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Parallel algorithms for the solution of variational inequalities

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One of the many ways of solving free-boundary problems is, when possible, to put them (perhaps after suitable transformations) in the framework of variational or quasi-variational inequalities. It then remains to solve them numerically, a task which has been studied by Glowinski, Lions, & Tremolieres [9] without reference to parallel algorithms.

On the other hand, systematic attempts to decompose the problems of the calculus of variations and of control theory have been made by Bensoussan, Lions, & Temam [4], using, among other things, ideas arising from splitting methods (see Marchuk [25] and the bibliography therein).

We propose here a general method for obtaining, in infinitely many ways, stable parallel algorithms for the solution of variational inequalities of evolution. This method was introduced in [12] for *equations* of evolution. We show here how it can be adapted to variational inequalities (what is needed from [12] is recalled here).

1. Introduction

We want to introduce parallel algorithms for the analysis and control of free-surface problems. We consider here some particular free-surface problems which can be expressed in the framework of variational inequalities. The formulation of variational inequalities will be (briefly) recalled in Section 2 below. For the time being, to fix ideas, let us recall merely that the classical Stefan free-surface problem *can* be formulated in the framework of variational inequalities. There is a huge literature devoted to parallel algorithms. Many of them are based on domain-decomposition methods, see P. L. Lions [24] Glowinski, Periaux, Shi, Widlund [10] and the bibliography therein. We have introduced in [12], and we have begun to develop in [13]–[16], a very general formulation of decomposition methods. This method is introduced and used here for the 'parallel solution' of variational inequalities. The general (abstract) decomposition is presented in Section 3, some examples being given in Section 4. Parallel algorithms are then given in Section 5. Section 6 presents further remarks and problems. We do not study here the stabilization and control of variational inequalities using the method of decomposition of Sections 3 and 5.

The numerical solution of variational inequalities has been studied by Glowinski, Lions, & Tremolieres [9], but without decomposition and parallelism.

Systematic methods of decomposition of problems of the calculus of variations and of control have been introduced by Bensoussan, Lions, & Temam [4], based on the decomposition of operators related to splitting methods (see the bibliography therein) and on the splitting of constraints, such as introduced by Lions & Temam [23]. All these methods could be combined with those introduced here, but the benefits would be unclear. We note also that the penalty arguments used here could be replaced by Lagrange-multiplier techniques. This is developed in current work by Lions & Pironneau [20, 21].

2. Formulation of the variational inequalities of evolution

Variational inequalities of evolution were introduced by Lions & Stampacchia [22]. We give here a simple presentation of them.

We are given two real Hilbert spaces V and H such that

$$V \subset H, \quad V \text{ dense in } H, \quad V \to H \text{ being continuous.} \tag{2.1}$$

We identify H with its dual, so that if V' denotes the dual of V, then

$$V \subset H \subset V'. \tag{2.2}$$

We are also given a set $K \subset V$ such that

$$K \text{ is a closed convex subset of } V. \tag{2.3}$$

We do not restrict generality (it suffices to make a translation) by assuming that

$$0 \in K. \tag{2.4}$$

Let f be given such that

$$f \in L^2(0, T; V'). \tag{2.5}$$

We consider now a bilinear form

$$\begin{aligned} &u, \hat{u} \to a(u, \hat{u}) \text{ which is continuous on } V \times V, \\ &a(u, \hat{u}) \text{ is symmetric } \textit{or not}, \\ &a(u, u) \geqslant \alpha \|u\|_V^2, \quad \alpha > 0, \quad \forall u \in V \end{aligned} \tag{2.6}$$

(where we denote by $\|u\|_X$ the norm of u in X).

We are interested in the solution of the following variational inequalities of evolution: find u such that

$$\begin{aligned} &u \in L^2(0, T; V) \cap L^\infty(0, T; H), \qquad u(t) \in K \text{ a.e.}, \\ &\left(\frac{\partial u}{\partial t}, \hat{u} - u\right)_H + a(u, \hat{u} - u) \geqslant (f, \hat{u} - u) \quad \forall \hat{u} \in K, \\ &u(0) = 0. \end{aligned} \tag{2.7}$$

REMARK 2.1 The solution has to be thought of as being *a weak solution of* (2.7); otherwise the condition '$u(0) = 0$' in (2.7) is somewhat ambiguous. This condition becomes precise if we add the condition

$$\frac{\partial u}{\partial t} \in L^2(0, T; V'), \tag{2.8}$$

but this condition can be too restrictive. We can introduce weak solutions in the following form.

