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A two-phase free boundary with a logarithmic term

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Abstract. We study minimizers of the energy functional

$$\int_{\Omega} |\nabla u|^2 + 2F(u)dx,$$

where $F'(u) \approx |u|^q \log u$ for some $-1 < q < 0$. We prove existence, optimal decay, and non-degeneracy of solutions, from free boundary points. Consequently, we derive the porosity property and an estimate on the Hausdorff dimension of the free boundary.

1. Introduction and statement of the results

1.1. Problem setting

For $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, consider the variational problem

$$\mathcal{J}(u) = \int_{\Omega} |\nabla u|^2 + 2F(u)dx \to \min, \tag{1.1}$$

among all functions in the class

$$K_\phi = \{u \in H^1(\Omega) : u = \phi \text{ on } \partial\Omega\}, \tag{1.2}$$

where $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$ as a boundary condition is given. In addition, F satisfies the following assumptions:

(H1) $F(0) = 0$ and there are positive constants $C, \gamma > 0$ such that

$$|F(u) - F(v)| \leq C|u - v|^\gamma.$$

(H2) F is differentiable at $u \neq 0$ and there is positive constant $\Lambda > 0$ such that

$$|f(u)| \leq \Lambda |u|^q (1 + |\log|u||) \quad \text{for } u \neq 0,$$

for some $q \in (-1, 0)$, where $f = F'$.

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(H3) There are positive constants $\lambda, \delta > 0$ such that

$$uf(u) \geq \lambda |u|^{1+q} \log |u|^{-1} \quad \text{for } 0 < |u| < \delta.$$

For our result concerning the existence and optimal decay, we just need assumptions (H1)–(H2). However, for non-degeneracy we shall additionally assume assumption (H3). These conditions can be met by a function like

$$F(u) = (\lambda_+ (u^+)^{1+q} + \lambda_- (u^-)^{1+q})\Big(\frac{1}{1+q} - \log |u|\Big),$$

where $u^\pm = \max\{\pm u, 0\}$ and $\lambda_\pm$ are positive parameters. The Euler–Lagrange equation associated with this example is

$$\Delta u = (1+q)(-\lambda_+ (u^+)^q \log u^+ + \lambda_- (u^-)^q \log u^-),$$

at least where u is away from zero, since the force term becomes unbounded at the origin.

Using the direct method of calculus of variations, we prove the existence of minimizers of $\mathcal{J}$ on Ω, restricted to the set K_ϕ. Note that generally there may exist more than one minimizer with given boundary values ϕ, since the functional $\mathcal{J}$ is not convex (see, e.g., [13]).

The minimizers of (1.1) will satisfy

$$\Delta u = f(u), \tag{1.3}$$

when $u \neq 0$. We define

$$\Omega^+ = \{x \in \Omega : u(x) > 0\}, \quad \Omega^- = \{x \in \Omega : u(x) < 0\}$$

as two different phases of u. The non-differentiability of the potential F at $u = 0$ implies the Euler–Lagrange equation associated to $\mathcal{J}$ to be singular along an a priori unknown interface

$$\Gamma^\pm = \partial\Omega^\pm \cap \Omega.$$

In this setting x_0 is a *one-phase* free boundary point if $x_0 \in (\Gamma^+ \setminus \Gamma^-) \cup (\Gamma^- \setminus \Gamma^+)$, and x_0 is a *two-phase* free boundary point if $x_0 \in \Gamma^+ \cap \Gamma^-$.

The two-phase points are of two different kinds, with the only distinction being whether $\nabla u = 0$ or not at these points. By $C^{1,\alpha}$-regularity of solutions and the implicit function theorem, for points of $\Gamma^+ \cap \Gamma^-$ such that $\nabla u(x_0) \neq 0$, there exists a small ρ such that $B_\rho(x_0) \cap \Gamma^+ = B_\rho(x_0) \cap \Gamma^-$ is a $C^{1,\alpha}$-surface. We denote the other parts of the free boundary by

$$\Gamma_0 := \{x \in \Gamma^+ \cup \Gamma^- : \nabla u(x) = 0\},$$

and study behavior of solution around Γ_0.

The main objective in this paper is to study the optimal regularity for the solution of problem (1.1) and then looking for the porosity of the free boundaries. We also show the Euler–Lagrange equation given by (1.3) holds.

1.2. Known results

Problem (1.1) when $F(u) = |u|^{1+q}$ and $q \in (-1, 1)$ has been well studied in the one-phase and two-phase settings, that is, when the minimizers are assumed to be non-negative or sign changing. It was shown that the positive solutions have the optimal regularity $C^{[\gamma],\gamma-[\gamma]}$, where $\gamma = 2/(1-q)$ in [12,13]. This result has been extended for the solution without the assumption $u \geq 0$ in [6] and for the system case in [7] when $0 < q < 1$; see also [5] when $F(u) = -|u|^{1+q}$. In addition, for $-1 < q < 0$ the optimal regularity has been given in [10]. When $q = 0$, the problem is the well-known two-phase obstacle problem

$$\Delta u = \lambda_+ \chi_{\Omega^+} - \lambda_- \chi_{\Omega^-} \quad \text{in } \Omega,$$

and has been studied in [16,17], where optimal $C^{1,1}_{\text{loc}}$ regularity has been proved.

