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Optimal Decay Rate of Solutions to Timoshenko System with Past History in Unbounded Domains

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Abstract. In this paper, we investigate the Cauchy problem for the Timoshenko system in thermo-elasticity, where the heat conduction is given by the Gurtin–Pipkin thermal law in one-dimensional space. We show an optimal decay rate of the L^2–norm of the solution with the rate of $(1+t)^{-\frac{1}{8}}$ which is better than $(1+t)^{-\frac{1}{12}}$ found in [6]. We also extend the recent results in [7, 8] and showed that those results are only particular cases of the one obtained here. Also, we prove that the decay rate is controlled by a crucial stability number α_g which depends on the parameters of the system.

Keywords. Timoshenko system, decay, regularity loss, heat conduction, Lyapunov functional
Mathematics Subject Classification (2010). Primary 35B37, 35L55, secondary 74D05, 93D15, 93D20

1. Introduction

In classical thermoelasticity, the heat conduction is given by Fourier's law, which assumes that the heat flux is proportional to the gradient of the temperature, written in one-dimensional space as

$$q(x,t) = -\kappa\theta_x(x,t), \tag{1}$$

where $\kappa > 0$ is the thermal conductivity. Equation (1) represents an instantaneous response to changes in the gradient of the temperature visible in the heat flux.

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Equation (1) together with the energy equation of the heat

$$\theta_t(x,t) + q_x(x,t) = 0, \tag{2}$$

leads to the diffusion equation (the heat equation)

$$\theta_t(x,t) - \kappa\theta_{xx}(x,t) = 0. \tag{3}$$

It is well known that the diffusion equation (3) has the unphysical property that if a sudden change of temperature is made at some point on the body, it will be felt instantly everywhere. Thus, we may say that diffusion gives rise to *infinite speed of propagation.* The attempt to correct the "paradox of instantaneous propagation of thermal disturbances" predicted by Fourier's theory of heat conduction has inspired the work of searching for new constitutive relations. Consequently, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made. Among these, Cattaneo's law, Gurtin's and Pipkin's theory, Jeffreys's law, Green's and Naghdi's theory and others. The common feature of these theories is that all lead to hyperbolic differential equation and permit transmission of heat flow as thermal waves at finite speed. The most obvious and simple generalization of Fourier's law is the Cattaneo (or Cattaneo–Maxwell) equation

$$\tau_q q_t(x,t) + q(x,t) = -\kappa\theta_x(x,t). \tag{4}$$

Equation (4) together with (3) leads to the damped wave equation (known as the telegraph equation):

$$\theta_{tt}(x,t) - \frac{\kappa}{\tau_q}\theta_{xx}(x,t) + \frac{1}{\tau_q}\theta_t(x,t) = 0. \tag{5}$$

Equation (5) is hyperbolic and it transmits waves of temperature with a finite speed equals to $\sqrt{\kappa\,\tau_q^{-1}}$. From the mathematical point of view, it is much easier to deal with the diffusion equation (3) than the damped wave equation (5), since equation (3) is a parabolic equation and has a smoothing effect. For this reason, and as we will see later, it is quit hard to treat problems involving Cattaneo's law of heat conduction, compared to those involving Fourier's law. Another important law of heat conduction is the Jeffreys law:

$$\tau_q q_t(x,t) + q(x,t) = -\kappa\theta_x(x,t) - \tau_q\kappa_1\theta_{tx}(x,t). \tag{6}$$

Equation (6) together with (2) leads to the equation

$$\theta_{tt}(x,t) - \frac{\kappa}{\tau_q}\theta_{xx}(x,t) + \frac{1}{\tau_q}\theta_t(x,t) - \kappa_1\theta_{txx}(x,t) = 0. \tag{7}$$

It is clear that if $\kappa_1 = 0$, then equation (6) reduces to the telegraph equation (5). On the other hand if $\kappa_1 = \kappa$, then equation (7) reduces to a diffusion equation

$$\phi_t(x,t) = \kappa_1 \phi_{xx}(x,t), \quad \phi(x,t) = \theta_t(x,t) + \frac{\kappa}{\tau_q \kappa_1}\theta(x,t). \tag{8}$$

Note that for τ_q sufficiently small, the Cattaneo equation (4) can be seen as a first-order approximation of a more general constitutive relation (single-phase-lagging model; Tzou [12]),

$$q(x, t+\tau_q) = -\kappa\theta_x(x,t). \tag{9}$$

Equation (9) states that the temperature gradient established at a point x at time t gives rise to a heat flux vector at x at a later time $t+\tau_q$. The delay time τ_q is interpreted as the relaxation time due to the fast-transient effects of thermal inertia (or small-scale effects of heat transport in time) and is called the phase-lag of the heat flux.

In [13], Tzou proposed a new theory of heat conduction which describes the interactions between phonons and electrons on the microscopic level as retarding sources causing a delayed response on the macroscopic scale. The physical meanings and the applicability of the dual-phase-lag model have been supported by the experimental results [14]. In this theory, the Fourier law is replaced by an approximation of the equation

$$q(x, t+\tau_q) = -\kappa\theta_x(x, t+\tau_\theta), \quad \tau_q > 0,\ \tau_\theta > 0, \tag{10}$$

where τ_q is the phase lag of the heat flux and τ_θ is the phase lag of the gradient of the temperature. According to equation (10), the temperature gradient at a point x of the material at time $t+\tau_\theta$ corresponds to the heat flux density vector at x at time $t+\tau_q$. The delay time τ_θ is interpreted as being caused by the micro-structural interactions such as phonon-electron interaction or phonon scattering, and is called the phase-lag of the temperature gradient (Tzou [13]).

If the two phase lags are equal, that is $\tau_q = \tau_\theta$, then, the relation (10) is identical with the classical Fourier law (1). While in the absence of the phase lag of the temperature gradient, $\tau_\theta = 0$ and by taking the first-order approximation for q, then equation (10) reduces to the Cattaneo law (4). In addition, for $\tau_\theta = \frac{\kappa_1}{\kappa}\tau_q$, and by taking the first order approximation for q and for θ, then we can recover Jeffreys's equation (6). Unfortunately, we cannot use the general form (10) as a more general heat conduction law since its combination with the classical energy equation (2) leads to an ill-posed problem (see [3]). However, as we have seen above, if we replace the delay expressions in (10) by their Taylor expansions at different orders, we obtain several heat conduction theories that will lead to well-posed equations.

The Cattaneo equation (4) can also be expressed as an integral over the history of the temperature gradient as

$$q(t) = -\frac{\kappa}{\tau_q}\int_{-\infty}^{t} e^{\frac{s-t}{\tau_q}}\theta_x(s)\,ds. \tag{11}$$

A more general form of equation (11) has been given by Gurtin and Pipkin [4]:

$$q(t) = -\kappa\int_{-\infty}^{t} g(t-s)\,\theta_x(s)\,ds, \tag{12}$$

where $g(s)$ is the heat flux relaxation function. In (12), the heat flux is determined by the integral over the history of the temperature gradient weighted against the relaxation function $g(s)$ called the heat flux kernel. The coupling between (12) and the energy equation (2), gives

$$\theta_t - \frac{1}{\beta}\int_0^{\infty} g(s)\theta_{xx}(t-s)\,ds = 0, \quad \beta = \frac{1}{\kappa}.$$

Many different constitutive models arise from different choices of $g(s)$. Equation (11) can easily be recovered from (12) by assuming

$$g(s) = \frac{1}{\tau_q}e^{-\frac{s}{\tau_q}}.$$

If we assume that θ_x is constant for all time and let $\kappa = \int_0^\infty g(s)\,ds$, then equation (12) reduces to the classical Fourier law (1). Also, the heat flux law of Jeffreys's type

$$q(t) = -\kappa\theta_x(t) - \frac{\kappa_1}{\tau_q}\int_{-\infty}^{t}\theta_x(s)\,e^{\frac{s-t}{\tau_q}}\,ds,$$

can be seen by letting

$$g(s) = \delta(s) + \frac{\kappa_1}{\tau_q\kappa}e^{-\frac{s}{\tau_q}}$$

in (12), where δ is the Dirac function. See [5] for more details.

