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Positive Solution of Lighthill-Type Equations

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Abstract. We study a unique solvability of Volterra integral equations of Lighthill's type subject to a positive solution on $[0, \infty)$. Also the asymptotics of the solution at the infinity is examined.

Keywords. Nonlinear equations, Volterra equations on semiaxis, positive solution, Lighthill's equation

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1. Introduction

In the present article we study the Lighthill-type Volterra integral equation

$$u(t) + \lambda \int_0^t (t-s)^{\alpha-1} s^\beta K(t,s) u^\mu(s) ds = f(t), \quad 0 < t < \infty, \tag{1}$$

depending on the real parameters $\lambda > 0$, $\mu \geq 1$, $0 < \alpha < 1$, $\beta > -1$. Assuming that the free term $f(t)$ and the coefficient function $K(t,s)$ are non-negative, we are interested in the existence and uniqueness of a non-negative solution $u_\star(t)$, $0 \leq t < \infty$, and in the behaviour of $u_\star(t)$ as $t \to \infty$.

A more general Volterra integral equation

$$u(t) + \lambda \int_0^t (t^\varrho - s^\varrho)^{\alpha-1} s^\beta K(t,s) u^\mu(s) ds = f(t), \quad 0 < t < \infty, \tag{2}$$

with $\varrho > 0$, obtains after the change of variables $t = \tau^{\frac{1}{\varrho}}$, $s = \sigma^{\frac{1}{\varrho}}$ the form of equation (1):

$$\widetilde{u}(\tau) + \frac{\lambda}{\varrho} \int_0^\tau (\tau - \sigma)^{\alpha-1} \sigma^{\beta'} \widetilde{K}(\tau, \sigma) \widetilde{u}^\mu(\sigma) d\sigma = \widetilde{f}(\tau), \quad 0 < \tau < \infty, \tag{3}$$

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where

$$\beta' = \tfrac{\beta+1}{\varrho} - 1, \quad \widetilde{u}(\tau) = u\big(\tau^{\frac{1}{\varrho}}\big) = u(t), \quad \widetilde{f}(\tau) = f\big(\tau^{\frac{1}{\varrho}}\big) = f(t), \\ \widetilde{K}(\tau,\sigma) = K\big(\tau^{\frac{1}{\varrho}}, \sigma^{\frac{1}{\varrho}}\big) = K(t,s). \tag{4}$$

This enables to reformulate results about equation (1) for equation (2).

Equation (2) is often met with the exponent $\alpha - 1 = -\frac{1}{\varrho}$, then $\varrho = \frac{1}{1-\alpha} > 1$. As well known, also the original Lighthill problem [5]

$$w^4(x) + \frac{1}{2}x^{-\frac{1}{2}} \int_0^x \left(x^{\frac{3}{2}} - \xi^{\frac{3}{2}}\right)^{-\frac{1}{3}} w'(\xi)d\xi = 0, \quad x > 0, \quad w(0) = 1, \tag{5}$$

can be transformed to an equation of type (1). Namely, with the help of the change of variables $x = t^{\frac{2}{3}}$, $\xi = s^{\frac{2}{3}}$, $u(t) = w\big(t^{\frac{2}{3}}\big)$, problem (5) takes the form

$$2t^{\frac{1}{3}}u^4(t) + \int_0^t (t-s)^{-\frac{1}{3}}u'(s)ds = 0, \quad u(0) = 1,$$

or

$$2t^{\frac{1}{3}}u^4(t) + \Gamma\left(\tfrac{2}{3}\right)(J^{\frac{2}{3}}u')(t) = 0, \quad u(0) = 1, \tag{6}$$

where

$$(J^\alpha v)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}v(s)ds, \quad t \ge 0, \quad \alpha > 0,$$

is the Riemann–Liouville operator having the properties (see e.g. [3])

$$(J^1 v)(t) = \int_0^t v(s)ds, \quad J^\alpha J^\beta = J^\beta J^\alpha = J^{\alpha+\beta} \quad \text{for } \alpha > 0,\ \beta > 0.$$

Applying $J^{\frac{1}{3}}$ to both sides of the equation in (6) we rewrite the problem (5) in the form

$$2\int_0^t (t-s)^{-\frac{2}{3}}s^{\frac{1}{3}}u^4(s)ds + \Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right)\left(J^1u'\right)(t) = 0, \quad u(0) = 1,$$

or

$$u(t) + \frac{\sqrt{3}}{\pi}\int_0^t (t-s)^{-\frac{2}{3}}s^{\frac{1}{3}}u^4(s)ds = 1, \quad t > 0,$$

since $\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right) = \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}$, $(J^1u')(t) = u(t) - u(0) = u(t) - 1$. Thus the Lighthill problem (5) is equivalent to equation (1) with

$$\lambda = \frac{\sqrt{3}}{\pi}, \quad \mu = 4, \quad \alpha = \beta = \frac{1}{3}, \quad f(t) \equiv 1, \quad K(t,s) \equiv 1. \tag{7}$$

The case $f(t) \equiv 1$, $K(t,s) \equiv 1$ in equation (1) gets a special interest.

2. Formulation of main results and comments

2.1. Positive solution to equation (1). Our basic results are formulated in Theorems 2.1 and 2.2. To characterise the monotony properties of f and K, we fix a further parameter $\gamma \in \mathbb{R}$. Introduce the following conditions:

(i) $\lambda > 0$, $\mu \geq 1$, $0 < \alpha < 1$, $\beta > -1$;

(ii) $-\infty < \gamma \leq 1 - \alpha$, $\mu\gamma < 1 + \beta$, $(\mu - 1)\gamma < \alpha + \beta$;

(iii) $f(t) \geq 0$ for $t > 0$, and $g(t) := t^\gamma f(t) \geq 0$ is continuous and monotone increasing for $t \geq 0$ (in particular, a finite limit $g(0) := \lim_{t\to 0} t^\gamma f(t)$ exists);

(iv) $K(t,s) \geq 0$ is continuous for $0 \leq s \leq t < \infty$, and $t^{\gamma+\alpha-1}K(t,s)$ is monotone decreasing w.r.t. t: $t_1^{\gamma+\alpha-1}K(t_1,s) \geq t_2^{\gamma+\alpha-1}K(t_2,s)$ for $0 \leq s \leq t_1 < t_2$;

(v) $f(t) > 0$ and $f(t)$ is continuously differentiable for $t > 0$;

(vi) $K(t,s) > 0$ for $0 < s < t < \infty$, and $K(t,s)$ is continuously differentiable as a function of two variables for $0 < s \leq t < \infty$.

Under conditions (i)–(iv), equation (1) takes with respect to $v(t) = t^\gamma u(t)$ the form

$$v(t) + \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v^\mu(s) ds = g(t), \quad 0 \leq t < \infty. \tag{8}$$

In the particular case $\gamma = 0$, equations (1) and (8) coincide, $f = g$, $u = v$, and conditions (i), (ii) take the form $\lambda > 0$, $\mu \geq 1$, $0 < \alpha < 1$, $\alpha + \beta > 0$.

For $\mu \notin \mathbb{N} := \{1, 2, \ldots\}$, only non-negative functions u and v have sense in equations (1) and (8); we are interested in non-negative solutions for any $\mu \geq 1$. Denote

$$C^+[0,\infty) = \{v \in C[0,\infty) : v(t) \geq 0 \text{ for } t \geq 0\},$$
$$C^+(0,\infty) = \{u \in C(0,\infty) : u(t) \geq 0 \text{ for } t > 0\}.$$

Theorem 2.1. *Assume* (i)–(iv). *Then equation* (8) *possesses a unique solution* $v_\star \in C^+[0,\infty)$*, and equation* (1) *possesses a unique solution* $u_\star \in C^+(0,\infty)$ *with* $t^\gamma u_\star(t)$ *in* $C^+[0,\infty)$*. It holds that* $0 \leq v_\star(t) \leq g(t)$ *for* $t \geq 0$*,* $v_\star(0) = g(0)$*, and* $0 \leq u_\star(t) \leq f(t)$ *for* $t > 0$*; if* $f \in C[0,\infty)$ *and* $\alpha+\beta > 0$ *then* $u_\star \in C[0,\infty)$*,* $u_\star(0) = f(0)$*.*

Under conditions (i)–(vi)*, the strict inequalities* $0 < v_\star(t) < g(t)$ *and* $0 < u_\star(t) < f(t)$ *hold for* $t > 0$*.*

Case $\mu = 1$ is special, then equation (8) is linear and under conditions (i)–(iv) the solution $v_\star$ is unique not only in $C^+[0,\infty)$ but also in $C[0,\infty)$; similarly, $u_\star$ is a unique solution of (1) in the class of all functions $u \in C(0,\infty)$ with $t^\gamma u(t)$ in $C[0,\infty)$.