However, there are few results for the regularity of the free boundary for problem (1.3). In [12], it has been shown that for $f(u) = (u^+)^q$, the free boundary $\partial\{u > 0\}$ has locally finite $\mathcal{H}^{n-1}$-Hausdorff dimension when $-1 < q < 1$. This implies that the non-coincident set $\{u > 0\}$ has locally finite perimeter and we are able to define the reduced part of the free boundary, $\partial_{\text{red}}\{u > 0\}$, where a tangent plane exists in a weak sense. Alt and Phillips showed in [2] that $\partial_{\text{red}}\{u > 0\}$ is a $C^{1,\alpha}$ surface. In the two-phase case, without any sign restriction on solutions and when $q = 0$, Shahgholian, Uraltseva, and Weiss in [14] (or [15] for dimension $n = 2$) have shown that near the so-called branching points the free boundaries Γ^+ and Γ^- are C^1 regular and tangent to each other (with an example showing that $\Gamma^\pm$ are not generally of class $C^{1,Dini}$). The same result holds for $q \in (0, 1)$ and $q \in (-1, 0)$ where $n = 2$; see [6,9].

Recently in [8], the authors investigated the following two-phase obstacle-like problem:

$$\Delta u = -\lambda_+ (\log u^+) \chi_{\{u>0\}} + \lambda_- (\log u^-) \chi_{\{u<0\}}.$$

In this work, using a monotonicity formula argument, they proved an optimal regularity result for solutions, which amounts to ∇u being log-Lipschitz. Also, in [4] the authors studied the one-phase case of this problem, when $u \geq 0$, and proved the optimal regularity and non-degeneracy.

In the rest of this paper, we shall develop the optimal regularity results in our setting.

1.3. Statement of the results

The results of this paper concern the existence and regularity of solutions of the above the free boundary.

Theorem 1.1. *Let K_ϕ be as in (1.2), where $\phi \in H^1(\Omega)$, and assume that (H2) holds. Then, $\mathcal{J}$ attains its minimum for some $u \in K_\phi$. Furthermore, if $\phi \in L^\infty(\Omega)$, then this minimum is bounded. More precisely, there exists c_1 such that $\|u\|_{L^\infty(\Omega)} \leq c_1$, where c_1 depend on n, q, Λ, ϕ, and Ω.*

We remark that the minimum is not necessarily unique. The next result concerns the growth/decay of solutions at the free boundary. Throughout this paper, we take

$$\beta = \frac{2}{1-q}.$$

Theorem 1.2. *Suppose that (H1) and (H2) hold. Then, any minimizer to the functional $\mathcal{J}$ satisfies*

$$\sup_{B_r(z)} |u| \leq c_2 r^\beta |\log r|^{\beta/2}, \quad r < 1/2, \tag{1.4}$$

for $z \in \Gamma_0$ and $B_r(z) \subset \Omega$. Here c_2 depends only on n, q, Λ.

Since the function F in (1.1) is not differentiable at $u = 0$, it is not straightforward to derive the Euler–Lagrange equation and the equation will be singular around $u = 0$. The main challenge is that we do not have any estimate on the $\mathcal{H}^{n-1}$-measure of the free boundary. From the next theorem, it follows that a minimizer of (1.1) can be represented by an equation.

Theorem 1.3. *Under assumptions (H1)–(H3), any minimizer to the functional $\mathcal{J}$ satisfies the following equation in the weak sense:*

$$\Delta u = f(u)\chi_{\{u\neq 0\}} \quad \text{in } \Omega. \tag{1.5}$$

The next results concern the analysis of the free boundary of the minimizers, a non-degeneracy property, porosity, and some estimate for the Hausdorff dimension.

Theorem 1.4. *Under assumptions (H1)–(H3), any minimizer to the functional $\mathcal{J}$ is non-degenerate in the following sense:*

$$\sup_{B_r(z)} u^\pm \geq c_3 r^\beta |\log r|^{\beta/2} \quad \text{for } r \leq r_0$$

for any $z \in \Gamma_0$, where $c_3 = c_3(n, q, \lambda)$ and r_0 is small enough.

Theorem 1.5. *Let u be a minimizer of (1.1) in K_ϕ. Then, for every compact set $K \subset \Omega$ we have that $\Gamma_0 \cap K$ is porous with porosity constant $\delta = \delta(n, \operatorname{dist}(K, \partial\Omega), c_2, c_3)$. In particular,*

$$|\Gamma_0 \cap K| = 0$$

for any $K \Subset \Omega$.

Theorem 1.6. *Let u be a minimizer of (1.1) in K_ϕ. For every compact set $K \subset \Omega$,*

$$\mathcal{H}^{n-2+\beta}(\Gamma_0 \cap K) = 0.$$

2. Existence (Proof of Theorem 1.1)

Most material in this section is fairly common, although somewhat technical.

Proof of Theorem 1.1. The proof of existence is divided into several steps. *Step 1: $\mathcal{J}$ is bounded from below.* Since $\phi \in K_\phi$, the admissible class (the set of all functions), where we minimize $\mathcal{J}$, is non-empty.