So, it is a quite natural and more general to investigate the the more general Gurtin and Pipkin heat conduction law (12). In this paper, we are interested on the coupling of the Timoshenko beam equations with the Gurtin and Pipkin heat conduction. Namely, we consider the system

$$\begin{aligned} \varphi_{tt} - (\varphi_x - \psi)_x &= 0,\\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x &= 0,\\ \theta_t - \frac{1}{\beta}\int_0^{\infty} g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{tx} &= 0, \end{aligned} \tag{13}$$

where the time variable $t \in (0, \infty)$, the space variable $x \in \mathbb{R}$, a and δ are strictly positive fixed constants. The constant $\beta > 0$ is equal to $\frac{1}{\kappa}$, where κ is the thermal conductivity as defined below. The memory kernel $g(s)$ is a convex summable function on $[0, \infty)$ with total mass of

$$1 = \int_0^\infty g(s)\, ds.$$

The functions $\varphi(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ are the transverse displacement, the rotation angle of the beam and the temperature difference, respectively. The system (13) is supplied with the following initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x). \end{cases} \tag{14}$$

Before, going on, let us recall some results related to our problem. The Timoshenko–Fourier system

$$\begin{aligned} \varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x &= 0, \\ \theta_t - \theta_{xx} + \delta\psi_{tx} &= 0 \end{aligned} \tag{15}$$

was first studied by Said-Houari and Kasimov in [9] and [10], where the authors proved in [10] that the solution $W = (\varphi_t, \psi_t, a\psi_x, \varphi_x - \psi, \theta)^T$ decays with the rate:

$$\|\partial_x^k W(t)\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}} \|W_0\|_{L^1} + Ce^{-ct}\|\partial_x^k W_0\|_{L^2}, \tag{16}$$

for $a = 1$, and

$$\|\partial_x^k W(t)\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}} \|W_0\|_{L^1} + C(1+t)^{-\frac{\ell}{2}} \|\partial_x^{k+\ell} W_0\|_{L^2}, \tag{17}$$

for $a \neq 1$. While, in [9], they showed that the same decay estimates can be obtained with the optimal decay rate $(1+t)^{-\frac{1}{4}-\frac{k}{2}}$ instead of $(1+t)^{-\frac{1}{12}-\frac{k}{6}}$, provided that an additional frictional damping term of the form $\lambda\psi_t(x,t)$ is considered in the second equation of (15). They also investigated the Timoshenko–Cattaneo system

$$\begin{aligned} \varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x &= 0, \\ \theta_t + q_x + \delta\psi_{tx} &= 0, \\ \tau_q q_t + q + \theta_x &= 0. \end{aligned} \tag{18}$$

and showed in [10] that the same decay rates as (16) and (17) hold for the solution of (18), but the decay rate is controlled by a new number (found first in [11])

$$\alpha = (\tau_q - 1)(1 - a^2) - \tau_q\delta^2 \tag{19}$$

rather than by a. In fact, they proved that the estimate (16) is obtained under the assumption $\alpha = 0$ which is exactly the assumption $a = 1$ for $\tau_q = 0$ (Timoshenko–Fourier system). The decay rates in [10] has been improved recently in [7], for the Timoshenko–Fourier and in [8] for the Timoshenko–Cattaneo. In both cases, the authors showed an optimal decay rate of the L^2-norm of the solution of the form $(1+t)^{-\frac{1}{8}}$. Their method was based on refinement of the Lyapunov functionals in [10] and on the eigenvalues expansion technique.

The authors of this paper considered system (13) in [6], and proved that the same number

$$\alpha_g := \left(\frac{\beta}{g(0)} - 1\right)(1 - a^2) - \delta^2 \frac{\beta}{g(0)}, \tag{20}$$

which controls the behavior of the solution in bounded domains [2], also plays a role in unbounded situation and affects the decay rate of the solution. In addition, we showed the decay estimates of the solution for $\alpha_g = 0$ and $\alpha_g \neq 0$ and proved that the L^2-norm of the solution decays with the rate $(1+t)^{-\frac{1}{12}}$ for initial data with suitable regularity properties.

The goal of this paper is first: to improve the decay rate obtained in [6]. In fact we showed that the L^2-norm of the solution decays with the optimal rate $(1+t)^{-\frac{1}{8}}$ rather than $(1+t)^{-\frac{1}{12}}$. Second, this result extends those in [7,8] and showed that they are only particular case of the one obtained here.

This paper is organized as follows: In Section 2, we state the problem. Section 3 is devoted to the energy method in the Fourier space and to the construction of the Lyapunov functionals. In Section 4, we prove the main estimates of the solution in the energy space.

2. Statement of the problem

We are studying the Cauchy problem of the Timoshenko system with Gurtin–Pipkin heat conduction for the heat flux

$$\begin{aligned} \varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x &= 0, \\ \theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{tx} &= 0. \end{aligned} \tag{21}$$

Following [1], we introduce the new variable

$$\eta(x,t,s) = \int_0^s \theta(x, t-\sigma)\,d\sigma = \int_{t-s}^t \theta(x,\sigma)\,d\sigma, \quad s \geq 0,\ t \geq 0. \tag{22}$$

Differentiating (22) with respect to t yields that η satisfies the supplementary equation

$$\eta_t(s) = -\eta_s(s) + \theta(t), \quad \eta(0) = 0, \quad \forall\, t \geq 0, \tag{23}$$

which has to be added to system (21). Then, we define the operator $T\eta = -\eta'$. From (23), we get the following equation:

$$\eta_t = T\eta + \theta. \tag{24}$$

Also, we define $\mu(s) = -g'(s)$ and assume that μ satisfies the following two assumptions:

(M1) μ is a nonnegative nonincreasing and absolutely continuous function on $\mathbb{R}^+$ such that

$$\mu(0) = \lim_{s\to 0} \mu(s) \in (0, \infty).$$

(M2) There exists $\nu > 0$ such that the differential inequality

$$\mu'(s) + \nu\mu(s) \leq 0$$

holds for almost every $s > 0$.

With all these new variables, we rewrite system (21) as:

$$\begin{aligned} \varphi_{tt} - (\varphi_x - \psi)_x &= 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + \delta\theta_x &= 0, \\ \theta_t - \frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\, ds + \delta\psi_{tx} &= 0, \\ \eta_t &= T\eta + \theta. \end{aligned} \tag{25}$$

To rewrite the system as a first-order (with respect to t) differential system, we define new variables, as follows:

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t.$$

Hence, system (21) takes the form

$$\begin{aligned} v_t - u_x + y &= 0, \\ u_t - v_x &= 0, \\ z_t - ay_x &= 0, \\ y_t - az_x - v + \delta\theta_x &= 0, \\ \theta_t - \frac{1}{\beta}\int_0^\infty \mu(s)\eta_{xx}(s)\, ds + \delta y_x &= 0, \\ \eta_t &= T\eta + \theta. \end{aligned} \tag{26}$$

Now, we define the solution

$$U(x,t) = (v, u, z, y, \theta, \eta)^T. \tag{27}$$

Hence, the initial conditions can be written as

$$U_0(x) = U(x,0) = U_0(v_0, u_0, z_0, y_0, \theta_0, \eta_0)^T. \tag{28}$$

Now, we may define

$$g(0) = \int_0^\infty \mu(s)\, ds. \tag{29}$$

Lemma 2.1. *The following inequality holds:*

$$\left| \int_0^\infty \mu(s)\hat{\eta}(s,t)\, ds \right|^2 \leq g(0) \int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds. \tag{30}$$

Proof. We have by using Hölder's inequality

$$\begin{aligned} \left| \int_0^\infty \mu(s)\hat{\eta}(s,t)\, ds \right|^2 &= \left| \int_0^\infty (\mu(s))^{\frac{1}{2}} (\mu(s))^{\frac{1}{2}} \hat{\eta}(s,t)\, ds \right|^2 \\ &\leq \left| \left(\int_0^\infty \mu(s)\, ds \right)^{\frac{1}{2}} \left(\int_0^\infty \mu(s)(\hat{\eta}(s,t))^2 ds \right)^{\frac{1}{2}} \right|^2 \\ &= \left(\int_0^\infty \mu(s)\, ds \right) \int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds \\ &= g(0) \int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Before closing this section, we introduce the following lemma, which will be used later in the proof of our main result and it can be proved as in [6].

Lemma 2.2. *For all $k \geq 0$, $c \geq 0$, there exists a constant $C > 0$ such that for all $t \geq 0$ the following estimate holds:*

$$\int_{|\xi|\leq 1} |\xi|^k e^{-c|\xi|^4 t} d\xi \leq C(1+t)^{-\frac{k+1}{4}}, \quad \xi \in \mathbb{R}. \tag{31}$$

3. The energy method in the Fourier space

Our goal in this section is to obtain some decay estimates of the Fourier image of the energy of (26). To achieve this, we use the energy method in the Fourier space and build some appropriate Lyapunov functionals, which lead eventually to our desired estimates.