Note that for $\gamma > 0$ condition (iii) does not exclude the possibility that $f(t) \to 0$ as $t \to \infty$, and then $u_\star(t) \to 0$ as $t \to \infty$. In certain cases, the behaviour of $u_\star(t)$ for large t can be determined more precisely:

Theorem 2.2. *Let condition* (i) *be fulfilled, and let* $K(t,s) \equiv 1$, $f(t) \equiv bt^r$ *with* $b = \mathrm{const} > 0$ *and* $r \in \mathbb{R}$ *such that*

$$r \geq \alpha + \beta - \mu(1-\alpha), \quad r > -(1-\alpha), \quad (\mu-1)r + \alpha + \beta > 0 \tag{9}$$

(*so in case* $\mu = 1$ *we assume in particular that* $\alpha + \beta > 0$).

Then also conditions (ii)–(vi) *are fulfilled for* $\gamma := \frac{\alpha+\beta-r}{\mu}$, *and by Theorem* 2.1, *equation* (1) *possesses a unique solution* $u_\star \in C^+(0,\infty)$ *with* $t^\gamma u_\star(t)$ *in* $C^+[0,\infty)$. *Moreover, for* $t > 0$ *it holds that*

$$u_\star(t) \leq c_r t^{-\gamma} = c_r t^{\frac{r-(\alpha+\beta)}{\mu}}, \quad c_r = \left(\frac{b}{\lambda B(\alpha,\, 1-\alpha+r)}\right)^{\frac{1}{\mu}}, \tag{10}$$

where $B(\alpha,\alpha') = \int_0^1 (1-x)^{\alpha-1} x^{\alpha'-1} dx$, $\alpha, \alpha' > 0$, *is the Euler beta function. Estimate* (10) *is unimprovable for large* t *in the following sense: for any* $\theta < 1$ *and any* $T \gg 0$, *there exists a* $t_{\theta,T} \geq T$ *such that* $u_\star(t_{\theta,T}) > \theta c_r t_{\theta,T}^{-\gamma}$.

If (9) *and* $r < \alpha + \beta$ *hold, then* $u_\star(t) \to 0$ *as* $t \to \infty$, *due to* (10).

If $\alpha + \beta > 0$ *and* $r \geq \max\{0,\, \alpha+\beta-\mu(1-\alpha)\}$, *then* (9) *is fulfilled, and by Theorem* 2.1, $u_\star \in C[0,\infty)$, $u_\star(0) = b$ *if* $r = 0$, $u_\star(0) = 0$ *if* $r > 0$.

In case $f(t) \equiv 1$ the representation $f(t) = bt^r$ holds with $b = 1$, $r = 0$, $B(\alpha,\, 1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}$, and Theorem 2.2 yields the following

Corollary 2.3. *Assume that* $f(t) \equiv 1$, $K(t,s) \equiv 1$, $\lambda > 0$, $\mu \geq 1$, $0 < \alpha < 1$ *and*

$$0 < \alpha + \beta \leq \mu(1-\alpha). \tag{11}$$

Then equation (1) *possesses a unique solution* $u_\star \in C^+[0,\infty)$. *It holds* $u_\star(0) = 1$, $0 < u_\star(t) < 1$ *for* $t > 0$, *and* $u_\star(t) \to 0$ *as* $t \to \infty$ *with the following unimprovable for large* t *estimate:*

$$u_\star(t) \leq \left(\frac{\sin(\alpha\pi)}{\lambda\pi}\right)^{\frac{1}{\mu}} t^{-\frac{\alpha+\beta}{\mu}}. \tag{12}$$

For the Lighthill data (7), condition (11) is fulfilled, and the unimprovable for large t estimate (12) takes the form

$$u_\star(t) \leq \left(\frac{1}{2}\right)^{\frac{1}{4}} t^{-\frac{1}{6}} \approx 0.841\, t^{-\frac{1}{6}}, \quad 0 < t < \infty,$$

so the convergence $u_\star(t) \to 0$ as $t \to \infty$ is slow. Lighthill [5] presents asymptotic expansions of the solution to problem (5) for $x = 0$ and for large x; the latter one is extremely effective but assumes certain rather strong regularity of the solution for large x not justified in [5] analytically. See also the asymptotic expansions [8] for $t = 0$ of the solution to (1) with $f(t) \equiv 1$, $K(t,s) \equiv 1$ in cases $\mu = 2$ and $\mu = 4$.

Under assumptions of Corollary 2.3, asymptotic expansions of the solution $u_\star(t)$ to equation (1) for large t are of greatest interest in numerical solving the equation. The justification of those is a challenging open problem, it needs a further study of the decay properties of $u_\star(t)$ and its derivatives for large t.

The proofs of Theorems 2.1 and 2.2 are presented in Sections 3 and 4, respectively. The last Section 5 is devoted to a further open problem concerning the monotone decrease of the solution $u_\star(t)$ to equation (1) under assumptions of Corollary 2.3.

2.2. Comments. We quote some known results in the direction of Theorem 2.1 and comment on conditions (i)–(iv).

1. In [4] and in [1, Section 6.4.2], the Volterra–Hammerstein integral equation

$$u(t) + \int_0^t k(t-s)G(s,u(s))ds = f(t), \quad t > 0,$$

is examined; see also [6]. Below we analyse the relations between our Theorem 2.1 and [1, Theorem 6.4.2], in which the existence of a continuous solution $u_\star(t)$, $0 \le u_\star(t) \le f(t)$ for $0 \le t < \infty$, is established under the following conditions:

(i_0) $f = f(t)$ is continuous and positive for $0 \le t < \infty$;

(ii_0) $k = k(t)$ is continuous and positive for $0 < t < \infty$, with $k \in L^1(0,1)$;

(iii_0) $G = G(s,u)$ is locally Lipschitz continuous in $u \in \mathbb{R}$ uniformly for $0 < s < \infty$, is integrable in s on any interval $(0,t)$ for any $u \in \mathbb{R}$, and is such that $uG(s,u) > 0$ holds for all (s,u), $u \neq 0$;

(iv_0) $\frac{f(t')}{f(t)} \le \frac{k(t'-s)}{k(t-s)}$ whenever $0 \le s \le t' < t$.

Let us emphasize that f is assumed to be strictly positive; the proof [1] of the formulated theorem gives up if $f(0) = 0$.

Theorem 2.1 concerns more specific equation (1) with the convolution kernel $k(t-s) = (t-s)^{\alpha-1}$, $0 < \alpha < 1$, and the nonlinearity $G(s,u) = s^\beta u^\mu$, $\beta > -1$, $\mu \ge 1$. On the other hand, the integral in the equation (1) contains additionally a coefficient function $K(t,s) \ge 0$ depending on t; in the case $K(t,s) \equiv 1$, condition (iv) is fulfilled due to (ii). The proof of [1, Theorem 6.4.2] needs values of $G(s,u)$ for $u < 0$, whereas in the formulation of Theorem 2.1 and its proof in Section 3 we use only $u \in \mathbb{R}^+$. Moreover, for $k(t) = t^{\alpha-1}$ and $G(s,u) = s^\beta u^\mu$, conditions (i)–(iv) of Theorem 2.1 are less restrictive than (i_0), (iii_0), (iv_0). Namely, for $\beta < 0$, condition (iii_0) is violated since $\frac{\partial G(s,u)}{\partial u} = \mu s^\beta u^{\mu-1}$ is not bounded as $s \to 0$. Further, as easily seen, condition (iv_0) with $k(t) = t^{\alpha-1}$ is equivalent to the monotone increase of the function $f(t)$, whereas (i)–(iii) allow $f(t)$ to be somewhere strictly decreasing (but so that $t^\gamma f(t)$ is monotone increasing). Finally, the equality $f(0)=0$ is not excluded by conditions (i)–(iv), whereas (i_0) is violated in the case $f(0)=0$.

A summary is that in case $k(t)=t^{\alpha-1}$, $G(s,u)=s^{\beta}u^{\mu}$, the claims of Theorem 2.1 are essentially more flexible than those in [1, Theorem 6.4.2]. This flexibility enables us to establish the unimprovable for large t estimate (10) in certain cases of interest.