We consider smooth functions $\hat{u}$ such that

$$\begin{aligned} &\hat{u} \in L^2(0, T; V), \qquad \frac{\partial \hat{u}}{\partial t} \in L^2(0, T; V'), \\ &\hat{u}(t) \in K \text{ for a.e. } t, \qquad \hat{u}(0) = 0. \end{aligned} \tag{2.9}$$

Then, if u satisfies (2.7) and is supposed to be smooth enough, we have (we write $(u, \hat{u})$ instead of $(u, \hat{u})_H$)

$$\int_0^T \left[\left(\frac{\partial \hat{u}}{\partial t}, \hat{u} - u \right) + a(a, \hat{u} - u) - (f, \hat{u} - u) \right] \mathrm{d}t$$
$$= \int_0^T \left[\left(\frac{\partial u}{\partial t}, \hat{u} - u \right) + a(u, \hat{u} - u) - (f, \hat{u} - u) \right] \mathrm{d}t$$
$$+ \int_0^T \left(\frac{\partial (\hat{u} - u)}{\partial t}, \hat{u} - u \right) dt.$$

The last term equals $\frac{1}{2} \|\hat{u}(T) - u(T)\|_H^2$ (since $u(0) = 0$, $\hat{u}(0) = 0$), so that

$$\int_0^T \left[\left(\frac{\partial \hat{u}}{\partial t}, \hat{u} - u \right) + a(u, \hat{u} - u) - (f, \hat{u} - u) \right] \mathrm{d}t \geqslant 0 \tag{2.10}$$

for all $\hat{u}$ satisfying (2.9).

We then define a *weak solution* of (2.7) as a function u such that

$$u \in L^2(0, T; V), \quad u(t) \in K \text{ a.e.}, \tag{2.11}$$

and which satisfies (2.10) for all $\hat{u}$ satisfying (2.9). See Lions [17] and a simple presentation in [18].

REMARK 2.2 Let us show how formally (2.7) 'follows' from (2.10). Let us take, in (2.10),

$$\hat{u} = \theta w + (1 - \theta) u, \quad 0 < \theta < 1,$$

where w is smooth and satisfies conditions analogous to those of (2.9).

Then, after dividing by θ, we have

$$\int_0^T \left[\left(\theta \frac{\partial w}{\partial t} + (1 - \theta) \frac{\partial u}{\partial t}, w - u \right) + a(u, w - u) - (f, w - u) \right] \mathrm{d}t \geqslant 0.$$

Letting $\theta \to 0$, we obtain

$$\int_0^T \left[\left(\frac{\partial u}{\partial t}, w - u \right) + a(u, w - u) - (f, w - u) \right] \mathrm{d}t \geqslant 0 \tag{2.12}$$

for all $w \in L^2(0, T; V)$ such that $w(t) \in K$ a.e..

By taking

$$w = \begin{cases} \hat{u} & \text{in a neighbourhood } \sigma \text{ of } t_o \in]0, T[, \\ u & \text{elsewhere,} \end{cases}$$

and letting $|\sigma| \to 0$, we obtain (2.7).

REMARK 2.3 For the proofs of existence and uniqueness of the solution of (2.7) or (2.10), we refer to Lions & Stampacchia [22], and to the books [17, 18].

REMARK 2.4 The formulation of Stefan's problem in the framework of variational inequalities is due to Duvaut [5]. For simple proofs, see for instance [18] or the recent course [28]. Very many free-boundary problems in the framework of variational inequalities are introduced and studied by Duvaut & Lions [6], Baiocchi & Capelo [1], Kinderlehrer & Stampacchia [11], Elliott & Ockendon [7], Friedmann [8], Meirmanov [26], and Rodrigues [27].

REMARK 2.5 The methods which follow apply to all the variational inequalities introduced in these references—with the exception of non-local problems: see Section 6.