Defining

$$\mathcal{J}_0 := \inf_{u \in K_\phi} \mathcal{J}(u),$$

we shall prove that $\mathcal{J}_0 > -\infty$. By (H2), we have

$$|F(u)| \leq \int_0^{|u|} \Lambda t^q (1 + |\log t|)\, dt \leq C_1 |u|^{1+q} (1 + |\log |u||) \leq C_1 (1 + |u|^{1+q+a}), \quad (2.1)$$

for some constant C_1 and $a > 0$. Thus,

$$\mathcal{J}(u) \geq \|\nabla u\|^2_{L^2(\Omega)} - C_1 \int_\Omega 1 + |u|^{1+q+a} dx \geq \|\nabla u\|^2_{L^2(\Omega)} - \widetilde{C}_1 (1 + \|u\|^{1+q+a}_{L^2(\Omega)}). \quad (2.2)$$

Using Poincaré's inequality, there is some positive constant C_2 such that

$$\|u - \phi\|^2_{L^2(\Omega)} \leq C_2 \|\nabla(u - \phi)\|^2_{L^2(\Omega)} \leq 2C_2 (\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla \phi\|^2_{L^2(\Omega)}).$$

Hence,

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\Omega)} &\geq \frac{1}{2C_2} \|u - \phi\|^2_{L^2(\Omega)} - \|\nabla \phi\|^2_{L^2(\Omega)} \\
&= \frac{1}{2C_2} \|u\|^2_{L^2(\Omega)} + \frac{1}{2C_2} \|\phi\|^2_{L^2(\Omega)} - \frac{1}{C_2} \int_\Omega u\phi\, dx - \|\nabla \phi\|^2_{L^2(\Omega)} \\
&\geq \frac{1}{2C_2} \|u\|^2_{L^2(\Omega)} + \frac{1}{2C_2} \|\phi\|^2_{L^2(\Omega)} - \frac{1}{C_2} \|\phi\|_{L^\infty(\Omega)} \|u\|_{L^1(\Omega)} - \|\nabla \phi\|^2_{L^2(\Omega)}.
\end{aligned}$$

Inserting this estimate into (2.2), we obtain

$$\mathcal{J}(u) \geq \frac{1}{2C_2} \|u\|^2_{L^2(\Omega)} - \widetilde{C}_2 (1 + \|u\|_{L^2(\Omega)} + \|u\|^{1+q+a}_{L^2(\Omega)}) \geq C_3,$$

where $C_3 = \min_{t \geq 0} \left\{ \frac{1}{2C_2} t^2 - \widetilde{C}_2 (1 + t + t^{1+q+a}) \right\}$ is bounded, since $1 + q + a < 2$. We thus conclude that $\mathcal{J}$ is bounded from below.

Step 2: Existence of a minimizer. To prove existence of a minimizer, let $v_j \in \phi + H^1_0(\Omega)$ be a minimizing sequence for $\mathcal{J}$ on K_ϕ. Obviously, for j large enough, we have

$$\mathcal{J}_0 \leftarrow \mathcal{J}(v_j) \leq \mathcal{J}_0 + 1.$$

We show that $\{v_j\}$ is bounded in $H^1(\Omega)$. Using (2.2), we have

$$\mathcal{J}(v_j) \geq \|\nabla v_j\|^2_{L^2(\Omega)} - \widetilde{C}_1 (1 + \|v_j\|^{1+q+a}_{L^2(\Omega)}) \quad (2.3)$$

which, along with Poincaré's inequality, leads us to

$$\begin{aligned}\|v_j\|^2_{L^2(\Omega)} &\le 2(\|v_j-\phi\|^2_{L^2(\Omega)}+\|\phi\|^2_{L^2(\Omega)})\\ &\le 4C_2(\|\nabla v_j\|^2_{L^2(\Omega)}+\|\nabla\phi\|^2_{L^2(\Omega)})+2\|\phi\|^2_{L^2(\Omega)}.\end{aligned}\tag{2.4}$$

Combining (2.3) and (2.4) gives us

$$\mathcal{J}(v_j)\ge\|\nabla v_j\|^2_{L^2(\Omega)}-C_3(1+\|\nabla v_j\|^{1+q+a}_{L^2(\Omega)}).$$

Since $1+q+a<2$, we have that $\{v_j\}$ is bounded in $H^1(\Omega)$, implying that there is a function u such that

$$\begin{cases} v_j\to u & \text{weakly in } H^1(\Omega),\\ v_j\to u & \text{in } L^2(\Omega),\\ v_j\to u & \text{a.e. in } \Omega.\end{cases}$$

By lower semicontinuity of the functional and uniform continuity of F, we have

$$\int_\Omega|\nabla u|^2dx\le\liminf_{j\to\infty}\int_\Omega|\nabla v_j|^2dx,$$
$$\int_{B_1}F(u)dx\le\liminf_{j\to\infty}\int_{B_1}F(v_j)dx.$$

Hence, $\mathcal{J}(u)\le\liminf_{j\to\infty}\mathcal{J}(v_j)=\mathcal{J}_0$, which means u is a minimizer.