Theorem 2.1 admits different generalizations for the Volterra–Uryson equation

$$u(t)+\int_0^t (t-s)^{\alpha-1}s^{\beta}G(t,s,u(s))ds=f(t),\quad t\geq 0.$$

In particular, it possesses a unique solution $u_\star\in C[0,\infty)$, $0\leq u_\star(t)\leq f(t)$ for $0\leq t<\infty$, provided that the following conditions are fulfilled:

- $0<\alpha<1$, $\alpha+\beta>0$;
- $f\in C[0,\infty)$ is monotone increasing, $f(0)\geq 0$, $f(t)>0$ for $t>0$;
- $G\in C(\triangle_f)$, $G(t,s,u)\geq 0$ for $(t,s,u)\in\triangle_f$, $G(t,s,0)\equiv 0$, where

$$\triangle_f=\left\{(t,s,u)\in\mathbb{R}^3: 0\leq s\leq t<\infty,\ 0\leq u\leq f(s)\right\};$$

- $G(t,s,u)$ is locally Lipschitz continuous in u:

$$\sup_{0\leq s\leq t\leq T,\ 0\leq u,v\leq f(s),\ u\neq v}\frac{|G(t,s,u)-G(t,s,v)|}{|u-v|}<\infty,\quad \text{for all } T>0;$$

- $G(t,s,u)$ is monotone increasing in u and monotone decreasing in t.

The proof of this theorem (corresponding to $\gamma=0$ in Theorem 2.1) and its generalizations will be presented in a future work. Also the case of more general convolution kernel $k(t-s)$ instead of $(t-s)^{\alpha-1}$ will be treated.

2. In the proof of Theorem 2.1, the following comments on condition (ii) are important. The change of variables $s=tx$ (then $ds=tdx$) yields for $\alpha,\alpha'>0$ the formula

$$\int_0^t (t-s)^{\alpha-1}s^{\alpha'-1}ds=\int_0^1 (1-x)^{\alpha-1}x^{\alpha'-1}dx\, t^{\alpha+\alpha'-1}=B(\alpha,\alpha')t^{\alpha+\alpha'-1},\quad t>0.$$

Thus

$$\int_0^t t^{\gamma}(t-s)^{\alpha-1}s^{\beta-\mu\gamma}ds=B(\alpha,1+\beta-\mu\gamma)t^{\alpha+\beta-(\mu-1)\gamma},\quad t>0. \tag{13}$$

Condition (ii) implies that $1+\beta-\mu\gamma>0$ (so the integral in the l.h.s. of (13) converges) and $\alpha+\beta-(\mu-1)\gamma>0$ (so the integral is small for small $t>0$). Denote by V_γ the integral operator of equation (8):

$$(V_\gamma v)(t)=\int_0^t t^{\gamma}(t-s)^{\alpha-1}s^{\beta-\mu\gamma}K(t,s)v^{\mu}(s)ds,\ 0<t\leq T,\quad (V_\gamma v)(0)=0, \tag{14}$$

so $v + \lambda V_\gamma v = g$ is equation (8) and $u + \lambda V_0 u = f$ is equation (1). Taking into account (ii) and the continuity, hence also the boundedness of $K(t,s) \geq 0$ for $0 \leq s \leq t \leq T$ with any $T > 0$, see (iv), we obtain with the help of (13) that V_γ is well-defined on the sets

$$C^+[0,T] := \{v \in C[0,T] : v(t) \geq 0 \text{ for } 0 \leq t \leq T\}, \quad T > 0,$$

and $V_\gamma : C^+[0,T] \to C^+[0,T]$ is a continuous operator w.r.t. the standard norm $\|v\|_{C[0,T]} = \max_{0\leq t\leq T} |v(t)|$ in $C[0,T]$.

On the other hand, V_γ becomes discontinuous if (ii) is violated so strongly that $\alpha + \beta < 0$ in case $\mu = 1$ or $\gamma > \min\left\{\frac{1+\beta}{\mu}, \frac{\alpha+\beta}{\mu-1}\right\}$ in case $\mu > 1$; if $\gamma \geq \frac{1+\beta}{\mu}$ and $v(0) > 0$ then $V_\gamma v$ is even undefined (the integral in (14) diverges).

In the cases $\mu = 1$, $\alpha + \beta = 0$, $\gamma \leq 1 - \alpha$ and $\mu > 1$, $\gamma = \frac{\alpha+\beta}{\mu-1} < \frac{1+\beta}{\mu}$, $\frac{\alpha+\beta}{\mu-1} \leq 1 - \alpha$, equation (8) belongs to the class of cordial Volterra integral equations [9] and needs a very different treatment compared with Theorems 2.1, 2.2 and the proof argument of those. The length restriction on a paper does not allow us to treat these cases in the present work.

3. One can see from the proof of Theorem 2.1 (see Section 3) that the solution $v_\star \in C^+[0,\infty)$ of (8) and the solution $u_\star \in C^+(0,\infty)$ of (1) with $t^\gamma u_\star(t)$ in $C[0,\infty)$ are unique also on finite intervals $[0,T]$, $T > 0$. This excludes the quenching [1] and the blow-up of positive solutions in finite time.

4. Under condition (i)–(iv), the case of the free term $f(t) \equiv 1$ in equation (1) becomes not so restrictive, from the point of view of solvability of (1), as it may seem at the first look. For instance, in the case $\gamma = 0$, $f(0) > 0$, dividing both sides of (1) by $f(t)$, we obtain w.r.t. $w(t) := \frac{u(t)}{f(t)}$ the equation

$$w(t) + \lambda \int_0^t (t-s)^{-\alpha} s^\beta K_f(t,s) w^\mu(s) ds = 1, \quad 0 \leq t < \infty,$$

in which the coefficient function $K_f(t,s) = \frac{K(t,s) f^\mu(s)}{f(t)}$ maintains the property (iv), and Theorem 2.1 applied to this equation yields the solvability of (1) for general $f(t)$. Here we observe a certain flexibility of conditions (i)–(iv) for equation (1).

5. Observe that in (v) and (vi) we assume the differentiability of $f(t)$ for $t > 0$, not for $t \geq 0$, and the differentiability of $K(t,s)$ for $0 < s \leq t < \infty$, not for $0 \leq s \leq t < \infty$. As a consequence, the differentiability of $\widetilde{f}(\tau) = f\left(\tau^{\frac{1}{\varrho}}\right)$ for $0 < \tau < \infty$ and of $\widetilde{K}(\tau,\sigma) = K\left(\tau^{\frac{1}{\varrho}}, \sigma^{\frac{1}{\varrho}}\right)$ for $0 < \sigma \leq \tau < \infty$ follows. This is exploited in Section 2.3 when equation (2) is presented in the form (3), (4).

2.3. Positive solution to equation (2). Recall that with the change of variables $t = \tau^{\frac{1}{\varrho}}$, $s = \sigma^{\frac{1}{\varrho}}$ equation (2) takes the form (3) which is of type (1). The results of Section 2.1 can be reformulated for equation (3) remembering relations (4), in particular, that now $\beta' = \frac{\beta+1}{\varrho} - 1$ in (3) plays the role of β in (1). Observe that $\beta' > -1$ if and only if $\beta > -1$, and that

$$\alpha + \beta' = \frac{\beta + 1 - \varrho(1-\alpha)}{\varrho}, \quad \beta' + 1 = \frac{\beta+1}{\varrho},$$

$$\tau^\gamma \widetilde{f}(\tau) = t^{\varrho\gamma} f(t), \quad \tau^\gamma \widetilde{u}(\tau) = t^{\varrho\gamma} u(t), \quad \tau^{\gamma+\alpha-1}\widetilde{K}(\tau,\sigma) = t^{\varrho(\gamma+\alpha-1)} K(t,s).$$

The counterparts of conditions (i)–(iv) take for (2) the following form:

(i′) $\lambda > 0$, $\mu \ge 1$, $0 < \alpha < 1$, $\beta > -1$, $\varrho > 0$;
(ii′) $-\infty < \gamma \le 1-\alpha$, $\varrho\mu\gamma < \beta + 1$, $\varrho(\mu-1)\gamma < \beta + 1 - \varrho(1-\alpha)$;
(iii′) $t^{\varrho\gamma} f(t) \ge 0$ is continuous and monotone increasing for $t > 0$;
(iv′) $K(t,s) \ge 0$ is continuous for $0 \le s \le t < \infty$, and $t^{\varrho(\gamma+\alpha-1)}K(t,s)$ is monotone decreasing w.r.t. argument t.

Conditions (v) and (vi) need not to be modified. The following result is a direct cosequence of Theorem 2.1.

Corollary 2.4. *Under conditions* (i′)–(iv′) *equation* (2) *possesses a unique solution* $u_\star \in C^+(0,\infty)$ *with* $t^{\varrho\gamma}u_\star(t)$ *in* $C^+[0,\infty)$*; it holds that* $0 \le u_\star(t) \le f(t)$ *for* $t > 0$*. If* $\varrho(1-\alpha) < \beta+1$ *and* $f \in C[0,\infty)$ *then* $u_\star \in C[0,\infty)$, $u_\star(0) = f(0)$*. Under conditions* (i′)–(iv′), (v) *and* (vi) *it holds that* $0 < u_\star(t) < f(t)$ *for* $t > 0$.

