Interfaces Free Bound. (Online first)
DOI 10.4171/IFB/509

Tangential contact between free and fixed boundaries for variational solutions to variable-coefficient Bernoulli-type free boundary problems

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Abstract. In this paper, we show that, given appropriate boundary data, the free boundaries of minimizers of functionals of type $J(v; A, \lambda_+, \lambda_-, \Omega) = \int_\Omega (\langle A(x)\nabla v, \nabla v\rangle + \Lambda(v))\, dx$ and the fixed boundary touch each other in a tangential fashion. We extend the results of Karakhanyan, Kenig, and Shahgholian [Calc. Var. Partial Differential Equations 28 (2007), 15–31] to the case of variable coefficients. We prove this result via classification of the global profiles, as per Karakhanyan, Kenig, and Shahgholian [Calc. Var. Partial Differential Equations 28 (2007), 15–31].

1. Introduction

The objective of this paper is to study the behavior of free boundary near the fixed boundary of domain for minimizers of Bernoulli-type functionals with Hölder continuous coefficients

$$J(v; A, \lambda_+, \lambda_-, \Omega) = \int_\Omega \big(\langle A(x)\nabla v, \nabla v\rangle + \Lambda(v)\big)\, dx, \tag{1.1}$$

where A is an elliptic matrix with Hölder continuous entries, and $\Lambda(v) = \lambda_+\chi_{\{v>0\}} + \lambda_-\chi_{\{v\le 0\}}$. We prove that if the value of boundary data and its derivative at a point are equal to zero (i.e., it satisfies the (DPT) condition mentioned below), then the contact of free boundary and the fixed boundary is tangential.

Several authors have extended the works of Alt, Caffarelli, and Friedman [2] on free boundary problems with constant coefficients to the case of variable coefficients. For example, the works of Argiolas and Ferrari [4, 14] which extend the seminal works [8, 9] to the case of x dependent coefficients. See also the recent work of Ferrari and Lederman [15] for the case of variable exponents. For the non-homogenous case with constant coefficients, one can refer to the work of De Silva, Ferrari, and Salsa [10].

Boundary interactions of free boundaries have gained significant attention in recent years. Whenever there are two media involved, the interactions of their respective diffusions can be modeled by free boundary problems. Often, free boundary of solution and

2020 Mathematics Subject Classification. Primary 49J05; Secondary 35B65, 35Q92, 35Q35.
Keywords. Variational calculus, Bernoulli-type free boundary problems, boundary behavior, Alt–Caffarelli–Friedman-type minimizers.

fixed boundary of set come in contact. In applications, the dam problem [3] and jets, wakes, and cavities [5] model phenomena which involve understanding of free boundary and fixed boundary.

Very recently, the works of Indrei [19, 20] study the interactions of free boundaries and fixed boundaries for fully non-linear obstacle problems. We refer to [17], where the authors shed more light into the angle of contact between fixed boundary and free boundary for one-phase Bernoulli problem.

We refer the reader to the work of Kenig, Karakhanyan, and Shahgholian [22, 23] where they deal with constant-coefficient Bernoulli-type free boundary problems. The case of variable coefficients in this paper brings forward some new difficulties (for example, the proof of Proposition 3.9). As it is common by now, our strategy in this article is to classify blowups of minimizers. We prove that the blowups and also their positivity sets converge to a global solution in $\mathcal{P}_\infty$ (cf. Definition 2.5). In Section 2, we list the assumptions and set some notations, and then in Section 3, we prove that blowups of minimizers converge to those of global solutions (cf. Definition 2.6). In the last section, we prove our main result.

2. Setting up the problem

We consider the following class of functions which we denote as $\mathcal{P}_r(\alpha, M, \lambda, \mathcal{D}, \mu)$. Before definition, we set the following notations:

$$B_R^+ := \{x \in B_R \text{ such that } x_N > 0\},$$
$$B_R' := \{x \in B_R \text{ such that } x_N = 0\}.$$

For $x \in \mathbb{R}^N$, we denote $x' \in \mathbb{R}^{N-1}$ as the projection of x on the plane $\{x_N = 0\}$; we denote the tangential gradient ∇' as follows:

$$\nabla' u := \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{N-1}}\right).$$

We define the affine space set $H^1_\phi(B_R^+)$ as follows:

$$H^1_\phi(B_R^+) = \{v \in H^1(B_R^+) : v - \phi \in H^1_0(B_R^+)\}.$$

For a given function $v \in H^1(B_2^+)$, we denote $F(v)$ as

$$F(v) := \partial\{v > 0\},$$

and Id is the notation for $N \times N$ identity matrix. For a given function v, we define $v^\pm$ as

$$v^+ := \max(0, v),$$
$$v^- := \max(0, -v).$$

Definition 2.1. A function $u \in H^1(B^+_{2/r})$ is said to belong to the class $\mathcal{P}_r(\alpha, M, \lambda_\pm, \mathcal{D}, \mu)$ if there exist symmetric matrix coefficients $A \in C^\infty(B^+_{2/r})^{N\times N}$, $\phi \in C^{1,\alpha}(B^+_{2/r})$, $\lambda_\pm > 0$, $0 < \mu < 1$, and $\mathcal{D} > 0$ such that the following hold:

(P1) $\|A\|_{L^\infty(B^+_{2/r})} \le M$, $\|\nabla\phi\|_{L^\infty(B^+_{2/r})} \le M$, $[A]_{C^\alpha(B^+_{2/r})}, [\nabla\phi]_{C^{1,\alpha}(B^+_{2/r})} \le r^\alpha M$, and $|\phi(x')| \le M r^{1+\alpha}|x'|^{1+\alpha}$ $(x' \in B'_{2/r})$. ϕ satisfies the following Degenerate Phase Transition condition (DPT):

$$\forall x' \in B'_{2/r} \text{ such that } \phi(x') = 0; \text{ then } |\nabla'\phi(x')| = 0. \tag{DPT}$$

(P2) $A(0) = \mathrm{Id}$, $\mu|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \frac{1}{\mu}|\xi|^2$ for all $x \in B^+_{2/r}$ and $\xi \in \mathbb{R}^N$.

(P3) $0 < \lambda_- < \lambda_+$.

(P4) u minimizes $J(\cdot; A, \lambda_+, \lambda_-, B^+_{2/r})$ (cf. (1.1)); that is, for every $u - v \in H^1_0(B^+_{2/r})$,

$$\int_{B^+_{2/r}} \big(\langle A(x)\nabla u, \nabla u\rangle + \Lambda(u)\big)\, dx \le \int_{B^+_{2/r}} \big(\langle A(x)\nabla v, \nabla v\rangle + \Lambda(v)\big)\, dx$$

$$(\Lambda(s) = \lambda_+\chi_{\{s>0\}} + \lambda_-\chi_{\{s\le 0\}}) \quad \text{and} \quad 0 \in F(u) \cap \overline{B^+_{2/r}}.$$

(P5) $u \in H^1_\phi(B^+_{2/r})$.

(P6) There exists $0 < r_0$ such that for all $0 < \rho \le r_0$ we have

$$\frac{|B^+_\rho(0) \cap \{u > 0\}|}{|B^+_\rho(0)|} > \mathcal{D}. \tag{2.1}$$

Remark 2.2. In fact, the functions $u \in \mathcal{P}_r(\alpha, M, \lambda_\pm, \mathcal{D}, \mu)$ carry more regularity than being only a Sobolev function. They are Hölder continuous in $\overline{B^+_{2/r}}$ (cf. Lemma 3.2).

Remark 2.3. We have assumed the coefficient matrices A to be of the class $C^\infty(B^+_2)$. However, all the estimates in this paper depend only on the C^α norms of A. Primarily, the reason we imposed the assumption $A \in C^\infty(B^+_2)^{N\times N}$ is to ensure that we can represent solutions to PDEs $-\mathrm{div}(A(s)\nabla u) = 0$ in terms of co-normal derivatives of Green's function (cf. (3.26) and (3.35)) which follows from [21, equation (1.12)]. This representation (in (3.26) and (3.35)) is also true for the case $A \in C^\alpha(B^+_2)^{N\times N}$; a formal proof of this fact will be present in our forthcoming articles. This way, one can replace the assumption $A \in C^\infty(B^+_2)^{N\times N}$ (in Definition 2.1) by $A \in C^\alpha(B^+_2)^{N\times N}$.

Remark 2.4. The assumption (P2), i.e., $A(0) = \mathrm{Id}$, does not compromise the generality of the problem. Indeed, assume that $A(0) \ne \mathrm{Id}$. Since $A(x)$ is a symmetric matrix for all $x \in B^+_R$, we have

$$\begin{aligned} J(u; B^+_R) &= \int_{B^+_R} \big(\langle A(x)\nabla u, \nabla u\rangle + \Lambda(u)\big)\, dx \\ &= \int_{B^+_R} \big(|A^{1/2}(x)\nabla u(x)|^2 + \Lambda(u)(x)\big)\, dx. \end{aligned}$$

Under the change of variables $y \to A(0)^{-1/2}x$, the functional $J(\cdot; B_R^+)$ is transformed into

$$J(u; B_R^+) = |\det(A(0))|^{1/2} \int_{\mathcal{B}_R^+} \big(|\mathcal{A}^{1/2}(y)\nabla v|^2 + \Lambda(v)\big)\, dy,$$

where

$$\begin{aligned} \mathcal{A}^{1/2}(y) &:= A^{1/2}(A(0)^{1/2}y)A(0)^{-1/2} \quad \text{(note that } \mathcal{A}(0) = \mathrm{Id}),\\ v(y) &:= u(A(0)^{1/2}y),\\ \mathcal{B}_R^+ &:= A(0)^{-1/2}B_R^+. \end{aligned}$$

Then, we can reformulate the minimization problem to minimize the functional

$$\mathcal{J}(v; \mathcal{B}_R^+) := \int_{\mathcal{B}_R^+} \big(|\mathcal{A}^{1/2}(y)\nabla v|^2 + \Lambda(v)\big)\, dy = \int_{\mathcal{B}_R^+} \big(\langle \mathcal{A}(y)\nabla v, \nabla v\rangle + \Lambda(v)\big)\, dy.$$

Moreover, under the (linear) transformation $x \to A(0)^{-1/2}x$, all the assumptions (P1)–(P6) remain structurally unchanged.

