambda^\alpha t) \tag{2}$$

is also a solution to (1) with the initial data

$$u_{0,\lambda} = \lambda^{2\alpha-2} u_0(\lambda x), v_{0,\lambda} = \lambda^{\alpha-2} v_0(\lambda x). \tag{3}$$

We can show that the spaces pair $\mathcal{FN}^{s_1}_{q,\mu,r} \times \mathcal{FN}^{s_2}_{q,\mu,r}$ with $s_1 = 2 - 2\alpha + d - \frac{d-\mu}{q}$ and $s_2 = 2 - \alpha + d - \frac{d-\mu}{q}$ are scaling invariant spaces (see Remark 2.3 for details).

In order to solve the equation (1), we consider the following equivalent integral system

$$\begin{cases} v(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_0 + \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} u(\tau)) d\tau := v_1 + v_2, \\ u(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} u_0 + \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla \cdot (u \nabla v_1) d\tau \\ \qquad + \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla \cdot (u \nabla v_2) d\tau, \end{cases} \tag{4}$$

where $e^{-t(-\Delta)^{\frac{\alpha}{2}}} \doteq \mathcal{F}^{-1} e^{-t|\xi|^\alpha} \mathcal{F}$.

Our results can be formulated as follows.

Theorem 1.1. *Let* $1 < \alpha \le 2$, $\max\{d - (3-2\alpha)q, 0\} < \mu < d$, $1 \le q < \infty$, $r \in [1, \infty]$ *and* $(u_0, v_0) \in \mathcal{FN}^{s_1}_{q,\mu,r} \times \mathcal{FN}^{s_2}_{q,\mu,r}$. *Then for any fixed* $\rho_0 > \frac{\alpha}{\alpha-1}$, $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$, *if* (u_0, v_0) *satisfies*

$$\|u_0\|_{\mathcal{FN}^{s_1}_{q,\mu,r}} + \|v_0\|_{\mathcal{FN}^{s_2}_{q,\mu,r}} \le \varepsilon,$$

then the equation (1) *possesses a unique global mild solution*

$$(u, v) \in X_\infty^{\rho_0 - \rho_0'}(s_1) \times X_\infty^{\rho_0 - \rho_0'}(s_2)$$

and

$$(u, v) \in C(0, \infty; \mathcal{FN}^{s_1}_{q,\mu,r}) \times C(0, \infty; \mathcal{FN}^{s_2}_{q,\mu,r}),$$

where

$$X_\infty^{\rho_0 - \rho_0'}(s) \triangleq X_\infty^{\rho_0}(s) \cap X_\infty^{\rho_0'}(s),$$

and

$$X_\infty^{\rho_0}(s) \triangleq \mathcal{L}^{\rho_0}\left((0,\infty); \mathcal{FN}^{s+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right), \; X_\infty^{\rho_0'}(s) \triangleq \mathcal{L}^{\rho_0'}\left((0,\infty); \mathcal{FN}^{s+\frac{\alpha}{\rho_0'}}_{q,\mu,r}\right).$$

Remark 1.2. Theorem 1.1 extends most results mentioned in the background for (1), even in the case of the classical dissipation $\alpha = 2$. On the other hand, we employ an auxiliary norm of Chemin type based on $\mathcal{FN}_{q,\mu,r}$-spaces in order to estimate the nonlinear terms in (1).

Remark 1.3. Theorem 1.1 provides a larger initial data class for (1) for global well-posedness. In fact, there are no inclusion relations between the spaces $\mathcal{FN}^{\mu+\beta}_{1,\mu,\infty}$ and $\mathcal{B}^\beta_2$ which are introduced in Section 2 ($\beta \in \mathbb{R}$ and $0 \le \mu < d$).

Remark 1.4. From the definition of Fourier–Besov–Morrey space, it is easy to see $\mathcal{FN}^{2-2\alpha}_{1,0,2} = \mathcal{B}^{2-2\alpha}_2$ and $\mathcal{FN}^{2-\alpha}_{1,0,2} = \mathcal{B}^{2-\alpha}_2$. In Theorem 1.1, we cannot take the case $q = 1$ and $\mu = 0$, so our results cannot contain [17, Theorem 1.1].

The paper is organized as follows. Section 2 presents some definitions and properties of Fourier–Besov-Morrey spaces and the Littlewood–Paley decomposition. In Section 3, we will give the proof of the linear estimates of the fractional heat semigroup. In Section 4, estimates for the bilinear term are proved. In Section 5, using the estimates obtained in Section 3 and Section 4, we will employ the Banach fixed point theorem to prove Theorem 1.1.

2. Preliminaries

In this section, we give some notations and recall basic properties about Fourier–Besov–Morrey spaces that will be used throughout the paper.

The Fourier–Besov–Morrey spaces were introduced in [7] and are constructed via a type of localization on Morrey spaces.