Step 3: L^∞ estimate. Let $M=\max(\|\phi\|_{L^\infty(\Omega)},e^{-1/q})$ and define $\bar{u}=\max(u,M)$. Since u is a minimizer and $F(u)$ is differentiable at $u\neq 0$, u satisfies (1.3) in $\{|u|>0\}$. Then,

$$\Delta\bar{u}=f(u)\ge-\Lambda|u|^q(1+\log|u|)\ge-\Lambda M^q(1+\log M)=:-\mathcal{K}\quad\text{when } u>M.$$

Applying the maximum principle to the positive subharmonic function $\bar{u}(x)+\frac{\mathcal{K}}{2n}|x|^2$, we get

$$\max_\Omega\bar{u}\le\max_{\partial\Omega}\Big(\bar{u}(x)+\frac{\mathcal{K}}{2n}|x|^2\Big)\le M+\mathcal{K}C(n,\Omega)=c_1(n,q,\Lambda,\Omega,\|\phi\|_{L^\infty(\Omega)}).$$

So, u is bounded above. By a similar argument with $\underline{u}=\min(u,-M)$, we obtain a bound from below for u. Therefore, we find the estimate

$$\|u\|_{L^\infty(\Omega)}\le\max(\|\bar{u}\|_{L^\infty(\Omega)},\|\underline{u}\|_{L^\infty(\Omega)})\le c_1. \qquad\blacksquare$$

Proposition 2.1. *Under the assumptions of Theorem 1.1, for any $\Omega'\Subset\Omega$, there is constant c_4 depending only on n,q,Λ,c_1,Ω, and Ω' such that*

$$\|u\|_{C^{1,\alpha}(\Omega')}\le c_4\quad\textit{for some } 0<\alpha<1.$$

Proof. We apply [10, Theorem 4.2], which gives a uniform upper bound in the $C^{1,\alpha}(\Omega')$-norm for minimizers of

$$\int_\Omega |\nabla u|^2 + 2G(u)\,dx, \tag{2.5}$$

when $g = G'$ satisfies $|g(u)| \le K|u|^p$ for some $p \in (-1, 0)$ and $K > 0$. This upper bound depends only on K, p, n, α, and $\|u\|_{L^\infty(\Omega)}$. Using Step 4 in the proof of Theorem 1.1, we know that a minimizer of (1.1) is uniformly bounded by c_1. Now let

$$g(u) = \begin{cases} f(u) & \text{for } |u| \le 2c_1, \\ f(2c_1)|\frac{u}{2c_1}|^{q-1} & \text{for } u \ge 2c_1, \\ f(-2c_1)|\frac{u}{2c_1}|^{q-1} & \text{for } u \le -2c_1. \end{cases}$$

Notice that the condition $|g(u)| \le K|u|^{q-\varepsilon}$ holds for some $0 < \varepsilon < 1 + q$. The constant K depends only on Λ, c_1, and ε. We claim that u is also a minimizer of (2.5). Thus, the statement in the proposition holds, due to [10, Theorem 4.2].

In order to show our claim, assume that v is minimizer of (2.5) so that $v = u = \phi$ on $\partial\Omega$ and

$$\int_\Omega |\nabla v|^2 + 2G(v)\,dx < \int_\Omega |\nabla u|^2 + 2G(u)\,dx. \tag{2.6}$$

We show that $\|v\|_{L^\infty(\Omega)} \le 2c_1$. Indeed, in the set $\{v > c_1\}$ we have

$$\Delta v = g(v) \ge -\mathcal{K},$$

where $\mathcal{K}$ is the constant defined in Step 3 of the proof of Theorem 1.1. (Note that $c_1 > M$.) Thus,

$$\max_\Omega v \le \max_{\partial\{v>c_1\}} \left(v(x) + \frac{\mathcal{K}}{2n}|x|^2\right) \le c_1 + \mathcal{K}C(n,\Omega) \le 2c_1.$$

Hence, $G(v) = F(v)$ as well as $G(u) = F(u)$ and, by (2.6),

$$\int_\Omega |\nabla v|^2 + 2F(v)\,dx < \int_\Omega |\nabla u|^2 + 2F(u)\,dx,$$

which contradicts that u is a minimizer. ∎

3. Optimal growth (Proof of Theorem 1.2)

The main idea to prove the optimal growth in (1.4) is using scaling and blow-up, and arriving at a contradiction. One of the key ideas in studying the infinitesimal properties of the free boundary is to make an infinite "zoom-in" or "blow-up" at a free boundary point.

More specifically, given a minimizer u of (1.1) and $0 < r < 1$, define the rescaling

$$u_{r,z}(x) := \frac{u(rx + z)}{\mu(r)}, \quad \forall x \in \Omega_{r,z} = \frac{1}{r}(\Omega - z)$$

for free boundary point z and $\mu(r) := r^\beta |\log r|^{\beta/2}$.

Proof of Theorem 1.2. We claim that for any local minimizer u and any free boundary point z, we have

$$\sup_{B_r(z)} |u| \leq C\mu(r), \quad 0 < r < 1/2,$$

which would imply the statement of Theorem 1.2.