In the absence of ambiguity on values of α, M, $\lambda_\pm$, $\mathcal{D}$, and μ, we use the notation $\mathcal{P}_r$ in place of $\mathcal{P}_r(\alpha, M, \lambda_\pm, \mathcal{D}, \mu)$. If $\phi \in C^{1,\alpha}(B_2^+)$ satisfies (DPT), from [7, Lemma 10.1], we know that $\phi^\pm|_{B_2'} \in C^{1,\alpha}(B_2')$ and also

$$\|\phi^\pm\|_{C^{1,\alpha}(B_2')} \le \|\phi\|_{C^{1,\alpha}(B_2')}.$$

Given $v \in H^1(B_R^+)$ and $r > 0$, we define the blowup $v_r \in H^1(B_{R/r}^+)$ as follows:

$$v_r(x) := \frac{1}{r}v(rx). \tag{2.2}$$

For the coefficient matrix A, $A^r(x)$ is defined as follows:

$$A^r(x) := A(rx).$$

One can check that if $u \in \mathcal{P}_1(\alpha, M, \lambda_\pm, \mathcal{D}, \mu)$, then $u_r \in \mathcal{P}_r(\alpha, M, \lambda_\pm, \mathcal{D}, \mu)$. Indeed if $u \in \mathcal{P}_1$ and u is a minimizer of the functional J (cf. (P4))

$$J(v; A, \lambda_+, \lambda_-, B_2^+) := \int_{B_2^+} \big(\langle A(x)\nabla v, \nabla v\rangle + \Lambda(v)\big)\, dx,$$
$$(\Lambda(s) = \lambda_+\chi_{\{s>0\}} + \lambda_-\chi_{\{s\le 0\}})$$

with boundary data $\phi \in C^{1,\alpha}(B_2^+)$ (i.e., $u \in H^1_\phi(B_2^+)$), then, by simple change of variables, we can check that $u_r \in H^1_{\phi_r}(B_{2/r}^+)$ (this verifies (P5)) and u_r minimizes

$$J(v; A^r, \lambda_+, \lambda_-, B_{2/r}^+) := \int_{B_{2/r}^+} \big(\langle A^r(x)\nabla v, \nabla v\rangle + \Lambda(v)\big)\, dx,$$
$$(\Lambda(s) = \lambda_+\chi_{\{s>0\}} + \lambda_-\chi_{\{s\le 0\}}).$$

Moreover, if A and ϕ satisfy conditions (P1) and (P2) for $r = 1$, then A^r and ϕ_r satisfy (P1) and (P2) for r. (P3) and (P6) remain invariant under the change of variables. Therefore, $u_r \in \mathcal{P}_r$.

In order to study the blowup limits ($\lim_{r\to 0} u_r$) of functions $u \in \mathcal{P}_1(\alpha, M, \lambda_\pm, \mathcal{D}, \mu)$, we define a class of global solutions $\mathcal{P}_\infty(C, \lambda_\pm)$. Let us set the following notation before giving the definition:

$$\Pi := \{x \in \mathbb{R}^N : x_N = 0\}.$$

Definition 2.5 (Global solution). We say that $u \in H^1(\mathbb{R}^N_+)$ belongs to the class of functions $\mathcal{P}_\infty(C, \lambda_\pm)$; that is, u is a global solution if there exist $C > 0$ and $0 < \lambda_+ < \lambda_-$ such that

(G1) $|u(x)| \le C|x|$ for all $x \in \mathbb{R}^N_+$,

(G2) u is continuous up to the boundary Π,

(G3) $u = 0$ on Π,

(G4) and for every ball $B_r(x_0)$, u is a minimizer of $J(\cdot; \mathrm{Id}, \lambda_+, \lambda_-, B_r(x_0) \cap \mathbb{R}^N_+)$ (cf. (1.1)); that is,

$$\int_{B_r(x_0)\cap\mathbb{R}^N_+} \big(|\nabla u|^2 + \Lambda(u)\big)\, dx \le \int_{B_r(x_0)\cap\mathbb{R}^N_+} \big(|\nabla v|^2 + \Lambda(v)\big)\, dx.$$

Here, $(\Lambda(s) = \lambda_+\chi_{\{s>0\}} + \lambda_-\chi_{\{s\le 0\}})$ and for every $v \in H^1(B_r(x_0) \cap \mathbb{R}^N_+)$ such that $u - v \in H^1_0(B_r(x_0) \cap \mathbb{R}^N_+)$.

Our main result intends to show that for a minimizer u of $J(\cdot; A, \lambda_+, \lambda_-, B_2^+)$ with A, $\lambda_\pm$, and u satisfying the properties (P1)–(P6), the free boundary of every such minimizer touches the flat part of fixed boundary tangentially at the origin. For this, we prove that as we approach closer and closer to the origin, the free boundary points cannot lie completely inside any cone which is perpendicular to the flat boundary and has its tip at the origin. The main result in this paper is stated below.

Theorem 2.6. *There exist a constant ρ_0 and a modulus of continuity σ such that if*

$$u \in \mathcal{P}_1(\alpha, M, \lambda_\pm, \mathcal{D}, \mu),$$

then

$$F(u) \cap B^+_{\rho_0} \subset \{x : x_N \le \sigma(|x|)|x|\}.$$

Here, σ depends only on α, M, $\lambda_\pm$, $\mathcal{D}$, μ.

3. Blowup analysis

The following is a classical result (cf. [1, Remark 4.2]); we present the proof for the case of variable coefficients.

Lemma 3.1. *Given a bounded and strictly elliptic matrix $A(x)$ and a non-negative continuous function w such that $\operatorname{div}(A(x)\nabla w) = 0$ in $\{w > 0\} \cap B_2^+$ in weak sense (i.e., in the distributional sense in the context of Sobolev spaces), then $w \in H^1_{\text{loc}}(B_2^+)$ and $\operatorname{div}(A(x)\nabla w) \geq 0$ in weak sense in B_2^+.*

Proof. Let $D \Subset B_2^+$ and $\eta \in C_c^\infty(B_2^+)$ be cutoff function for D. That is, $\eta \in C_c^\infty(B_2^+)$ be such that

$$\eta(x) = \begin{cases} 1 & \text{in } D, \\ 0 & \text{on } \partial B_2^+. \end{cases}$$

Since $\operatorname{div}(A(x)\nabla w) = 0$ in $\{w > 0\} \cap B_2^+$, we have

$$\begin{aligned} 0 &= \int_{B_2^+} \langle A(x)\nabla w, \nabla((w-\varepsilon)^+\eta^2)\rangle\, dx \\ &= \int_{B_2^+\cap\{w>\varepsilon\}} \eta^2 \langle A(x)\nabla w, \nabla w\rangle\, dx + \int_{B_2^+\cap\{w>\varepsilon\}} w\langle A(x)\nabla w, \nabla\eta^2\rangle\, dx \\ &\quad - \varepsilon \int_{B_2^+\cap\{w>\varepsilon\}} \langle A(x)\nabla w, \nabla\eta^2\rangle\, dx \end{aligned}$$

which implies

$$\begin{aligned} \int_{B_2^+\cap\{w>\varepsilon\}} \langle A(x)\nabla w, \nabla w\rangle \eta^2\, dx &\leq \int_{B_2^+\cap\{w>\varepsilon\}} \left| w\langle A(x)\nabla w, \nabla\eta^2\rangle \right| dx \\ &\quad + \varepsilon \int_{B_2^+\cap\{w>\varepsilon\}} \left| \langle A(x)\nabla w, \nabla\eta^2\rangle \right| dx. \end{aligned}$$

By the choice of η and ellipticity of the matrix A, we obtain, using Young's inequality,

$$\begin{aligned} &\mu \int_{B_2^+\cap\{w>\varepsilon\}} |\nabla w|^2 \eta^2\, dx \\ &\quad \leq \frac{1}{\mu} \left| \int_{B_2^+\cap\{w>\varepsilon\}} \eta w |\nabla w||\nabla\eta|\, dx \right| + \frac{\varepsilon}{\mu} \left| \int_{B_2^+\cap\{w>\varepsilon\}} \eta |\nabla w||\nabla\eta|\, dx \right| \\ &\quad \leq C_1(\mu) \Bigg[\frac{1}{\delta} \int_{B_2^+\cap\{w>\varepsilon\}} w^2 |\nabla\eta|^2\, dx + \delta \int_{B_2^+\cap\{w>\varepsilon\}} \eta^2 |\nabla w|^2\, dx \\ &\qquad + \delta\varepsilon \int_{B_2^+\cap\{w>\varepsilon\}} \eta^2 |\nabla w|^2\, dx + \frac{\varepsilon}{\delta} \int_{B_2^+\cap\{w>\varepsilon\}} |\nabla\eta|^2\, dx \Bigg]. \end{aligned}$$

After choosing $\delta > 0$ very small and rearranging the terms in the equation above, since $\eta = 1$ in D, we finally get

$$\begin{aligned} \int_{D\cap\{w>\varepsilon\}} |\nabla w|^2\, dx &\leq \int_{B_2^+\cap\{w>\varepsilon\}} |\nabla w|^2 \eta^2\, dx \\ &\leq C(\mu) \Bigg[\int_{B_2^+\cap\{w>\varepsilon\}} w^2 |\nabla\eta|^2\, dx + \int_{B_2^+\cap\{w>\varepsilon\}} |\nabla\eta|^2\, dx \Bigg]. \end{aligned}$$

As $\varepsilon \to 0$, we obtain

$$\begin{aligned}\int_D |\nabla w|^2\,dx &= \int_{\{w>0\}\cap D} |\nabla w|^2\,dx = \lim_{\varepsilon\to 0}\int_{\{w>\varepsilon\}\cap D} |\nabla w|^2 \\ &\le C(\mu)\lim_{\varepsilon\to 0}\left[\int_{B_2^+\cap\{w>\varepsilon\}} w^2|\nabla\eta|^2\,dx + \int_{B_2^+\cap\{w>\varepsilon\}} |\nabla\eta|^2\,dx\right] \\ &\le C(\mu,D)\left[\int_{B_2^+\cap \mathrm{supp}(\eta)} w^2\,dx + 1\right].\end{aligned}$$

Since $w \in C(B_2^+)$, therefore w is uniformly bounded in $\mathrm{supp}(\eta)$, and therefore,

$$\int_D \left(|\nabla w|^2 + |w^2|\right)dx \le C(\mu,D)\left[\int_{B_2^+\cap\mathrm{supp}(\eta)} w^2\,dx + 1\right] \le C(\mu, D, \|w\|_{L^\infty(\mathrm{supp}(\eta))}).$$

Now, for $0 \le \varphi \in C_c^\infty(B_2^+)$, consider the test function

$$v = \varphi\Big(1 - \Big(\min\Big(2 - \frac{w}{\varepsilon}, 1\Big)\Big)^+\Big)$$

$$\int_{B_2^+} \langle A(x)\nabla w, \nabla\varphi\rangle\,dx = \int_{B_2^+} \Big\langle A(x)\nabla w, \nabla\Big(\varphi\Big(\Big(2-\frac{w}{\varepsilon}\Big)\wedge 1\Big)^+\Big)\Big\rangle\,dx.$$

We can easily check that $v \ge 0$ in B_2^+ and $v \in H_0^1(B_2^+)$; in particular,

$$\varphi\Big(\Big(2-\frac{w}{\varepsilon}\Big)\wedge 1\Big)^+ = \begin{cases} \varphi, & x \in \{w \le \varepsilon\}, \\ \varphi\cdot\big(2-\frac{w}{\varepsilon}\big), & x \in \{\varepsilon < w \le 2\varepsilon\}, \\ 0, & x \in \{w > 2\varepsilon\}. \end{cases}$$