3. Decomposition method

We introduce N couples of Hilbert spaces V_i and H_i, and N convex sets K_i:

$$V_i \subset H_i \subset V_i' \quad (i = 1, \dots, N), \tag{3.1}$$

$$K_i \subset V_i,\ K_i \text{ closed convex subset of } V_i,\ \text{non empty.} \tag{3.2}$$

We are given linear operators r_i such that

$$\begin{aligned} &r_i \in \mathcal{L}(H; H_i) \cap \mathcal{L}(V; V_i) \quad (i = 1, \dots, N), \\ &r_i \text{ maps } K \text{ into } K_i. \end{aligned} \tag{3.3}$$

We are also given a family of Hilbert spaces H_{ij} such that

$$H_{ij} = H_{ji} \quad \forall i, j \in [1, \dots, N], \tag{3.4}$$

and a family of operators r_{ij} such that

$$r_{ij} \in \mathcal{L}(H_j; H_{ij}). \tag{3.5}$$

The following hypotheses are made:

$$r_j r_{ji} \varphi = r_i r_{ij} \varphi \quad \forall \varphi \in V, \tag{3.6}$$

$$\begin{aligned} &\text{if } N \text{ elements } u_i \text{ are given such that} \\ &u_i \in K_i \quad \forall i, \qquad r_{ij} u_j = r_{ji} u_i \quad \forall i, j \\ &\text{then there exists } u \in K \text{ such that} \\ &u_i = r_i u, \quad \text{and moreover} \quad \|u\|_V^2 \leqslant c \left(\sum_{i=1}^{N} \|u_i\|_{V_i}^2 \right). \end{aligned} \tag{3.7}$$

REMARK 3.1 Examples are given in Section 4.

REMARK 3.2 If $K_i = V_i$ for all i, we are in the situation of equations (see [13, 14]).

REMARK 3.3 The hypothesis

$$K_i = V_i \text{ for a subset of } [1, \dots, N] \tag{3.8}$$

is perfectly acceptable!

We now proceed with the decomposition of the problem. We introduce the following bilinear forms:

$$c_i(u_i, \hat{u}_i) \text{ is continuous, } \textit{symmetric}\text{, on } H_i \times H_i\text{, and it satisfies} \\ c_i(u_i, u_i) \geqslant \gamma_i \|u_i\|_{H_i}^2, \gamma_i > 0, \forall u_i \in H_i, \tag{3.9}$$

$$a_i(u_i, \hat{u}_i) \text{ is continuous, symmetric } \textit{or not}\text{, on } V_i \times V_i\text{, and it satisfies} \\ a_i(u_i, u_i) \geqslant \alpha_i \|u_i\|_{V_i}^2, \alpha_i > 0, \forall u_i \in V_i, \tag{3.10}$$

We assume that

$$\sum_{i=1}^{N} c_i(r_i u, r_i \hat{u}) = (u, \hat{u})_H \quad \forall u, \hat{u} \in H, \tag{3.11}$$

$$\sum_{i=1}^{N} a_i(r_i u, r_i \hat{u}) = a(u, \hat{u}) \quad \forall u, \hat{u} \in V. \tag{3.12}$$

Finally, we assume that the function f is also 'decomposed' as follows:

$$\text{we are given functions } f_i \in L^2(0, T; V_i') \text{ such that} \\ \sum_{i=1}^{N} (f_i, r_i \hat{u}) = (f, \hat{u}) \quad \forall \hat{u} \in V. \tag{3.13}$$

We are now ready to introduce the *decomposed approximation*.

We look for functions u_i $(i = 1, \ldots, N)$ such that

$$c_i\left(\frac{\partial u_i}{\partial t}, \hat{u}_i - u_i\right) + a_i(u_i, \hat{u}_i - u_i) + \frac{1}{\varepsilon} \sum_j (r_{ji} u_i - r_{ij} u_j, r_{ji}(\hat{u}_i - u_i))_{H_{ij}} \\ \geqslant (f_i, \hat{u}_i - u_i) \quad \forall \hat{u}_i \in K_i, \tag{3.14}$$

$$u_i \in L^2(0, T; V_i), \qquad u_i(t) \in K_i \text{ a.e.}, \qquad u_i(0) = 0. \tag{3.15}$$

REMARK 3.4 Each of the variational inequalities (3.14) has to be thought of in its weak formulation, as introduced in Section 2.

