A Note on Leighton's Variational Lemma for Fractional Laplace Equations

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Abstract. In this note, we establish Leighton's variational lemma for fractional Laplace equations. We use the classical techniques to establish this variational lemma. We also point out several questions concerning the zeros of the solutions to fractional Laplace equations which play an important role in the qualitative theory of fractional Laplace equations.

Keywords. Fractional Laplacian, variational methods, Leighton's variational lemma

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1. Introduction

In this note, we are interested to establish Leighton's variational lemma for the fractional Laplace equation

$$(-\Delta)^s u - a(x)u = 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{in } \mathbb{R}^n \backslash \Omega, \tag{1}$$

where $a \in L^\infty(\Omega)$, Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary, $n > 2s \, (0 < s < 1)$ and $(-\Delta)^s$ stands for the fractional Laplacian. The choice of $n > 2s$ is specified later.

In the recent years, there have been a good amount of research works on the fractional Laplace equations dealing with existence, multiplicity and regularity questions but to the best of our knowledge, there are not many results available which deal with qualitative behavior of the solutions such as Sturm–Picone theorem. We refer to a very recent paper [6] which deals with qualitative behaviours of fractional equations.

Let us recall that in 1836, Sturm [22] established the first important comparison theorem which deals with a pair of linear ODEs

$$lx \equiv (p_1(t)x'(t))' + q_1(t)x(t) = 0. \tag{2}$$

$$Ly \equiv (p_2(t)y'(t))' + q_2(t)y(t) = 0, \tag{3}$$

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on a bounded interval (t_1, t_2), where p_1, p_2, q_1, q_2 are real-valued continuous functions and $p_1(t) > 0$, $p_2(t) > 0$ on $[t_1, t_2] \subset (0, \infty)$. The original Sturm's comparison theorem [22] reads as

Theorem 1.1 (Sturm's comparison theorem)**.** *Suppose* $p_1(t) = p_2(t)$ *and* $q_1(t) > q_2(t)$, $\forall t \in (t_1, t_2)$. *If there exists a nontrivial real solution* y *of* (3) *such that* $y(t_1) = 0 = y(t_2)$, *then every real solution of* (2) *has at least one zero in* (t_1, t_2).

In 1909, Picone [18] modified Sturm's theorem. The modification reads as

Theorem 1.2 (Sturm–Picone theorem)**.** *Suppose that* $p_2(t) \geq p_1(t)$ *and* $q_1(t) \geq q_2(t)$, $\forall t \in (t_1, t_2)$. *If there exists a nontrivial real solution* y *of* (3) *such that* $y(t_1) = 0 = y(t_2)$, *then every real solution of* (2) *unless a constant multiple of* y *has at least one zero in* (t_1, t_2).

In 1962, Leighton [15] proved a comparison theorem to the above pair of equations (2), (3). He showed that Sturm and Sturm–Picone theorems may be regarded as special cases of this theorem. In order to prove his theorem, he defined the quadratic functionals associated with (2) and (3) as follows:

$$j(u) \quad = \int_{t_1}^{t_2} [p_1(t)(u'(t))^2 - q_1(t)(u(t))^2] \, dt.$$
$$J(u) \quad = \int_{t_1}^{t_2} [p_2(t)(u'(t))^2 - q_2(t)(u(t))^2] \, dt,$$

where the domain D of j and J is defined to be the set of all real-valued functions $u \in C^1[t_1, t_2]$ such that $u(t_1) = u(t_2) = 0$ (t_1, t_2 are consecutive zeros of u). The variation of $j(u)$ is defined as $V(u) = J(u) - j(u)$, i.e.,

$$V(u) = \int_{t_1}^{t_2} \left[(p_2(t) - p_1(t))(u'(t))^2 + (q_1(t) - q_2(t))(u(t))^2 \right] dt. \qquad (4)$$

Now, Leighton's theorem reads as follows:

Theorem 1.3. (Leighton's theorem) Suppose there exists a nontrivial real solution u of $Lu = 0$ in (t_1, t_2) such that $u(t_1) = u(t_2) = 0$ and $V(u) \geq 0$, then every real solution of $lv = 0$ unless a constant multiple of u has at least one zero in (t_1, t_2).