Therefore, we have

$$\begin{aligned}&\int_{B_2^+} \langle A(x)\nabla w, \nabla\varphi\rangle\,dx \\ &= \int_{B_2^+} \Big\langle A(x)\nabla w, \nabla\Big(\varphi\Big(\Big(2-\frac{w}{\varepsilon}\Big)\wedge 1\Big)^+\Big)\Big\rangle\,dx \\ &= \int_{B_2^+\cap\{w\le\varepsilon\}} \langle A(x)\nabla w, \nabla\varphi\rangle\,dx + \int_{B_2^+\cap\{\varepsilon<w\le 2\varepsilon\}} \Big\langle A(x)\nabla w, \nabla\Big(\varphi\Big(2-\frac{w}{\varepsilon}\Big)\Big)\Big\rangle\,dx \\ &= \int_{B_2^+\cap\{w\le\varepsilon\}} \langle A(x)\nabla w, \nabla\varphi\rangle\,dx + 2\int_{B_2^+\cap\{\varepsilon<w\le 2\varepsilon\}} \langle A(x)\nabla w, \nabla\varphi\rangle\,dx \\ &\quad - \frac{2}{\varepsilon}\int_{B_2^+\cap\{\varepsilon<w\le 2\varepsilon\}} \langle A(x)\nabla w, \nabla(w\varphi)\rangle\,dx \\ &\le C(\mu)\int_{B_2^+\cap\{\varepsilon<u\le 2\varepsilon\}} |\nabla w||\nabla\varphi|\,dx + \frac{2}{\varepsilon}\int_{B_2^+\cap\{\varepsilon<w\le 2\varepsilon\}} w\langle A(x)\nabla w, \nabla\varphi\rangle\,dx \\ &\quad - \frac{2}{\varepsilon}\int_{B_2^+\cap\{\varepsilon<w\le 2\varepsilon\}} \varphi\langle A(x)\nabla w, \nabla w\rangle\,dx \\ &\le C(\mu)\int_{B_2^+\cap\{\varepsilon<w\le 2\varepsilon\}} |\nabla w||\nabla\varphi|\,dx.\end{aligned}$$

The last term goes to zero as $\varepsilon \to 0$. Therefore, we can say that

$$\int_{B_2^+} \langle A(x)\nabla w, \nabla\varphi\rangle \, dx \le 0.$$

We conclude the proof from [13, Theorem 1.39] (see also [24, Theorem 6.22]). ■

Lemma 3.2 (Hölder continuity). *If $u \in \mathcal{P}_1$, then $u \in C^{0,\alpha_0}(\overline{B_2^+})$ for some $0 < \alpha_0 < 1$. In fact,*

$$\|u\|_{C^{\alpha_0}(B_2^+)} \le C(\mu, \lambda_\pm)\|u\|_{L^\infty(B_2^+)}.$$

Proof. The functional $J(\cdot; A, \lambda_+, \lambda_-, B_2^+)$ satisfies the hypothesis of [16, Theorem 7.3] and $\phi \in C^{1,\alpha}(\overline{B_2^+})$; therefore, Lemma 3.2 follows from the arguments in [16, Section 7.8], where the boundary regularity is treated. ■

Remark 3.3. Since every function $u \in \mathcal{P}_1$ is continuous, therefore the positivity set $\{u > 0\}$ is an open set.

Corollary 3.4. *If $u \in \mathcal{P}_1$, then $u^\pm$ are A-subharmonic.*

Proof. The claim follows directly from Lemmas 3.2 and 3.1. ■

Lemma 3.5. *If $u \in \mathcal{P}_1$, then*

$$|u(x)| \le C(\mu, [A]_{C^\alpha(B_2^+)}, N)M\,|x| \quad \text{in } B_1^+. \tag{3.1}$$

Proof. Let w be such that

$$\begin{cases} \operatorname{div}(A(x)w) = 0 & \text{in } B_2^+, \\ w = \phi^+ & \text{in } \partial B_2^+. \end{cases}$$

Since u is A-subharmonic in B_2^+ (cf. Corollary 3.4), by maximum principle, if $x \in B_1^+$ we have

$$\begin{aligned} u^+(x) \le w(x) &\le w(x) - w(x') + w(x') \\ &\le \|\nabla w\|_{L^\infty(B_1^+)}|x - x'| + |\phi^+(x')| \\ &= \|\nabla w\|_{L^\infty(B_1^+)} x_N + |\phi^+(x')| \\ &\le (\|\nabla w\|_{L^\infty(B_1^+)} + M)|x| \quad \forall x \in B_1^+. \end{aligned} \tag{3.2}$$

In the last inequality, we have used (P1). Now, we prove that the term $\|\nabla w\|_{L^\infty(B_1^+)}$ is uniformly bounded.

From [6, Theorem 2], we have

$$\|\nabla w\|_{L^\infty(B_1^+)} \le C\big(\mu, [A]_{C^\alpha(B_2^+)}, N\big)\big[\|w\|_{L^\infty(B_2^+)} + \|\phi^+\|_{C^{1,\alpha}(B_2')}\big]. \tag{3.3}$$

By comparison principle,

$$\|w\|_{L^\infty(B_2^+)} = \|w\|_{L^\infty(\partial B_2^+)} = \|\phi^+\|_{L^\infty(\partial B_2^+)} \le M.$$

Plugging this information in (3.3), we get

$$\|\nabla w\|_{L^\infty(B_1^+)} \le C(\mu, [A]_{C^\alpha(B_2^+)}, N)M,$$

and then using (3.2), we obtain

$$u^+(x) \le C(\mu, [A]_{C^\alpha(B_2^+)}, N)M|x| \quad \forall x \in B_1^+. \tag{3.4}$$

And analogously,

$$u^-(x) \le C(\mu, [A]_{C^\alpha(B_2^+)}, N)M|x| \quad \forall x \in B_1^+. \tag{3.5}$$

By adding (3.4) and (3.5), we prove (3.1). ∎

Remark 3.6. We can check that, for every $u \in \mathcal{P}_1$, $u_r \in \mathcal{P}_r$ and u_r is A^r-subharmonic and satisfies (3.1) in $B_{1/r}^+$. That is,

$$|u_r(x)| \le C(\mu, [A]_{C^\alpha(B_2^+)}, N)M\,|x|, \quad x \in B_{1/r}^+.$$

Lemma 3.7 (Uniform H^1 bounds). *Let $u \in \mathcal{P}_1$. Then, for $R > 0$ such that $2R \le \frac{2}{r}$, we have*

$$\int_{B_R^+} |\nabla u_r|^2\,dx \le C(N, \lambda, \mu, R, M).$$

Proof. Since $u_r \in \mathcal{P}_r$, from (P4), we can say that u_r is a minimizer of $J(\cdot; A^r, \lambda_\pm, B_{2R}^+)$ with boundary data ϕ_r. Here, A^r and ϕ_r satisfy conditions (P1) and (P2). Precisely speaking, u_r is the minimizer of the following functional:

$$J(v; A^r, \lambda_\pm, B_{2R}^+) := \int_{B_{2R}^+} \big(\langle A^r(x)\nabla v, \nabla v\rangle + \Lambda(v)\big)\,dx.$$

Here, $(\Lambda(v) = \lambda_+\chi_{\{v>0\}} + \lambda_-\chi_{\{v\le 0\}})$. Consider $h \in H^1(B_{2R}^+)$ as a harmonic replacement

$$\begin{cases} \operatorname{div}(A^r(x)\nabla h) = 0 & \text{in } B_{2R}^+ \\ h - u_r \in H_0^1(B_{2R}^+). \end{cases}$$

In other words, h is the minimizer of $\int_{B_{2R}^+}\langle A^r(x)\nabla h, \nabla h\rangle\,dx$ in the set $H^1_{u_r}(B_{2R}^+)$.

From the minimality of u_r and the choice of h, we have

$$\begin{aligned}\int_{B_{2R}^+} &\langle A^r(x)\nabla(u_r - h), \nabla(u_r - h)\rangle\,dx \\ &= \int_{B_{2R}^+} \langle A^r(x)\nabla(u_r - h), \nabla(u_r + h - 2h)\rangle\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{B_{2R}^+} \langle A^r(x)\nabla(u_r - h), \nabla(u_r + h)\rangle \, dx - 2\int_{B_{2R}^+} \langle A^r(x)\nabla(u_r - h), \nabla h\rangle \, dx \\
&= \int_{B_{2R}^+} \big(\langle A^r(x)\nabla u_r, \nabla u_r\rangle - \langle A^r(x)\nabla h, \nabla h\rangle\big) \, dx \quad \text{(since } h \text{ is } A^r\text{-harmonic in } B_R^+\text{)} \\
&\le \int_{B_{2R}^+} (\Lambda(h) - \Lambda(u_r)) \, dx \le C(N, \lambda, R).
\end{aligned}
$$

Using the ellipticity of A, we get

$$
\int_{B_R^+} |\nabla(u_r - h)|^2 \, dx \le \int_{B_{2R}^+} |\nabla(u_r - h)|^2 \, dx \le C(N, \lambda, \mu, R).
$$

Expanding the left-hand side, we get

$$
\begin{aligned}
\int_{B_R^+} |\nabla u_r|^2 \, dx &\le \int_{B_R^+} |\nabla u_r|^2 \, dx + \int_{B_R^+} |\nabla h|^2 \, dx \\
&\le C(N, \lambda, \mu, R) + 2\int_{B_R^+} \nabla u_r \cdot \nabla h \, dx \\
&\le C(N, \lambda, \mu, R) + \varepsilon \int_{B_R^+} |\nabla u_r|^2 \, dx + \frac{1}{\varepsilon} \int_{B_R^+} |\nabla h|^2 \, dx.
\end{aligned}
$$

By choosing $\varepsilon = \frac{1}{8}$, we are left with

$$
\int_{B_R^+} |\nabla u_r|^2 \, dx \le C(N, \lambda, \mu, R)\bigg(1 + \int_{B_R^+} |\nabla h|^2 \, dx\bigg).
$$

To close the argument, we need to show that $\int_{B_R^+} |\nabla h|^2 \, dx$ is uniformly bounded, indeed from [6, Theorem 2], $\|\nabla h\|_{L^\infty(B_R^+)} \le C(\mu, M)$. ■

Lemma 3.8 (Compactness). *Let $r_j \to 0^+$, and a sequence $\{v_j\} \in \mathcal{P}_1$. Then, the blowups $u_j := (v_j)_{r_j}$ (as defined in (2.2)) converge (up to subsequence) uniformly in B_R^+ and weakly in $H^1(B_R^+)$ to some limit for any $R > 0$. Moreover, if u_0 is such a limit of u_j in the above-mentioned topologies, then u_0 belongs to $\mathcal{P}_\infty$.*

Proof. We fix $R > 0$; since $v_j \in \mathcal{P}_1$, therefore $u_j \in \mathcal{P}_{r_j}$, and as argued in the proof of previous lemma, the functions u_j are minimizers of the functional $J(\cdot; A_j, \lambda_\pm, B_R^+)$ for j sufficiently large that $R < \frac{1}{r_j}$. We set the notation for the functional J_j as

$$
J_j(v) := J(v; A_j, \lambda_\pm, B_R^+) = \int_{B_R^+} (\langle A_j(x)\nabla v, \nabla v\rangle + \Lambda(v)) \, dx, \quad v \in H^1_{u_j}(B_R^+).
$$

We require to rescale the boundary data in the same way as we do to v_j, we also denote the boundary values for $u_j \in \mathcal{P}_{r_j}$ as ϕ_j. Here, the sequences $A_j \in C^\alpha(B_{2/r_j}^+)^{N\times N}$ and $\phi_j \in C^{1,\alpha}(B_{2/r_j})$ satisfy conditions (P1) and (P2) with

$$
r = r_j, \quad \Lambda(v) := \lambda_+ \chi_{\{v>0\}} + \lambda_- \chi_{\{v\le 0\}}.
$$

We set the following notation for the functional J_0:

$$J_0(v; B_R^+) := \int_{B_R^+} \big(|\nabla v|^2 + \Lambda(v)\big)\, dx. \tag{3.6}$$

From Lemma 3.2, we know that $u_j \in C^{\alpha_0}(\overline{B_{2/r_j}^+})$ which implies $C^{\alpha_0}(\overline{B_R^+})$. In particular, $\|u_j\|_{C^{0,\alpha_0}(B_R^+)} \le C(\mu, \lambda_\pm)$. Hence, u_j is a uniformly bounded and equicontinuous sequence in $\overline{B_R^+}$; we can apply Arzela–Ascoli theorem to show that u_j uniformly converges to a function $u_0 \in C^{0,\alpha_0}(\overline{B_R^+})$.