Definition 2.1. Let $1 \le q \le \infty$ and $0 \le \mu < d$. The homogeneous Morrey space $\mathcal{M}_{q,\mu}$ is the set of all functions $f \in L^q(B_r(x_0))$ such that

$$\|f\|_{q,\mu} = \sup_{x_0\in\mathbb{R}^d, R>0} R^{-\frac{\mu}{q}} \|f\|_{L^q(R(x_0))} < \infty, \tag{5}$$

where $B_R(x_0)$ is the open ball in $\mathbb{R}^d$ centered at x_0 and with radius $R > 0$.

When $q = 1$, the L^1-norm in (5) is understood as the total variation of the measure f on $B_R(x_0)$ and $\mathcal{M}_{1,\mu}$ as a subspace of Radon measures. When $\mu = 0$, we have $\mathcal{M}_{q,0} = L^q$.

In what follows, we introduce homogeneous Littlewood–Paley decomposition. For more detail, we refer the reader to [1].

Let $f \in S'(\mathbb{R}^d)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx,$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be such that $0 \le \varphi \le 1$ and $\text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^d : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi(2^{-j}\xi), \quad \psi_j(\xi) = \sum_{k\le j-1} \varphi_k(\xi),$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi_0(x).$$

We now present some frequency localization operators:

$$\dot{\Delta}_j f = \varphi_j(D)f = 2^{dj} \int_{\mathbb{R}^d} h(2^j y) f(x-y)dy,$$

and

$$\dot{S}_j f = \sum_{k\le j-1} \dot{\Delta}_k f = \psi_j(D)f = 2^{dj} \int_{\mathbb{R}^d} g(2^j y) f(x-y)dy.$$

From the definition, one easily derives that

$$\dot{\Delta}_j \dot{\Delta}_k f = 0, \quad \text{if } |j-k| \ge 2,$$
$$\dot{\Delta}_j(\dot{S}_{k-1} f \dot{\Delta}_k f) = 0, \quad \text{if } |j-k| \ge 5.$$

The following Bony paraproduct decomposition will be applied throughout the paper.

$$uv = \dot{T}_u v + \dot{T}_v u + R(u,v),$$

where

$$\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\dot{\Delta}}_j v, \quad \tilde{\dot{\Delta}}_j v = \sum_{|j'-j|\le 1} \dot{\Delta}_{j'} v.$$

Now let us give the definition of Fourier–Besov–Morrey space, see [7].

Definition 2.2. Let $1 \le r, q \le \infty, 0 \le \mu < d$ and $s \in \mathbb{R}$. The Fourier–Besov–Morrey space $\mathcal{FN}^s_{q,\mu,r}$ is defined as the set of all distributions $f \in \mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials, such that $\varphi_k \hat{f} \in \mathcal{M}_{q,\mu}$, for all $k \in \mathbb{Z}$, and

$$\|f\|_{\mathcal{FN}^s_{q,\mu,r}} \overset{\text{def}}{=} \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{ksr} \|\varphi_k \hat{f}\|^r_{q,\mu}\right)^{\frac{1}{r}}, & r < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{ks} \|\varphi_k \hat{f}\|_{q,\mu}, & r = \infty. \end{cases} \tag{6}$$

The space $\mathcal{FN}^s_{q,\mu,r}$ endowed with the norm (6) is a Banach space and contains homogeneous functions of degree $-d = s - d + \frac{d-\mu}{q}$. If $\mu = 0$, then $\mathcal{FN}^s_{q,0,r}$ is the Fourier–Besov space $\mathcal{FB}^s_{q,r}$. When $r = 1$, then the Fourier–Besov space $\mathcal{FB}^s_{q,r}$ is the Fourier–Herz space $\dot{\mathcal{B}}^s_q$.

Remark 2.3. The space pair $\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r} \times \mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}$ is a critical space with respect to the scaling property for equations (1). Similarly to [13], for $u_{0,\lambda} = \lambda^{2\alpha-2}u_0(\lambda x)$, its Fourier transform is $\lambda^{2\alpha-2-d}\hat{u}_0(\lambda^{-1}\xi)$. Let $h_j(\xi) \stackrel{\text{def}}{=} \varphi(2^{-j+[\log_2\lambda]-\log_2\lambda}\xi)\lambda^{2\alpha-2-d}\hat{u}_0(\lambda^{-1}\xi)$. By change of variable, we get