REMARK 3.5 In (3.14), ε is positive and small. The corresponding term in (3.14) is a penalty term.

REMARK 3.6 In the examples, $\|r_{ji}\|$ is a sparse matrix. For a given i, the only j used in (3.14) are those such that

$$r_{ji} \neq 0$$

(they are the 'neighbours' of i).

One can prove

THEOREM 3.1 The set of (decomposed) variational inequalities (3.14), (3.15) admits a unique solution $u_i = u_i^\varepsilon$ $(i = 1, \ldots, N)$. Further, as $\varepsilon \to 0$, one has

$$u_i^\varepsilon \to u_i \quad \text{in} \quad L^2(0, T; V_i) \text{ weakly} \tag{3.16}$$

and

$$u_i = r_i u, \tag{3.17}$$

where u is the solution of (2.7) (actually of (2.10)).

We now present a sketch of the proof.

Step 1: A priori estimates

We can assume, without loss of generality, that $0 \in K_i$. Therefore taking $\hat{u}_i = 0$ in (3.14) is allowed (for a complete proof, the technical details are much more complicated. One has to work first on approximations of (3.14), by using (other) penalty arguments; see the bibliographical references). This simplifiction gives (we write u_i instead of u_i^ε for the time being)

$$c_i\left(\frac{\partial u_i}{\partial t}, u_i\right) + a_i(u_i, u_i) + \frac{1}{\varepsilon} X_i \leqslant (f_i, u_i) \quad (i = 1, \ldots, N), \tag{3.18}$$

where

$$X_i = \sum_j (r_{ji} u_i - r_{ij} u_j, r_{ji} u_i)_{H_{ij}}. \tag{3.19}$$

We can write

$$X_i = \frac{1}{2} \sum_j \|r_{ji} u_i - r_{ij} u_j\|_{H_{ij}}^2 + \frac{1}{2} \sum_j \|r_{ji} u_i\|_{H_{ij}}^2 - \frac{1}{2} \sum_j \|r_{ij} u_i\|_{H_{ij}}^2. \tag{3.20}$$

But one verifies easily that

$$\sum_i X_i = \frac{1}{2} \sum_{i,j} \|r_{ji} u_i - r_{ij} u_j\|_{H_{ij}}^2. \tag{3.21}$$

Therefore by integration in t, in the interval $(0, t)$, of (3.18), and by summing in i, using (3.21), we obtain

$$\begin{aligned} &\frac{1}{2} \sum_i c_i(u_i(t)) + \sum \int_0^t a_i(u_i(s))\,\mathrm{d}s + \\ &+ \frac{1}{2\varepsilon} \sum_{i,j} \int_0^t \|r_{ji} u_i(s) - r_{ij} u_j(s)\|_{H_{ij}}^2\,\mathrm{d}s \leqslant \sum_i \int_0^t (f_i, u_i)\,\mathrm{d}s. \end{aligned} \tag{3.22}$$

Step 2

It follows easily from (3.22), (3.9), and (3.10) that, as $\varepsilon \to 0$ (and we now use the notation u_i^ε),

$$\begin{aligned} &u_i^\varepsilon \text{ remains in a bounded set of } L^2(0, T; V_i) \cap L^\infty(0, T; H_i), \\ &u_i^\varepsilon(t) \in K_i, \end{aligned} \tag{3.23}$$

$$\frac{1}{\sqrt{\varepsilon}}(r_{ji}u_i^\varepsilon - r_{ij}u_j^\varepsilon) \text{ remains in a bounded set of } L^2(0, T; H_{ij}). \tag{3.24}$$

Therefore we can extract a subsequence, still denoted by u_i^ε, such that

$$\begin{aligned} &u_i^\varepsilon \to u_i \quad \text{in} \quad L^2(0, T; V_i) \text{ weakly,} \\ &u_i(t) \in K_i, \end{aligned} \tag{3.25}$$

and, by virtue of (3.24), we have

$$r_{ji}u_i = r_{ij}u_j \quad \forall i, j. \tag{3.26}$$

Notice that we have not used the fact that u_i^ε remains in a bounded set of $L^\infty(0, T; H_i)$.