Suppose there is no such constant C. Then, we would be able to find a sequence of local minimizers u_j, a sequence of real positive numbers $r_j \searrow 0$, and a sequence z^j of free boundary points Γ_0 such that

$$\sup_{B_r(z^j)} |u_j| \leq j\mu(r), \quad \forall\, 0 < r_j \leq r < 1, \tag{3.1}$$

and such that for $r = r_j$,

$$\sup_{B_{r_j}(z^j)} |u_j| = j\mu(r_j). \tag{3.2}$$

Now define

$$\tilde{u}_j(x) := \frac{1}{j} u_{r_j,z^j}(x) = \frac{u_j(r_j x + z^j)}{j\mu(r_j)}, \quad \forall x \in \Omega_{r_j,z^j}.$$

Then, it follows from (3.1)–(3.2) that

$$\tilde{u}_j(0) = |\nabla \tilde{u}_j(0)| = 0; \qquad \sup_{B_1} |\tilde{u}_j| = 1; \qquad \sup_{B_R} |\tilde{u}_j| \leq R^\beta, \quad \forall\, R \geq 1. \tag{3.3}$$

Note that we have used the inequality $\mu(Rr_j) \leq R^\beta \mu(r_j)$ for $R \geq 1$.

Since u_j is a local minimizer of the functional $\mathcal{J}$, $\tilde{u}_j$ will be a local minimizer of

$$\mathcal{J}_j(v) := \int_{\Omega_{r_j,z^j}} |\nabla v|^2 + 2F_j(v)\, dx, \tag{3.4}$$

where

$$F_j(v) = \left(\frac{r_j}{j\mu(r_j)}\right)^2 F(j\mu(r_j)v).$$

For any R fixed, we apply Proposition 2.1 to $\tilde{u}_j$ as a minimizer of $\mathcal{J}_j$ in B_{2R}. Hence, the sequence $\tilde{u}_j$ is uniformly bounded in $C^{1,\alpha}(B_R)$. Up to a subsequence, we may assume that $\tilde{u}_j$ converges to $\tilde{u}_0$ locally in $C^1(\mathbb{R}^n)$. By (3.3), we have the following similar properties:

$$\tilde{u}_0(0) = |\nabla \tilde{u}_0(0)| = 0; \qquad \sup_{B_1} |\tilde{u}_0| = 1; \qquad \sup_{B_R} |\tilde{u}_0| \leq R^\beta, \quad \forall\, R \geq 1. \tag{3.5}$$

We claim that $\tilde{u}_0$ is a harmonic function which, along with (3.5), violates Liouville's theorem and thus, we have a contradiction. To close the argument, we shall prove this claim by showing that $\tilde{u}_0$ is a minimizer of

$$\int_{B_R} |\nabla v|^2\, dx$$

for any R. Assume that $v = \tilde{u}_0$ on ∂B_R and define $v_j := v + \tilde{u}_j - \tilde{u}_0$. So, $v_j = \tilde{u}_j$ on ∂B_R. Since $\tilde{u}_j$ is a minimizer of (3.4), for sufficiently large j we have $B_R \subset \Omega_{r_j,z^j}$ and so

$$\int_{B_R} |\nabla \tilde{u}_j|^2 + 2F_j(\tilde{u}_j)\, dx \le \int_{B_R} |\nabla v_j|^2 + 2F_j(v_j)\, dx. \tag{3.6}$$

On the other hand, we recall from estimate (2.1) that $|F(u)| \le C_1 |u|^{1+q}(1 + |\log |u||)$ and get

$$\begin{aligned}|F_j(v)| &= \Big(\frac{r_j}{j\mu(r_j)}\Big)^2 |F(j\mu(r_j)v)| \le C_1 \frac{r_j^2}{(j\mu(r_j))^{1-q}}(1 + |\log(j\mu(r_j)v)|) \\ &\le \frac{C_1}{j^{1-q}|\log r_j|}\Big(1 + \log j + \beta|\log r_j| + \frac{\beta}{2}|\log|\log r_j|| + |\log|v||\Big).\end{aligned}$$

Hence,

$$F_j(v) \to 0 \quad \text{uniformly for } |v| \le C.$$

Passing (3.6) to the limit, we get

$$\int_{B_R} |\nabla \tilde{u}_0|^2 dx \le \int_{B_R} |\nabla v|^2 dx.$$

■

Corollary 3.1. *Let u be a minimizer of $\mathcal{J}$. Then, there is a constant $\tilde{c}_2 = \tilde{c}_2(n, q, \Lambda, c_2)$ such that*

$$\sup_{B_r(z)} |\nabla u| \le \tilde{c}_2 r^{\beta-1} |\log r|^{\beta/2}, \quad r < 1/4, \tag{3.7}$$

for $z \in \Gamma_0$ and $B_r(z) \subset \Omega$.