$$\begin{aligned}
\|h_j(\xi)\|_{q,\mu} &= \lambda^{2\alpha-2-d}\|\varphi(2^{-j+[\log_2\lambda]-\log_2\lambda}\xi)\hat{u}_0(\lambda^{-1}\xi)\|_{q,\mu}\\
&= \lambda^{2\alpha-2-d}\sup_{x_0\in\mathbb{R}^d, R>0} R^{-\frac{\mu}{q}}\left(\int_{B_R(x_0)}|\varphi(2^{-j+[\log_2\lambda]-\log_2\lambda}\xi)\widehat{u_0}(\lambda^{-1}\xi)|^q\,d\xi\right)^{\frac{1}{q}}\\
&= \lambda^{2\alpha-2-d+\frac{d-\mu}{q}}\sup_{x_0\in\mathbb{R}^d, R>0}(\lambda^{-1}R)^{-\frac{\mu}{q}}\left(\int_{B_{\lambda^{-1}R}(\lambda^{-1}x_0)}|\varphi(2^{-j+[\log_2\lambda]})\widehat{u_0}(\xi)|^q\,d\xi\right)^{\frac{1}{q}}\\
&= \lambda^{2\alpha-2-d+\frac{d-\mu}{q}}\|\varphi_{j-[\log_2\lambda]}\hat{u}_0\|_{q,\mu},
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\{2^{j(2-2\alpha+d-\frac{d-\mu}{q})}\|h_j(\xi)\|_{q,\mu}\}\|_{\ell^r}\\
&= \|\{2^{j(2-2\alpha+d-\frac{d-\mu}{q})}\lambda^{2\alpha-2-d+\frac{d-\mu}{q}}\|\varphi_{j-[\log_2\lambda]}\hat{u}_0(\zeta)\|_{q,\mu}\}\|_{\ell^r}\\
&= \|\{2^{(\log_2\lambda-[\log_2\lambda])(2\alpha-2-d+\frac{d-\mu}{q})}2^{(j-[\log_2\lambda])(2-2\alpha+d-\frac{d-\mu}{q})}\|\varphi_{j-[\log_2\lambda]}\hat{u}_0(\zeta)\|_{q,\mu}\}\|_{\ell^q}\\
&\approx \|u_0\|_{\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}}.
\end{aligned}$$

Recalling that $\varphi_j(\xi)\hat{u}_{0,\lambda}(\xi) = \sum_{|k-j|\le 2}\varphi_j(\xi)h_k(\xi)$, it is easy to deduce that

$$\|u_{0,\lambda}\|_{\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}} \approx \|u_0\|_{\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}}.$$

Similarly, we have

$$\|v_{0,\lambda}\|_{\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}} \approx \|v_0\|_{\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}}.$$

In the following lemma, we give some important properties about Morrey spaces, which can be found in [7].

Lemma 2.4. *Let $1 \le q_1, q_2, q_3 < \infty$ and $0 \le \mu_1, \mu_2, \mu_3 < d$.*

(i) *(Hölder's inequality) Let $\frac{1}{q_3} = \frac{1}{q_2} + \frac{1}{q_1}$ and $\frac{\mu_3}{q_3} = \frac{\mu_2}{q_2} + \frac{\mu_1}{q_1}$. If $f_i \in \mathcal{M}_{q_i,\mu_i}$ for $i = 1, 2$, then $f_1 f_2 \in \mathcal{M}_{q_3,\mu_3}$ and*

$$\|f_1 f_2\|_{q_3,\mu_3} \le \|f_1\|_{q_1,\mu_1}\|f_2\|_{q_2,\mu_2}. \tag{7}$$

(ii) (*Young's inequality*) *If* $\varphi \in L^1(\mathbb{R}^d)$ *and* $g \in \mathcal{M}_{q_1,\mu_1}$, *then*

$$\|\varphi * g\|_{q_1,\mu_1} \leq \|\varphi\|_1 \|g\|_{q_1,\mu_1}, \tag{8}$$

where $*$ *denotes the standard convolution operator.*

(iii) (*Bernstein-type inequality*) *Let* $q_2 \leq q_1$ *be such that* $\frac{d-\mu_1}{q_1} \leq \frac{d-\mu_2}{q_2}$. *If* $A > 0$ *and supp*$(\hat{f}) \subset \{\xi \in \mathbb{R}^d : |\xi| \leq A2^j\}$, *then*

$$\|\xi^\gamma \hat{f}\|_{q_2,\mu_2} \leq C 2^{j|\gamma| + j\left(\frac{d-\mu_2}{q_2} - \frac{d-\mu_1}{q_1}\right)} \|\hat{f}\|_{q_1,\mu_1}, \tag{9}$$

where γ *is a multi-index,* $j \in \mathbb{Z}$ *and* $C > 0$ *is a constant independent of* j, ξ *and* f.

Now, we list a time-dependent space based on $\mathcal{FN}^s_{q,\mu,r}$, which was introduced in [6].