Observe that $f(t) = bt^r$ implies $\widetilde{f}(\tau) = f\big(\tau^{\frac{1}{\varrho}}\big) = b\tau^{r'}$, $r' = \frac{r}{\varrho}$. The reformulation of Theorem 2.2 and Corollary 2.3 for equation (2) through (3), (4) yields the following results.

Corollary 2.5. *Assume* (i′) *and*

$$r \ge \beta+1-\varrho(\mu+1)(1-\alpha), \quad r > -\varrho(1-\alpha), \quad (\mu-1)r > \varrho(1-\alpha)-1-\beta. \tag{15}$$

Then equation (2) *with* $K(t,s) \equiv 1$, $f(t) = bt^r$, $b = \mathrm{const} > 0$ *possesses a unique solution* $u_\star \in C^+(0,\infty)$ *with* $t^{\varrho\gamma}u_\star(t)$ *in* $C^+[0,\infty)$*. The following unimprovable for large* t *estimate holds true:*

$$u_\star(t) \le \left(\frac{\varrho b}{\lambda B(\alpha, 1-\alpha+\frac{r}{\varrho})}\right)^{\frac{1}{\mu}} t^{\frac{r+\varrho(1-\alpha)-(\beta+1)}{\mu}}, \quad 0 < t < \infty,$$

$$\frac{r+\varrho(1-\alpha)-(\beta+1)}{\mu} < r.$$

In particular, $u_\star(t) \to 0$ *as* $t \to \infty$ *if and only if* (15) *is complemented by the condition* $r < \beta + 1 - \varrho(1-\alpha)$.

If $\varrho(1-\alpha) < \beta+1$ *and* $r \ge \max\{0, \beta+1-\varrho(\mu+1)(1-\alpha)\}$ *then* $u_\star \in C[0,\infty)$ *with* $u_\star(0) = b$ *in case* $r = 0$, $u_\star(0) = 0$ *in case* $r > 0$.

Corollary 2.6. *Assume that* $f(t) \equiv 1$, $K(t,s) \equiv 1$, $\lambda > 0$, $\mu \geq 1$, $0 < \alpha < 1$, $\beta > -1$, $\varrho > 0$ *and*

$$\frac{\beta+1}{\mu+1} \leq \varrho(1-\alpha) < \beta + 1. \tag{16}$$

Then equation (2) *possesses a unique solution* $u_\star \in C^+[0,\infty)$. *It holds that* $u_\star(0) = 1$, $0 < u_\star(t) < 1$ *for* $t > 0$, *and* $u_\star(t) \to 0$ *as* $t \to \infty$ *with the following unimprovable for large* t *estimate:*

$$u_\star(t) \leq \left(\frac{\varrho \sin(\alpha\pi)}{\lambda\pi}\right)^{\frac{1}{\mu}} t^{\frac{\varrho(1-\alpha)-(\beta+1)}{\mu}}, \quad 0 < t < \infty. \tag{17}$$

According to (16), (17), the most fast decay $u_\star(t) = O(t^{-\varrho(1-\alpha)})$ is achieved for $\varrho(1-\alpha) = \frac{\beta+1}{\mu+1}$, i.e. $\beta = \varrho(\mu+1)(1-\alpha) - 1$.

In the standard case $\varrho(1-\alpha) = 1$, condition (16) reduces to $0 < \beta \leq \mu$, and estimate (17) takes the form

$$u_\star(t) \leq \left(\frac{\varrho \sin(\alpha\pi)}{\lambda\pi}\right)^{\frac{1}{\mu}} t^{-\frac{\beta}{\mu}}, \quad 0 < t < \infty.$$

3. Proof of Theorem 2.1

3.1. Local solvability of equation (1). Assume the conditions (i)–(iv). The operator $V_\gamma : C^+[0,T] \to C^+[0,T]$ defined in (14) is then continuous. For $v \in C^+[0,T]$ it holds that $K(t,s)v^\mu(s) \geq 0$, $c_{T,v} := \max_{0\leq s\leq t\leq T} K(t,s)v^\mu(s) < \infty$, and together with (13) we obtain

$$0 \leq (V_\gamma v)(t) \leq c_{T,v} B(\alpha, 1+\beta-\mu\gamma) t^{\alpha+\beta-(\mu-1)\gamma}, \quad 0 < t \leq T,$$

where $1 + \beta - \mu\gamma > 0$ and $\alpha + \beta - (\mu-1)\gamma > 0$ by (ii).

Introduce also the operator $A_\gamma : C^+[0,T] \to C[0,T]$ by

$$A_\gamma v = g - \lambda V_\gamma v, \quad v \in C^+[0,T],$$

with (fixed) $g \in C^+[0,T]$, $g(t) = t^\alpha f(t)$, see (iii). The equality $A_\gamma v = v$ holds if and only if $v + \lambda V_\gamma v = g$. Thus $v_\star \in C^+[0,T]$ is a solution to equation (8) on $[0,T]$ if and only if $v_\star$ is a fixed point of A_γ. The existence of a unique fixed point $v_\star$ of operator A_γ on a sufficiently small interval $[0,t_0]$ can be established with the help of the Banach fixed point principle. Treating a closed subset Ω of a Banach space X as a complete metric space with the distance function $\operatorname{dist}(v_1,v_2) := \|v_1 - v_2\|_X$, the standard Banach fixed point principle takes the following formulation.

Banach fixed point principle. *If an operator* A *maps a closed subset* Ω *of a Banach space* X *into* Ω *itself and is contractive in* Ω, *i.e.* A *satisfies with some* $q < 1$ *the condition* $\|Av_1 - Av_2\|_X \leq q\,\|v_1 - v_2\|_X$, *for all* $v_1, v_2 \in \Omega$, *then* A *possesses in* Ω *a unique fixed point* $v_\star$.

Lemma 3.1. *Assume* (i)–(iv). *Take a* $q \in (0,1)$ *and a (possibly large)* $T > 0$. *Let* $t_0 > 0$ *be sufficiently small so that* $t_0 \le T$ *and*

$$\lambda\mu\, g(T)^{\mu-1}\kappa_T\, B(\alpha, 1+\beta-\mu\gamma)t_0^{\alpha+\beta-(\mu-1)\gamma} \le q, \tag{18}$$

where

$$\kappa_T = \max_{0\le s\le t\le T} K(t,s).$$

Then equation (8) *possesses a unique solution* $v_\star \in C^+[0,t_0]$, $0 \le v_\star(t) \le g(t)$ *for* $0 \le t \le t_0$, *hence equation* (1) *possesses a unique solution* $u_\star \in C^+(0,t_0]$ *with* $t^\gamma u_\star(t) = v_\star(t)$ *in* $C^+[0,t_0]$. *In the case* $\mu = 1$ *the solution* $v_\star$ *of* (8) *is unique in the whole* $C[0,t_0]$.

Proof. Clearly $0 \le V_\gamma v \le V_\gamma g$ for $0 \le v \le g$. With the help of (18), taking into account that $g \in C[0,T]$ is monotone increasing and $g(t) \ge 0$ by condition (iii), we get for $0 \le t \le t_0$ that

$$\begin{aligned}(\lambda V_\gamma g)(t) &= \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) g^\mu(s) ds \\ &\le \lambda\,\kappa_T\, t^\gamma \int_0^t (t-s)^{\alpha-1} s^{\beta-\mu\gamma} ds\, g^{\mu-1}(T) g(t) \\ &\le \lambda\kappa_T\, g(T)^{\mu-1} B(\alpha, 1+\beta-\mu\gamma) t_0^{\alpha+\beta-(\mu-1)\gamma} g(t) \\ &\le \frac{q}{\mu} g(t) \\ &\le g(t).\end{aligned}$$

Thus $\lambda V_\gamma \Omega_{t_0} \subset \Omega_{t_0}$ where

$$\Omega_{t_0} := \{v \in C[0,t_0] :\, 0 \le v(t) \le g(t) \text{ for } 0 \le t \le t_0\}$$

is a closed subset of the space $X = C[0,t_0]$. Also $A_\gamma \Omega_{t_0} \subset \Omega_{t_0}$. Indeed, for $v \in \Omega_{t_0}$ we have $0 \le \lambda(V_\gamma v)(t) \le g(t)$, $0 \le t \le t_0$, that implies $(A_\gamma v)(t) = g(t) - \lambda(V_\gamma v)(t) \ge 0$ and $(A_\gamma v)(t) = g(t) - \lambda(V_\gamma v)(t) \le g(t)$ for $0 \le t \le t_0$, i.e. $A_\gamma v \in \Omega_{t_0}$.