Proof. Let $u_r(x) := u(z + rx)/\mu(r)$. It will be a minimizer of

$$\mathcal{J}_r(v) := \int_{B_1} |\nabla v|^2 + 2F_r(v) dx,$$

where $F_r(v) := r^2 F(\mu(r)v)/\mu(r)^2$. We have the following estimate for derivative of F_r at $v \ne 0$:

$$\begin{aligned}|F_r'(v)| &\le r^2 |f(\mu(r)v)|/\mu(r) \le \Lambda r^2 \mu(r)^{q-1}|v|^q(1 + |\log|\mu(r)v||) \\ &\le 3\Lambda |v|^q (1 + |\log|v||).\end{aligned}$$

By (1.4), $\|u_r\|_{L^\infty(B_1)} \le c_2$. This, together with Proposition 2.1, entails that $\|u_r\|_{C^{1,\alpha}(B_{1/2})}$ is uniformly bounded for all $r < 1/2$. This proves (3.7). ■

4. Euler–Lagrange equation

In this section we derive the Euler–Lagrange equation for minimizers of (1.1) (see Theorem 1.3). First, by the following lemma, we show that the minimizer is a sort of subsolution:

Lemma 4.1. *If u is a minimizer of* (1.1)*, then* $u^\pm := \max(\pm u, 0)$ *satisfies*

$$\Delta u^\pm \geq \pm f(u)\chi_{\{\pm u>0\}} \quad \text{in } \Omega$$

in the weak sense.

Proof. Given a non-negative test function $\phi \in C_0^\infty(\Omega)$, choose a sequence $\varepsilon_n \to 0$ such that the level sets $\{u = \varepsilon_n\}$ are smooth. (The existence of this sequence is established by virtue of Sard's theorem.)

$$\begin{aligned} -\int_\Omega \nabla u^+ \cdot \nabla \phi \, dx &= \lim_{\varepsilon_n \to 0} -\int_{\{u>\varepsilon_n\}} \nabla u \cdot \nabla \phi \, dx \\ &= \lim_{\varepsilon_n \to 0} \left(\int_{\{u>\varepsilon_n\}} f(u)\phi \, dx + \int_{\{u=\varepsilon_n\}} |\nabla u| \phi \, d\mathcal{H}^{n-1} \right) \\ &\geq \limsup_{\varepsilon_n \to 0} \int_{\{u>\varepsilon_n\}} f(u)\phi \, dx = \int_{\{u>0\}} f(u)\phi \, dx, \end{aligned}$$

where in the last line we have used the monotone convergence theorem. (Condition (H3) implies that $f(u) > 0$ for $0 < u < \delta$ which shows that $f(u)\phi\chi_{\{u>\varepsilon_n\}}$ is increasing.) A similar argument holds for u^-. ∎

Now by the following lemma, we show that the right-hand side of (1.5) is integrable:

Lemma 4.2. *If u is a minimizer of* (1.1)*, then* $f(u)\chi_{\{u\neq 0\}} \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. Let $\Omega' \Subset \Omega$ be an arbitrary compact set and $\phi \in C_0^\infty(\Omega)$ be a cut-off function with $0 \leq \phi \leq 1$ and $\phi = 1$ in Ω'. By (H3), since u is bounded and $f(u) \geq 0$ in $\{0 < u < \delta\}$, we have

$$\begin{aligned} \int_{\Omega' \cap \{0<u\leq\delta\}} |f(u)| dx &= \int_{\Omega' \cap \{0<u\leq\delta\}} f(u)\phi dx \\ &\leq \int_{\Omega \cap \{0<u\leq\delta\}} f(u)\phi dx \\ &\leq \int_{\{0<u\}} f(u)\phi dx + C_1 \\ &\leq \int_\Omega -\nabla u^+ \cdot \nabla \phi dx + C_1 < \infty, \end{aligned}$$

where we have used Lemma 4.1 in the last line. Using the same argument, we get that $f(u)$ is integrable in $\Omega' \cap \{-\delta < u < 0\}$. Since $f(u)$ is continuous in $|u| > \delta$ and u is bounded, we get the desired result. ∎

Proof of Theorem 1.3. The Euler–Lagrange equation given by (1.5) follows trivially by the first variations in the set $\{|u| > 0\}$. To check the equation in the set $\{u = 0\}$, first consider some point x_0 that $\nabla u(x_0) \neq 0$. Since $u \in C^{1,\alpha}$, $\{u = 0\} \cap B_r(x_0) = \Gamma^+ \cap B_r(x_0) = \Gamma^- \cap B_r(x_0)$ is a C^1 hypersurface for some small r. Then,

$$\begin{aligned}\int_{B_r} \nabla u \cdot \nabla \phi \, dx &= \int_{\Omega^+ \cap B_r} \nabla u \cdot \nabla \phi \, dx + \int_{\Omega^- \cap B_r} \nabla u \cdot \nabla \phi \, dx \\ &= -\int_{\Omega^+ \cap B_r} f(u)\phi \, dx + \int_{\Gamma^+ \cap B_r} -|\nabla u| \phi \, d\mathcal{H}^{n-1} \\ &\quad - \int_{\Omega^- \cap B_r} f(u)\phi \, dx + \int_{\Gamma^- \cap B_r} |\nabla u| \phi \, d\mathcal{H}^{n-1} \\ &= -\int_{\{u \neq 0\} \cap B_r} f(u)\phi \, dx,\end{aligned}$$

for a test function $\phi \in C_0^\infty(B_r(x_0))$. Thus, (1.5) holds in $B_r(x_0)$.