It is easy to see that Theorems 1.1 and 1.2 are special cases of Leighton's theorem. We point out that the proof of Leighton's theorem heavily depends on a lemma so-called Leighton's variational lemma, which is stated as follows:

Lemma 1.4 (Leighton's variational lemma)**.** *If there exists a function* $u \in D$, *not identically zero, such that* $J(u) \leq 0$, *then every real solution of* $Lv = 0$ *except a constant multiple of* u *vanishes at some point of* (t_1, t_2).

We mention that all the above comparison theorems have been extended to a pair of linear elliptic partial differential equations of type

$$lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij} D_j u) + cu \ = 0. \tag{5}$$

$$Lv \equiv \sum_{i,j=1}^{n} D_i(A_{ij} D_j v) + Cv = 0, \tag{6}$$

in $\Omega \subset \mathbb{R}^n$, where Ω is a bounded domain with smooth boundary, a_{ij}, A_{ij}, c, C are real and continuous on $\overline{\Omega}$ and the matrices a_{ij} and A_{ij} are symmetric and positive definite in Ω.

In 1955, Hartman and Wintner [13] extended Sturm–Picone theorem (Theorem 1.2) to (5), (6) and their theorem reads as follows:

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary has a piecewise continuous unit normal. Suppose $a_{ij} - A_{ij}$ is positive semidefinite and $C \geq c$ on $\overline{\Omega}$. If there exists a nontrivial solution u of $lu = 0$ in Ω such that $u = 0$ on $\partial\Omega$, then every solution of $Lv = 0$ vanishes at some point of $\overline{\Omega}$.*

In 1965, Clark and Swanson [3] obtained a analog of Leighton's theorem (Theorem 1.3) using the variation of lu, which is defined as

$$V(u) = \int_{\Omega} \left[\sum_{i,j=1}^{n} (a_{ij} - A_{ij}) D_i u D_j u + (C - c)u^2 \right] dx.$$

Their theorem reads as follows:

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary has a piecewise continuous unit normal. Suppose $a_{ij} - A_{ij}$ is positive semidefinite and $C \geq c$ on $\overline{\Omega}$. If there exists a nontrivial solution u of $lu = 0$ in Ω such that $u = 0$ on $\partial\Omega$ and $V(u) \geq 0$, then every solution of $Lv = 0$ vanishes at some point of $\overline{\Omega}$.*

Again, it is easy to see that Theorem 1.5 is a special case of Theorem 1.6 and the proof of Theorem 1.6 depends on the following n-dimensional version of Lemma 1.4. Let us define the quadratic functional associated with (6):

$$M(u) = \int_{\Omega} \left[\sum_{i,j=1}^{n} (A_{ij} D_i u D_j u - Cu^2) \right] dx,$$

where the domain $\mathcal{D}$ of M is defined to be the set of all real-valued continuous functions on $\overline{\Omega}$ which vanish on the boundary and have uniformly continuous first partial derivatives in Ω.

Lemma 1.7 (*n*-dimensional version of Leighton's variational lemma). *If there exists $u \in \mathcal{D}$ not identically zero such that $M(u) \leq 0$, then every solution v of $Lv = 0$ vanishes at some point of $\overline{\Omega}$.*