With the help of condition (18) we get also that A_γ is a contraction on Ω_{t_0}. Indeed, $A_\gamma v_1 - A_\gamma v_2 = \lambda V_\gamma v_2 - \lambda V_\gamma v_1$, and for $v_1, v_2 \in \Omega_{t_0}$, $0 \le t \le t_0$, we have that $|v_1^\mu(s) - v_2^\mu(s)| \le \mu g(T)^{\mu-1} |v_1(s) - v_2(s)|$, $0 \le s \le T$, and

$$\begin{aligned}&|(A_\gamma v_1)(t) - (A_\gamma v_2)(t)| = \lambda\, |(V_\gamma v_1)(t) - (V_\gamma v_2)(t)| \\ &\le \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s)\, |v_1^\mu(s) - v_2^\mu(s)|\, ds \\ &\le \lambda\mu\kappa_T g(T)^{\mu-1} B(\alpha, 1+\beta-\mu\gamma) t_0^{\alpha+\beta-(\mu-1)\gamma}\, \|v_1 - v_2\|_{C[0,t_0]} \\ &\le q\, \|v_1 - v_2\|_{C[0,t_0]}\,,\end{aligned}$$

hence

$$\|A_\gamma v_1 - A_\gamma v_2\|_{C[0,t_0]} \le q \, \|v_1 - v_2\|_{C[0,t_0]} \, .$$

By the Banach principle, operator A_γ has a unique fixed point $v_\star \in \Omega_{t_0}$, thus equation (8) possesses a unique solution $v_\star \in C^+[0,t_0]$ (note that $0 \le v_\star(t) \le g(t)$ for any solution $v_\star \in C^+[0,t_0]$ of (8), i.e. $v_\star \in \Omega_{t_0}$), and equation (1) possesses a unique continuous solution $u_\star \in C^+(0,t_0]$ with $t^\gamma u_\star(t)$ in $C^+[0,t_0]$.

In the case $\mu = 1$, $V_\gamma \in \mathcal{L}(C[0,t_0])$ is a linear compact Volterra integral operator with the spectrum $\{0\}$, therefore a solution $v_\star \in C^+[0,t_0]$ of (8) is unique in the whole $C[0,t_0]$. □

3.2. Local extension of the solution. Assuming that the solution $v_\star(t)$ of (8) is already determined on $[0,t_1]$, $t_0 \le t_1 < T$, with $t_0 > 0$ and T from Lemma 2.1, we rewrite equation (8) for $t_1 \le t \le T$ in the form

$$v(t) + \lambda(V_{1,\gamma} v)(t) = g_1(t), \quad \text{or} \quad A_{1,\gamma} v = v, \tag{19}$$

where

$$\begin{aligned}
(V_{1,\gamma} v)(t) &:= \int_{t_1}^{t} t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v^\mu(s) ds, && v \in C^+[t_1,T],\\
g_1(t) &:= g(t) - \lambda \int_0^{t_1} t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v_\star^\mu(s) ds, && t \ge t_1,\\
A_{1,\gamma} v &:= g_1 - \lambda V_{1,\gamma} v, && v \in C^+[t_1,T].
\end{aligned}$$

Lemma 3.2. *Assume* (i)–(iv). *Take a $q \in (0,1)$ and a (possibly large) $T > 0$. Assume that the solution $v_\star(t)$, $0 \le v_\star(t) \le g(t)$, of equation* (8) *is already determined on $[0,t_1]$ for some $t_1 \in [t_0,T)$, $t_0 > 0$ from Lemma* 2.1. *Denote*

$$c_{t_0,T} = \max_{t_0 \le s \le t \le T} t^\gamma s^{\beta-\mu\gamma}, \quad \kappa_T = \max_{0 \le s \le t \le T} K(t,s).$$

Then $g_1(t_1) \ge 0$, $g_1(t)$ is monotone increasing for $t \ge t_1$, and $\lambda(V_{1,\gamma} g_1)(t) \le g_1(t)$ for $t_1 \le t \le t_2$ with $t_2 > t_1$ such that

$$\lambda \mu c_{t_0,T} \, \kappa_T \, g(T)^{\mu-1} \frac{(t_2-t_1)^\alpha}{\alpha} = q \tag{20}$$

provided that (20) *yields $t_2 \le T$; if* (20) *yields $t_2 > T$ we reduce t_2 to $t_2 = T$. As a consequence, the solution $v_\star(t)$ of equation* (8) *has a unique continuous extension from $[0,t_1]$ to $[0,t_2]$, such that $0 \le v_\star(t) \le g(t)$ for $0 \le t \le t_2$. Respectively, the solution $u_\star(t) = t^{-\gamma} v_\star(t)$ of equation* (1) *has a unique continuous extension from $(0,t_1]$ to $(0,t_2]$ such that $0 \le u_\star(t) \le f(t)$ for $0 < t \le t_2$.*

In this formulation, we have not tried to achieve a maximal possible length $t_2 - t_1$ of the extension step, for us it is more important that $t_2 - t_1$ is independent of the position of t_1 in $[t_0,T)$ so far as $t_2 < T$.

Proof. Let us confirm that g_1 has the properties asserted in the Lemma. First of all,

$$g_1(t_1) = g(t_1) - \lambda \int_0^{t_1} t_1^\gamma (t_1 - s)^{\alpha-1} s^{\beta-\mu\gamma} K(t_1, s) v_\star^\mu(s) ds = v_\star(t_1) \geq 0.$$

Further, $g_1(t)$ is monotone increasing for $t \geq t_1$ since $g(t)$ has this property by (iii), whereas

$$-\int_0^{t_1} t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v_\star^\mu(s) ds = -\int_0^{t_1} t^{1-\alpha} (t-s)^{\alpha-1} s^{\beta-\mu\gamma} t^{\gamma+\alpha-1} K(t,s) u_\star^\mu(s) ds$$

is monotone increasing since $t^{1-\alpha}(t-s)^{\alpha-1}$ is monotone decreasing w.r.t. t, and the same is true $t^{\gamma+\alpha-1}K(t,s)$ by condition (iv). Finally, it holds that $(\lambda V_{1,\gamma} g_1)(t) \leq g_1(t)$, $t_1 \leq t \leq T$, due to the monotone increase of $g_1(t)$, inequality $0 \leq g_1(t) \leq g(t)$ and (20):

$$\begin{aligned}
(\lambda V_{1,\gamma} g_1)(t) &= \lambda \int_{t_1}^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) g_1^\mu(s) ds \\
&\leq \lambda c_{t_0,T}\, \kappa_T \int_{t_1}^t (t-s)^{\alpha-1} ds\, g(T)^{\mu-1} g_1(t) \\
&= \lambda c_{t_0,T}\, \kappa_T\, g(T)^{\mu-1} \frac{(t-t_1)^\alpha}{\alpha} g_1(t) \\
&= \tfrac{1}{\mu} q g_1(t) \\
&\leq g_1(t), \quad t_1 \leq t \leq t_2.
\end{aligned}$$

In a same way as in the proof of Lemma 3.1 we conclude that $\lambda V_{1,\gamma}$ and $A_{1,\gamma}$ map the closed set

$$\Omega_{t_1,t_2} := \{v \in C[t_1,t_2] :\ 0 \leq v(t) \leq g_1(t) \text{ for } t_1 \leq t \leq t_2\} \subset C[t_1,t_2] =: X$$

into itself. Let us check that $A_{1,\gamma}$ is contractive on Ω_{t_1,t_2}. Again $A_{1,\gamma}v_1 - A_{1,\gamma}v_2 = \lambda V_{1,\gamma}v_2 - \lambda V_{1,\gamma}v_1$, and for $v_1, v_2 \in \Omega_{t_1,t_2}$, $t_1 \leq t \leq t_2$ we get that

$$\begin{aligned}
&|(A_{1,\gamma}v_1)(t) - (A_{1,\gamma}v_2)(t)| \\
&= \lambda\, |(V_{1,\gamma}v_1)(t) - (V_{1,\gamma}v_2)(t)| \\
&\leq \lambda \int_{t_1}^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s)\, |v_1^\mu(s) - v_2^\mu(s)|\, ds \\
&\leq \lambda\mu c_{t_0,T}\, \kappa_T \int_{t_1}^t (t-s)^{\alpha-1} ds \max_{t_1 \leq s \leq t_2} \max\left\{v_1^{\mu-1}(s), v_2^{\mu-1}(s)\right\} \|v_1 - v_2\|_{C[t_1,t_2]} \\
&\leq \lambda\mu c_{t_0,T}\, \kappa_T \frac{(t_2-t_1)^\alpha}{\alpha} g(T)^{\mu-1} \|v_1 - v_2\|_{C[t_1,t_2]} \\
&= q\, \|v_1 - v_2\|_{C[t_1,t_2]},
\end{aligned}$$

thus

$$\|A_{1,\gamma}v_1 - A_{1,\gamma}v_2\|_{C[t_1,t_2]} \leq q\, \|v_1 - v_2\|_{C[t_1,t_2]}.$$