In order to check the equation at Γ_0, choose $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta(t) = 0$ for $t \leq 1$ and $\eta(t) = 1$ for $t \geq 2$. Now define

$$w_k(x) = \eta\Big(\frac{k}{\log|\log d(x)|}\Big),$$

where $d(x) := \mathrm{dist}(x, \{u = |\nabla u| = 0\})$. This device is due to Ahlfors [1]; see also [3]. The main properties of the sequence w_k are that $w_k = 0$ in a neighborhood of $\{u = |\nabla u| = 0\}$; $\lim_{k\to\infty} w_k(x) = 1$ for $x \notin \{u = |\nabla u| = 0\}$; and for a positive constant C,

$$|\nabla w_k(x)| \leq \frac{C}{k}(d(x)|\log d(x)|)^{-1}. \tag{4.1}$$

For any test function $\phi \in C_0^\infty(B_r(x_0))$, we compute

$$\begin{aligned}-\int_\Omega \nabla u \cdot \nabla \phi \, dx &= \lim_{k\to\infty} -\int_\Omega (\nabla u \cdot \nabla \phi) w_k \, dx \\ &= \lim_{k\to\infty} \int_\Omega f(u)\phi w_k + (\nabla u \cdot \nabla w_k)\phi \, dx.\end{aligned}$$

The first integral in the right-hand side converges to

$$\int_\Omega f(u)\phi \chi_{\{u \neq 0\}} \, dx,$$

due to the dominated convergence theorem and recalling that $f(u) \in L^1_{\mathrm{loc}}$. In the rest of the paper, we are going to show that the second term vanishes in the limit. This completes the proof.

For any $x \in \Omega$ close enough to $\{u = |\nabla u| = 0\}$, let $y \in \partial\{|u| > 0\}$ be any point such that $d(x) = |x - y|$ is realized. By Theorem 1.2 and Corollary 3.1,

$$|u(x)| \leq c_2 d(x)^\beta |\log d(x)|^{\beta/2}, \quad |\nabla u(x)| \leq \tilde{c}_2 d(x)^{\beta-1} |\log d(x)|^{\beta/2}. \tag{4.2}$$

Applying estimates (4.1) and (4.2), along with the fact $1 < \beta < 2$, we get

$$\begin{aligned}\left|\int_\Omega (\nabla u \cdot \nabla w_k)\phi\, dx\right| &\leq \int_{\{d(x)>0\}} \frac{C}{k} d(x)^{\beta-2} |\log d(x)|^{(\beta-2)/2}\, dx \\ &\leq \int_\Omega \frac{C}{k} |u(x)|^{(\beta-2)/\beta}\, dx \\ &\leq \int_{\{|u|\leq\delta\}} \frac{C}{k\lambda} f(u)|\log|u^{-1}||^{-1}\, dx \\ &\quad + \int_{\{|u|>\delta\}} \frac{C}{k} |u(x)|^{(\beta-2)/\beta}\, dx \\ &\leq \int_{\{|u|\leq\delta\}} \frac{C}{k\lambda} f(u)|\log\delta|^{-1}\, dx + \int_{\{|u|>\delta\}} \frac{C}{k}\delta^q\, dx \to 0. \qquad \blacksquare\end{aligned}$$

5. Analysis of the free boundary

In this section, we prove non-degeneracy, porosity, and the estimate for the Hausdorff dimension of the free boundary; see Theorems 1.4, 1.5, and 1.6, respectively.

Proof of Theorem 1.4. Using the optimal growth estimate given by (1.4), we have that, for $r < r_0 < 1$ ($r_0 = r_0(\beta)$),

$$|u(x)| \leq \sup_{B_r(z)} |u| \leq c_2 r^\beta |\log r|^{\beta/2}, \quad \forall\, x \in B_r(z),$$

where $z \in \Gamma_0$. Choosing r_0 small enough, we assume

$$c_2 r^\beta |\log r|^{\beta/2} \leq r.$$

In particular, $\log|u(x)| \leq \log r$. From condition (H3) for $r < \delta$ when $u(x) > 0$, we have

$$\begin{aligned}\Delta u = f(u) &\geq \lambda u^q \log u^{-1} \geq \lambda u^q \log(r^{-1}) \geq \lambda c_2^q (r^\beta |\log r|^{\beta/2})^q \log(r^{-1}) \\ &= \lambda c_2^q r^{\beta-2} |\log r|^{\beta/2}.\end{aligned}$$

Similarly, in the case $u(x) < 0$, we have

$$\Delta u = f(u) \leq \lambda |u|^q \log|u| \leq \lambda c_2^q (r^\beta |\log r|^{\beta/2})^q \log r = -\lambda c_2^q r^{\beta-2} |\log r|^{\beta/2}.$$

For an arbitrary point $y \in \Omega^+$ (close to z), set $w(x) = c|x-y|^2$, where

$$0 < c = \lambda c_2^q r^{\beta-2} |\log r|^{\beta/2}/2n.$$

Then, $\Delta(u-w) \geq 0$ in $B_r(y) \cap \Omega^+$ and, using the maximum principle, we conclude that

$$0 \leq u(y) \leq \sup_{B_r(y)\cap\Omega^+} (u-w) \leq \sup_{\partial B_r(y)\cap\Omega^+} (u-w) = \sup_{\partial B_r(y)\cap\Omega^+} u - cr^2.$$