Since $u_j = \phi_j$ on B'_R, from (P1), we have $|\phi_j(x)| \le M r_j^{1+\alpha}|x|^{1+\alpha}$ for $x \in B'_R$; therefore, $|\phi_j(x)| \le C(M, \alpha) r_j^{1+\alpha} R^{1+\alpha}$. Hence, $\phi_j \to 0$ uniformly on B'_R. We have

$$u_0 = \lim_{j\to\infty} u_j = \lim_{j\to\infty} \phi_j = 0 \quad \text{on } B'_R.$$

Thus, u_0 satisfies (G2) and (G3) inside the domain $\overline{B_R^+}$. Also, from Lemma 3.7, we have

$$\int_{B_R^+} |\nabla u_j|^2\, dx \le C(N, \lambda_\pm, \mu, R, M, \alpha). \tag{3.7}$$

Then, by the linear growth condition (cf. Remark 3.6), u_j also satisfies

$$|u_j(x)| \le C(\mu, \alpha) M |x| \quad x \in B_R^+.$$

Hence, passing to the limit, we have $|u_0(x)| \le C(\mu, \alpha) M |x|$, $\forall x \in B_R^+$. In other words, u_0 satisfies (G1) in $\overline{B_R^+}$. Moreover, we have

$$\int_{B_R^+} |u_j|^2\, dx \le C(\mu, \alpha, M) \int_{B_R^+} |x|^2\, dx \le C(\mu, \alpha, M, N, R). \tag{3.8}$$

Thus, (3.7) and (3.8) imply that u_j is a bounded sequence in $H^1(B_R^+)$. Hence, up to a subsequence, $u_j \rightharpoonup u_0$ weakly in $H^1(B_R^+)$. We rename the subsequence again as u_j.

We have found a blowup limit up to a subsequence u_0 and have shown that u_0 satisfies (G1), (G2), and (G3) in $\overline{B_R^+}$. In order to show that $u_0 \in \mathcal{P}_\infty$, it only remains to verify that u_0 satisfies (G4); i.e., u_0 is a local minimizer of $J_0(\cdot; B_R^+)$ for all $R > 0$ (cf. (3.6)). For that, we first claim that

$$\int_{B_R^+} \big(|\nabla u_0|^2 + \Lambda(u_0)\big)\, dx \le \liminf_{j\to\infty} \int_{B_R^+} \big(\langle A_j(x)\nabla u_j, \nabla u_j\rangle + \Lambda(u_j)\big)\, dx. \tag{3.9}$$

Indeed, let us look separately at the term $J_j(u_j)$ on the right-hand side of the above equation

$$J_j(u_j) = \int_{B_R^+} \big(\langle A_j(x)\nabla u_j, \nabla u_j\rangle + \lambda_+ \chi_{\{u_j>0\}} + \lambda_- \chi_{\{u_j\le 0\}}\big)\, dx.$$

We rewrite the first term as follows:

$$\int_{B_R^+} \langle A_j(x)\nabla u_j, \nabla u_j\rangle\, dx = \int_{B_R^+} \langle (A_j(x) - \mathrm{Id})\nabla u_j, \nabla u_j\rangle\, dx + \int_{B_R^+} |\nabla u_j|^2\, dx. \quad (3.10)$$

From (P1) and (P2), we have for all $x \in B_R^+$

$$\|A_j(x) - \mathrm{Id}\,\|_{L^\infty(B_R^+)} \leq M r_j^\alpha |x|^\alpha \leq C(M,R,\alpha) r_j^\alpha \to 0 \quad \text{as } j \to \infty.$$

Therefore, $A_j \to \mathrm{Id}$ uniformly and $\|\nabla u_j\|_{L^2(B_2^+)}$ is bounded (cf. (3.7)). Hence, the first term on the right-hand side of (3.10) tends to zero as $j \to \infty$. Thus, from (3.10) and by weak lower semi-continuity of H^1 norm, we have

$$\int_{B_R^+} |\nabla u_0|^2\, dx \leq \liminf_{j\to\infty} \int_{B_R^+} |\nabla u_j|^2\, dx = \liminf_{j\to\infty} \int_{B_R^+} \langle A_j(x)\nabla u_j, \nabla u_j\rangle\, dx. \quad (3.11)$$

For the second term, we claim that

$$\int_{B_R^+} \lambda_+\chi_{\{u_0>0\}} + \lambda_-\chi_{\{u_0\leq 0\}}\, dx \leq \liminf_{j\to\infty} \int_{B_R^+} \lambda_+\chi_{\{u_j>0\}} + \lambda_-\chi_{\{u_j\leq 0\}}\, dx. \quad (3.12)$$

To see this, we first show that, for almost every $x \in B_R^+$, we have

$$\lambda_+\chi_{\{u_0>0\}}(x) + \lambda_-\chi_{\{u_0\leq 0\}}(x) \leq \liminf_{j\to\infty}(\lambda_+\chi_{\{u_j>0\}}(x) + \lambda_-\chi_{\{u_j\leq 0\}}(x)). \quad (3.13)$$

Indeed, let $x_0 \in B_R^+ \cap (\{u_0 > 0\} \cup \{u_0 < 0\})$. Then, by the uniform convergence of u_j to u_0, we can easily see that $u_j(x_0)$ attains the sign of $u_0(x_0)$ for sufficiently large value of j. Hence, (3.13) holds in $\{u_0 > 0\} \cup \{u_0 < 0\}$.

Now, assume that $x_0 \in B_R^+ \cap \{u_0 = 0\}$. Then, the left-hand side of (3.13) is equal to

$$\lambda_+\chi_{\{u_0>0\}}(x_0) + \lambda_-\chi_{\{u_0\leq 0\}}(x_0) = \lambda_-.$$

Regarding the right-hand side of (3.13), we see that

$$\lambda_+\chi_{\{u_j>0\}}(x_0) + \lambda_-\chi_{\{u_j\leq 0\}}(x_0) = \begin{cases} \lambda_+ & \text{if } u_j(x_0) > 0, \\ \lambda_- & \text{if } u_j(x_0) \leq 0. \end{cases}$$

Since $\lambda_- < \lambda_+$ (cf. (P3)), the right-hand side in the equation above is always greater than or equal to λ_-. Then,

$$\lambda_+\chi_{\{u_0>0\}}(x_0) + \lambda_-\chi_{\{u_0\leq 0\}}(x_0) = \lambda_- \leq \liminf_{j\to\infty}(\lambda_+\chi_{\{u_j>0\}}(x_0) + \lambda_-\chi_{\{u_j\leq 0\}}(x_0)).$$

Thus, (3.13) is proven for all $x \in B_R^+$, and hence, (3.12) holds by Fatou's lemma.

By adding (3.11) and (3.12) and [12, Theorem 3.127], we obtain (3.9). Now, we will use (3.9) to prove the minimality of u_0 for the functional $J_0(\cdot; B_R^+)$ (cf. 3.6).

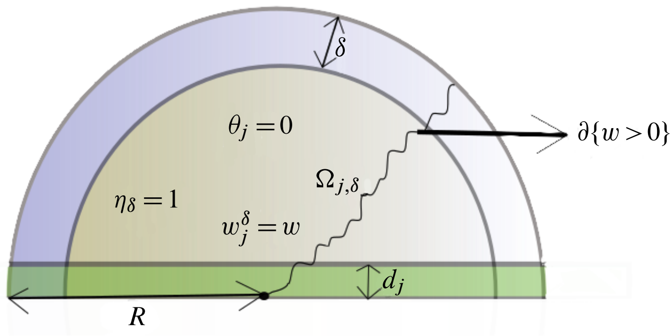

Figure 1. Curvy line represents the free boundary of w.

Pick any $w \in H^1(B_R^+)$ such that $u_0 - w \in H_0^1(B_R^+)$. We construct an admissible competitor w_j^δ to compare the minimality of u_j for the functional $J_j(\cdot; B_R^+)$. Then, we intend to use (3.9).

In this direction, we define two cutoff functions $\eta_\delta : \mathbb{R}^N \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}$ as follows:

$$\eta_\delta(x) := \begin{cases} 1, & x \in B_{R-\delta}, \\ 0, & x \in \mathbb{R}^N \setminus B_R, \end{cases} \qquad \theta(t) := \begin{cases} 1, & |t| \leq 1/2, \\ 0, & |t| \geq 1. \end{cases}$$

We can take $|\nabla \eta_\delta| \leq \frac{C(N)}{\delta}$. We define $\theta_j(x) = \theta(\frac{x_N}{d_j})$ for a sequence $d_j \to 0$, which will be suitably chosen in later steps of the proof. Let w_j^δ be a test function defined as

$$w_j^\delta := w + (1 - \eta_\delta)(u_j - u_0) + \eta_\delta \theta_j \phi_j.$$

Since the function $w_j^\delta - u_j$ is continuous in $\overline{B_R^+}$ and is pointwise equal to zero on ∂B_R^+, which is a Lipschitz surface in $\mathbb{R}^N$, therefore $u_j - w_j^\delta \in H_0^1(B_R^+)$. For further steps, the reader can refer to Figure 1.