With the help of Banach fixed point principle we again obtain that the equation (19) has a unique solution $v_\star \in C[t_1, t_2]$, $0 \le v_\star(t) \le g_1(t) \le g(t)$ for $t_1 \le t \le t_2$. Thus the solution $v_\star(t)$ of equation (8) is extended from $[0, t_1]$ to $[0, t_2]$, the extension is unique, and $0 \le v_\star(t) \le g(t)$ for $0 \le t \le t_2$. □

3.3. Extension of the solution to $[0, \infty)$**.** Under conditions of Lemma 3.2 $h := t_2 - t_1$ is independent of the position t_1 in $[t_0, T)$ so far as (20) implies that $t_2 < T$. Hence, with a finite number of extension steps of length h we obtain that the solution $v_\star(t)$ of equation (8) has a unique continuous extension from $[0, t_0]$ to $[0, T]$, $0 \le v_\star(t) \le g(t)$ for $0 \le t \le T$. Since $T > 0$ is arbitrary, $v_\star(t)$ has a unique continuous extension to $[0, \infty)$, $0 \le v_\star(t) \le g(t)$ for $0 \le t < \infty$, and the solution $u_\star(t) = t^{-\gamma} v_\star(t)$ of equation (1) has a unique continuos extension from $(0, t_0]$ to $(0, \infty)$, with $0 \le u_\star(t) \le f(t)$ for $0 < t < \infty$. In the case $\mu = 1$ this extension is unique in $C[0, \infty)$.

Equality $v_\star(0) = g(0)$ holds, since $(V_\gamma v)(0) = 0$ for $v \in C^+[0, T]$.

Conditions $f \in C^+[0, \infty)$, $\alpha + \beta > 0$ imply that $u_\star \in C^+[0, \infty)$, $u_\star(0) = f(0)$. Indeed, $u_\star(t) = t^{-\gamma} v_\star(t) \le t^{-\gamma} g(t) = f(t) \le c$ for $0 \le t \le 1$, hence $(V_0 u_\star)(t) \le c^\mu B(\alpha, \beta + 1) t^{\alpha+\beta} \to 0$ as $t \to 0$, so the claim follows from equality $u_\star + \lambda V_0 u_\star = f$ (the fact that $u_\star$ is the solution of (1)).

3.4. Differentiability of the solution. In the proof of strict inequalities $0 < u_\star(t) < f(t)$, $0 < v_\star(t) < g(t)$, we need the differentiability of $u_\star(t)$ and $v_\star(t)$ for $t > 0$.

Lemma 3.3. *Under conditions* (i)–(vi) *it holds that* $u_\star, v_\star \in C^1(0, \infty)$.

Proof. Let us prove that $v_\star(t)$ is continuously differentiable at a given point $t_1 > 0$. Fix some $t_0 \in (0, t_1)$ and $T > t_1$. For $t_0 \le t \le T$, rewrite equation (8) in the form

$$v(t) + \lambda \int_{t_0}^{t} t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v^\mu(s) ds = g_0(t), \quad t_0 \le t \le T, \tag{21}$$

where

$$g_0(t) := g(t) - \lambda \int_0^{t_0} t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v_\star^\mu(s) ds, \quad t_0 \le t \le T.$$

By (i)–(vi), the coefficient function $t^\gamma s^{\beta-\mu\gamma} K(t,s)$ in (21) and its first order derivatives w.r.t. t and s are continuous for $t_0 \le s \le t \le T$, whereas $g_0 \in C^1[t_0 + \delta, T]$, $\delta \in (0, T)$, implying [2,7] that $v_\star \in C^1(t_0 + \delta, T)$. In particular, $v_\star$ and $u_\star$ are continuously differentiable in a neighbourhood of t_1. Since $t_1 > 0$ is arbitrary, we obtain that $v_\star \in C^1(0, \infty)$, $u_\star \in C^1(0, \infty)$. □

The behaviour of $v_\star'(t)$ can be examined also near $t = 0$: under conditions (i)–(vi) and $\alpha + \beta > 0$ it holds $|v_\star'(t)| \le c t^{\frac{\alpha+\beta}{\mu} - 1}$ for $0 < t \le 1$, thus $v_\star' \in L^1(0, 1)$. (This fact is redundant in the proof of Theorem 2.1 but useful in Section 5.)

3.5. Strict inequalities $0 < v_\star(t) < g(t)$, $0 < u_\star(t) < f(t)$ for $t > 0$. Assume conditions (i)–(vi). We already know that $0 \le v_\star(t) \le g(t)$ and we lead to a contradiction the possibility that $v_\star(t_1) = 0$ at some $t_1 > 0$. We may assume that $v_\star(t) > 0$ for $t_0 \le t < t_1$ with some $t_0 \in (0, t_1)$ since the equality $v_\star(t) = 0$ for $0 \le t \le t_1$ contradicts the condition $g(t) > 0$ for $t > 0$, see (vi), and we can take the zero t_1 of $v_\star(t)$ so that $v_\star(t) > 0$ on some interval $[t_0, t_1)$. Then

$$\lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v_\star^\mu(s) ds = g(t) - v_\star(t) < g(t) \quad \text{for } t_0 \le t < t_1,$$

$$\lambda \int_0^{t_1} t_1^\gamma (t_1-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t_1,s) v_\star^\mu(s) ds = g(t_1),$$

and since g is monotone increasing, it holds for $t_0 \le t < t_1$ that

$$\int_0^{t_1} t_1^\gamma (t_1-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t_1,s) v_\star^\mu(s) ds > \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v_\star^\mu(s) ds,$$

or,

$$\begin{aligned} &\int_0^t \left[t_1^\gamma (t_1-s)^{\alpha-1} K(t_1,s) - t^\gamma (t-s)^{\alpha-1} K(t,s) \right] s^{\beta-\mu\gamma} v_\star^\mu(s) ds \\ &+ \int_t^{t_1} t_1^\gamma (t_1-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t_1,s) v_\star^\mu(s) ds > 0. \end{aligned}$$

We obtain a desired contradiction showing that, for $t < t_1$ close to t_1, actually the inverse inequality holds, i.e.

$$\begin{aligned} &\int_t^{t_1} t_1^\gamma (t_1-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t_1,s) v_\star^\mu(s) ds \\ &< \int_0^t \left[t^\gamma (t-s)^{\alpha-1} K(t,s) - t_1^\gamma (t_1-s)^{\alpha-1} K(t_1,s) \right] s^{\beta-\mu\gamma} v_\star^\mu(s) ds \end{aligned} \tag{22}$$

We present an upper bound for the l.h.s. of (22) and a lower bound for the r.h.s. such that for $t < t_1$ close to t_1, inequality "$<$" is valid for these bounds.

Upper bound for the l.h.s. of (22). By Lemma 3.3, $v_\star \in C^1(0,\infty)$, hence $v_\star(s) = v_\star(s) - v_\star(t_1) \le c(t_1 - s)$, $v_\star^\mu(s) \le c^\mu (t_1-s)^\mu$, implying for the l.h.s. of (22) the estimate

$$\int_t^{t_1} t_1^\gamma (t_1-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t_1,s) v_\star^\mu(s) ds \le c' \int_t^{t_1} (t_1-s)^{\alpha-1+\mu} ds = c''(t_1-t)^{\mu+\alpha} \tag{23}$$

for $t_0 \le t < t_1$. Note that $\mu + \alpha > 1$.