From the above research works, it is very clear that Leighton's variational lemma plays an important role to establish and extend Sturm–Picone's comparison theorem. Motivated by the above research works and by an increasing interest on fractional Laplace equations in recent years, see for instance [2, 4, 5, 12, 16, 17, 19, 20, 27], it is natural to ask whether is there any Sturm–Picone's type theorem for a pair of fractional Laplace equations or not? To answer this question, the first step is whether one can establish Leighton's variational lemma for fractional Laplace equation. In this note, we answer this question. More precisely, we establish Leighton's variational lemma for fractional Laplace equations. Due to the nonlocal nature of the operator, it is not clear whether Sturm–Picone theorem holds for a pair of linear fractional Laplace equation or not. Since the classical proof of Sturm–Picone theorem for a pair of ordinary differential equations/systems as well as elliptic partial differential equations steadily rests on Leighton's lemma, so it has been extended in different directions, see, for instance, Jaroš et al. [11], Komkov [14], Došlý and Jaroš [7], see [26] for a generalization of Leighton's variational lemma for nonlinear differential equations and the earlier developments on this area. The results of [26] are used and extended to more general equations by A. Tiryaki [24, 25] and in his other papers and the references cited therein.

Let us recall some elementary definitions. We denote by S the Schwartz space of rapidly decreasing functions, defined as follows:

$$S = \{u \in C^{\infty}(\mathbb{R}^n) : \text{ for any } \alpha, \beta \in \mathbb{N}_0^n, \ \sup_{x \in \mathbb{R}^n} |x^{\beta} D^{\alpha} u(x)| < \infty\}.$$

Let $s \in (0, 1)$ be fixed. The fractional Laplacian of $u \in S$ is defined for $x \in \mathbb{R}^n$ as follows:

$$(-\Delta)^s u(x) = C_{n,s} \ P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = C_{n,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where

$$C_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos \zeta}{|\zeta|^{n+2s}} d\zeta \right)^{-1},$$

which is a normalization constant, see [5, Section 3].

One can also write the above singular integral as follows:

$$(-\Delta)^s u(x) = -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad \forall x \in \mathbb{R}^n, \ u \in S(\mathbb{R}^n),$$

see [5]. When $s < \frac{1}{2}$ and $u \in C^{0,\alpha}(\mathbb{R}^n)$ with $\alpha > 2s$, or if $u \in C^{1,\alpha}(\mathbb{R}^n)$, $1 + 2\alpha > 2s$, the above integral is well-defined.

Following [21], suppose X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map

$$(x, y) :\longrightarrow \frac{g(x) - g(y)}{|x - y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^{2n}\backslash(\mathcal{C}\,\Omega \times \mathcal{C}\,\Omega)),$$

where $\mathcal{C}\,\Omega = \mathbb{R}^n\backslash\Omega$. Moreover,

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n\backslash\Omega\}.$$

Let $Q = \mathbb{R}^{2n}\backslash(\mathcal{C}\,\Omega) \times (\mathcal{C}\,\Omega)$. The space X is endowed with the norm defined as

$$||g||_X = ||g||_{L^2(\Omega)} + \left(\int_Q \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}}dxdy\right)^{\frac{1}{2}}. \tag{7}$$

One can see easily that $||.||_X$ is a norm on X. Using a sort of Poincaré–Sobolev inequality for functions in X_0, one can see that

$$||u||_{X_0} = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}dxdy\right)^{\frac{1}{2}} \tag{8}$$

is a norm on X_0 and is equivalent to the norm defined in (7). Since $v \in X_0$ so $v = 0$ a.e. in $\mathbb{R}^n\backslash\Omega$ and therefore the integral in (8) can be extended to all $\mathbb{R}^{2n}$. It is easy to see that $(X_0, ||.||_{X_0})$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{X_0} := \int\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dxdy.$$

The plan of this note is as follows. We recall and prove some preparatory results in Section 1. Section 2 deals with the proof of Leighton's variational lemma which is followed by a few remarks and questions.

The following is an embedding lemma which is used further.

Lemma 1.8. *If $v \in X$, then $v \in H^s(\Omega)$. Moreover,*

$$||v||_{H^s(\Omega)} \leq ||v||_X.$$

Also, if $v \in X_0$, then $v \in H^s(\mathbb{R}^n)$ and

$$||v||_{H^s(\Omega)} \leq ||v||_{H^s(\mathbb{R}^n)} \leq ||v||_X.$$

For a proof of the above lemma, we refer the reader to [20]. The next theorem deals with the embedding of fractional Sobolev space. One can see the paper of Di Nezza et al. [5] for the complete details.