Let $\Omega_{\delta,j} = B_R^+ \cap \{\theta_j = 0\} \cap \{\eta_\delta = 1\}$, and $\mathcal{R}_{\delta,j} = B_R^+ \setminus \Omega_{\delta,j}$; by observing that $w_j^\delta = w$ on $\Omega_{\delta,j}$, we see that

$$\begin{aligned} |\{w_j^\delta > 0\} \cap B_R^+| &= |\{w_j^\delta > 0\} \cap \Omega_{\delta,j}| + |\{w_j^\delta > 0\} \cap \mathcal{R}_{\delta,j}| \\ &= |\{w > 0\} \cap \Omega_{\delta,j}| + |\{w_j^\delta > 0\} \cap \mathcal{R}_{\delta,j}| \\ &= |\{w > 0\} \cap (B_R^+ \setminus \mathcal{R}_{\delta,j})| + |\{w_j^\delta > 0\} \cap \mathcal{R}_{\delta,j}| \\ &= |\{w > 0\} \cap B_R^+| - |\{w > 0\} \cap \mathcal{R}_{\delta,j}| + |\{w_j^\delta > 0\} \cap \mathcal{R}_{\delta,j}|. \end{aligned}$$

From the above discussions, we have

$$|\{w > 0\} \cap B_R^+| - |\mathcal{R}_{\delta,j}| \leq |\{w_j^\delta > 0\} \cap B_R^+| \leq |\{w > 0\} \cap B_R^+| + |\mathcal{R}_{\delta,j}|.$$

Since we know that $\lim_{\delta\to 0}(\lim_{j\to 0} |\mathcal{R}_{\delta,j}|) = 0$, we have

$$\lim_{\delta \to 0} \Big(\lim_{j \to \infty} |\{w_j^\delta > 0\} \cap B_R^+| \Big) = |\{w > 0\} \cap B_R^+|, \tag{3.14}$$

and similarly,

$$\lim_{\delta\to 0}\big(\lim_{j\to\infty}|\{w_j^\delta \le 0\}\cap B_R^+|\big) = |\{w\le 0\}\cap B_R^+|. \tag{3.15}$$

Given $u_j \in \mathcal{P}_{r_j}$ and $w_j^\delta - u_j \in H_0^1(B_R^+)$, from the minimality of u_j for the functional J_j, we have

$$\begin{aligned}\int_{B_R^+}&\big(\langle A_j(x)\nabla u_j, \nabla u_j\rangle + \lambda_+\chi_{\{u_j>0\}} + \lambda_-\chi_{\{u_j\le 0\}}\big)\,dx\\ &\le \int_{B_R^+}\big(\langle A_j(x)\nabla w_j^\delta, \nabla w_j^\delta\rangle + \lambda_+\chi_{\{w_j^\delta>0\}} + \lambda_-\chi_{\{w_j^\delta\le 0\}}\big)\,dx,\end{aligned}$$

and from (3.9), we obtain

$$\begin{aligned}\int_{B_R^+}&\big(|\nabla u_0|^2 + \lambda_+\chi_{\{u_0>0\}} + \lambda_-\chi_{\{u_0\le 0\}}\big)\,dx\\ &\le \liminf_{j\to\infty}\int_{B_R^+}\big(\langle A_j(x)\nabla w_j^\delta, \nabla w_j^\delta\rangle + \lambda_+\chi_{\{w_j^\delta>0\}} + \lambda_-\chi_{\{w_j^\delta\le 0\}}\big)\,dx\\ &\le \limsup_{j\to\infty}\int_{B_R^+}\big(\langle A_j(x)\nabla w_j^\delta, \nabla w_j^\delta\rangle + \lambda_+\chi_{\{w_j^\delta>0\}} + \lambda_-\chi_{\{w_j^\delta\le 0\}}\big)\,dx.\end{aligned} \tag{3.16}$$

From the same reasoning as for the justification of (3.11), we have

$$\limsup_{j\to\infty}\int_{B_R^+}\langle A_j(x)\nabla w_j^\delta, \nabla w_j^\delta\rangle\,dx = \limsup_{j\to\infty}\int_{B_R^+}|\nabla w_j^\delta|^2\,dx.$$

Therefore, rewriting (3.16) as

$$\begin{aligned}\int_{B_R^+}&\big(|\nabla u_0|^2 + \lambda_+\chi_{\{u_0>0\}} + \lambda_-\chi_{\{u_0\le 0\}}\big)\,dx\\ &\le \limsup_{j\to\infty}\int_{B_R^+}\big(|\nabla w_j^\delta|^2\,dx + \lambda_+\chi_{\{w_j^\delta>0\}} + \lambda_-\chi_{\{w_j^\delta\le 0\}}\big)\,dx,\end{aligned} \tag{3.17}$$

we claim that

$$\lim_{\delta\to 0}\left(\limsup_{j\to\infty}\int_{B_R^+}|\nabla w_j^\delta|^2\,dx\right) = \int_{B_R^+}|\nabla w|^2\,dx. \tag{3.18}$$

To obtain the claim above, we prove that

$$\lim_{\delta\to 0}\left(\limsup_{j\to\infty}\int_{B_R^+}|\nabla(w_j^\delta - w)|^2\,dx\right) = 0.$$

From the definition of w_j^δ, we know that

$$w_j^\delta - w = (1-\eta_\delta)(u_j - u_0) + \eta_\delta\theta_j\phi_j.$$

Therefore, we have

$$
\begin{aligned}
&\int_{B_R^+} |\nabla(w_j^\delta - w)|^2\,dx \\
&\quad\le C\left(\int_{B_R^+} |\nabla((1-\eta_\delta)(u_j-u_0))|^2\,dx + \int_{B_R^+} |\nabla(\theta_j\eta_\delta\phi_j)|^2\,dx\right) \\
&\quad\le C(N)\left(\int_{B_R^+} (1-\eta_\delta)^2|\nabla(u_j-u_0)|^2\,dx + \frac{1}{\delta^2}\int_{B_R^+} |u_j-u_0|^2\,dx \right.\\
&\qquad\qquad \left. + \int_{B_R^+} |\nabla(\theta_j\eta_\delta\phi_j)|^2\,dx\right). \qquad (3.19)
\end{aligned}
$$

Let us consider the first term on the right-hand side. We know that $\int_{B_R^+} |\nabla(u_j - u_0)|^2\,dx$ is uniformly bounded in $j \in \mathbb{N}$ (cf. (3.7)). Therefore,

$$
\begin{aligned}
&\lim_{\delta\to 0}\left(\limsup_{j\to\infty}\int_{B_R^+} (1-\eta_\delta)^2|\nabla(u_j-u_0)|^2\,dx\right) \\
&\quad= \big(\lim_{\delta\to 0}\|1-\eta_\delta\|^2_{L^\infty(B_R^+)}\big)\left(\limsup_{j\to\infty}\int_{B_R^+}|\nabla(u_j-u_0)|^2\,dx\right) \\
&\quad\le C(N,\lambda,\mu,R,M,\alpha)\lim_{\delta\to 0}\|1-\eta_\delta\|_{L^\infty(B_R^+)} = 0. \qquad (3.20)
\end{aligned}
$$

Regarding the second term, since $|u_j - u_0|$ tends to zero in $L^2(B_R^+)$ as $j \to \infty$, therefore the second term also tends to zero as $j \to \infty$. We write

$$
\lim_{\delta\to 0}\left(\lim_{j\to\infty}\frac{1}{\delta^2}\int_{B_R^+}|u_j-u_0|^2\,dx\right) = 0. \qquad (3.21)
$$

Lastly, we claim that

$$
\lim_{j\to\infty}\int_{B_R^+}|\nabla(\theta_j\eta_\delta\phi_j)|^2\,dx = 0. \qquad (3.22)
$$

Indeed, we have

$$
\begin{aligned}
\int_{B_R^+}|\nabla(\theta_j\eta_\delta\phi_j)|^2\,dx &\le C\left(\int_{B_R^+}|\nabla\eta_\delta|^2(\theta_j\phi_j)^2\,dx + \int_{B_R^+}|\nabla\phi_j|^2(\eta_\delta\theta_j)^2\,dx\right.\\
&\qquad\left. + \int_{B_R^+}|\nabla\theta_j|^2(\eta_\delta\phi_j)^2\,dx\right). \qquad (3.23)
\end{aligned}
$$

Since $\eta_\delta, \theta_j \le 1$, $|\nabla\eta_\delta| \le \frac{C(N)}{\delta}$, and $\|\nabla\phi_j\|_{L^\infty(B_R^+)} \le M$ (cf. (P1)), we obtain

$$
\begin{aligned}
&\int_{B_R^+}|\nabla\eta_\delta|^2(\theta_j\phi_j)^2\,dx + \int_{B_R^+}|\nabla\phi_j|^2(\eta_\delta\theta_j)^2\,dx \\
&\quad\le C(N)\big|\{\theta_j \ne 0\}\cap B_R^+\big|\left(\frac{1}{\delta^2} + M^2\right). \qquad (3.24)
\end{aligned}
$$

We know that $|\{\theta_j \neq 0\} \cap B_R^+| \to 0$ as $j \to \infty$; hence, from (3.24), the first and second terms in (3.23) tend to zero as $j \to \infty$. The last term in (3.23) also tends to zero as $j \to \infty$; indeed from (P1), we have

$$[\nabla \phi_j]_{C^\alpha(B_R^+)} \leq r_j^\alpha M.$$

Since $\phi_j(0) = 0$, therefore we have $|\phi_j| \leq R^\alpha [\nabla \phi_j]_{B_R^+} \leq R^\alpha r_j^\alpha M$ in B_R^+. Also, observing that $|\nabla \theta_j| \leq \frac{1}{d_j}$, $\eta_\delta \leq 1$ in B_R^+, we have

$$\int_{B_R^+} |\nabla \theta_j|^2 (\eta_\delta \phi_j)^2 \, dx \leq M R^{2\alpha} \frac{r_j^{2\alpha}}{d_j^2} |B_R^+|.$$

If we choose a sequence $d_j \to 0^+$ such that we also have

$$\frac{r_j^\alpha}{d_j} \to 0,$$

the third term in (3.23) tends to zero as $j \to \infty$. Plugging in the estimates above (3.20), (3.21), and (3.22) in (3.19), we obtain the claim (3.18).

From equations (3.14), (3.15), and (3.18), we obtain that the right-hand side of (3.17) is equal to $J_0(w; B_R^+)$; therefore, u_0 is a minimizer of $J_0(\cdot; B_R^+)$. That is,

$$J_0(u_0; B_R^+) \leq J_0(w; B_R^+)$$

for every $w \in H^1(B_R^+)$ such that $u_0 - w \in H_0^1(B_R^+)$. Since the inequality above (which corresponds to (G4)) and other verified properties of u_0 (i.e., (G1), (G2), and (G3) in B_R^+) hold for every $R > 0$, therefore $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+)$ satisfies all the properties in Definition 2.5. Hence, $u_0 \in \mathcal{P}_\infty$. ∎

After proving that the (subsequential) limits of blowups are global solutions, we proceed to show that the positivity sets (and hence the free boundaries) of blowups converge in certain sense to that of blowup limit. For this, we will need to establish that the minimizers $u \in \mathcal{P}_r$ are non-degenerate near the free boundary. In the proof below, we adapt the ideas from [2].

Proposition 3.9 (Non-degeneracy near the free boundary). *Let $u \in \mathcal{P}_{r_0}$ for some $r_0 > 0$ and $x_0 \in B^+_{2/r_0}$. Then, for every $0 < \kappa < 1$, there exists a constant $c(\mu, N, \kappa, \lambda_\pm) > 0$ such that, for all $B_r(x_0) \subset B^+_{\frac{2}{r_0}}$, we have*

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u^+ \, d\mathcal{H}^{N-1}(x) \; < \; c(\mu, N, \kappa, \lambda_\pm) \implies u^+ = 0 \quad \text{in } B_{\kappa r}(x_0).$$

Proof. We fix $x_0 \in \{u > 0\} \cap B_2^+$ and $r > 0$ such that $B_r(x_0) \subset B_2^+$. We denote $\gamma := \frac{1}{r} \fint_{B_r(x_0)} u^+ \, dx$. We know from Lemma 3.2 that the set $\{u > 0\}$ is open. Also, since $u \in \mathcal{P}_{r_0}$, there exists $A \in C^\infty(B^+_{2/r_0})^{N \times N}$, $\varphi \in C^{1,\alpha}(B^+_{2/r_0})$, $\lambda_\pm$ satisfying (P1)–(P6).