Applying the Fourier transform to (26), we get

$$\hat{v}_t - i\xi\hat{u} + \hat{y} = 0, \tag{32}$$

$$\hat{u}_t - i\xi\hat{v} = 0, \tag{33}$$

$$\hat{z}_t - ai\xi\hat{y} = 0, \tag{34}$$

$$\hat{y}_t - ai\xi\hat{z} - \hat{v} + \delta i\xi\hat{\theta} = 0, \tag{35}$$

$$\hat{\theta}_t + \frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds + \delta i\xi\hat{y} = 0, \tag{36}$$

$$\hat{\eta}_t = T\hat{\eta} + \hat{\theta}. \tag{37}$$

Together with the initial data, written in terms of the solution vector $\hat{U}(\xi,t) = (\hat{v},\hat{u},\hat{z},\hat{y},\hat{\phi},\hat{\rho},\hat{\theta},\hat{\eta})^T(\xi,t)$, as

$$\hat{U}(\xi,0) = \hat{U}_0(\xi). \tag{38}$$

The energy functional $\hat{E}(\xi,t)$ associated to the system (32)–(38) is defined as follows:

$$\hat{E}(\xi,t) = \frac{1}{2}\left\{|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2 + \frac{\xi^2}{\beta}\int_0^\infty \mu(s)|\hat{\eta}(t,s)|^2\,ds\right\}. \tag{39}$$

Lemma 3.1. *Let $(\hat{v},\hat{u},\hat{z},\hat{y},\hat{\theta},\hat{\eta})$ be the solution of* (32)–(38)*, then the energy $\hat{E}\,(\xi,t)$ given by* (39) *is a nonincreasing function and satisfies, for all $t \geq 0$,*

$$\frac{d}{dt}\hat{E}(\xi,t) = \frac{\xi^2}{2\beta}\int_0^\infty \mu'(s)|\hat{\eta}(t,s)|^2 ds. \tag{40}$$

Proof. Multiplying equation (32) by $\bar{\hat{v}}$, equation (33) by $\bar{\hat{u}}$, equation (34) by $\bar{\hat{z}}$, equation (35) by $\bar{\hat{y}}$ and equation (36) by $\bar{\hat{\theta}}$ and taking the real parts, we get

$$\frac{1}{2}\frac{d}{dt}(|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\theta}|^2) = -\operatorname{Re}\left\{\frac{\xi^2}{\beta}\bar{\hat{\theta}}(t,\xi)\int_0^\infty \mu(s)\hat{\eta}(t,s)ds\right\}. \tag{41}$$

Taking the conjugate of equation (37), then multiplying the resulting equation by $\mu(s)\hat{\eta}(\xi,t,s)$ and taking the integration with respect to s, we obtain

$$\begin{aligned}&\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_t(\xi,t,s)\,ds\\&= -\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_s(\xi,t,s)\,ds + \int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\theta}}(\xi,t)\,ds.\end{aligned}$$

Hence, we have

$$
\begin{aligned}
&-\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\theta}}(\xi,t)\,ds\right\}\\
&=-\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_t(\xi,t,s)\,ds\right\}\\
&\quad-\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_s(\xi,t,s)\,ds\right\} \qquad (42)\\
&=-\frac{1}{2}\frac{d}{dt}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)|\hat{\eta}(\xi,t,s)|^2\,ds\right\}\\
&\quad-\operatorname{Re}\left\{\frac{\xi^2}{\beta}\int_0^\infty \mu(s)\hat{\eta}(\xi,t,s)\bar{\hat{\eta}}_s(\xi,t,s)\,ds\right\}.
\end{aligned}
$$

Integrating the second term in the right-hand side of (42) by parts and using the assumption (M1) and (22), we have

$$
-\frac{\xi^2}{\beta}\operatorname{Re}\left\{\int_0^\infty \mu(s)\bar{\hat{\eta}}_s(\xi,t,s)\hat{\eta}(\xi,t,s)\,ds\right\}=\frac{\xi^2}{2\beta}\int_0^\infty \mu'(s)|\hat{\eta}(\xi,t,s)|^2 ds.
$$

Hence, collecting (41) and (42), then (40) holds. □

Proposition 3.2. *Let $\hat{U}(\xi,t)=(\hat{v},\hat{u},\hat{z},\hat{y},\hat{\theta},\hat{\eta})$ be the solution of* (32)–(38) *and*

$$
\alpha_g=\left(\frac{\beta}{g(0)}-1\right)(1-a^2)-\frac{\beta}{g(0)}\delta^2. \qquad (43)
$$

Then, there exist two positive constants, C and c, such that for all $t\geq 0$:

$$
\hat{E}(\xi,t)\leq C\hat{E}(\xi,0)e^{-c\rho(\xi)t}, \qquad (44)
$$

where

$$
\rho(\xi)=\begin{cases}\dfrac{\xi^4}{(1+\xi^2)^2}, & \text{if } \alpha_g=0,\\[2mm] \dfrac{\xi^4}{(1+\xi^2)^4}, & \text{if } \alpha_g\neq 0.\end{cases} \qquad (45)
$$

We are going to prove Proposition 3.2 by means of several lemmas. Following [10], we define the functional

$$
\begin{aligned}
\mathscr{B}_1(\xi,t)&=\operatorname{Re}\left\{-\frac{\beta}{g(0)}\hat{v}\bar{\hat{y}}-\frac{\beta}{g(0)}a\hat{u}\bar{\hat{z}}+\left(\frac{1}{\delta^2}-\frac{a^2}{\delta^2}+\frac{\beta}{g(0)}\right)\delta\bar{\hat{\theta}}\hat{u}\right\}\\
&\quad+\frac{1-a^2}{\delta g(0)}\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right).
\end{aligned} \qquad (46)
$$

Then, we have the following lemma, which has been proved in [6, Lemma 3.2].

Lemma 3.3. *The functional $\mathscr{B}_1(\xi,t)$ satisfies*

$$\begin{aligned}
&\frac{d}{dt}\mathscr{B}_1(\xi,t)-\frac{\beta}{g(0)}|\hat{y}|^2+\frac{\beta}{g(0)}|\hat{v}|^2\\
&=\frac{\alpha_g}{\delta\beta}\operatorname{Re}\Big(\xi^2\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s)ds\Big)+\alpha_g\operatorname{Re}(i\xi\bar{\hat{u}}\hat{y}) \qquad (47)\\
&\quad+\frac{1-a^2}{\delta g(0)}\operatorname{Re}\Big(i\xi\int_0^\infty\mu(s)\hat{y}\bar{\hat{\eta}}(s,t)\,ds\Big)+\frac{1-a^2}{\delta g(0)}\operatorname{Re}\Big(i\xi\int_0^\infty\mu'(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\Big),
\end{aligned}$$

where

$$\alpha_g:=\left(\frac{\beta}{g(0)}-1\right)(1-a^2)-\delta^2\frac{\beta}{g(0)}. \qquad (48)$$

Following [8], we introduce the functional

$$\mathscr{B}_2(\xi,t)=\operatorname{Re}\Big\{-\delta i\xi\hat{u}\bar{\hat{v}}-\delta\xi^2\big(\hat{v}\bar{\hat{y}}+a\bar{\hat{z}}\hat{u}\big)-\bar{\hat{\theta}}\hat{u}-i\xi\bar{\hat{\theta}}\hat{y}-(a^2-1)\xi^2\bar{\hat{\theta}}\hat{u}\Big\}. \qquad (49)$$

Then, we have the following lemma.

Lemma 3.4. *The functional $\mathscr{B}_2(\xi,t)$ satisfies*

$$\begin{aligned}
&\frac{d}{dt}\mathscr{B}_2(\xi,t)+\delta\xi^2|\hat{u}|^2\\
&=(1-a^2-\delta^2)\operatorname{Re}(i\xi^3\bar{\hat{v}}\hat{\theta})-a\operatorname{Re}(\xi^2\hat{z}\bar{\hat{\theta}})+\delta\xi^2|\hat{\theta}|^2+\operatorname{Re}\Big(\frac{\xi^2}{\beta}\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\Big) \qquad (50)\\
&\quad+(a^2-1)\operatorname{Re}\Big(\frac{\xi^4}{\beta}\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\Big)+\operatorname{Re}\Big(\frac{i\xi^3}{\beta}\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\Big).
\end{aligned}$$

Proof. Multiplying equation (32) by $-\bar{\hat{y}}$ and equation (35) by $-\bar{\hat{v}}$, adding the results and taking the real part, we have

$$-\frac{d}{dt}\operatorname{Re}(\hat{v}\bar{\hat{y}})-|\hat{y}|^2+|\hat{v}|^2=-\operatorname{Re}(i\xi\hat{u}\bar{\hat{y}})-\operatorname{Re}(ai\xi\bar{\hat{v}}\hat{z})+\operatorname{Re}(\delta i\xi\hat{\theta}\bar{\hat{v}}). \qquad (51)$$