Letting $y \to z$, we arrive at

$$\sup_{\partial B_r(z)} u^+ \geq cr^2 = c_3 r^\beta |\log r|^{\beta/2},$$

where $c_3 = \lambda c_2^q/2n$. For u^-, we can argue in the same way. ■

Before coming to the proof of porosity of the free boundary $\partial\{|u| > 0\}$, we revisit the definition of porosity; for more details, see [11]. We say a measurable set $E \subset \mathbb{R}^n$ is *porous* with a porosity constant $0 < \delta < 1$ if every ball $B = B_r(x)$ contains a smaller ball $B' = B_{\delta r}(y)$ such that

$$B_{\delta r}(y) \subset B_r(x) \setminus E.$$

We say E is *locally porous* in an open set Ω if $E \cap K$ is porous for any $K \Subset \Omega$.

Proof of Theorem 1.5. Let

$$\tau := \min\{r_0, \operatorname{dist}(K, \partial\Omega)\},$$

where r_0 is the constant from Theorem 1.4. Now consider $x \in \Gamma_0 \cap K$ with ball $B_r(x)$. If $0 < r < \tau$, the non-degeneracy property (Theorem 1.4) implies the existence of $y \in \partial B_{r/2}(x)$ such that

$$u(y) \geq c_3 \Big(\frac{r}{2}\Big)^\beta \Big|\log \frac{r}{2}\Big|^{\beta/2}.$$

Using Theorem 1.2, we have

$$c_3 \Big(\frac{r}{2}\Big)^\beta \Big|\log \frac{r}{2}\Big|^{\beta/2} \leq u(y) \leq c_2 d(y)^\beta |\log d(y)|^{\beta/2}, \tag{5.1}$$

where $d(y) = \operatorname{dist}(y, \Gamma_0)$. On the other hand, the function $s \mapsto -s \log s$ is increasing in the interval $(0, s_0)$ for some s_0. Let

$$r_1 := \min\{\tau, s_0\}.$$

Then, if $0 < r < r_1$, we have $d(y) \leq \frac{r}{2} < r_1$. Thus,

$$d(y)|\log d(y)| \leq \frac{r}{2}\Big|\log \frac{r}{2}\Big|.$$

Back to (5.1), we see that

$$c_3 \Big(\frac{r}{2}\Big)^\beta \Big|\log \frac{r}{2}\Big|^{\beta/2} \leq c_2 d(y)^{\beta/2} \Big(\frac{r}{2}\Big|\log \frac{r}{2}\Big|\Big)^{\beta/2}.$$

Hence,

$$\delta r \leq d(y), \quad \text{where } \delta := \frac{1}{2}\Big(\frac{c_3}{c_2}\Big)^{2/\beta},$$

and $B_{\delta r}(y) \subset \Omega \setminus \Gamma_0$.

The porosity implies that the upper density of the free boundary at every point $x \in \Gamma_0$ is less than one, that is,

$$\limsup_{r\to 0} \frac{|\Gamma_0 \cap B_r(x)|}{|B_r|} \le 1 - \delta^n < 1.$$

This completes the proof. ■

Proof of Theorem 1.6. Consider a covering $\Gamma_0 \subset \bigcup_{i\in I} B_r(y_i)$ for some small r such that no more than N balls from this covering overlap (N just depends on n). Applying Theorem 1.5, one gets $|B_r| \le \delta^{-n}|B_r \setminus \Gamma_0|$. This implies

$$\sum_{i\in I} r^n = C \sum_{i\in I} |B_r(y_i)| \le C\delta^{-n} \sum_{i\in I} |B_r(y_i) \setminus \Gamma_0| \le C\delta^{-n} N \big|\{x : 0 < d(x) \le r\}\big|,$$

where $d(x) := \operatorname{dist}(x, \Gamma_0)$.

By applying Theorem 1.2, we get

$$\begin{aligned}
r^{\beta-2}\big|\{x : 0 < d(x) \le r\}\big| &\le \int_{\{0<d(x)\le r\}} \Big(\frac{1}{d(x)}\Big)^{2-\beta} dx \\
&\le C \int_{\{0<d(x)\le r\}} |u(x)|^q |\log d(x)|^{1-\beta/2} dx \\
&\le C\lambda^{-1} \int_{\{0<d(x)\le r\}} |f(u)| \frac{|\log d(x)|^{1-\beta/2}}{|\log |u||} dx \\
&\le C\lambda^{-1} \int_{\{0<d(x)\le r\}} \frac{|f(u)|\,|\log d(x)|^{1-\beta/2}}{|\log(c_2 d(x)^\beta |\log d(x)|^{\beta/2})|} dx \\
&\le C\lambda^{-1} \int_{\{0<d(x)\le r\}} \frac{|f(u)|\,|\log d(x)|^{1-\beta/2}}{\beta|\log d(x)| - |\log c_2| - \beta|\log|\log d(x)||/2} dx \\
&\le \frac{2C\lambda^{-1}}{|\log r|} \int_\Omega |f(u)| dx \to 0, \qquad \text{as } r \to 0.
\end{aligned}$$

The conclusion follows from the fact that $|\{x : 0 < d(x) \le r\}| = o(r^{2-\beta})$ as $r \to 0$. ■

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