Theorem 1.9 ([5]). *The following embeddings are continuous:*

(1) $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, $2 \le q \le \frac{2n}{n-2s}$,   *if* $n > 2s$,

(2) $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, $2 \le q \le \infty$,   *if* $n = 2s$,

(3) $H^s(\mathbb{R}^n) \hookrightarrow C_b^j(\mathbb{R}^n)$,   *if* $n < 2(s-j)$.

Moreover, for any $R > 0$ *and any* $p \in [1, 2_*(s))$, $2_*(s) = \frac{2n}{n-2s}$, *the embedding* $H^s(B_R) \hookrightarrow\hookrightarrow L^p(B_R)$ *is compact, where*

$$C_b^j(\mathbb{R}^n) = \{u \in C^j(\mathbb{R}^n) : D^k u \text{ is bounded on } \mathbb{R}^n \text{ for } |k| \le j\}.$$

Next, we deal with the following important inequality which is obtained using the so-called Picone's inequality. This is given partly in [8]. Since it is short and interesting, so we reproduce it here.

Lemma 1.10. *For* $u, v \in X_0$ *with* $v > 0$ *a.e. in* $\mathbb{R}^n$, *we have*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v . (-\Delta)^{\frac{s}{2}} \left(\frac{u^2}{v}\right) dx \le \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx \tag{9}$$

and equality holds if and only if u *is a constant multiple of* v.

Proof. We use the following simple inequality to establish (9). This inequality is called Picone's inequality, see Lemma 6.2 for $p = 2$ in [1], see also [9].
For any $a, b, c, d \in \mathbb{R}$, $c > 0$, $d > 0$, we have

$$(c - d)\left(\frac{a^2}{c} - \frac{b^2}{d}\right) \le (a - b)^2 \tag{10}$$

and equality holds if and only if $bc = ad$. The above inequality is also true even when $c < 0$, $d < 0$.

By definition, we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v . (-\Delta)^{\frac{s}{2}} \left(\frac{u^2}{v}\right) dx$$

$$= \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[v(x) - v(y)]\left(\frac{(u(x))^2}{v(x)} - \frac{(u(y))^2}{v(y)}\right)}{|x-y|^{n+2s}} dx dy$$

$$\le \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|(u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy \quad (\text{by } (10))$$

$$= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx.$$

This completes the proof. □

Remark 1.11. We remark that if $u \equiv 0$ equality trivially holds in (9) but v is not necessarily a multiple of u. This says that the statement "u is a constant multiple of v" in Lemma 1.10 is not equivalent to "v is a constant multiple of u".

2. Leighton's variational lemma

In this section, we prove Leighton's variational lemma. In order to do so, we associate a "quadratic functional", namely, $j(u)$ corresponding to (1), which is defined as follows:

$$j(u) = \int_{\mathbb{R}^n} \left[|(-\Delta)^{\frac{s}{2}} u|^2 - a(x)u^2 \right] dx, \quad u \in X_0.$$

It is easy to see that $j(u)$ is nothing but the second variation of the energy functional

$$E : X_0 \longrightarrow \mathbb{R}$$

defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{2} \int_{\Omega} a(x)u^2 dx$$

associated with (1). If $u \in X_0$ satisfies $j(u) \leq 0$, it simply says that u is a local maxima of E. Let u be an unstable solution of (1), then it will satisy $j(u) \leq 0$, see [10].

Theorem 2.1 (Weaker version of Leighton's variational lemma). *Let $2s < n < 4s$ $0 < s < 1$. If there exists a function $u \in X_0$ such that $j(u) < 0$, then every solution v of*

$$(-\Delta)^s v = a(x)v \quad in \ \Omega; \quad v = 0 \ in \ \mathbb{R}^n \backslash \Omega, \ a \in L^{\infty}(\Omega) \qquad (11)$$

vanishes at some point of Ω.