Multiplying equation (33) by $-a\bar{\hat{z}}$ and equation (34) by $-a\bar{\hat{u}}$, adding the results and taking the real part, we find

$$-\frac{d}{dt}\operatorname{Re}(a\hat{u}\bar{\hat{z}})=-\operatorname{Re}(ai\xi\hat{v}\bar{\hat{z}})-\operatorname{Re}(a^2i\xi\bar{\hat{u}}\hat{y}). \qquad (52)$$

Multiplying equation (33) by $\delta\bar{\hat{\theta}}$ and equation (36) by $\delta\bar{\hat{u}}$, adding the results and taking the real part, we obtain

$$\frac{d}{dt}\operatorname{Re}(\delta\bar{\hat{\theta}}\hat{u})=\operatorname{Re}(\delta i\xi\bar{\hat{\theta}}\hat{v})-\operatorname{Re}\left(\frac{\delta\xi^2}{\beta}\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s)\,ds\right)-\operatorname{Re}(\delta^2i\xi\bar{\hat{u}}\hat{y}). \qquad (53)$$

Next, multiplying (32) and (33) by $-i\xi\bar{\hat{u}}$ and $i\xi\bar{\hat{v}}$, respectively. Then, adding the results and taking the real part, we obtain

$$\frac{d}{dt}\operatorname{Re}\left(i\xi\hat{u}\bar{\hat{v}}\right)+\xi^2|\hat{v}|^2=\xi^2|\hat{u}|^2+\operatorname{Re}(i\xi\bar{\hat{u}}\hat{y}). \tag{54}$$

Multiplying equation (35) and (36) by $i\xi\bar{\hat{\theta}}$ and $-i\xi\bar{\hat{y}}$, respectively. Adding the results and taking the real part, we obtain

$$\begin{aligned}&-\frac{d}{dt}\operatorname{Re}\left(i\xi\bar{\hat{\theta}}\hat{y}\right)-\operatorname{Re}\left(\frac{i\xi^3}{\beta}\bar{\hat{y}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)\\&+\delta\xi^2|\hat{y}|^2+\operatorname{Re}(a\xi^2\bar{\hat{\theta}}\hat{z})-\operatorname{Re}(i\xi\bar{\hat{\theta}}\hat{v})-\delta\xi^2|\hat{\theta}|^2=0.\end{aligned} \tag{55}$$

Now, computing $\delta\xi^2((51)+(52))-\frac{1}{\delta}\big(1+(a^2-1)\xi^2\big)(53)-\delta(54)+(55)$, then, we obtain (50). This completes the proof of Lemma 3.4. □

Now, we need to eliminate the term $(1-a^2-\delta^2)\xi^3\operatorname{Re}(i\bar{\hat{v}}\hat{\theta})$ in (50), to do so we define the functional

$$\mathscr{B}_3(\xi,t)=\mathscr{B}_2(\xi,t)+\frac{\xi^2}{g(0)}(1-a^2-\delta^2)\operatorname{Re}\left(i\bar{\hat{v}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right).$$

and we have the following lemma.

Lemma 3.5. *The functional $\mathscr{B}_3(\xi,t)$ satisfies:*

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_3(\xi,t)+\delta\xi^2|\hat{u}|^2\\&=-a\xi^2\operatorname{Re}(\hat{z}\bar{\hat{\theta}})+\delta\xi^2|\hat{\theta}|^2+\frac{\xi^3}{g(0)}(1-a^2-\delta^2)\operatorname{Re}\left(i\bar{\hat{v}}\int_0^\infty \mu'(s)\hat{\eta}(s,t)\,ds\right)\\&\quad+\frac{\xi^2}{\beta}\operatorname{Re}\left(\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)+\xi^4\frac{\alpha_g}{\beta}\operatorname{Re}\left(\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)\\&\quad+\xi^3\frac{(\alpha_g-a^2)}{\beta}\operatorname{Re}\left(i\bar{\hat{y}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right),\end{aligned} \tag{56}$$

Proof. We multiply (32) and (37) by $-i\xi\mu(s)\bar{\hat{\eta}}(s,t)$ and $i\xi\mu(s)\bar{\hat{v}}$, respectively. Adding the results and taking the integration with respect to s then, taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\operatorname{Re}\left(i\xi\bar{\hat{v}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)\\&=\operatorname{Re}\left(\xi^2\int_0^\infty \mu(s)\hat{u}\bar{\hat{\eta}}(s,t)\,ds\right)+\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{y}\bar{\hat{\eta}}(s,t)\,ds\right)\\&\quad+\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)T\hat{\eta}\bar{\hat{v}}\,ds\right)+\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{\theta}\bar{\hat{v}}\,ds\right).\end{aligned} \tag{57}$$

Furthermore, using (38), the last term in the above identity can be written as $\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\bar{\hat{\theta}}\hat{v}\,ds\right) = \mathrm{Re}\left(i\xi\bar{\hat{\theta}}\hat{v}\int_0^\infty \mu(s)\,ds\right) = g(0)\,\mathrm{Re}(i\xi\bar{\hat{\theta}}\hat{v})$. Also, an integration by parts leads to $\mathrm{Re}\left(i\xi\int_0^\infty \mu(s)T\hat{\eta}\bar{\hat{v}}\,ds\right) = \mathrm{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s,t)\bar{\hat{v}}\,ds\right)$. Hence, (57) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}\left(i\xi\bar{\hat{v}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)\\&= \mathrm{Re}\left(\xi^2\int_0^\infty \mu(s)\hat{u}\bar{\hat{\eta}}(s,t)\,ds\right) + \mathrm{Re}\left(i\xi\int_0^\infty \mu(s)\hat{y}\bar{\hat{\eta}}(s,t)\,ds\right)\\&\quad + \mathrm{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s,t)\bar{\hat{v}}\,ds\right) + g(0)\,\mathrm{Re}(i\xi\bar{\hat{\theta}}\hat{v}).\end{aligned} \tag{58}$$

Now, we compute $\frac{\xi^2}{g(0)}(1-a^2-\delta^2)$(58)+(50), then we have (56). This completes the proof of Lemma 3.5. □

Define the functional

$$\mathscr{B}_4(\xi,t) = \mathrm{Re}(i\xi\hat{y}\bar{\hat{\theta}}). \tag{59}$$

Then, we have the following lemma, where its proof has been given in [6, Lemma 3.4]

Lemma 3.6. *The functional $\mathscr{B}_4(\xi,t)$ can be estimated as follows:*

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_4(\xi,t)+(\delta-\epsilon_1)\xi^2|\hat{y}|^2\\&\le\epsilon_1'\frac{\xi^2}{1+\xi^2}|\hat{z}|^2+C(\epsilon_1')(1+\xi^2)|\hat{\theta}|^2+\epsilon_1'\xi^2|\hat{v}|^2+C(\epsilon_1)\xi^2 g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2 ds,\end{aligned} \tag{60}$$

where ϵ_1 and ϵ_1' are two arbitrary positive constants.

Following [8], we define the functional

$$\mathscr{B}_5(\xi,t) = -\delta\xi\,\mathrm{Re}\left(i\bar{\hat{y}}\hat{z}\right) + a\Big(-\xi\,\mathrm{Re}\left(i\bar{\hat{\theta}}\hat{y}\right) - \mathrm{Re}\left(\bar{\hat{\theta}}\hat{u}\right)\Big) - \frac{\delta}{a}\mathrm{Re}\left(a\bar{\hat{z}}\hat{u}\right). \tag{61}$$

Thus, we have the following lemma.

Lemma 3.7. *The functional $\mathscr{B}_5(\xi,t)$ satisfies:*

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_5(\xi,t) + \delta a\xi^2|\hat{z}|^2\\&= a\delta\xi^2|\hat{\theta}|^2 + (\delta^2-a^2)\xi^2\,\mathrm{Re}(\bar{\hat{z}}\hat{\theta}) + \frac{a\xi^3}{\beta}\mathrm{Re}\left(i\bar{\hat{y}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right)\\&\quad + \frac{a\xi^2}{\beta}\mathrm{Re}\left(\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s,t)\,ds\right).\end{aligned} \tag{62}$$

Proof. Multiplying (34) by $i\xi\bar{\hat{y}}$ and (35) by $-i\xi\bar{\hat{z}}$, we get

$$\frac{d}{dt}\operatorname{Re}\left(i\xi\bar{\hat{y}}\hat{z}\right) + a\xi^2|\hat{y}|^2 - a\xi^2|\hat{z}|^2 = -\operatorname{Re}(i\xi\hat{v}\bar{\hat{z}}) - \delta\xi^2\operatorname{Re}(i\xi\hat{\theta}\bar{\hat{z}}) \tag{63}$$

Now, computing $-\delta(63) - \frac{a}{\delta}(53) + \frac{\delta}{a}(52) + a(55)$, then (62) holds. □

As in [6], define the functional, then we have (see [6, Lemma 3.5])

$$\mathscr{B}_6(\xi,t) = -\xi^2\operatorname{Re}\left(\hat{\theta}\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\,ds\right). \tag{64}$$

Lemma 3.8. *Let $\hat{U}(\xi,t)$ be the solution of (32)–(38). Then, the functional $\mathscr{B}_6(\xi,t)$ satisfies for all $t \geq 0$,*

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_6(\xi,t) + (g(0)-\epsilon_6)\xi^2|\hat{\theta}|^2\\ &\leq \epsilon_5\frac{\xi^4}{1+\xi^2}|\hat{y}|^2 + C(\epsilon_5)(1+\xi^2)g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\\ &\quad + C(\epsilon_6)g'(0)\int_0^\infty \xi^2\mu'(s)|\hat{\eta}(s,t)|^2\,ds,\end{aligned} \tag{65}$$

where ϵ_5 and ϵ_6 are arbitrary positive constants.