Definition 2.5. Let $1 \leq \rho \leq \infty, 0 < T < \infty$ and $I = (0,T)$. The Banach spaces $\mathcal{L}^\rho(I, \mathcal{FN}^s_{q,\mu,r})$ consist of all the Bochner measurable functions from I to $\mathcal{FN}^s_{q,\mu,r}$, whose norms are given by

$$\|f\|_{\mathcal{L}^\rho(I;\mathcal{FN}^s_{q,\mu,r})} \overset{\text{def}}{=} \left(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\varphi_j \hat{f}\|^r_{L^\rho(I;\mathcal{M}_{q,\mu})}\right)^{\frac{1}{r}} = \left(\sum_{j\in\mathbb{Z}} 2^{jsr} \Big\| \|\varphi_j \hat{f}\|_{q,\mu} \Big\|^r_{L^\rho(I)}\right)^{\frac{1}{r}}. \tag{10}$$

In order to prove the existence of solution for (1), we need the following classical result on the existence of solutions for abstract equations with bilinear structure. Its proof can be found in [5, p. 189, Lemma 5].

Lemma 2.6. *Let* $(X, \|\cdot\|_X)$ *be an abstract Banach space with norm* $\|\cdot\|$ *and* $L : X \to X$ *be a linear operator such that, for any* $x \in X$

$$\|L(x)\| \leq \theta \|x\|$$

and $B : X \times X \to X$ *is a bilinear operator such that, for any* $x_1, x_2 \in X$,

$$\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X,$$

for some constant $\eta > 0$. *Then, for any* $0 \leq \theta < 1$ *and for any* $y \in X$ *such that*

$$4\eta\|y\| < (1-\theta)^2$$

the equation

$$x = y + B(x,x) + L(x)$$

has a solution $x \in X$. *In particular, the solution is such that*

$$\|x\| \leq \frac{2\|y\|}{1-\theta},$$

and is the unique one such that

$$\|x\| < \frac{1-\theta}{2\eta}.$$

3. Linear estimates in Fourier–Besov–Morrey spaces

In this section, we will establish some crucial estimates in the proof of Theorem 1.1. We now consider the following linear estimates for the fractional heat semigroup $\{e^{-t(-\Delta)^{\frac{\alpha}{2}}}\}_{t\geq 0}$.

Lemma 3.1. *Let $I=(0,T)$, $s\in\mathbb{R}$, $q,r,\rho\in[1,\infty]$ and $0\leq\mu<d$. There exists a constant $C>0$ such that*

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r}\right)}\leq C\|u_0\|_{\mathcal{FN}^s_{q,\mu,r}}, \tag{11}$$

where $u_0\in\mathcal{FN}^s_{q,\mu,r}$.

Proof. Since $\operatorname{supp}\varphi_j\subset\{2^{j-1}\leq|\xi|\leq 2^{j+1}\}$, we get

$$\|\mathcal{F}[\Delta_j e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0]\|_{q,\mu}=\|\varphi_j e^{-t|\xi|^\alpha}\widehat{u_0}\|_{q,\mu}\leq e^{-t2^{\alpha(j-1)}}\|\varphi_j\widehat{u_0}\|_{q,\mu}.$$

Then, by the Minkowski inequality, we have

$$\begin{aligned}\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\|_{\mathcal{L}^\rho(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r})}&\leq\left\|\left\{2^{j(s+\frac{\alpha}{\rho})}\left(\int_0^T e^{-t\rho 2^{\alpha(j-1)}}\,dt\right)^{\frac{1}{\rho}}\|\varphi_j\widehat{u_0}\|_{q,\mu}\right\}\right\|_{\ell^r}\\&\leq\left\|\left\{2^{j(s+\frac{\alpha}{\rho})}\left(\frac{1-e^{-T\rho 2^{(j-1)\alpha}}}{\rho 2^{\alpha(j-1)}}\right)^{\frac{1}{\rho}}\|\varphi_j\widehat{u_0}\|_{q,\mu}\right\}\right\|_{\ell^r}\\&\leq C\|u_0\|_{\mathcal{FN}^s_{q,\mu,r}}.\end{aligned}$$

□

Lemma 3.2. *Let $I=(0,T)$, $s\in\mathbb{R}$, $q,r,\rho\in[1,\infty]$, $0\leq\mu<d$ and $1\leq\rho_1\leq\rho$. There exists a constant $C>0$ such that*

$$\left\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}f(\tau)d\tau\right\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r}\right)}\leq C\|f\|_{\mathcal{L}^{\rho_1}\left(I;\mathcal{FN}^{s-\alpha+\frac{\alpha}{\rho_1}}_{q,\mu,r}\right)}, \tag{12}$$

for all $f\in\mathcal{L}^{\rho_1}\left(I;\mathcal{FN}^{s-\alpha+\frac{\alpha}{\rho_1}}_{q,\mu,r}\right)$.