It follows from (3.25), (3.26), and the hypothesis (3.7) that

$$u_i = r_i u_*, \qquad u_*(t) \in K \text{ a.e.}, \qquad u_* \in L^2(0, T; V). \tag{3.27}$$

It remains to show that $u_* = u$, the solution of (2.7) or (2.10).

Step 3

We use the weak formulation recalled in Section 2. To avoid slight technical difficulties, we further *weaken* (2.10), by writing it (in a perfectly legitimate way!)

$$\int_0^T \left[\left(\frac{\partial \hat{u}}{\partial t}, \hat{u} - u\right) + a(\hat{u}, \hat{u} - u) - (f, \hat{u} - u)\right] \mathrm{d}t \geqslant 0 \tag{3.28}$$

for all $\hat{u}$ satisfying (2.9).

We introduce $\hat{u}_i$ such that

$$\begin{aligned} &\hat{u}_i \in L^2(0, T; V_i), \qquad \frac{\partial \hat{u}_i}{\partial t} \in L^2(0, T; V_i'), \\ &\hat{u}_i(t) \in K_i \text{ for a.e. } t, \qquad \hat{u}_i(0) = 0, \end{aligned} \tag{3.29}$$

and we replace (3.24) by its (very) weak form

$$\begin{aligned} &\int_0^T \left[c_i\left(\frac{\partial \hat{u}_i}{\partial t}, \hat{u}_i - u_i^\varepsilon\right) + a_i(\hat{u}_i, \hat{u}_i - u_i^\varepsilon) + \frac{1}{\varepsilon}\sum_j (r_{ji}\hat{u}_i - r_{ij}\hat{u}_j, r_{ji}(\hat{u}_i - u_i^\varepsilon))_{H_{ij}}\right] \mathrm{d}t \\ &\geqslant \int_0^T (f_i, \hat{u}_i - u_i^\varepsilon)\, \mathrm{d}t. \end{aligned} \tag{3.30}$$

Let us now assume that

$$\begin{aligned} &\hat{u}_i = r_i\varphi, \\ &\varphi \in L^2(0, T; V), \qquad \frac{\partial \varphi}{\partial t} \in L^2(0, T; V'), \qquad \varphi(0) = 0, \qquad \varphi(t) \in K. \end{aligned} \tag{3.31}$$

Since $r_{ji}r_i\varphi = r_{ij}r_j\varphi$, the $\frac{1}{\varepsilon}$ terms in (3.30) drop out, so that

$$\begin{aligned} &\int_0^T \left[c_i\left(\frac{\partial}{\partial t}(r_i\varphi), r_i\varphi - u_i^\varepsilon\right) + a_i(r_i\varphi, r_i\varphi - u_i^\varepsilon)\right] \mathrm{d}t \\ &\qquad\qquad \geqslant \int_0^T (f_i, r_i\varphi - u_i^\varepsilon)\, \mathrm{d}t. \end{aligned} \tag{3.32}$$

We can pass to the limit in ε in (3.32). Because of (3.27), we obtain

$$\int_0^T \left[c_i\left(\frac{\partial}{\partial t}(r_i\varphi), r_i\varphi - r_i u_*\right) + a_i(r_i\varphi, r_i\varphi - r_i u_*)\right] \mathrm{d}t \geqslant \int_0^T (f_i, r_i\varphi - r_i u_*)\,\mathrm{d}t. \tag{3.33}$$

Summing (3.33) in i and using (3.11), (3.12), and (3.13), we obtain

$$\int_0^T [(\frac{\partial\varphi}{\partial t}, \varphi - u_*) + a(\varphi, \varphi - u_*) - (f, \varphi - u_*)]\,\mathrm{d}t \geqslant 0$$

so that (by uniqueness) $u_* = u$. □

4. Examples

Let Ω be an open set of $\mathbb{R}^d (d = 1, 2, 3$ in the applications). Let us consider, with the notation of Lions & Magenes [19].