3.1. Proof of Proposition 3.2. We divide the proof of Proposition 3.2 into two cases, according to the value of α_g.

Case one: $\alpha_g = 0$. Set $\alpha_g = 0$ in (47), then, we obtain

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_1(\xi,t) - \frac{\beta}{g(0)}|\hat{y}|^2 + \frac{\beta}{g(0)}|\hat{v}|^2\\ &= \frac{1-a^2}{\delta g(0)}\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\hat{y}\bar{\hat{\eta}}(s,t)\,ds\right) + \frac{1-a^2}{\delta g(0)}\operatorname{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right).\end{aligned} \tag{66}$$

Now, applying Young's inequality, we find, for any $\epsilon_3 > 0$,

$$\begin{aligned}\left|\operatorname{Re}\left(i\xi\int_0^\infty \mu(s)\bar{\hat{\eta}}(s,t)\hat{y}\,ds\right)\right| &\leq \epsilon_3|\hat{y}|^2 + C(\epsilon_3)\xi^2\left|\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right|^2\\ &\leq \epsilon_3|\hat{y}|^2 + C(\epsilon_3)g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\end{aligned}$$

and

$$\begin{aligned}\left|\operatorname{Re}\left(i\xi\int_0^\infty \mu'(s)\hat{\eta}(s)\bar{\hat{v}}\,ds\right)\right| &\leq \epsilon_3|\hat{v}|^2 + C(\epsilon_3)\xi^2\left|\int_0^\infty \mu'(s)\hat{\eta}(s)\,ds\right|^2\\ &\leq \epsilon_3|\hat{v}|^2 + C(\epsilon_3)g'(0)\int_0^\infty \xi^2\mu'(s)|\hat{\eta}(s,t)|^2\,ds.\end{aligned}$$

Hence, taking the above estimates into account, then (66) can be estimated as:

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_1(\xi,t)+\left(\frac{\beta}{g(0)}-\epsilon_3\right)|\hat{v}|^2\\&\le C(\epsilon_3)\left(|\hat{y}|^2+g'(0)\int_0^\infty\xi^2\mu'(s)|\hat{\eta}(s,t)|^2\,ds+g(0)\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\right).\end{aligned}\tag{67}$$

Also, substituting $\alpha_g=0$ in (56) we have

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_3(\xi,t)+\delta\xi^2|\hat{u}|^2\\&=-a\xi^2\operatorname{Re}(\hat{z}\bar{\hat{\theta}})+\delta\xi^2|\hat{\theta}|^2+\frac{\xi^3}{g(0)}(1-a^2-\delta^2)\operatorname{Re}\left(i\bar{\hat{v}}\int_0^\infty\mu'(s)\hat{\eta}(s,t)\,ds\right)\\&\quad+\frac{\xi^2}{\beta}\operatorname{Re}\left(\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right)-\xi^3\frac{a^2}{\beta}\operatorname{Re}\left(i\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right).\end{aligned}\tag{68}$$

Next, we define the functional

$$\mathscr{B}_7(\xi,t)=\mathscr{B}_3(\xi,t)+a\mathscr{B}_5(\xi,t).$$

Adding the above estimates (68) and (62), we get

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_7(\xi,t)+\delta\xi^2|\hat{u}|^2+\delta a^2\xi^2|\hat{z}|^2\\&=a(\delta^2-a^2-1)\xi^2\operatorname{Re}(\hat{z}\bar{\hat{\theta}})+\delta(a^2+1)\xi^2|\hat{\theta}|^2\\&\quad+\frac{\xi^3}{g(0)}(1-a^2-\delta^2)\operatorname{Re}\left(i\bar{\hat{v}}\int_0^\infty\mu'(s)\hat{\eta}(s,t)\,ds\right)\\&\quad+\frac{a^2+1}{\beta}\xi^2\operatorname{Re}\left(\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right).\end{aligned}\tag{69}$$

Applying Young's inequality, we have $|a(\delta^2-a^2-1)\operatorname{Re}(\xi^2\hat{z}\bar{\hat{\theta}})|\le\epsilon_2\xi^2\,|\hat{z}|^2+C(\epsilon_2)\xi^2|\hat{\theta}|^2$. Also, Young's inequality together with (30) gives

$$\left|\frac{a^2+1}{\beta}\operatorname{Re}\left(\xi^2\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right)\right|\le\epsilon_2'\xi^2|\hat{u}|^2+C(\epsilon_2')g(0)\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds,$$

Similarly,

$$\begin{aligned}&\left|\frac{1}{g(0)}(1-a^2-\delta^2)\operatorname{Re}\left(i\xi^3\bar{\hat{v}}\int_0^\infty\mu'(s)\hat{\eta}(s,t)\,ds\right)\right|\\&\le\epsilon_2'\frac{\xi^4}{1+\xi^2}|\hat{v}|^2+C(\epsilon_2')(1+\xi^2)g'(0)\int_0^\infty\xi^2\mu'(s)|\hat{\eta}(s,t)|^2ds.\end{aligned}$$

Inserting the above estimates into (69), then we get

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_7(\xi,t)+(\delta-\epsilon_2')\xi^2|\hat{u}|^2+(\delta a^2-\epsilon_2)\xi^2|\hat{z}|^2\\&\leq C(\epsilon_2)\xi^2|\hat{\theta}|^2+\epsilon_2'\frac{\xi^4}{1+\xi^2}|\hat{v}|^2+C(\epsilon_2')(1+\xi^2)g(0)\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2ds\\&\quad+C(\epsilon_2')(1+\xi^2)g'(0)\int_0^\infty\xi^2\mu'(s)|\hat{\eta}(s,t)|^2ds,\end{aligned}\tag{70}$$

Now, we are ready to define the Lyapunov functional

$$\begin{aligned}L_1(\xi,t)=&\frac{\gamma_1}{1+\xi^2}\mathscr{B}_7(\xi,t)+\gamma_2\frac{\xi^2}{(1+\xi^2)^2}\mathscr{B}_4(\xi,t)\\&+\frac{\gamma_3}{1+\xi^2}\mathscr{B}_6(\xi,t)+\gamma_4\frac{\xi^4}{(1+\xi^2)^2}\mathscr{B}_1(\xi,t),\end{aligned}\tag{71}$$

where $\gamma_1,\ \gamma_2,\ \gamma_3$ and γ_4 are positive constants to be fixed later.