Proof. Thanks to $\operatorname{supp}\varphi_j\subset\{2^{j-1}\leq|\xi|\leq 2^{j+1}\}$ and the Young inequality, we have

$$\begin{aligned}\left\|\varphi_j\mathcal{F}\left[\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}f(\tau)d\tau\right]\right\|_{\mathcal{L}^\rho(I;\mathcal{M}_{q,\mu})}&\leq\|\int_0^t e^{-(t-\tau)2^{\alpha(j-1)}}\|\varphi_j\hat{f}\|_{q,\mu}d\tau\|_{L^\rho(I)}\\&\leq\|e^{-t2^{\alpha(j-1)}}\chi_{[0,T]}\|_{L^\theta}\|\varphi_j\hat{f}\|_{L^{\rho_1}(I;\mathcal{M}_{q,\mu})}\\&\leq\left(\frac{1-e^{-T2^{(j-1)\alpha}\theta}}{\theta 2^{\alpha(j-1)}}\right)^{\frac{1}{\theta}}\|\varphi_j\hat{f}\|_{L^{\rho_1}(I;\mathcal{M}_{q,\mu})}\\&\leq C2^{-j\alpha(1+\frac{1}{\rho}-\frac{1}{\rho_1})}\|\varphi_j\hat{f}\|_{L^{\rho_1}(I;\mathcal{M}_{q,\mu})},\end{aligned}$$

where $1+\frac{1}{\rho}=\frac{1}{\theta}+\frac{1}{\rho_1}$.

This inequality together with the definition of the norm $\|\cdot\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r}\right)}$ yield

$$\left\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}f(\tau)d\tau\right\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r}\right)} \le C\left\|\left\{2^{j(s+\frac{\alpha}{\rho})}2^{-j\alpha(1+\frac{1}{\rho}-\frac{1}{\rho_1})}\|\varphi_j\hat{f}\|_{L^{\rho_1}(I;\mathcal{M}_{q,\mu})}\right\}\right\|_{\ell^r} \le C\|f\|_{\mathcal{L}^{\rho_1}\left(I;\mathcal{FM}^{s-\alpha+\frac{\alpha}{\rho_1}}_{q,\mu,r}\right)}. \qquad \square$$

Remark 3.3. If we take $\rho_1=1$, then (12) becomes

$$\left\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}f(\tau)d\tau\right\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r}\right)} \le C\|f\|_{\mathcal{L}^1(I;\mathcal{FN}^s_{q,\mu,r})}.$$

If we take $\rho_1=\rho$, then (12) becomes

$$\left\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}f(\tau)d\tau\right\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{q,\mu,r}\right)} \le C\|f\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}-\alpha}_{q,\mu,r}\right)}.$$

4. Bilinear estimates in Fourier–Besov–Morrey spaces

Lemma 4.1. *Let $I=(0,T)$, $s\in\mathbb{R}$, $q,r\in[1,\infty]$, $\max\{d-(3-2\alpha+d)q,0\}<\mu<d$, $\rho_0>\frac{\alpha}{\alpha-1}$ and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. There exists a constant $C>0$ such that*

$$\begin{aligned}&\|\nabla\cdot(f\nabla g)\|_{\mathcal{L}^1\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right)}\\ &\le C\left(\|f\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right)}\|g\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}_{q,\mu,r}\right)}\right.\\ &\qquad\left.+\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right)}\|f\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}_{q,\mu,r}\right)}\right)\end{aligned}$$

for all $f\in\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right)\cap\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}_{q,\mu,r}\right)$ and $g\in\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right)\cap\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}_{q,\mu,r}\right)$.

Proof. Using the following paraproduct formula due to J. M. Bony [2],

$$\begin{aligned}\dot\Delta_j(f\nabla g) &= \sum_{|k-j|\le 4}\dot\Delta_j\left(\dot S_{k-1}f\dot\Delta_k(\nabla g)\right)+\sum_{|k-j|\le 4}\dot\Delta_j\left(\dot S_{k-1}(\nabla g)\dot\Delta_k f\right)\\ &\quad+\sum_{k\ge j-2}\sum_{|k-k'|\le 1}\dot\Delta_j\left(\dot\Delta_k f\dot\Delta_{k'}(\nabla g)\right)\\ &:= I_1+I_2+I_3.\end{aligned}$$

Then, by the triangle inequality in ℓ^r and in $\mathcal{M}_{q,\mu}$, it follows that

$$\begin{aligned}&\|\nabla\cdot(f\nabla g)\|_{\mathcal{L}^1\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right)}\le\|f\nabla g\|_{\mathcal{L}^1\left(I;\mathcal{FN}^{3-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right)}\\ &\le\left\|\left\{2^{j(3-2\alpha+d-\frac{d-\mu}{q})}\|\hat I_1\|_{L^1(I;\mathcal{M}_{q,\mu})}\right\}\right\|_{\ell^r}+\left\|\left\{2^{j(3-2\alpha+d-\frac{d-\mu}{q})}\|\hat I_2\|_{L^1(I;\mathcal{M}_{q,\mu})}\right\}\right\|_{\ell^r}\\ &\quad+\left\|\left\{2^{j(3-2\alpha+d-\frac{d-\mu}{q})}\|\hat I_3\|_{L^1(I;\mathcal{M}_{q,\mu})}\right\}\right\|_{\ell^r}\\ &:= J_1+J_2+J_3.\end{aligned}$$