Proof. Since $v \in X_0$ is a solution to (11) so it satisfies

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v . (-\Delta)^{\frac{s}{2}} \phi \, dx = \int_{\Omega} a(x)v\phi \, dx, \quad \forall \, \phi \in X_0. \qquad (12)$$

Suppose we assume that $v \neq 0$ in Ω. By Lemma 1.8 and Theorem 1.9, we have $v \in L^2(\mathbb{R}^n)$. Since $a \in L^{\infty}(\Omega)$, so by [19, Proposition 1.4], we have $v \in C^{\beta}(\mathbb{R}^n) \cap X_0$, $0 < \beta < 1$ and, in particular, v is bounded. Without any loss of generality, we may assume that $v > 0$ in Ω (the case when $v < 0$ in Ω can be dealt similarly).

Since $u \in X_0$, again using the similar arguments as above, we have $u \in C^{\gamma}(\mathbb{R}^n) \cap X_0$, $0 < \gamma < 1$ and, in particular, u is bounded. Since $v = 0$ in $\mathbb{R}^n \backslash \Omega$, so we cannot take $\frac{u^2}{v}$ as a test function in (12). Now for any given small $\epsilon > 0$, it is easy to see that $\frac{u^2}{v+\epsilon} \in X_0$ and let us take $\phi = \frac{u^2}{v+\epsilon}$ as a test function in (12), which yields

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} v . (-\Delta)^{\frac{s}{2}} \frac{u^2}{v+\epsilon} dx = \int_{\Omega} a(x)v \frac{u^2}{v+\epsilon} dx,$$

which is same as

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}}\frac{u^2}{v+\epsilon}dx = \int_\Omega a(x)v\frac{u^2}{v+\epsilon}dx,$$

i.e.,

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}}\frac{u^2}{v+\epsilon}dx - \int_\Omega a(x)v\frac{u^2}{v+\epsilon}dx \\
&\leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}u|^2 dx - \int_\Omega a(x)v\frac{u^2}{v+\epsilon}dx \quad \text{(from Lemma 1.10)} \\
&= j(u) \quad \text{(letting } \epsilon \to 0 \text{ in the above inequality)},
\end{aligned}
$$

which is a contradiction and this completes the proof. $\qquad\square$

Remark 2.2. In order to obtain the L^∞ regularity of u, in view of [19, Proposition 1.4(iii)], we have assumed that $2s < n < 4s$.

Next, the assumption $j(u) < 0$ in the above theorem is weakened to $j(u) \leq 0$ and we state and prove a stronger version of Leighton's variational lemma.

Theorem 2.3 (Stronger version of Leighton's variational lemma). *Let $2s < n < 4s$, $0 < s < 1$. If there exists a function $u \in X_0$ not identically zero such that $j(u) \leq 0$, then every solution v of (11) except a constant multiple of u vanishes at some point of Ω.*

Proof. Since $v \in X_0$ is a solution to (11) so it satisfies

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}v.\,(-\Delta)^{\frac{s}{2}}\phi\,dx = \int_\Omega a(x)v\phi\,dx, \quad \forall\,\phi \in X_0. \tag{13}$$

Suppose we assume that $v \neq 0$ in Ω. By Lemma 1.8 and Theorem 1.9, we have $v \in L^2(\mathbb{R}^n)$. Since $a \in L^\infty(\Omega)$, so by [19, Proposition 1.4], we have $v \in C^\beta(\mathbb{R}^n) \cap X_0$, $0 < \beta < 1$ and, in particular, v is bounded. Without any loss of generality, we may assume that $v > 0$ in Ω (the case when $v < 0$ in Ω can be dealt similarly).