$$V = H_0^1(\Omega) \subset H = L^2(\Omega), \tag{4.1}$$

$$K = \{v | v \geqslant 0 \quad \text{in} \quad \Omega, v \in V\}, \tag{4.2}$$

$$(u, v) = \int_\Omega uv\,\mathrm{d}x, \qquad a(u, v) = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x. \tag{4.3}$$

For these choices of the data, variational inequalities (2.7) becomes, in an explicit form:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u - f \geqslant 0, \\ &u \geqslant 0, \\ &\left(\frac{\partial u}{\partial t} - \Delta u - f\right) u = 0 \quad \text{in} \quad \Omega \times (0, T), \end{aligned} \tag{4.4}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{4.5}$$

$$u|_{t=0} = 0. \tag{4.6}$$

This is (after a transformation of the unknown as by Duvaut [5]; see [18] or [28]) the Stefan problem (with *one* phase). The free surface (resp. free region, resp. mushy region) is defined by

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} - \Delta u - f = 0 \quad \text{in} \quad \{u > 0\}. \tag{4.7}$$

We now decompose this problem, using the tools introduced in Section 3. We consider an *overlapping covering* of Ω, consisting of sets Ω_i such that

$$\Omega = \cup\Omega_i, \tag{4.8}$$

$$\text{for all } i, \text{ there exists at least one } j \text{ (a neighbour of } i\text{) such that} \quad \Omega_i \cap \Omega_j \neq \emptyset. \tag{4.9}$$

We introduce *partitions of unity* :

$$\begin{aligned} &\rho_i \text{ are defined in } \Omega_i, \\ &\text{(and can be extended by 0 outside } \Omega_i), \\ &\rho_i \geqslant 0 \quad \text{in} \quad \Omega_i, \\ &\sum_{i=1}^{N} \rho_i = 1 \quad \text{in} \quad \Omega, \end{aligned} \tag{4.10}$$

$$\sigma_i \text{ has the same properties as } \rho_i. \tag{4.11}$$

REMARK 4.1 One can have $\sigma_i = \rho_i$ or not. The hypothesis $\sigma_i = \rho_i$ is needed for precise error estimates, as will be shown elsewhere.

We now introduce

$$H_i = \left\{ u_i | \int_{\Omega_i} \rho_i u_i^2 \, \mathrm{d}x < \infty \right\}, \tag{4.12}$$

$$V_i = \left\{ u_i | \int_{\Omega_i} \sigma_i |\nabla u_i|^2 \, \mathrm{d}x < \infty, u_i = 0 \quad \text{on} \quad \partial\Omega_i \cap \partial\Omega \right\}. \tag{4.13}$$

REMARK 4.2 If $\bar{\Omega}_i \subset \Omega$, there are no boundary conditions in (4.13). If $\partial\Omega_i \cap \partial\Omega \neq \emptyset$, the condition $u_i = 0$ on $\partial\Omega_i \cap \partial\Omega$ does make sense.

We now define

$$c_i(u_i, \hat{u}_i) = \int_{\Omega_i} \rho_i u_i \hat{u}_i \, \mathrm{d}x, \tag{4.14}$$

$$a_i(u_i, \hat{u}_i) = \int_{\Omega_i} \sigma_i \nabla u_i \cdot \nabla \hat{u}_i \, \mathrm{d}x. \tag{4.15}$$

We introduce next

$$H_{ij} = L^2(\Omega_i \cap \Omega_j), \tag{4.16}$$

$$\begin{aligned} r_i u &= \text{ restriction of } u \in H \text{ (resp. V) to } \Omega_i, \\ r_{ji} u_i &= \text{ restriction of } u_i \in L^2(\Omega_i) \text{ to } \Omega_j. \end{aligned} \tag{4.17}$$

One defines

$$K_i = \{u_i | u_i \in V_i, u_i \geqslant 0 \quad \text{in} \quad \Omega_i\}. \tag{4.18}$$

Let us assume that

$$f \in L^2(\Omega \times (0, T))$$

and let τ_i be still another decomposition of unity. If we set

$$f_i = \tau_i f, \tag{4.19}$$

then all the hypotheses of Section 3 are satisfied, and one can use the decomposition (3.14).