Lower bound for the r.h.s. of (22). Fix a small $\delta \in (0, t_1 - t_0)$; the condition about the smallness of $\delta > 0$ will be formulated later. Due to the monotone decrease of functions $t^{1-\alpha}(t-s)^{\alpha-1}$ and $t^{\gamma+\alpha-1}K(t,s)$ w.r.t. t (see (iv)), we have for $t_0+\delta<t<t_1$ that

$$\begin{aligned}
&t^{\gamma}(t-s)^{\alpha-1}K(t,s)-t_1^{\gamma}(t_1-s)^{\alpha-1}K(t_1,s)\\
&=t^{1-\alpha}(t-s)^{\alpha-1}t^{\gamma+\alpha-1}K(t,s)-t_1^{1-\alpha}(t_1-s)^{\alpha-1}t_1^{\gamma+\alpha-1}K(t_1,s)\\
&=\left[t^{1-\alpha}(t-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right]t^{\gamma+\alpha-1}K(t,s)\\
&\quad+t_1^{1-\alpha}(t_1-s)^{\alpha-1}\left[t^{\gamma+\alpha-1}K(t,s)-t_1^{\gamma+\alpha-1}K(t_1,s)\right]\\
&\geq\left[t^{1-\alpha}(t-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right]t^{\gamma+\alpha-1}K(t,s)\\
&\geq 0,
\end{aligned}$$

so for $t\in(t_0+\delta,t_1]$ we can estimate the r.h.s. of (22) from below as follows:

$$\begin{aligned}
&\int_0^t\left[t^{\gamma}(t-s)^{\alpha-1}K(t,s)-t_1^{\gamma}(t_1-s)^{\alpha-1}K(t_1,s)\right]s^{\beta-\mu\gamma}v_\star^{\mu}(s)ds\\
&\geq\int_{t_0}^{t-\delta}\left[t^{1-\alpha}(t-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right]t^{\gamma+\alpha-1}K(t,s)s^{\beta-\mu\gamma}v_\star^{\mu}(s)ds\\
&\geq c_\delta\int_{t_0}^{t-\delta}\left[t^{1-\alpha}(t-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right]ds\\
&=c_\delta\varphi_\delta(t),
\end{aligned}$$

where due to (vi) and the positiveness of $v_\star(s)$ on $[t_0,t_1)$,

$$c_\delta:=\min_{t_0\leq s\leq t\leq t_1-\delta}t^{\gamma+\alpha-1}K(t,s)s^{\beta-\mu\gamma}v_\star^{\mu}(s)>0$$

and

$$\begin{aligned}
\varphi_\delta(t)&:=\int_{t_0}^{t-\delta}\left[t^{1-\alpha}(t-s)^{\alpha-1}-t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right]ds\\
&=\frac{t^{1-\alpha}}{\alpha}\left((t-t_0)^{\alpha}-\delta^{\alpha}\right)+\frac{t_1^{1-\alpha}}{\alpha}\left((t_1-t+\delta)^{\alpha}-(t_1-t_0)^{\alpha}\right).
\end{aligned}$$

Clearly, $\varphi_\delta(t_1)=0$. From the equality

$$\varphi_\delta'(t)=\frac{1-\alpha}{\alpha}t^{-\alpha}\left((t-t_0)^{\alpha}-\delta^{\alpha}\right)+t^{1-\alpha}(t-t_0)^{\alpha-1}-t_1^{1-\alpha}(t_1-t+\delta)^{\alpha-1}$$

we see that $\varphi_\delta'(t)\to-\infty$ as $t\to t_1$, $\delta\to 0$; we can fix a sufficiently small $\delta>0$ so that $\varphi_\delta'(t)\leq-1$ for $0<t_1-t\leq\delta$ (this is the smallness condition on δ mentioned above). Then for $t_1-\delta\leq t<t_1$, with certain $\tau\in(t,t_1)$,

$$\varphi_\delta(t)=\varphi_\delta(t)-\varphi_\delta(t_1)=\varphi_\delta'(\tau)(t-t_1)\geq t_1-t.$$

The summary is that for $t_1-\delta\leq t<t_1$ it holds

$$\int_0^t\left[t^{\gamma}(t-s)^{\alpha-1}K(t,s)-t_1^{\gamma}(t_1-s)^{\alpha-1}K(t_1,s)\right]s^{\beta-\mu\gamma}v_\star^{\mu}(s)ds\geq c_\delta(t_1-t).$$

Comparing this with (23) we see that for $t < t_1$ close to t_1, inequality (22) really holds true.

This completes the proof of inequality $v_\star(t) > 0$ for $t > 0$. From the equality

$$v_\star(t) + \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) v_\star^\mu(s) ds = g(t)$$

we observe also that $v_\star(t) < g(t)$ for $t > 0$, since the integral term is strictly positive due to the positiveness of $v_\star(t)$ for $t > 0$ and the positiveness of $K(t,s)$ for $0 < s < t$, see (vi).

Finally, inequalities $0 < u_\star(t) < f(t)$ for $t > 0$ follow from inequalities $0 < v_\star(t) < g(t)$. The proof of Theorem 2.1 is complete. $\square$

4. Proof of Theorem 2.2

4.1. Positive solution of equation (1). Assume the conditions of Theorem 2.2, in particular (i) and that $f(t) \equiv bt^r$, $K(t,s) \equiv 1$. Observe that assumption (9) is a reformulation of condition (ii) for $\gamma = \frac{\alpha+\beta-r}{\mu}$. Indeed,

$$\begin{aligned} \frac{\alpha+\beta-r}{\mu} &\leq 1-\alpha && \text{iff } r \geq \alpha+\beta-\mu(1-\alpha), \\ \mu\frac{\alpha+\beta-r}{\mu} &< 1+\beta && \text{iff } r > -(1-\alpha), \\ (\mu-1)\frac{\alpha+\beta-r}{\mu} &< \alpha+\beta && \text{iff } (\mu-1)r+\alpha+\beta > 0. \end{aligned}$$

Further,

$t^\gamma f(t) = bt^{\gamma+r}$ is monotone increasing iff $\frac{\alpha+\beta-r}{\mu} + r \geq 0$, and
$t^{\gamma+\alpha-1} K(t,s) = t^{\gamma+\alpha-1}$ is monotone decreasing iff $\frac{\alpha+\beta-r}{\mu} + \alpha - 1 \leq 0$;

both inequalities are fulfilled due to (9). Thus also conditions (iii) and (iv) are fulfilled. By Theorem 2.1 equation (1) possesses a unique non-negative solution $u_\star \in C(0,\infty)$ with $t^\gamma u_\star(t)$ in $C^+[0,\infty)$.

4.2. Comparison of solutions. The proof of estimate (10) is based on the comparison of solutions to equation (8) corresponding to different free terms g.

Lemma 4.1. *Assume* (i)–(iv). *Let $\overline{v} \in C^+[0,\infty)$ and $\overline{g} := \overline{v} + \lambda V_\gamma \overline{v}$ be such that $\overline{g}(t) - g(t) \geq 0$ and $\overline{g}(t) - g(t)$ is monotone increasing for $t \geq 0$. Then $0 \leq v_\star(t) \leq \overline{v}(t)$ for $t \geq 0$ where $v_\star \in C^+[0,\infty)$ is the (unique) solution of equation* (8).

Proof. Rewrite the identity

$$\overline{v}(t) - v_\star(t) + \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K(t,s) \left[\overline{v}^\mu(s) - v_\star^\mu(s)\right] ds \equiv \overline{g}(t) - g(t)$$

in the form

$$\overline{v}(t) - v_\star(t) + \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta-\mu\gamma} K_\star(t,s) \left[\overline{v}(s) - v_\star(s)\right] ds \equiv \overline{g}(t) - g(t)$$

where $K_\star(t,s) = K(t,s)\kappa_\star(s)$, $\kappa_\star(s) \equiv 1$ in case $\mu = 1$ and

$$\kappa_\star(s) := \begin{cases} \dfrac{\overline{v}^\mu(s) - v_\star^\mu(s)}{\overline{v}(s) - v_\star(s)} & \text{if } \overline{v}(s) \neq v_\star(s) \\ \mu v_\star^{\mu-1}(s) & \text{if } \overline{v}(s) = v_\star(s) \end{cases}, \quad 0 \le s < \infty, \quad \text{in case } \mu > 1.$$

Observe that $\kappa_\star \in C[0,\infty)$ and $\kappa_\star(s) \ge 0$ for $s \ge 0$ since $\overline{v}(s) - v_\star(s)$ and $\overline{v}^\mu(s) - v_\star^\mu(s)$ are of the same sign for $\overline{v}(s) \neq v_\star(s)$, whereas on a possible subinterval where $\overline{v}(s) = v_\star(s)$ the values of $\kappa_\star(s)$ could be arbitrarily chosen; the choice $\kappa_\star(s) = \mu v_\star^{\mu-1}(s)$ yields the continuity of $\kappa_\star$. We see that $v(t) = \overline{v}(t) - v_\star(t)$ is a solution of the equation

$$v(t) + \lambda \int_0^t t^\gamma (t-s)^{-\alpha} s^{\beta-\mu\gamma} K_\star(t,s) v(s) ds = \overline{g}(t) - g(t),$$

which we can treat as an equation type (8) with $\mu' = 1$, $\beta' = \beta - (\mu - 1)\gamma$:

$$v(t) + \lambda \int_0^t t^\gamma (t-s)^{\alpha-1} s^{\beta'-\mu'\gamma} K_\star(t,s) v^{\mu'}(s) ds = \overline{g}(t) - g(t). \tag{24}$$