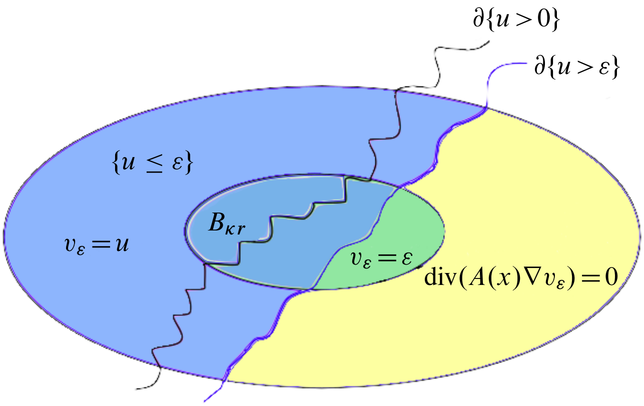

Figure 2. Graph of v_ε.

Therefore, u solves the PDE $\mathrm{div}(A(x)\nabla u) = 0$ in $\{u > 0\} \cap B^+_{2/r_0}$. By elliptic regularity theory, u is locally smooth, ($C^{1,\alpha}_{\mathrm{loc}}(\{u > 0\} \cap B^+_{2/r_0})$). Then, for almost every $\varepsilon > 0$, (smooth) $B_r \cap \partial\{u > \varepsilon\}$ is a $C^{1,\alpha}$ surface. Pick one such small $\varepsilon > 0$, and we consider the test function v_ε given by

$$\begin{cases} \mathrm{div}(A(x)\nabla v_\varepsilon) = 0 & \text{in } (B_r(x_0) \setminus B_{\kappa r}(x_0)) \cap \{u > \varepsilon\}, \\ v_\varepsilon = u & \text{in } B_r(x_0) \cap \{u \leq \varepsilon\}, \\ v_\varepsilon = \varepsilon & \text{in } B_{\kappa r}(x_0) \cap \{u > \varepsilon\}, \\ v_\varepsilon = u & \text{on } \partial B_r(x_0). \end{cases}$$

Refer to Figure 2 for a pictorial understanding of definition v_ε. The function v_ε defined above belongs to $H^1(B_r(x_0))$; thanks to [11, Theorem 3.44] ([11, Theorem 3.44] is proven for C^1 domains, but the proof can also be adapted for Lipschitz domains [13, Theorem 4.6]). We intend to show that v_ε is bounded in $H^1(B_r(x_0))$. This ensures the existence of limit $\lim_{\varepsilon\to 0} v_\varepsilon$ exists in weak sense in $H^1(B_r(x_0))$ and strong sense in $L^2(B_r(x_0))$. Let G be the Green function for $L(v) = \mathrm{div}(A(x)\nabla v)$ in the ring $B_r(x_0) \setminus B_{\kappa r}(x_0)$. Then, if there is a function w such that

$$\begin{cases} \mathrm{div}(A(x)\nabla w) = 0 & \text{in } B_r(x_0) \setminus B_{\kappa r}(x_0), \\ w = u & \text{on } \partial B_r(x_0) \cap \{u > \varepsilon\}, \\ w = \varepsilon & \text{elsewhere on } \partial(B_r(x_0) \setminus B_{\kappa r}(x_0)), \end{cases} \tag{3.25}$$

we can also write that $w - \varepsilon = (u - \varepsilon)^+$ on $\partial B_r(x_0)$ and $w - \varepsilon = 0$ on $\partial B_{\kappa r}(x_0)$. From boundary elliptic regularity (cf. [6, Theorem 2]), $(w - \varepsilon) \in C^{1,\alpha}(\overline{B_{r-\eta}(x_0) \setminus B_{\kappa r}(x_0)})$ for every $\eta \in (0, (1-\kappa)r)$. Consider $\bar{x} \in \partial B_{\kappa r}(x_0)$ and any sequence $\{x_k\} \subset B_r(x_0) \setminus B_{\kappa r}(x_0)$ such that $x_k \to \bar{x} \in \partial B_{\kappa r}(x_0)$. We have

$$\nabla(w - \varepsilon)(\bar{x}) = \lim_{k\to\infty} \nabla(w - \varepsilon)(x_k).$$

By Green representation formulae for $w-\varepsilon$ in (3.25) (cf. [21, equation (1.12)]), we have

$$(w-\varepsilon)(x_k) = \int_{\partial(B_r(x_0)\setminus B_{\kappa r}(x_0))} (u-\varepsilon)^+(y)(A(y)\nabla_y G(x_k,y))\cdot \nu_y \, d\sigma(y). \tag{3.26}$$

Therefore,

$$\begin{aligned}
&\nabla(w-\varepsilon)(\bar{x}) \\
&= \lim_{k\to\infty} \nabla(w-\varepsilon)(x_k) \\
&= \lim_{k\to\infty} \nabla_x \Bigg(\int_{\partial(B_r(x_0)\setminus B_{\kappa r}(x_0))} (u-\varepsilon)^+(y)(A(y)\nabla_y G(x,y))\cdot \nu_y \, d\sigma(y) \Bigg)\Bigg|_{x=x_k} \\
&= \lim_{k\to\infty} \int_{\partial B_r(x_0)\cap\{u>\varepsilon\}} (u-\varepsilon)^+(y)(A(y)\nabla_x\nabla_y G(x_k,y))\cdot \nu_y \, d\sigma(y) \quad \forall \bar{x}\in\partial B_{\kappa r}(x_0),
\end{aligned} \tag{3.27}$$

where ν_y is the unit outer normal vector at a point y on the boundary. Computations in (3.27) are justified by the regularity and estimates given in [26, Theorem 3] (see also the classical paper by Grüter and Widman [18, Theorem 3.3 (vi)]). As a matter of fact, we can proceed further to obtain for $\bar{x}\in\partial B_{\kappa r}(x_0)$

$$\begin{aligned}
&|\nabla w(\bar{x})| \\
&\le C(\mu,N) \lim_{k\to\infty} \int_{\partial B_r(x_0)\cap\{u>\varepsilon\}} (u-\varepsilon)^+(y) |\nabla_x\nabla_y G(x_k,y)| \, d\mathcal{H}^{N-1}(y) \\
&\le \frac{C(\mu,N)}{(1-\kappa)^N}\frac{1}{r} \fint_{\partial B_r} (u-\varepsilon)^+ \, d\mathcal{H}^{N-1}(y) \le C(\mu,N,\kappa)\gamma \quad \text{on } \partial B_{\kappa r}(x_0).
\end{aligned} \tag{3.28}$$

We can easily check by the respective definitions that $w\ge v_\varepsilon$ on $\partial(B_r(x_0)\setminus B_{\kappa r}(x_0))$. Moreover, since $w\in C(\overline{B_r(x_0)\setminus B_{\kappa r}(x_0)})$ and $w\ge\varepsilon$ on $\partial(B_r(x_0)\setminus B_{\kappa r}(x_0))$ by the maximum principle (recall $\operatorname{div}(A(x)\nabla w)=0$ in $B_r(x_0)\setminus B_{\kappa r}(x_0)$), we conclude that $w\ge v_\varepsilon$ in $B_r(x_0)\setminus B_{\kappa r}(x_0)$. In particular, $w\ge v_\varepsilon$ on ∂D_ε, where

$$D_\varepsilon := (B_r(x_0)\setminus B_{\kappa r}(x_0))\cap\{u>\varepsilon\}.$$

By comparison principle, we know that $w\ge v_\varepsilon$ in D_ε, and since $w=v_\varepsilon=\varepsilon$ on $\partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}$, hence from (3.28)

$$|\nabla v_\varepsilon| \le |\nabla w| \le C(\mu,N,\kappa)\gamma \quad \text{on } \partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}. \tag{3.29}$$

Given that $\operatorname{div}(A(x)\nabla v_\varepsilon)=0$ in D_ε, we have, by divergence theorem and (3.29),

$$\begin{aligned}
\int_{D_\varepsilon} (A(x)\nabla v_\varepsilon)\cdot\nabla(v_\varepsilon-u)\,dx &= \int_{\partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}} (u-v_\varepsilon)(A(x)\nabla v_\varepsilon)\cdot\nu(y)\,d\mathcal{H}^{N-1}(y) \\
&\le C(\mu)\int_{\partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}} |u-\varepsilon||\nabla v_\varepsilon|\,d\mathcal{H}^{N-1}(y) \\
&\le C(\mu,N,\kappa)\gamma \int_{\partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}} |u-\varepsilon|\,d\mathcal{H}^{N-1}(y) \\
&=: M_0(u).
\end{aligned}$$

The justification for the use of divergence theorem in D_ε can be found in [2, equation (3.4)]. From the calculations above, we can write

$$
\begin{aligned}
&\int_{D_\varepsilon} (A(x)\nabla v_\varepsilon) \cdot \nabla (v_\varepsilon - u)\, dx \le M_0 \\
\Rightarrow &\int_{D_\varepsilon} (A(x)\nabla v_\varepsilon) \cdot \nabla v_\varepsilon \, dx \le M_0 + \int_{D_\varepsilon} (A(x)\nabla v_\varepsilon) \cdot \nabla u\, dx \\
\Rightarrow &\ \mu \int_{D_\varepsilon} |\nabla v_\varepsilon|^2\, dx \le M_0 + \frac{1}{\mu} \int_{D_\varepsilon} |\nabla v_\varepsilon||\nabla u|\, dx \\
\Rightarrow &\ \mu \int_{D_\varepsilon} |\nabla v_\varepsilon|^2\, dx \le M_0 + \frac{\varepsilon_0}{2\mu} \int_{D_\varepsilon} |\nabla v_\varepsilon|^2\, dx + \frac{1}{2\varepsilon_0 \mu} \int_{D_\varepsilon} |\nabla u|^2\, dx.
\end{aligned}
$$

Putting very small $\varepsilon_0 > 0$ in the last inequality, we have

$$
\int_{D_\varepsilon} |\nabla v_\varepsilon|^2\, dx \le M_0 + C(\mu) \int_{D_\varepsilon} |\nabla u|^2\, dx =: M_1(u).
$$

Since $v_\varepsilon = \varepsilon$ in $B_{\kappa r}(x_0) \cap \{u > \varepsilon\}$ which implies that $\nabla v_\varepsilon = 0$ in $B_{\kappa r}(x_0) \cap \{u > \varepsilon\}$ and $v_\varepsilon = u$ in $B_r(x_0) \setminus D_\varepsilon$,

$$
\int_{B_r(x_0)} |\nabla v_\varepsilon|^2\, dx = \int_{D_\varepsilon} |\nabla v_\varepsilon|^2\, dx + \int_{B_r(x_0) \setminus D_\varepsilon} |\nabla u|^2\, dx =: M_2(u). \tag{3.30}
$$

By the definition of v_ε, $0 < v_\varepsilon \le u$ on ∂D_ε and $\operatorname{div}(A(x)\nabla v_\varepsilon) = \operatorname{div}(A(x)\nabla u) = 0$ in D_ε, therefore, by comparison principle, $0 < v_\varepsilon < u$ in D_ε. In the set $B_{\kappa r}(x_0) \cap \{u > \varepsilon\}$, we have $v_\varepsilon = \varepsilon < u$ and $v_\varepsilon = u$ in $B_r(x_0) \cap \{u \le \varepsilon\}$. Overall, we have $0 < v_\varepsilon \le u$ in $B_r(x_0) \cap \{u > \varepsilon\}$. Therefore,

$$
\begin{aligned}
\int_{B_r(x_0)} |v_\varepsilon|^2\, dx &\le \int_{B_r(x_0) \cap \{u \le \varepsilon\}} |u|^2\, dx + \int_{B_r(x_0) \cap \{u > \varepsilon\}} |u|^2\, dx \\
&= \int_{B_r(x_0)} |u|^2\, dx.
\end{aligned} \tag{3.31}
$$

Hence, from (3.30) and (3.31), v_ε is bounded in $H^1(B_r(x_0))$. Therefore, up to a subsequence, there exists a limit $v = \lim_{\varepsilon \to 0} v_\varepsilon$ in weak H^1 sense such that v satisfies the following:

$$
\begin{cases}
\operatorname{div}(A(x)\nabla v) = 0 & \text{in } (B_r(x_0) \setminus B_{\kappa r}(x_0)) \cap \{u > 0\}, \\
v = u & \text{in } B_r(x_0) \cap \{u \le 0\}, \\
v = 0 & \text{in } B_{\kappa r}(x_0) \cap \{u > 0\}, \\
v = u & \text{on } \partial B_r(x_0).
\end{cases} \tag{3.32}
$$

We verify the above properties (3.32) of v at the end of this proof.