By Bernstein-type inequality (9) with $\vartheta=0,(q_2,\mu_2)=(1,0)$ and $(q_1,\mu_1)=(q,\mu)$, we have

$$\|\varphi_k\hat h\|_{L^1}\le C2^{k(d-\frac{d-\mu}{q})}\|\varphi_k\hat h\|_{q,\mu}.$$

Thus

$$\begin{aligned}&\|\hat I_1\|_{L^1(I;\mathcal{M}_{q,\mu})}\\ &\le\int_I\sum_{|k-j|\le 4}\left\|\left(\widehat{\dot S_{k-1}f}*\widehat{\dot\Delta_k(\nabla g)}\right)\right\|_{q,\mu}dt\\ &\le\int_I\sum_{|k-j|\le 4}\left(\sum_{k'<k-2}\|\varphi_{k'}\hat f\|_{L^1}\right)\|\varphi_k\widehat{(\nabla g)}\|_{q,\mu}\,dt\\ &\le C\sum_{|k-j|\le 4}\int_I\left(\sum_{k'<k-2}2^{k'(d-\frac{d-\mu}{q})}\|\varphi_{k'}\hat f\|_{q,\mu}\right)2^k\|\varphi_k\hat g\|_{q,\mu}\,dt\\ &\le C\sum_{|k-j|\le 4}2^k\left(\sum_{k'<k-2}2^{k'(d-\frac{d-\mu}{q})}\|\varphi_{k'}\hat f\|_{L^{\rho_0}(I;\mathcal{M}_{q,\mu})}\right)\|\varphi_k\hat g\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})}\\ &\le C\|f\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right)}\sum_{|k-j|\le 4}2^k\left(\sum_{k'<k-2}2^{k'(2\alpha-2-\frac{\alpha}{\rho_0})r'}\right)^{\frac{1}{r'}}\|\varphi_k\hat g\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})}\\ &\le C\|f\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}_{q,\mu,r}\right)}\sum_{|k-j|\le 4}2^{k(2\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\hat g\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})},\end{aligned}$$

where we have used the condition $\rho_0 > \frac{\alpha}{\alpha-1}$ in the last inequality.

Therefore, by the Young inequality, we can estimate

$$\begin{aligned}
J_1 &\le C\|f\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\quad\times \left\| 2^{j(3-2\alpha+d-\frac{d-\mu}{q})} \left\{ \sum_{|k-j|\le 4} 2^{k(2\alpha-1-\frac{\alpha}{\rho_0})} \|\varphi_k \hat{g}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})} \right\} \right\|_{\ell^r} \\
&\le C\|f\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\quad\times \left\| \left\{ \sum_{|k-j|\le 4} 2^{(j-k)(3-2\alpha+d-\frac{d-\mu}{q})} 2^{k(2+d-\frac{d-\mu}{q}-\frac{\alpha}{\rho_0})} \|\varphi_k \hat{g}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})} \right\} \right\|_{\ell^r} \\
&\le C\|f\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \|g\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)},
\end{aligned}$$

where we have used the fact that $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$.

Similarly,

$$\begin{aligned}
&\|\hat{I}_2\|_{L^1(I;\mathcal{M}_{q,\mu})} \\
&\le \int_I \sum_{|k-j|\le 4} \left\| \left(\widehat{\dot{S}_{k-1}(\nabla g)} * \widehat{\dot{\Delta}_k(f)} \right) \right\|_{q,\mu} dt \\
&\le \int_I \sum_{|k-j|\le 4} \left(\sum_{k'<k-2} \|\varphi_{k'} \widehat{\nabla g}\|_{L^1} \right) \|\varphi_k \hat{f}\|_{q,\mu}\, dt \\
&\le C \sum_{|k-j|\le 4} \int_I \left(\sum_{k'<k-2} 2^{k'(1+d-\frac{d-\mu}{q})} \|\varphi_{k'} \hat{g}\|_{q,\mu} \right) \|\varphi_k \hat{f}\|_{q,\mu}\, dt \\
&\le C \sum_{|k-j|\le 4} \left(\sum_{k'<k-2} 2^{k'(1+d-\frac{d-\mu}{q})} \|\varphi_{k'} \hat{g}\|_{L^{\rho_0}(I;\mathcal{M}_{q,\mu})} \right) \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \sum_{|k-j|\le 4} \left(\sum_{k'<k-2} 2^{k'(\alpha-1-\frac{\alpha}{\rho_0})r'} \right)^{\frac{1}{r'}} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \sum_{|k-j|\le 4} 2^{k(\alpha-1-\frac{\alpha}{\rho_0})} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})},
\end{aligned}$$

where we again have used the fact that $\rho_0 > \frac{\alpha}{\alpha-1}$ in the last inequality.