Since $u \in X_0$, again using the similar arguments as above, we have $u \in C^\gamma(\mathbb{R}^n) \cap X_0$, $0 < \gamma < 1$ and, in particular, u is bounded. Since $v = 0$ in $\mathbb{R}^n \backslash \Omega$, so we can not take $\frac{u^2}{v}$ as a test function in (13). Now for any given small $\epsilon > 0$, it is easy to see that $\frac{u^2}{v+\epsilon} \in X_0$ and let us take $\phi = \frac{u^2}{v+\epsilon}$ as a test function in (13), which yields

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}v.\,(-\Delta)^{\frac{s}{2}}\frac{u^2}{v+\epsilon}dx = \int_\Omega a(x)v\frac{u^2}{v+\epsilon}dx,$$

which is same as

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}}\frac{u^2}{v+\epsilon}dx = \int_\Omega a(x)v\frac{u^2}{v+\epsilon}dx,$$

i.e.,

$$0 = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}} \frac{u^2}{v+\epsilon} dx - \int_{\Omega} a(x)v \frac{u^2}{v+\epsilon} dx. \qquad (14)$$

Now, from Lemma 1.10, we have

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}} \frac{u^2}{v+\epsilon} dx \le \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}u|^2 dx.$$

We claim that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}} \frac{u^2}{v+\epsilon} dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}u|^2 dx. \qquad (15)$$

Suppose, if not, then by an application of Lemma 1.10, we have $u = c_\epsilon(v+\epsilon)$ for some $c_\epsilon \in \mathbb{R}$. From this, if $v \not\equiv 0$, fixed $p \in \Omega \cap \{v \neq 0\}$ and one can see that $c_\epsilon = \frac{u_p}{v(p)+\epsilon}$, which implies that c_ϵ converges to some $c_0 \in \mathbb{R}$ as $\epsilon \to 0$, and then $u = c_0 v$. We note that $c_0 \neq 0$, because if $c_0 = 0$, then $u \equiv 0$, which is a contradiction as we have assumed that $u \not\equiv 0$ and therefore $v = \frac{1}{c_0}u$, which is not possible and therefore (15) holds. Now, (14) and (15) yield that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}(v+\epsilon).\,(-\Delta)^{\frac{s}{2}} \frac{u^2}{v+\epsilon} dx - \int_{\Omega} a(x)v \frac{u^2}{v+\epsilon} dx \\
&< \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}u|^2 dx - \int_{\Omega} a(x)v \frac{u^2}{v+\epsilon} dx \\
&= j(u) \quad (\text{letting } \epsilon \to 0 \quad \text{in the above inequality}),
\end{aligned}$$

which is a contradiction and this completes the proof. $\qquad \square$

The following remarks and questions are in order:

Remark 2.4. In the present remark, we show that Theorem 2.3 does not hold when the condition $v = 0$ in $\mathbb{R}^n\backslash\Omega$ is replaced by the more classical $v = 0$ on $\partial\Omega$. This shows a concrete difference with respect to the classical case. We show this by constructing a counterexample.

Let us take $n = 1$, $\Omega = (-1,1)$ and a be the first eigenvalue of the fractional Laplacian with homogeneous data. Let $u \in X_0$ be the first eigenfunction associated with a, i.e.,

$$\left(-\frac{d^2}{dx^2}\right)^s u = au \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}\backslash\Omega.$$

It is easy to check that $j(u) = 0$. Let v be the minimizer of j with the condition that $v = v_0$ outside $(-1,1)$ with

$$v_0(x) = \begin{cases} 1, & \text{if } |x| \in (3,4), \\ 0, & \text{otherwise.} \end{cases}$$

Such v satisfies the equation in $(-1, 1)$ and we have $v_0(\pm 1) = 0$, (i.e., $v = 0$ on $\partial\Omega$). Since v is a minimizer of j so v does not vanish in Ω. Nevertheless, it is not possible that v is a multiple of u, otherwise, the difference $w = v - u$ would satisfy $\left(-\frac{d^2}{dx^2}\right)^s u = 0$ in $(-1, 1)$ with w vanishing in $\mathbb{R}\backslash((-1, 1)\cup(3, 4)\cup(-4, -3))$ and w positive in $(3, 4)\cup(-4, -3)$. But then, calculating $\left(-\frac{d^2}{dx^2}\right)^s w(0)$, one can see that

$$0 = P.V. \int_{\mathbb{R}} \frac{w(0) - w(y)}{|y|^{1+2s}} dy = -P.V. \int_{\mathbb{R}} \frac{w(y)}{|y|^{1+2s}} dy < 0,$$

which is a contradiction.