On the other hand, assumption (M2), leads to

$$\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\leq\frac{1}{\nu}\int_0^\infty(-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds.\tag{72}$$

Consequently, taking the derivative of $L_1(\xi,t)$ with respect to t, using (60), (65), (67) and (70) and keeping in mind (72), we get

$$\begin{aligned}&\frac{d}{dt}L_1(\xi,t)+\frac{\xi^4}{(1+\xi^2)^2}\Big[(\delta-\epsilon_1)\gamma_2-\epsilon_5\gamma_3-C(\epsilon_3)\gamma_4\Big]|\hat{y}|^2\\&+\frac{\xi^4}{(1+\xi^2)^2}\Big[\Big(\frac{\beta}{g(0)}-\epsilon_3\Big)\gamma_4-\epsilon_2'\gamma_1-\epsilon_1'\gamma_2\Big]|\hat{v}|^2\\&+\frac{\xi^2}{1+\xi^2}\Big[(g(0)-\epsilon_6)\gamma_3-C(\epsilon_2)\gamma_1-C(\epsilon_1')\gamma_2\Big]|\hat{\theta}|^2\\&+\frac{\xi^2}{1+\xi^2}(\delta-\epsilon_2')\gamma_1|\hat{u}|^2+\frac{\xi^2}{1+\xi^2}\Big[(\delta a^2-\epsilon_2)\gamma_1-\epsilon_1'\gamma_2\Big]|\hat{z}|^2\\&-C_1\int_0^\infty(-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds\\&\leq 0,\end{aligned}\tag{73}$$

where C_1 is a generic positive constant that depends on ϵ_i, γ_j and ν, yet is independent on t and ξ. Now, we choose the constants in (73) very carefully in order to make all the coefficients (except the last one) in (73) positive. Let us fix $\epsilon_1,\ \epsilon_2,\ \epsilon_2',\ \epsilon_3$ and ϵ_6 small enough such that

$$\epsilon_1<\delta,\quad\epsilon_2<\delta a^2,\quad\epsilon_2'<\delta,\quad\epsilon_3<\frac{\beta}{g(0)},\quad\epsilon_6<g(0).$$

We take $\gamma_1 = 1$ and choose γ_4 large enough such that $\gamma_4 > \frac{\epsilon_2'}{\frac{\beta}{g(0)}-\epsilon_3}$. Now, take γ_2 large enough such that $\gamma_2 > \frac{C(\epsilon_3)\gamma_4}{\delta-\epsilon_1}$. Next, choose ϵ_1' small enough such that

$$\epsilon_1' < \min\left\{\frac{(\delta a^2 - \epsilon_2)}{\gamma_2}, \frac{\left(\frac{\beta}{g(0)} - \epsilon_3\right)\gamma_4 - \epsilon_2'}{\gamma_2}\right\}$$

Furthermore, we select γ_3 large enough such that $\gamma_3 > \frac{C(\epsilon_2)+C(\epsilon_3)\gamma_2}{g(0)-\epsilon_6}$. Finally, we choose ϵ_5 small enough such that

$$\epsilon_5 < \frac{(\delta - \epsilon_1)\gamma_2 - C(\epsilon_1')\gamma_4}{\gamma_3}.$$

Consequently, we deduce that there exists a positive constant $\eta_1 > 0$ such that

$$\frac{d}{dt}L_1(\xi,t) + \eta_1 Q_1(\xi,t) \le C_1 \int_0^\infty (-\mu'(s))\xi^2|\hat{\eta}(s,t)|^2\,ds, \tag{74}$$

where

$$Q_1(\xi,t) = \frac{\xi^4}{(1+\xi^2)^2}\left(|\hat{v}|^2 + |\hat{y}|^2\right) + \frac{\xi^2}{1+\xi^2}\left(|\hat{u}|^2 + |\hat{z}|^2 + |\hat{\theta}|^2\right). \tag{75}$$

It is straightforward to see that

$$Q_1(\xi,t) \ge \frac{\xi^4}{(1+\xi^2)^2}\left(|\hat{v}|^2 + |\hat{y}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{\theta}|^2\right). \tag{76}$$

Now, we define the Lyapunov functional

$$\mathscr{L}_1(\xi,t) = N\hat{E}(\xi,t) + L_1(\xi,t), \tag{77}$$

where N is a large positive constant that will be chosen later on. Using (40) and (74), the functional $\mathscr{L}_1(\xi,t)$ satisfies the estimate

$$\frac{d}{dt}\mathscr{L}_1(\xi,t) + \eta_1 Q_1(\xi,t) + \left(\frac{N}{2\beta} - C_1\right)\int_0^\infty (-\mu'(s))\xi^2|\hat{\eta}(t,s)|^2\,ds \le 0. \tag{78}$$

By choosing N large enough such that $N > 2\beta C_1$, and exploiting the estimate (72), we deduce from (39) and (76) that there exists a positive constant η_2 such that

$$\frac{d}{dt}\mathscr{L}_1(\xi,t) + \eta_2\frac{\xi^4}{(1+\xi^2)^2}\hat{E}(\xi,t) \le 0, \quad \forall t \ge 0. \tag{79}$$

Now, using (77) and (71), together with the definitions of all functionals involved in (71), we deduce that there exist two positive constants β_1 and β_2, such that, for all $t \ge 0$,

$$\beta_1\hat{E}(\xi,t) \le \mathscr{L}_1(\xi,t) \le \beta_2\hat{E}(\xi,t). \tag{80}$$

Combining (79) and (80), we find that for all $t \geq 0$, we have

$$\frac{d}{dt}\mathscr{L}_1(\xi,t) \leq -\frac{\eta_2}{\beta_2}\frac{\xi^4}{(1+\xi^2)^2}\mathscr{L}_1(\xi,t), \quad \forall t \geq 0. \tag{81}$$

Applying Gronwall's lemma and using (80) once again, then (44) holds.

Case two: $\alpha_g \neq 0$**.** In this case, we estimate the terms involving α_g in (47) as follows:

$$|\alpha_g \operatorname{Re}(i\xi\bar{\hat{u}}\hat{y})| \leq \frac{\epsilon_3'}{2}\xi^2|\hat{u}|^2 + C(\epsilon_3')|\hat{y}|^2,$$

$$\left|\frac{\alpha_g}{\delta\beta}\operatorname{Re}\left(\xi^2\bar{\hat{u}}\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right)\right| \leq \frac{\epsilon_3'}{2}\xi^2|\hat{u}|^2 + C(\epsilon_3')g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds,$$

where we have applied Young's inequality, for an arbitrary $\epsilon_3' > 0$. Hence, taking the above estimates into account, then (47) can be written as:

$$\begin{aligned}\frac{d}{dt}\mathscr{B}_1(\xi,t) + \left(\frac{\beta}{g(0)} - \epsilon_3\right)|\hat{v}|^2 &\leq \epsilon_3'\xi^2|\hat{u}|^2 + C(\epsilon_3,\epsilon_3')|\hat{y}|^2\\ &\quad + C(\epsilon_3,\epsilon_3')g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\\ &\quad + C(\epsilon_3)g'(0)\int_0^\infty \xi^2\mu'(s)|\hat{\eta}(s,t)|^2\,ds.\end{aligned} \tag{82}$$

Also, we have from [6] (see the estimate [6, (3.27)])

$$\begin{aligned}&\frac{d}{dt}\operatorname{Re}(i\xi\hat{y}\bar{\hat{\theta}}) + \delta\xi^2(|\hat{y}|^2 - |\hat{\theta}|^2) - \frac{1}{\beta}\operatorname{Re}\left(i\xi^3\bar{\hat{y}}\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right)\\ &= -\operatorname{Re}(a\xi^2\bar{\hat{\theta}}\hat{z}) + \operatorname{Re}(i\xi\bar{\hat{\theta}}\hat{v}).\end{aligned} \tag{83}$$

Applying Young's inequality, we have, for any $\epsilon_1, \epsilon_1' > 0$,

$$|\operatorname{Re}(a\xi^2\bar{\hat{\theta}}\hat{z})| \leq \epsilon_1'\frac{\xi^4}{1+\xi^2}|\hat{z}|^2 + C(\epsilon_1')(1+\xi^2)|\hat{\theta}|^2,$$

$$|\operatorname{Re}(i\xi\bar{\hat{\theta}}\hat{v})| \leq \epsilon_1'\frac{\xi^2}{1+\xi^2}|\hat{v}|^2 + C(\epsilon_1')(1+\xi^2)|\hat{\theta}|^2.$$

and

$$\left|\operatorname{Re}\left(i\frac{\xi^3}{\beta}\bar{\hat{y}}\int_0^\infty \mu(s)\hat{\eta}(s)\,ds\right)\right| \leq \epsilon_1\xi^2|\hat{y}|^2 + C(\epsilon_1)\xi^4 g(0)\int_0^\infty \mu(s)|\hat{\eta}(s,t)|^2 ds.$$

Plugging the above estimates into (83), we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{B}_4(\xi,t) + (\delta-\epsilon_1)\xi^2|\hat{y}|^2 &\leq C(\epsilon_1)\xi^2 g(0)\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\\ &\quad + C(\epsilon_1')(1+\xi^2)|\hat{\theta}|^2 + \epsilon_1'\frac{\xi^4}{1+\xi^2}|\hat{z}|^2 + \epsilon_1'\frac{\xi^2}{1+\xi^2}|\hat{v}|^2,\end{aligned} \tag{84}$$

On the other hand, we can modify the estimate (65) as follows (see the estimate [6, (3.51)]):

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_6(\xi,t)+(g(0)-\epsilon_6)\xi^2|\hat{\theta}|^2\\&\le \epsilon_5\frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2+C(\epsilon_5)g(0)(1+\xi^2)^2\int_0^\infty \xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds\\&\quad +C(\epsilon_6)g'(0)\int_0^\infty \xi^2\mu'(s)|\hat{\eta}(s,t)|^2\,ds,\end{aligned} \tag{85}$$