Therefore, by the Young inequality, we get

$$\begin{aligned}
J_2 &\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\quad \times \left\| \left\{ 2^{j(3-2\alpha+d-\frac{d-\mu}{q})} \sum_{|k-j|\le 4} 2^{k(\alpha-1-\frac{\alpha}{\rho_0})} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})} \right\} \right\|_{\ell^r} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\quad \times \left\| \left\{ \sum_{|k-j|\le 4} 2^{(j-k)(3-2\alpha+d-\frac{d-\mu}{q})} 2^{k(2-\alpha+d-\frac{d-\mu}{q}-\frac{\alpha}{\rho_0})} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I;\mathcal{M}_{q,\mu})} \right\} \right\|_{\ell^r} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \|f\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}.
\end{aligned}$$

For J_3, the Young inequality, the Bernstein inequality together with the Hölder inequality imply

$$\begin{aligned}
&\|\hat{I}_3\|_{L^1(I;\mathcal{M}_{q,\mu})} \\
&\le \int_I \sum_{k\ge j-2} \left\| \varphi_j \left((\varphi_k \hat{f}) * \Big(\sum_{|k-k'|\le 1} \varphi_{k'} \widehat{\nabla g} \Big) \right) \right\|_{q,\mu} dt \\
&\le \int_I \sum_{k\ge j-2} \|\varphi_k \hat{f}\|_{q,\mu} \left(\sum_{|k-k'|\le 1} 2^{k'} \|\varphi_{k'} \hat{g}\|_{L^1} \right) dt \\
&\le C \sum_{k\ge j-2} \int_I \|\varphi_k \hat{f}\|_{q,\mu} \left(\sum_{|k-k'|\le 1} 2^{k'} 2^{k'(d-\frac{d-\mu}{q})} \|\varphi_{k'} \hat{g}\|_{q,\mu} \right) dt \\
&\le C \sum_{k\ge j-2} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I,\mathcal{M}_{q,\mu})} \left(\sum_{|k-k'|\le 1} 2^{k'} 2^{k'(d-\frac{d-\mu}{q})} \|\varphi_{k'} \hat{g}\|_{L^{\rho_0}(I,\mathcal{M}_{q,\mu})} \right) \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \sum_{k\ge j-2} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I,\mathcal{M}_{q,\mu})} \left(\sum_{|k-k'|\le 1} 2^{k'(\alpha-1-\frac{\alpha}{\rho_0})r'} \right)^{\frac{1}{r'}} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \sum_{k\ge j-2} 2^{k(\alpha-1-\frac{\alpha}{\rho_0})} \|\varphi_k \hat{f}\|_{L^{\rho_0'}(I,\mathcal{M}_{q,\mu})}.
\end{aligned}$$

Hence, using the Young inequality again, one has

$$\begin{aligned}
J_3 &\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\quad\times \left\|\left\{2^{j(3-2\alpha+d-\frac{d-\mu}{q})}\sum_{k\ge j-2}2^{k(\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\hat{f}\|_{L^{\rho_0'}(I,\mathcal{M}_{q,\mu})}\right\}\right\|_{\ell^r} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\quad\times \left\|\left\{\sum_{k\ge j-2}2^{(j-k)(3-2\alpha+d-\frac{d-\mu}{q})}2^{k(2-\alpha+d-\frac{d-\mu}{q}-\frac{\alpha}{\rho_0})}\|\varphi_k\hat{f}\|_{L^{\rho_0'}(I,\mathcal{M}_{q,\mu})}\right\}\right\|_{\ell^r} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\sum_{2\ge k}2^{k(3-2\alpha+d-\frac{d-\mu}{q})}\|f\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)} \\
&\le C\|g\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\|f\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)},
\end{aligned}$$

where the condition $\mu>d-(3-2\alpha+d)q$ ensures that the series $\sum_{2\ge k}2^{k(3-2\alpha+d-\frac{d-\mu}{q})}$ converges. This finishes the proof of Lemma 4.1. □

5. Proof of Theorem 1.1

In this section, we will apply Lemma 2.6 together with the estimates established in Section 3 and 4 to prove Theorem 1.1.