Let us use the function v as a test function with respect to minimality condition on u in $B_r(x_0)$; we have

$$
\int_{B_r(x_0)} (\langle A(x)\nabla u, \nabla u\rangle + \Lambda(u))\, dx \le \int_{B_r(x_0)} (\langle A(x)\nabla v, \nabla v\rangle + \Lambda(v))\, dx.
$$

Since $v = u$ in $\{u \le 0\}$ and $\{v > 0\} \subset \{u > 0\}$, the integration in the set $\{u \le 0\}$ gets canceled from both sides, and we are left with the terms mentioned below.

Setting $D_0 := (B_r(x_0) \setminus B_{\kappa r}(x_0)) \cap \{u > 0\}$, we have

$$\begin{aligned}
&\int_{B_r(x_0)\cap\{u>0\}} \big(\langle A(x)\nabla u, \nabla u\rangle - \langle A(x)\nabla v, \nabla v\rangle\big)\, dx \\
&\quad \le \int_{B_r(x_0)\cap\{u>0\}} (\Lambda(v) - \Lambda(u))\, dx \\
&\quad = \int_{B_{\kappa r}(x_0)\cap\{u>0\}} (\Lambda(v) - \Lambda(u))\, dx \\
&\quad = \lambda_0 |B_{\kappa r}(x_0) \cap \{u > 0\}| \quad (\lambda_0 := -(\lambda_+ - \lambda_-)).
\end{aligned}$$

We have the second equality above because

$$\chi_{\{v>0\}} = \chi_{\{u>0\}} \quad \text{in } D_0.$$

Since $v = 0$ in $\{u > 0\} \cap B_{\kappa r}$, we have

$$\begin{aligned}
&\int_{B_{\kappa r}(x_0)\cap\{u>0\}} \langle A(x)\nabla u, \nabla u\rangle\, dx \\
&\quad + \int_{D_0} (\langle A(x)\nabla u, \nabla u\rangle - \langle A(x)\nabla v, \nabla v\rangle)\, dx \le \lambda_0 |B_{\kappa r}(x_0) \cap \{u > 0\}|.
\end{aligned}$$

Using the ellipticity of A and shuffling the terms in the above equation, we obtain

$$\begin{aligned}
&\int_{B_{\kappa r}(x_0)\cap\{u>0\}} (\mu|\nabla u|^2 - \lambda_0)\, dx \\
&\quad \le \int_{D_0} (\langle A(x)\nabla v, \nabla v\rangle - \langle A(x)\nabla u, \nabla u\rangle)\, dx \\
&\quad = \int_{D_0} (\langle A(x)\nabla(v - u), \nabla(v + u)\rangle)\, dx \\
&\quad = \int_{D_0} (\langle A(x)\nabla(v - u), \nabla(u - v + 2v)\rangle)\, dx \\
&\quad \le 2 \int_{D_0} \langle A(x)\nabla v, \nabla(v - u)\rangle\, dx \\
&\quad \le 2 \liminf_{\varepsilon\to 0} \int_{D_0} \langle A(x)\nabla v_\varepsilon, \nabla(v_\varepsilon - u)\rangle \\
&\quad = 2 \liminf_{\varepsilon\to 0} \int_{D_\varepsilon} \langle A(x)\nabla v_\varepsilon, \nabla(v_\varepsilon - u)\rangle\, dx \quad (\text{since } v_\varepsilon = u \text{ in } D_\varepsilon \setminus D_0) \\
&\quad = 2 \liminf_{\varepsilon\to 0} \int_{\partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}} (u - \varepsilon)(A(x)\nabla v_\varepsilon) \cdot \nu\, dx \\
&\quad \le \frac{2}{\mu} \liminf_{\varepsilon\to 0} \int_{\partial B_{\kappa r}(x_0)\cap\{u>\varepsilon\}} (u - \varepsilon)\big|\nu \cdot \nabla v_\varepsilon\big|\, dx := M.
\end{aligned} \tag{3.33}$$

The second-to-last equality in the above calculation is obtained from integration by parts; its justification can be found in [2, equation (3.4)]. From (3.33) and (3.29), and using the trace inequality in $H^1(B_{\kappa r})$, we have (for some different constant $C(\kappa)$)

$$\begin{aligned}
M &\leq C(\mu, N, \kappa)\gamma \int_{\partial B_{\kappa r(x_0)}} u^+ \, d\mathcal{H}^{N-1}(x) \\
&\leq C(\mu, N, \kappa)\gamma \int_{B_{\kappa r(x_0)}} \left(|\nabla u^+| + \frac{1}{r} u^+ \right) dx \\
&\leq C(\mu, N, \kappa)\gamma \Bigg[|B_{\kappa r(x_0)} \cap \{u > 0\}|^{1/2} \left(\int_{B_{\kappa r(x_0)}} |\nabla u^+|^2 \, dx \right)^{1/2} \\
&\qquad\qquad + \frac{1}{r} \sup_{B_{\kappa r}(x_0)} (u^+) \big| \{B_{\kappa r}(x_0) \cap \{u > 0\}\} \big| \Bigg] \\
&\leq C(\mu, N, \kappa)\gamma \Bigg[\frac{1}{2\sqrt{-\lambda_0}} \int_{B_{\kappa r}(x_0) \cap \{u>0\}} |\nabla u^+|^2 \, dx \\
&\quad + \frac{\sqrt{-\lambda_0}}{2} |B_{\kappa r(x_0)} \cap \{u > 0\}| + \frac{1}{r} \sup_{B_{\kappa r}(x_0)} (u^+) \int_{B_{\kappa r}(x_0) \cap \{u>0\}} 1 \, dx \Bigg] \\
&= \frac{C(\mu, N, \kappa)\gamma}{2\sqrt{-\lambda_0}} \left(\int_{B_{\kappa r}(x_0) \cap \{u>0\}} |\nabla u^+|^2 - \lambda_0 \, dx \right) \\
&\quad + \frac{C(\mu, N, \kappa)\gamma}{\lambda_0 r} \sup_{B_{\kappa r}(x_0)} (u^+) \int_{B_{\kappa r}(x_0) \cap \{u>0\}} \lambda_0 \, dx.
\end{aligned} \tag{3.34}$$

We have used Hölder's inequality and then Young's inequality above. From Lemma 3.1, u^+ is A-subharmonic in $B_r(x_0)$. If G' is the Green's function for

$$L'(v) = \operatorname{div}(A(x)\nabla v) \quad \text{in } B_r(x_0),$$

then by comparison principle and Green's representation (cf. [21, equation (1.12)])

$$u^+(x) \leq \int_{\partial B_r(x_0)} u^+(y)(A(y)\nabla_y G'(x, y)) \cdot \nu_y \, d\mathcal{H}^{N-1}(y) \quad \forall x \in B_{\kappa r}(x_0). \tag{3.35}$$

Since for all $y \in \partial B_r(x_0)$ and $x \in B_{\kappa r}(x_0)$, we have

$$\frac{1}{|x - y|^{N-1}} \leq \frac{C(\kappa)}{r^{N-1}},$$

then, using the Green's function estimates (cf. [18, Theorem 3.3 (v)]), we get

$$\begin{aligned}
\sup_{B_{\kappa r}(x_0)} u^+ &\leq C(\mu) \int_{\partial B_r(x_0)} \frac{u^+(y)}{|x - y|^{N-1}} \, d\mathcal{H}^{N-1}(y) \\
&\leq C(\mu, \kappa, N) \fint_{\partial B_r(x_0)} u^+ \, d\mathcal{H}^{N-1}(y) = C(\mu, \kappa, N)\gamma r.
\end{aligned} \tag{3.36}$$

Using (3.33) and (3.36) in (3.34), we have

$$\begin{aligned}\mu \int_{B_{\kappa r}(x_0)\cap\{u>0\}} & \left(|\nabla u|^2 - \lambda_0\right) dx \\ & \leq \frac{C(\mu,\kappa,N)\gamma}{2\sqrt{-\lambda_0}} \int_{B_{\kappa r}(x_0)\cap\{u>0\}} \left(|\nabla u^+|^2 - \lambda_0\right) dx \\ & \quad + \frac{C(\mu,\kappa,N)\gamma}{\lambda_0 r} \sup_{B_{\kappa r}(x_0)} (u^+) \int_{B_{\kappa r}(x_0)\cap\{u>0\}} \lambda_0 \, dx \\ & \leq \frac{C(\mu,\kappa,N)\gamma}{\mu\sqrt{-\lambda_0}} \left(1 + \frac{C(\kappa)\gamma}{\sqrt{-\lambda_0}}\right) \mu \int_{B_{\kappa r}(x_0)\cap\{u>0\}} \left(|\nabla u|^2 - \lambda_0\right) dx.\end{aligned}$$

If γ is small enough, then

$$\int_{B_{\kappa r}(x_0)\cap\{u>0\}} \left(|\nabla u|^2 - \lambda_0\right) dx = 0.$$

In particular, $|\{u > 0\} \cap B_{\kappa r(x_0)}| = 0$, that is, $u^+ = 0$ almost everywhere in $B_{\kappa r}(x_0)$.

It remains to verify the properties of v in (3.32). Before looking at the proof, we observe that, for a given $\varphi \in C_c^\infty(D_0)$, there exists $\varepsilon_0 > 0$ such that $\varphi \in C_c^\infty(D_\varepsilon)$ for all $\varepsilon < \varepsilon_0$. Indeed, since $\mathrm{supp}(\varphi)$ is a compact set and $\bigcup_{\varepsilon>0} D_\varepsilon$ is a cover of $\mathrm{supp}(\varphi)$, then for a finite set $\{\varepsilon_1, \dots, \varepsilon_n\}$ we have $\mathrm{supp}(\varphi) \subset \bigcup_{i=1}^n D_{\varepsilon_i} \subset D_{\varepsilon_{\max}}$, where $\varepsilon_{\max} = \max(\varepsilon_1, \dots, \varepsilon_n)$. Therefore, $\varphi \in C_c^\infty(D_\varepsilon)$ for all $\varepsilon < \varepsilon_{\max}$.