The set of inequalities (3.14) can be made explicit. Let us define

$$P_i\{u_1,\ldots,u_N\} = \rho_i \frac{\partial u_i}{\partial t} - \sum_k \frac{\partial}{\partial x_k}\left(\sigma_i \frac{\partial u_i}{\partial x_k}\right) + \frac{1}{\varepsilon}(r_{ji}u_i - r_{ij}u_j). \tag{4.20}$$

Then

$$\begin{aligned} &P_i\{u_1,\ldots,u_N\} - f_i \geqslant 0, \\ &u_i \geqslant 0, \\ &(P_i\{u_1,\ldots,u_N\} - f_i)u_i = 0 \quad \text{in} \quad \Omega_i \times (0,T) \end{aligned} \tag{4.21}$$

with

$$u_i = 0 \quad \text{on} \quad \partial\Omega_i \cap \partial\Omega \tag{4.22}$$

and no other boundary conditions *if* σ_i *is* (suitably) *zero* on $\partial\Omega_i \backslash (\partial\Omega_i \cap \partial\Omega)$. (Otherwise there are some Neumann boundary conditions, since the test functions do not satisfy boundary conditions outside $\partial\Omega_i \cap \partial\Omega$).

REMARK 4.3 For a given i, only those j such that

$$\Omega_j \cap \Omega_i \neq \emptyset$$

appear in (4.21).

REMARK 4.4 The decomposition can be achieved in infinitely many ways.

REMARK 4.5 Several interfaces (actually N) appear. We do not know if this fact can be used to define a kind of mushy region.

REMARK 4.6 It is clear that the method presented for this particular example completely general, as far as the convex set K is defined by local constraints. See Section 6 below.

REMARK 4.7 All what has been said can be adapted to non-overlapping coverings (see [13]).

We now introduce a parallel algorithm based on the decomposition method introduced in Section 3.

5. Parallel algorithm

We introduce the time step Δt and a semi-discretization. We denote by u_i^n (what we hope is) an approximation of $u_i(n\Delta t)$.

We then define u_i^n by

$$\begin{aligned} &c_i\left(\frac{u_i^n - u_i^{n-1}}{\Delta t}, \hat{u}_i - u_i^n\right) + a_i(u_i^n, \hat{u}_i - u_i^n) \\ &+ \frac{1}{\varepsilon}\sum_j (r_{ji}u_i^n - r_{ij}u_j^{n-1}, r_{ji}(\hat{u}_i - u_i^n))_{H_{ij}} \geqslant (f_i^n, \hat{u}_i - u_i^n) \quad \forall \hat{u}_i \in K_i, \\ &u_i^n \in K_i \quad (n = 1, 2, \ldots), \end{aligned} \tag{5.1}$$

where

$$f_i^n = \frac{1}{\Delta t}\int_{(n-1)\Delta t}^{n\Delta t} f_i(t)\,\mathrm{d}t, \qquad u_i^o = 0. \tag{5.2}$$

REMARK 5.1 The algorithm (5.1) is parallel. Each u_i^n is computed through the solution of a stationary variational inequalities. Once the u_j^{n-1} are computed, in the computation of u_i^n, only those j such that $r_{ij} \neq 0$ are used.

REMARK 5.2 Methods other than the penalty method could be used as well—such as Lagrange-multiplier methods. This will be reported elsewhere (see Lions & Pironneau [21]).

Let us show now the *stability of the algorithm*. Replacing $\hat{u}_i$ by 0 in (5.1), we obtain

$$c_i(u_i^n - u_i^{n-1}, u_i^n) + a_i(u_i^n) + \frac{1}{\varepsilon}X_i^n \leqslant (f_i^n, u_i^n), \tag{5.3}$$

where

$$X_i^n = \sum_j (r_{ji}u_i^n - r_{ij}u_j^{n-1}, r_{ji}u_i^n)_{H_{ij}}. \tag{5.4}$$