Finally, we define

$$\mathscr{B}_8(\xi,t)=\mathscr{B}_3(\xi,t)+\mathscr{B}_5(\xi,t).$$

Summing up (56) and (62), we get

$$\begin{aligned}&\frac{d}{dt}\mathscr{B}_8(\xi,t)+\delta\xi^2|\hat{u}|^2+\delta a\xi^2|\hat{z}|^2\\&=(\delta^2-a^2-a)\xi^2\,\mathrm{Re}(\hat{z}\bar{\hat{\theta}})+\delta(a+1)\xi^2|\hat{\theta}|^2\\&\quad+\frac{\xi^3}{g(0)}(1-a^2-\delta^2)\,\mathrm{Re}\left(i\bar{\hat{v}}\int_0^\infty\mu'(s)\hat{\eta}(s,t)\,ds\right)\\&\quad+\frac{\xi^2}{\beta}\,\mathrm{Re}\left(\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right)+\xi^4\frac{(\alpha_g+a)}{\beta}\,\mathrm{Re}\left(\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right)\\&\quad+\xi^3\frac{(\alpha_g-a^2+a)}{\beta}\,\mathrm{Re}\left(i\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right).\end{aligned} \tag{86}$$

Applying Young's inequality we have $|(\delta^2-a^2-a)\,\mathrm{Re}(\xi^2\hat{z}\bar{\hat{\theta}})|\le \epsilon_2\xi^2\,|\hat{z}|^2+C(\epsilon_2)\xi^2|\hat{\theta}|^2$, and

$$\begin{aligned}&\left|\frac{(\alpha_g-a^2+a)}{\beta}\,\mathrm{Re}\left(i\xi^3\bar{\hat{y}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right)\right|\\&\le\epsilon_2'\frac{\xi^2}{1+\xi^2}|\hat{y}|^2+C(\epsilon_2')\xi^2(1+\xi^2)g(0)\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2ds.\end{aligned}$$

Similarly,

$$\begin{aligned}&\left|\frac{\xi^2+(\alpha_g+a)\xi^4}{\beta}\,\mathrm{Re}\left(\bar{\hat{u}}\int_0^\infty\mu(s)\hat{\eta}(s,t)\,ds\right)\right|\\&\le\epsilon_2'\xi^2|\hat{u}|^2+C(\epsilon_2')(1+\xi^2)^2g(0)\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2\,ds,\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\xi^3}{g(0)}(1-a^2-\delta^2)\,\mathrm{Re}\left(i\xi^3\bar{\hat{v}}\int_0^\infty\mu'(s)\hat{\eta}(s,t)\,ds\right)\right|\\&\le\epsilon_2'\frac{\xi^2}{(1+\xi^2)}|\hat{v}|^2+C(\epsilon_2')\xi^2(1+\xi^2)g'(0)\int_0^\infty\xi^2\mu'(s)|\hat{\eta}(s,t)|^2ds.\end{aligned}$$

Using the above estimates we write

$$\begin{aligned}
&\frac{d}{dt}\mathscr{B}_8(\xi,t)+(\delta-\epsilon_2')\xi^2|\hat{u}|^2+(\delta a-\epsilon_2)\xi^2|\hat{z}|^2\\
&\leq C(\epsilon_2)\xi^2|\hat{\theta}|^2+\epsilon_2'\frac{\xi^2}{1+\xi^2}|\hat{y}|^2+\epsilon_2'\frac{\xi^2}{(1+\xi^2)}|\hat{v}|^2\\
&\quad+C(\epsilon_2')(1+\xi^2)^2g(0)\int_0^\infty\xi^2\mu(s)|\hat{\eta}(s,t)|^2ds\\
&\quad+C(\epsilon_2')\xi^2(1+\xi^2)g'(0)\int_0^\infty\xi^2\mu'(s)|\hat{\eta}(s,t)|^2ds.
\end{aligned}\tag{87}$$

Now, we define the functional

$$\begin{aligned}
L_2(\xi,t)=\lambda_1\frac{\xi^2}{(1+\xi^2)^2}\mathscr{B}_8(\xi,t)+\lambda_2\frac{\xi^2}{(1+\xi^2)^2}\mathscr{B}_4(\xi,t)\\
+\frac{\lambda_3}{1+\xi^2}\mathscr{B}_6(\xi,t)+\lambda_4\frac{\xi^4}{(1+\xi^2)^3}\mathscr{B}_1(\xi,t),
\end{aligned}\tag{88}$$

where λ_1, λ_2, λ_3 and λ_4 are positive constants to be fixed later. Taking the derivative of $L_2(\xi,t)$ with respect to t and using the inequalities (82), (84), (85) and (87), we have

$$\begin{aligned}
&\frac{d}{dt}L_2(\xi,t)+\frac{\xi^4}{(1+\xi^2)^2}\Big[(\delta-\epsilon_1)\lambda_2-\epsilon_2'\lambda_1-\epsilon_5\lambda_3-C(\epsilon_3,\epsilon_3')\lambda_4\Big]|\hat{y}|^2\\
&+\frac{\xi^4}{(1+\xi^2)^2}\Big[(\delta-\epsilon_2')\lambda_1-\epsilon_3'\lambda_4\Big]|\hat{u}|^2+\frac{\xi^4}{(1+\xi^2)^2}\Big[(\delta a-\epsilon_2)\lambda_1-\epsilon_1'\lambda_2\Big]|\hat{z}|^2\\
&+\frac{\xi^4}{(1+\xi^2)^3}\Big[\Big(\frac{\beta}{g(0)}-\epsilon_3\Big)\lambda_4-\epsilon_2'\lambda_1-\epsilon_1'\lambda_2\Big]|\hat{v}|^2\\
&+\frac{\xi^2}{1+\xi^2}\Big[(g(0)-\epsilon_6)\lambda_3-C(\epsilon_2)\lambda_1-C(\epsilon_1')\lambda_2\Big]|\hat{\theta}|^2\\
&-C(1+\xi^2)\int_0^\infty\xi^2(-\mu'(s))|\hat{\eta}(s,t)|^2\,ds\\
&\leq 0,
\end{aligned}\tag{89}$$

where we have made use of (72). Here as before, C is a generic positive constant that depends on ϵ_i, γ_j and ν, yet is independent on t and ξ.

As we did in Case one, we fix ϵ_1, ϵ_2, ϵ_2', ϵ_3 and ϵ_6 as follows:

$$\epsilon_1<\delta,\quad \epsilon_2<\delta a,\quad \epsilon_2'<\delta,\quad \epsilon_3<\frac{\beta}{g(0)},\quad \epsilon_6<g(0).$$

Also, we choose λ_i as we did for γ_i. That is we fix $\lambda_1=1$ and choose λ_4 large enough such that $\lambda_4>\frac{\epsilon_2'}{\frac{\beta}{g(0)}-\epsilon_3}$. Once λ_4 is fixed, we choose ϵ_3' small enough such that $\epsilon_3' < \frac{\delta-\epsilon_2'}{\lambda_4}$. Next, we choose λ_2 large enough such that $\lambda_2 > \frac{\epsilon_2'+C(\epsilon_3,\epsilon_3')\lambda_4}{\delta-\epsilon_1}$. Then, we fix ϵ_1' small enough such that

$$\epsilon_1' < \min\left\{\frac{\delta a^2-\epsilon_2}{\lambda_2}, \frac{\left(\frac{\beta}{g(0)}-\epsilon_3\right)\lambda_4-\epsilon_2'}{\lambda_2}\right\}.$$

Now, we select λ_3 large enough such that $\lambda_3 > \frac{C(\epsilon_2)+C(\epsilon_1')\lambda_2}{g(0)-\epsilon_6}$. Finally, we take ϵ_5 small enough such that

$$\epsilon_5 < \frac{(\delta-\epsilon_1)\lambda_2-\epsilon_3\lambda_1-C(\epsilon_3,\epsilon_3')\lambda_4}{\lambda_1}.$$

Consequently, we deduce that there exists a positive constant $\eta_3 > 0$, such that

$$\frac{d}{dt}L_2(\xi,t)+\eta_3 Q_2(\xi,t) \le C(1+\xi^2)\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(s,t)|^2\,ds, \tag{90}$$

where

$$Q_2(\xi,t) = \frac{\xi^4}{(1+\xi^2)^2}\left(|\hat{u}|^2+|\hat{y}|^2+|\hat{z}|^2\right)+\frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2+\frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2. \tag{91}$$

It is straightforward to see that

$$Q_2(\xi,t) \ge \frac{\xi^4}{(1+\xi^2)^3}\left(|\hat{u}|^2+|\hat{z}|^2+|\hat{v}|^2+|\hat{y}|^2+|\hat{\theta}|^2\right). \tag{92}$$