Let $\rho_0 > \frac{\alpha}{\alpha-1}$ be any given real number and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. Let $X_T^{\rho_0-\rho_0'}(s)$ be the space defined in Theorem 1.1. It is clear that $X_T^{\rho_0-\rho_0'}(s)$ is a Banach space equipped with the norm

$$\|u\|_{X_T^{\rho_0-\rho_0'}(s)} = \|u\|_{\mathcal{L}^{\rho_0}\left((0,T);\mathcal{FN}_{q,\mu,r}^{s+\frac{\alpha}{\rho_0}}\right)} + \|u\|_{\mathcal{L}^{\rho_0'}\left((0,T);\mathcal{FN}_{q,\mu,r}^{s+\frac{\alpha}{\rho_0'}}\right)}.$$

Set

$$\begin{aligned}
y &\triangleq e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0, \\
L(u) &\triangleq \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}\nabla\cdot(u\nabla v_1)d\tau, \quad v_1 = e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, \\
B(u,w) &\triangleq \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}\nabla\cdot(u\nabla v_2)d\tau, \quad v_2 = \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}w d\tau.
\end{aligned}$$

Then the second equation of (4) can be rewritten as

$$u = y + B(u,u) + L(u).$$

It follows from Lemma 3.1 with $s = 2 - 2\alpha + d - \frac{d-\mu}{q}, I = [0,\infty)$ and $\rho = \rho_0$ (or ρ_0') that $\|y\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \le C\|u_0\|_{\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}}$ and $\|y\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)} \le C\|u_0\|_{\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}}$, which implies

$$\|y\|_{X_T^{\rho_0-\rho_0'}\left(2-2\alpha+d-\frac{d-\mu}{q}\right)} \le 2C\|u_0\|_{\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}}.$$

Applying Lemma 3.2 with $s = 2 - 2\alpha + d - \frac{d-\mu}{q}$ and $\rho_1 = 1$, Lemma 4.1 and Lemma 3.1 with $s = 2 - \alpha + d - \frac{d-\mu}{q}$ and $\rho = \rho_0$ (or ρ_0'), we get

$$\begin{aligned}
&\|L(u)\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)} \\
&\le C\|\nabla\cdot(u\nabla v_1)\|_{\mathcal{L}^{1}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}\right)} \\
&\le C\left(\|u\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\left\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\right\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\right. \\
&\quad\left.+\left\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\right\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\|u\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\right) \\
&\le C\left(\|u\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\right. \\
&\quad\left.+\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\|u\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\right) \\
&\le C\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\left(\|u\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}+\|u\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\right) \\
&\le C\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\|u\|_{X_T^{\rho_0-\rho_0'}\left(2-2\alpha+d-\frac{d-\mu}{q}\right)},
\end{aligned}$$

and

$$\|L(u)\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)} \leq C\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\|u\|_{X_T^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}.$$

This means

$$\|L(u)\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})} \leq 2C\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\|u\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}.$$

Also, we can estimate

$$\begin{aligned}
&\|B(u,w)\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\\
&\leq C\|\nabla\cdot(u\nabla v_2)\|_{\mathcal{L}^1\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}\right)}\\
&\leq C\Bigg(\|u\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\left\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}w(\tau))d\tau\right\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\\
&\quad+\left\|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}w(\tau))d\tau\right\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\|u\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\Bigg)\\
&\leq C\|u\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\|w\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\\
&\quad+C\|w\|_{\mathcal{L}^{\rho_0}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0}}\right)}\|u\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)}\\
&\leq C\|u\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}\|w\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}.
\end{aligned}$$

and

$$\|B(u,w)\|_{\mathcal{L}^{\rho_0'}\left(I;\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}+\frac{\alpha}{\rho_0'}}\right)} \leq C\|u\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}\|w\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}.$$

That is

$$\|B(u,w)\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})} \leq 2C\|u\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}\|w\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})}.$$

For $0<\varepsilon<1$, let $v_0\in\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}$ be such that $2C\|v_0\|_{\mathcal{FN}_{q,\mu,r}^{2-\alpha+d-\frac{d-\mu}{q}}}\leq\varepsilon$. Also, we consider $u_0\in\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}$ with small norm and $R>0$ satisfying

$$8C\|y\|_{X_\infty^{\rho_0-\rho_0'}(2-2\alpha+d-\frac{d-\mu}{q})} \leq 16C^2\|u_0\|_{\mathcal{FN}_{q,\mu,r}^{2-2\alpha+d-\frac{d-\mu}{q}}} < (1-\varepsilon)^2,$$

and $2C\|u_0\|_{\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}} < R$. Then Lemma 2.6 implies that (4) has a unique solution in $B_{\frac{2R}{1-\varepsilon}}(0)$, where $B_{\frac{2R}{1-\varepsilon}}(0)$ is the closed ball with center 0 and radius $\frac{2R}{1-\varepsilon}$ in $X^{\rho_0-\rho_0'}_{\infty}(2-2\alpha+d-\frac{d-\mu}{q})$.

By the same argument as in [17, p. 1147, Step 5], we can prove that

$$u \in L^\infty\left(0,\infty;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right) \quad \text{and} \quad v \in L^\infty\left(0,\infty;\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right).$$

A standard argument yields that

$$u \in C\left(0,\infty;\mathcal{FN}^{2-2\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right) \quad \text{and} \quad v \in C\left(0,\infty;\mathcal{FN}^{2-\alpha+d-\frac{d-\mu}{q}}_{q,\mu,r}\right),$$

and we are done.

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