The counterpart of condition (ii) reads as

$$\gamma \le 1 - \alpha, \quad \gamma < 1 + \beta', \quad \alpha + \beta' > 0,$$

and these inequalities immediately follow from corresponding inequalities in (ii): the second one means that $\gamma < 1 + \beta - (\mu - 1)\gamma$, or $\mu\gamma < 1 + \beta$, and the third one means that $\alpha + \beta - (\mu - 1)\gamma > 0$, or $(\mu - 1)\gamma < \alpha + \beta$. The monotone increase and continuity of $\overline{g}(t) - g(t) \ge 0$ is assumed; $K_\star(t,s)$ inherits the property (iv) from $K(t,s)$. So for equation (24) the counterparts of conditions (i)–(iv) are fulfilled, and due to the linearity of (24) its solution $\overline{v}(s) - v_\star(s)$ is unique in $C[0,\infty)$. By Theorem 2.1, $\overline{v}(s) - v_\star(s) \ge 0$ for $t \ge 0$, so $v_\star(s) \le \overline{v}(s)$ for $t \ge 0$. □

4.3. Estimate (10). Let us turn to equations (1) and (8) with $K(t,s) \equiv 1$, $f(t) \equiv bt^r$, $g(t) \equiv t^\gamma f(t) \equiv bt^{\gamma+r}$, where $b > 0$, r satisfies (9), and $\gamma = \frac{\alpha+\beta-r}{\mu}$. Function $v_\star(t) = t^\gamma u_\star(t)$ is the unique solution of (8) in $C^+[0,\infty)$; the constant function $\overline{v}(t) \equiv c_r$ with $c_r > 0$ defined in (10) is the solution of (8) corresponding to the free term $\overline{g}(t)$ which we are able to determine:

$$\overline{g}(t)=c_r+\lambda c_r^\mu\int_0^t t^\gamma(t-s)^{\alpha-1}s^{\beta-\mu\gamma}ds=c_r+\lambda c_r^\mu B(\alpha,1+\beta-\mu\gamma)t^{\gamma+\alpha+\beta-\mu\gamma}=c_r+bt^{\gamma+r}$$

$\Big(\alpha+\beta-\mu\gamma=r$ due to equality $\gamma:=\frac{\alpha+\beta-r}{\mu}$; $\gamma+r=\frac{\alpha+\beta+(\mu-1)r}{\mu}>0$ due to (9)$\Big)$. So $\overline{g}(t) - g(t) \equiv c_r > 0$ is monotone increasing. By Lemma 4.1

$$t^\gamma u_\star(t) = v_\star(t) \le \overline{v}(t) = c_r, \quad \text{or} \quad u_\star(t) \le c_r t^{-\gamma},$$

so estimate (10) really holds true.

It remains to prove that, in the sense of the Theorem, estimate (10) is unimprovable for large t. We lead to a contradiction the opposite claim that there exist a $\theta < 1$ and a $T > 0$ such that $u_\star(t) \le \theta c_r t^{-\gamma}$ for any $t \ge T$. Then for $t \ge 2T$ it holds that

$$\begin{aligned}
bt^r &= f(t) = u_\star(t) + \lambda\int_0^t (t-s)^{\alpha-1}s^\beta u_\star^\mu(s)ds \\
&\le \theta c_r t^{-\gamma} + \lambda\int_0^T (t-s)^{\alpha-1}s^\beta u_\star^\mu(s)ds + \lambda(\theta c_r)^\mu\int_T^t (t-s)^{\alpha-1}s^{\beta-\mu\gamma}ds \\
&\le \theta c_r t^{-\gamma} + \lambda(t-T)^{\alpha-1}\int_0^T s^\beta u_\star^\mu(s)ds + \lambda(\theta c_r)^\mu\int_0^t (t-s)^{\alpha-1}s^{\beta-\mu\gamma}ds \\
&= \theta c_r t^{-\gamma} + \lambda M(t-T)^{\alpha-1} + \lambda(\theta c_r)^\mu B(\alpha,1+\beta-\mu\gamma)t^{\alpha+\beta-\mu\gamma} \\
&\le \theta c_r t^{-\gamma} + \lambda M' t^{\alpha-1} + \lambda\theta^\mu c_r^\mu B(\alpha,1-\alpha+r)t^r \\
&= \theta c_r t^{-\gamma} + \lambda M' t^{\alpha-1} + \theta^\mu bt^r,
\end{aligned}$$

where we took into account that $\alpha+\beta-\mu\gamma = r$ and $1+\beta-\mu\gamma = 1-\alpha+r$ due to equality $\gamma := \frac{\alpha+\beta-r}{\mu}$; we exploited also the definition of c_r presented in (10); the meaning of constants M and M' is clear from the context. A conclusion is that

$$\begin{aligned}
bt^r &\le \frac{1}{1-\theta^\mu}\left(\theta c_r t^{-\gamma} + \lambda M' t^{\alpha-1}\right) && \text{for } t \ge 2T, \\
b &\le \frac{1}{1-\theta^\mu}\left(\theta c_r t^{-\gamma-r} + \lambda M' t^{\alpha-1-r}\right) \to 0 && \text{as } t \to \infty,
\end{aligned}$$

since $\alpha - 1 - r < 0$, $\gamma + r > 0$ due to (9). So $b = 0$ that contradicts the assumption $b > 0$. The proof of Theorem 2.2 is complete. $\square$

5. An open problem: monotony of the solution

We have been unsuccessful trying to prove the following conjecture.

Conjecture 5.1. Assume (i), and let $f(t) \equiv 1$, $K(t,s) \equiv 1$. Then the solution $u_\star(t) > 0$ of equation (1) is monotone decreasing for $t \geq 0$.

In case $f(t) \equiv 1$, $K(t,s) \equiv 1$, it follows from the monotone decrease of $u_\star(t)$ that $u_\star(t) \to 0$ as $t \to \infty$, since otherwise $\int_0^t (t-s)^{\alpha-1} u_\star^\mu(s)ds \to \infty$ as $t \to \infty$ and $u_\star$ cannot be a solution to equation (1).

In [5], for problem (5), the decrease of the solution is demonstrated by an asymptotical solving the problem for $x \leq 0.1$ and for $x \geq 0.5$, with some guess about the behaviour of the solution for $0.1 \leq x \leq 0.5$. Below we present two cases where Conjecture 5.1 can be established analytically feeding so the hope that it holds true also in general.

First note that integrating by parts and after that differentiating we get for $w \in C[0,\infty) \cap C^1(0,\infty)$ with $w' \in L^1(0,1)$ the differentiation formula

$$\frac{d}{dt} \int_0^t (t-s)^{\alpha-1} w(s)ds = w(0)t^{\alpha-1} + \int_0^t (t-s)^{\alpha-1} w'(s)ds, \quad t > 0. \tag{25}$$

1. Case $\beta = 0$. Then equation (1) with $f(t) \equiv 1$, $K(t,s) \equiv 1$ reads as

$$u(t) + \lambda \int_0^t (t-s)^{\alpha-1} u^\mu(s)ds = 1, \quad 0 < t < \infty.$$

Its solution $u_\star(t)$ satisfies $u_\star(0) = 1$. With the help of (25) we conclude that $w_\star(t) := -u_\star'(t)$ satisfies the equation

$$w(t) + \lambda\mu \int_0^t (t-s)^{\alpha-1} u_\star^{\mu-1}(s) w(s)ds = t^{\alpha-1}.$$

By Theorem 2.1 with $\gamma = 1-\alpha$, this equation with $K(t,s) = u_\star^{\mu-1}(s)$ and $f(t) = t^{\alpha-1}$ is uniquely solvable, $w_\star \in C(0,\infty)$, $0 < w_\star(t) < t^{\alpha-1}$ for $t > 0$. Hence $u_\star'(t) < 0$ for $t > 0$, thus $u_\star(t)$ is strictly decreasing for $t \geq 0$.

2. Case $\alpha = 1$. Then equation (1) with $f(t) \equiv 1$, $K(t,s) \equiv 1$, i.e. the equation

$$u(t) + \lambda \int_0^t s^\beta u^\mu(s)ds = 1, \quad 0 < t < \infty,$$

is equivalent to the Cauchy problem

$$u'(t) + \lambda t^\beta u^\mu(t) = 0, \quad u(0) = 1,$$

which can be solved analytically:

$$u_\star(t) = \exp\left(-\frac{\lambda t^{\beta+1}}{\beta+1}\right) \qquad \text{for } \mu = 1,$$

$$u_\star(t) = \left(1 + \frac{\lambda(\mu-1)}{\beta+1} t^{\beta+1}\right)^{-\frac{1}{\mu-1}} \qquad \text{for } \mu > 1,$$

We see that $u_\star(t)$ is strictly decreasing for $t \geq 0$.

Concerning Conjecture 5.1, it is natural to ask also the following more general question: For which f the solution $u_\star(t)$ of equation (1) with $K(t,s) \equiv 1$ is monotone decreasing for $t \geq t_\star$ with certain $t_\star \geq 0$? Of course, the same question is of interest also in case of more general $K(t,s)$.

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