Now, we define the functional

$$\mathscr{L}_2(\xi,t) = M(1+\xi^2)\hat{E}(\xi,t)+L_2(\xi,t), \tag{93}$$

where M is a large positive number that will be chosen later. Using (40) and (90), the functional $\mathscr{L}_2(\xi,t)$ satisfies the estimate

$$\frac{d}{dt}\mathscr{L}_2(\xi,t)+\eta_3 Q_2(\xi,t)+\left(\frac{M}{2\beta}-C\right)(1+\xi^2)\int_0^\infty \xi^2(-\mu'(s))|\hat{\eta}(t,s)|^2\,ds \le 0. \tag{94}$$

By choosing M large enough such that $M > 2C\beta$, we deduce from (39) and (92) that for M large enough, there exists a positive constant η_4 such that

$$\frac{d}{dt}\mathscr{L}_2(\xi,t)+\eta_4\frac{\xi^4}{(1+\xi^2)^3}\hat{E}(\xi,t) \le 0, \quad \forall t \ge 0. \tag{95}$$

Now, using (93) and (88), together with the definitions of all functionals involved in (88) for all $\xi \in \mathbb{R}$, we deduce that there exist two positive constants β_3 and β_4, such that, for all $t \ge 0$,

$$\beta_3(1+\xi^2)\hat{E}(\xi,t) \le \mathscr{L}_2(\xi,t) \le \beta_4(1+\xi^2)\hat{E}(\xi,t). \tag{96}$$

Combining (95) and (96), we have

$$\frac{d}{dt}\mathscr{L}_2(\xi,t) \leq -\frac{\eta_4}{\beta_4}\frac{\xi^4}{(1+\xi^2)^4}\mathscr{L}_2(\xi,t), \quad \forall t \geq 0. \tag{97}$$

Applying Gronwall's lemma and again using (94), then (44) holds. This completes the proof of Proposition 3.2.

4. The Decay estimate

In this section, we derive the decay rates of the energy of (25).

Theorem 4.1. *Let s be a nonnegative integer, $\alpha_g = \left(\frac{\beta}{g(0)} - 1\right)(1-a^2) - \delta^2\frac{\beta}{g(0)}$ as in (48), and assume that $E_s(0)$ and $\sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}$ are bounded. Then, the energy $E_k(t)$, defined by*

$$E_k(t) = \mathcal{E}_k(t) + \frac{1}{\beta}\int_0^\infty \mu(s)\int_{\mathbb{R}} |\partial_x^k \eta(x,t,s)|^2\, dxds, \tag{98}$$

where for all $k \geq 0$,

$$\mathcal{E}_k(t) := \frac{1}{2}\int_{\mathbb{R}} \left\{(\partial_x^k\varphi_t)^2 + (\partial_x^k\psi_t)^2 + (\partial_x^k(\varphi_x - \psi))^2 + a^2(\partial_x^k\psi_x)^2\right\}(x,t)\, dx, \tag{99}$$

satisfies the following decay estimates:

- *if $\alpha_g = 0$*

$$E_k(t) \leq C\sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}(1+t)^{-\frac{1}{4}-\frac{k}{2}} + Ce^{-ct}E_k(0), \tag{100}$$

- *if $\alpha_g \neq 0$*

$$E_k(t) \leq C\sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}(1+t)^{-\frac{1}{4}-\frac{k}{2}} + C(1+t)^{-\frac{\ell}{2}}E_{k+\ell}(0), \tag{101}$$

where k and ℓ are nonnegative integers satisfying $k+\ell \leq s$ and C and c are positive constants.

Remark 4.2. We should mention here that for $\alpha_g \neq 0$, the decay rate of the high-frequency part (the last term in (101)) is slower than the one obtained in [7,8] which is $(1+t)^{-\ell}$, but since ℓ is a positive integer, then for example to get the decay estimate of $(1+t)^{-\frac{1}{8}}$ of the L^2-norm, in both situations, the initial data should be in $L^1 \cap H^1$. Thus, the faster decay rate $(1+t)^{-\ell}$ obtained in [7,8] will not give any improvement for the regularity of the initial data (at least the regularity required for the decay rate of L^2-norm and the H^1-norm of the solution). However, there is a slightly difference for the regularity assumption for some higher-order terms such as the H^2-norm of the solution.

Proof of Theorem 4.1. **Case one:** $\alpha_g = 0$. In this case, using (45), we have

$$\rho(\xi) \geq \begin{cases} c\xi^4 & \text{for } |\xi| \leq 1, \\ c & \text{for } |\xi| \geq 1. \end{cases} \tag{102}$$

Applying the Plancherel theorem together with inequality (44), we have:

$$\begin{aligned} E_k(t) &= \int_{\mathbb{R}} |\xi|^{2k} \hat{E}(\xi,t)\,d\xi \\ &\leq C \int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\,d\xi \\ &= C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\,d\xi + C \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\,d\xi \\ &= I_1(t) + I_2(t). \end{aligned}$$

Here, we split the integral into two parts, so that $I_1(t)$ is the low-frequency part where $|\xi| \leq 1$ and $I_2(t)$ is the high-frequency part where $|\xi| \geq 1$. Using the first inequality in (102), we can estimate $I_1(t)$ as:

$$\begin{aligned} I_1(t) &= C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\,d\xi \\ &\leq C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\xi^4 t} \hat{E}(\xi,0)\,d\xi \\ &\leq C \sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\} \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\xi^4 t}\,d\xi. \end{aligned} \tag{103}$$

Finally, using Lemma 2.2, we obtain

$$I_1(t) \leq C \sup_{|\xi|\leq 1}\{\hat{E}(\xi,0)\}(1+t)^{-\frac{1}{4}-\frac{k}{2}}. \tag{104}$$

Using the second inequality of (102), we can find the estimate for $I_2(t)$ as follows:

$$I_2(t) = C\int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\,d\xi = Ce^{-ct}\int_{|\xi|\geq 1} |\xi|^{2k} \hat{E}(\xi,0)\,d\xi \leq Ce^{-ct}E_k(0). \tag{105}$$

Now, a collection of the estimates (104) and (105) shows that estimate (100) holds.

Case two: $\alpha_g \neq 0$ in this case $\rho(\xi)$ can be written as

$$\rho(\xi) \geq \begin{cases} c\xi^4 & \text{for } |\xi| \leq 1, \\ c\xi^{-4} & \text{for } |\xi| \geq 1. \end{cases} \tag{106}$$

Applying again the Plancherel theorem together with inequality (44), we have

$$\begin{aligned} E_k(t) &= \int_{\mathbb{R}} |\xi|^{2k} \hat{E}(\xi,t)\, d\xi \\ &\le C \int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\, d\xi \\ &= C \int_{|\xi|\le 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\, d\xi + C \int_{|\xi|\ge 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\, d\xi \\ &= J_1(t) + J_2(t). \end{aligned}$$

As we did before, we split the integral into two parts, $J_1(t)$ the low-frequency part where $|\xi| \le 1$ and $J_2(t)$ the high-frequency part where $|\xi| \ge 1$. Using the first inequality in (106) we can estimate $J_1(t)$ exactly as in Case one. Now, $J_2(t)$ can be estimated using the second inequality in (106) as

$$\begin{aligned} J_2(t) &= C \int_{|\xi|\ge 1} |\xi|^{2k} e^{-c\rho(\xi)t} \hat{E}(\xi,0)\, d\xi \\ &\le C \int_{|\xi|\ge 1} |\xi|^{2k} e^{c\xi^{-4}t} \hat{E}(\xi,0)\, d\xi \\ &\le C \sup_{|\xi|\ge 1} \{ |\xi|^{-2\ell} e^{c\xi^{-4}t} \} \int_{\mathbb{R}} |\xi|^{2(\ell+k)} \hat{E}(\xi,0)\, d\xi \\ &\le C(1+t)^{-\frac{\ell}{2}} E_{k+\ell}(0), \end{aligned} \tag{107}$$

where we have used the estimate $\sup_{|\xi|\ge 1}\{|\xi|^{-2\ell} e^{c|\xi|^{-4}t}\} \le C(1+t)^{-\frac{\ell}{2}}$ Now, adding estimates (104) and (107), then estimate (101) holds. □

Remark 4.3. As we have said in the introduction, the Timoshenko–Fourier and Timoshenko–Cattaneo are only particular cases of the result in this paper and since it has been proved in [7, 8], by using the eigenvalues expansion, that the decay rate $(1+t)^{-\frac{1}{8}}$ is optimal. Then, it is not possible to improve the decay in this paper for initial data $U_0 \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} U_0(x)\, dx \neq 0$.

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