Let us first verify that $\mathrm{div}(A(x)\nabla v) = 0$ in D_0. For this, let $\varphi \in C_c^\infty(D_0)$; then from the continuity of u, there exists a $\varepsilon_0 > 0$ such that $\mathrm{supp}(\varphi) \subset D_\varepsilon$ for all $\varepsilon < \varepsilon_0$; also, we have

$$\int_{D_0} \langle A\nabla v, \nabla\varphi\rangle \, dx = \int_{\mathrm{supp}(\varphi)} \langle A\nabla v, \nabla\varphi\rangle \, dx \tag{3.37}$$

since $\mathrm{supp}(\varphi) \subset D_\varepsilon$; from the definition of v_ε, we have

$$\int_{\mathrm{supp}(\varphi)} \langle A\nabla v_\varepsilon, \nabla\varphi\rangle \, dx = 0,$$

and we know that v is a weak limit of v_ε in $H^1(B_r(x_0))$; therefore, from (3.37), we have

$$\int_{D_0} \langle A\nabla v, \nabla\varphi\rangle \, dx = \int_{\mathrm{supp}(\varphi)} \langle A\nabla v, \nabla\varphi\rangle \, dx = \lim_{\varepsilon\to 0} \int_{\mathrm{supp}(\varphi)} \langle \nabla v_\varepsilon, \nabla\varphi\rangle \, dx = 0.$$

Hence, we show that $\mathrm{div}(A(x)\nabla v) = 0$ in D_0. To show that $v = 0$ in $B_{\kappa r}(x_0) \cap \{u > 0\}$, we now take the function $\varphi \in C_c^\infty(B_{\kappa r}(x_0) \cap \{u > 0\})$. From the same reasoning as above, we know that there exists an $\varepsilon_0 > 0$ such that $\mathrm{supp}(\varphi) \subset B_{\kappa r}(x_0) \cap \{u > \varepsilon\}$ for all $\varepsilon < \varepsilon_0$. From the definition of v_ε, we have

$$\int_{\mathrm{supp}(\varphi)} v_\varepsilon\varphi \, dx = \int_{\{u>\varepsilon\}\cap B_{\kappa r}(x_0)} v_\varepsilon\varphi \, dx = \varepsilon \int_{\{u>\varepsilon\}\cap B_{\kappa r}(x_0)} \varphi \, dx,$$

and in limit $\varepsilon \to 0$, from the above equation, we have

$$\int_{\text{supp}(\varphi)} v\varphi \, dx = \lim_{\varepsilon \to 0} \int_{\text{supp}(\varphi)} v_\varepsilon \varphi \, dx = 0,$$

and therefore, $v = 0$ a.e. in $B_{\kappa r}(x_0) \cap \{u > 0\}$. To prove that $v = u$ in $\{u \leq 0\}$, we observe that $\{u \leq 0\} \subset \{u \leq \varepsilon\}$; hence, from the definition of v_ε, we have

$$v_\varepsilon = u \quad \text{in } \{u \leq 0\}.$$

Since the weak limits maintain the equality (cf. [25, Lemma 3.14]), the claim follows in the limit $\varepsilon \to 0$. Apart from that, since $v_\varepsilon = u$ on $\partial B_r(x_0)$, therefore, from the conservation of traces in weak convergence, it follows that $v = u$ on $\partial B_r(x_0)$. This completes the proof of Proposition 3.9. ∎

Remark 3.10. In the proposition above, the constant is local in nature; this means that the value of the constant depends on the choice of compact set $K \subset\subset B_2^+$, where $x_0 \in K$.

Lemma 3.11. *Let u_0 and u_j be as in Lemma 3.8. Then, for a subsequence of u_j, for any $R > 0$, we have*

$$\chi_{\{u_k>0\}\cap B_R^+} \to \chi_{\{u_0>0\}\cap B_R^+} \quad \textit{a.e. in } B_R^+. \tag{3.38}$$

This in turn implies

$$\chi_{\{u_j>0\}\cap B_R^+} \to \chi_{\{u_0>0\}\cap B_R^+} \quad \textit{in } L^1(B_R^+). \tag{3.39}$$

Proof. From Lemma 3.8, we can consider a subsequence of u_j such that $u_j \to u_0$ in $L^\infty(B_R^+)$. Let $x \in B_R^+$. If $x \in \{u_0 > 0\} \cap B_R^+$ (or $\chi_{\{u_0>0\}\cap B_R^+}(x) = 1$), then $u_j(x) > \frac{u(x)}{2} > 0$ (or $\chi_{\{u_j>0\}\cap B_R^+}(x) = 1$) for k sufficiently large. Thus, we conclude that

$$\chi_{\{u_j>0\}\cap B_R^+}(x) \to \chi_{\{u_0>0\}\cap B_R^+}(x) \text{ as } j \to \infty \text{ for all } x \in \{u_0 > 0\} \cap B_R^+.$$

If $x \in \{u_0 \leq 0\}^o \cap B_R^+$ (or $\chi_{\{u_0>0\}\cap B_R^+}(x) = 0$), then there exists $\delta > 0$ such that

$$B_\delta(x) \subset \{u_0 \leq 0\} \cap B_R^+.$$

Thus, we have

$$\frac{1}{\delta} \fint_{\partial B_\delta(x)} u_0^+ \, d\mathcal{H}^{N-1} = 0.$$

Again, by the uniform convergence of u_j to u_0 in B_R^+ (cf. Lemma 3.8), we obtain

$$\frac{1}{\delta} \fint_{\partial B_\delta(x)} u_j^+ \, d\mathcal{H}^{N-1} \leq \frac{1}{2} c(\mu, N, \lambda_\pm) \quad \text{for } j \text{ sufficiently large.} \tag{3.40}$$

Here, $c(\mu, N, \lambda_\pm)$ is as in Proposition 3.9. This implies that $u_j \leq 0$ in $B_{\frac{\delta}{2}}(x)$ (cf. Proposition 3.9). In particular, $\chi_{\{u_j(x)\leq 0\}}(x) = 0$ for j sufficiently large. This way, we obtain

$$\chi_{\{u_j>0\}\cap B_R^+}(x) \to \chi_{\{u_0>0\}\cap B_R^+}(x) \text{ as } j \to \infty \text{ for all } x \in \{u_0 \leq 0\} \cap B_R^+. \tag{3.41}$$

From the representation theorem [2, Theorem 7.3], we know that

$$|\partial\{u_0 > 0\} \cap B_R^+| = 0.$$

From (3.40), (3.41), and the fact that

$$|\partial\{u_0 > 0\} \cap B_R^+| = 0,$$

we obtain the claim (3.38). Since

$$|\chi_{\{u_j>0\}\cap B_R^+}| \leq 1,$$

the claim (3.39) follows from Lebesgue's dominated convergence theorem. ■

4. The main result

We rephrase the notion of the tangential touch of the free boundary to the fixed boundary, which is equivalent to the tangential touch condition mentioned in the statement of Theorem 2.6.

In the proof of our main result, we will show that, given $u \in \mathcal{P}_1$, for every $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that

$$\partial\{u > 0\} \cap B_\rho^+ \subset B_\rho^+ \setminus K_\varepsilon \quad \forall 0 < \rho \leq \rho_\varepsilon,$$

where

$$K_\varepsilon := \left\{x \in \mathbb{R}_+^N : x_N \geq \varepsilon\sqrt{x_1^2 + \cdots + x_{N-1}^2}\right\}.$$

Proof of Theorem 2.6. We assume, by contradiction, that the free boundaries of functions in $\mathcal{P}_1$ do not touch the origin in a tangential fashion to the plane. Then, there exist $\varepsilon > 0$ and sequences $v_j \in \mathcal{P}_1$ and $x_j \in F(v_j) \cap K_\varepsilon$ such that $|x_j| \to 0$ as $j \to \infty$. Let $r_j = |x_j|$, and we consider the blowups $u_j := (v_j)_{r_j}$.

Let $u_0 := \lim_{j\to\infty} u_j$ as in Lemma 3.8. Also, let $x_0 \in \partial B_1^+ \cap K_\varepsilon$ be a limit up to a subsequence (still called x_j) such that

$$x_0 = \lim_{j\to\infty} \frac{x_j}{|x_j|}.$$

Since $x_j \in F(v_j)$, we have $v_j(x_j) = 0$. Therefore, on rescaling,

$$u_j\left(\frac{x_j}{r_j}\right) = \frac{1}{r_j} v_j(x_j) = 0.$$

In the limit as $j \to \infty$, we have

$$u_0(x_0) = \lim_{j\to\infty} u_j\left(\frac{x_j}{r_j}\right) = 0.$$

From the density assumption that u_j satisfy condition (2.1) and Lemma 3.11, we have for any given $R > 0$

$$\begin{aligned}\frac{|\{u_0 > 0\} \cap B_R^+|}{|B_R^+|} &= \fint_{B_R^+} \chi_{\{u_0>0\}}\, dx = \lim_{j\to\infty} \fint_{B_R^+} \chi_{\{u_j>0\}}\, dx \\ &= \lim_{j\to\infty} \frac{1}{|B_{Rr_j}^+|} \int_{B_{Rr_j}^+} \chi_{\{v_j>0\}}\, dx \\ &= \lim_{j\to\infty} \frac{|\{v_j > 0\} \cap B_{Rr_j}^+|}{|B_{Rr_j}^+|} > \mathcal{D}. \end{aligned} \tag{4.1}$$

We can see that the computations done in (4.1), in fact, show that the density property remains invariant under the blowup of any function v. This way, we conclude that the function $(u_0)_0$ which is the blowup limit of (u_0) (in particular, $(u_0)_0 := \lim_{r\to 0}(u_0)_r$) also satisfies

$$\frac{|\{(u_0)_0 > 0\} \cap B_R^+|}{|B_R^+|} > \mathcal{D} \quad \forall R > 0. \tag{4.2}$$

Now, we note that from Lemma 3.8 $u_0 \in \mathcal{P}_\infty$. Moreover, from (4.1), $u_0 \not\equiv 0$, and from [22, Theorem 4.2, Lemma 4.3], we have $u_0 \geq 0$; also, from (4.2), we conclude that $(u_0)_0 \not\equiv 0$. This way, again by [22, Theorem 4.9], we have $u_0(x) = c\, x_N^+$ for all $x \in \mathbb{R}_+^N$ for some constant $c > 0$.

Hence, the function u_0 cannot be equal to zero at any point in $\mathbb{R}_+^N$. But we have $x_0 \in \partial B_1^+ \cap K_\varepsilon$ and $u_0(x_0) = 0$. This leads to a contradiction. In order to construct the modulus of continuity σ, for $x \in \mathbb{R}_+^N$, we assign the value $\sigma(|x|)$ to be the maximum opening η of cone K_η such that $F(u) \cap B_r^+ \subset K_\eta^c$ for all $r \leq |x|$. In other words,

$$\begin{aligned}\sigma(|x|) &= \inf\{\eta \mid F(u) \cap B_r^+ \subset K_\eta^c,\ r \leq |x|\} \\ &= \sup\Big\{\frac{x_N}{|x'|} \mid x \in F(u) \cap B_r^+ \setminus \{0\},\ r \leq |x|\Big\}.\end{aligned}$$

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Received 3 July 2022; revised 15 March 2023.

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