Remark 2.5. It will be of interest to get a nonlinear version of Theorem 2.3 for a class of equations

$$(-\Delta)^s u - a(x)f(u) = 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{in } \mathbb{R}^n\backslash\Omega$$

under certain conditions on a and f.

Remark 2.6. Let us consider a pair of fractional Laplace equations:

$$lu \equiv (-\Delta)^s u - a(x)u = 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{in } \mathbb{R}^n\backslash\Omega. \tag{16}$$
$$Lv \equiv (-\Delta)^s v - b(x)v = 0 \quad \text{in } \Omega; \quad v = 0 \quad \text{in } \mathbb{R}^n\backslash\Omega, \tag{17}$$

where $a, b \in L^\infty(\Omega)$ and l and L are fractional partial differential operators, where Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary, $n > 2s$ $(0 < s < 1)$. Corresponding to (16) and (17), one can associate the "quadratic functionals" $j(u)$ and $J(u)$, respectively, for $u \in X_0$ defined as follows:

$$j(u) = \int_{\mathbb{R}^n} \left[|(-\Delta)^{\frac{s}{2}}u|^2 - a(x)u^2\right] dx,$$
$$J(u) = \int_{\mathbb{R}^n} \left[|(-\Delta)^{\frac{s}{2}}u|^2 - b(x)u^2\right] dx.$$

One can also define the variation $V(u) = J(u) - j(u) = \int_{\mathbb{R}^n}(a(x) - b(x))u^2 dx$. It is not clear whether using Theorem 2.3, the classical Sturm's comparison theorem holds for (16) and (17) or not. It would be of much interest to answer this question.

Remark 2.7. It would be interesting to check that Theorem 2.3 can be generalized to equations involving more general non-local operators than fractional Laplacian, like,

$$L_M u(x) = -\frac{1}{2}\int_{\mathbb{R}^n} [u(x + y) + u(x - y) - 2u(x)]M(y)\, dy, \quad x \in \mathbb{R}^n,$$

where $M : \mathbb{R}^n \backslash \{0\} \longrightarrow (0, \infty)$ is a function such that $kM \in L^1(\mathbb{R}^n)$, $k(x) = \min\{|x|^2, 1\}$, and there exists $\theta > 0$ such that

$$M(x) \geq \theta |x|^{-(n+2s)}, \quad s \in (0, 1), \ x \in \mathbb{R}^n \backslash \{0\}, \quad M(x) = M(-x),$$

see [2, 4] and the reference therein dealing with such fractional operators. A typical example of M is $M(x) = |x|^{-(n+2s)}$ and in this case $L_M = (-\Delta)^s$, the fractional Laplacian.

In view of previous remark, again it is not clear whether classical Sturm's comparison theorem holds for a pair of such kind of equations or not.

Remark 2.8. Let $\Omega_a = \{x \in \mathbb{R}^n : |x - x_0| < a\}$ denote the n-disc in $\mathbb{R}^n$, where x_0 is a fixed point in $\mathbb{R}^n$ and a is a positive number. Let Ω'_a denote the complement of Ω_a in $\mathbb{R}^n$. Following the definition of [23, p. 205], a function u is said to be oscillatory in $\mathbb{R}^n$ whenever u has a zero in $\mathbb{R}^n \cap \Omega'_a$ for all $a > 0$. In the context of fractional Laplacian, due to the nonlocal (global) nature of the operator, it is not adequate to prescribe the boundary values only on the boundary $\partial \Omega$ but we declare u to be zero in whole complement $\mathbb{R}^n \backslash \Omega$, see [16, p. 801] and many other papers. Now in view of the above, we say that a solution u of (1) is oscillatory if u has a zero in $\Omega \cap \Omega'_a$ for all $a > 0$, $\Omega'_a \subset \Omega$. It would be of interest to find out some sufficient conditions for the oscillation of every solution of (1). In that case, we call (1) as an oscillatory equation.

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