We observe that

$$c_i\left(\frac{u_i^n - u_i^{n-1}}{\Delta t}, u_i^n\right) = \frac{1}{2\Delta t}c_i(u_i^n - u_i^{n-1}) + \frac{1}{2\Delta t}(c_i(u_i^n) - c_i(u_i^{n-1})),$$

so that

$$\sum_{n=1}^{m} c_i\left(\frac{u_i^n - u_i^{n-1}}{\Delta t}, u_i^n\right) = \frac{1}{2\Delta t}c_i(u_i^m) + \frac{1}{2\Delta t}\xi_{im}, \tag{5.5}$$

where

$$\xi_{im} = \sum_{n=1}^{m} c_i(u_i^n - u_i^{n-1}). \tag{5.6}$$

We observe next that

$$X_i^n = \frac{1}{2}\sum \|r_{ji}u_i^n - r_{ij}u_j^{n-1}\|_{H_{ij}}^2 + \frac{1}{2}\sum_j \|r_{ji}u_i^n\|_{H_{ij}}^2 - \frac{1}{2}\sum_j \|r_{ij}u_j^{n-1}\|_{H_{ij}}^2. \tag{5.7}$$

If we define

$$Y^n = \frac{1}{2}\sum_{i,j} \|r_{ji}u_i^n\|_{H_{ij}}, \tag{5.8}$$

then

$$\sum_i X_i^n = Z^n + Y^n - Y^{n-1}, \tag{5.9}$$

where

$$Z^n = \frac{1}{2}\sum_{i,j} \|r_{ji}u_i^n - r_{ij}u_j^{n-1}\|_{H_{ij}}^2. \tag{5.10}$$

Consequently, by summing (5.3) in i and in n, we obtain

$$\frac{1}{2\Delta t}\sum_i c_i(u_i^m)+\frac{1}{2\Delta t}\sum_i \xi_{im}+\sum_{n=1}^m\sum_i a_i(u_i^n) + \frac{1}{\varepsilon}Y^m+\frac{1}{\varepsilon}\sum_{n=1}^m Z^n \leqslant \sum_{n=1}^m\sum_i (f_i^n, u_i^n). \tag{5.11}$$

But

$$(f_i^n, u_i^n) \leqslant \frac{1}{2}a_i(u_i^n)+\frac{c}{2}\|f_i^n\|^2_{V_i'},$$

so that (5.11), after multiplying by $2\Delta t$, gives

$$\sum_i c_i(u_i^m)+\sum_i \xi_{im}+\Delta t\sum_{n=1}^m\sum_i a_i(u_i^n) + \frac{2\Delta t}{\varepsilon}Y^m+\frac{2\Delta t}{\varepsilon}\sum_{n=1}^m Z^n \leqslant c\Delta t\sum_{n=1}^m \|f_i^n\|^2_{V_i'} \leqslant c_4; \tag{5.12}$$

hence stability follows.

REMARK 5.3 Of course other time-discretization schemes could be used in (5.1).

6. Remarks and problems

REMARK 6.1 An open problem. The previous methods do not apply (at least without new ideas) for nonlocal constraints—for example, for variational inequalities of the type

$$\left(\frac{\partial u}{\partial t}, \hat{u}-u\right)_H + a(u,\hat{u}-u)+j(\hat{u})-j(u) \geqslant (f,\hat{u}-u) \quad \forall \hat{u}\in V, \tag{6.1}$$
$$u(t)\in V, \qquad u(0)=0,$$

where $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and where for instance

$$j(\hat{u}) = \left(\int_\Omega |\nabla\hat{u}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \square \tag{6.2}$$

REMARK 6.2 Bingham's flow (see Lions & Duvaut [6]) is an example of physical interest of variational inequalities with nonlocal constraints, similar to (but more complicated than) the previous example of Remark 6.1.

REMARK 6.3 One can extend the methods of the present paper to some quasi-variational inequalities (see [2] or [1]).

REMARK 6.4 The examples of decomposition given here (Section 4) correspond to decomposition of domains. Other possibilities can be envisioned, such as multi-Galerkin methods, using replica equations (in our case, replica variational inequalities). We shall investigate this topic in future research. Replica equations have been introduced in the paper [16], dedicated to the memory of G. Stampacchia.

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