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$C^{1,\alpha}$-regularity for a class of degenerate/singular fully non-linear elliptic equations

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Abstract. We establish an optimal $C^{1,\alpha}$-regularity for viscosity solutions of degenerate/singular fully non-linear elliptic equations by finding minimal regularity requirements on the associated operator.

1. Introduction

In this paper, we study regularity of viscosity solutions to fully non-linear elliptic equations of the form

$$\mathcal{F}(x, Du, D^2u) = 0 \quad \text{in } B_1 \subset \mathbb{R}^n,\ n \geqslant 2, \tag{1.1}$$

where $\mathcal{F} \equiv \mathcal{F}(x, p, M) : B_1 \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ is an elliptic operator with respect to the matrix argument M (that is,whenever $(x, p, M) \in B_1 \times \mathbb{R}^n \times \mathcal{S}(n)$, $N \in \mathcal{S}(n)$ and $N \geqslant 0$,

$$\mathcal{F}(x, p, M + N) \leqslant \mathcal{F}(x, p, M) \tag{1.2}$$

is satisfied—see, for instance, [31,54]); whose ellipticity may degenerate or blow-up along the critical region $\{p = 0\}$; where $\mathcal{S}(n)$ denotes the set of $n \times n$ real symmetric matrices; u stands for a real-valued continuous unknown function defined on the unit ball B_1; and $Du = (\frac{\partial u}{\partial x_i})$ and $D^2u = (\frac{\partial^2 u}{\partial x_i \partial x_j})$ denote, respectively, the gradient and Hessian of u. Indeed, we are interested in partial differential equations of the form in (1.1) such that $D_M\mathcal{F}(x, p, M) \sim \Phi(x, |p|)$.

A primary model example we keep in mind concerns equations of the form

$$\Phi(x, |Du|)F(D^2u) = f(x) \quad \text{in } B_1 \tag{1.3}$$

for a (λ, Λ)-elliptic operator F and a suitable function $\Phi : B_1 \times (0, \infty) \to (0, \infty)$ satisfying minimal conditions which will be specified in a few lines. In fact, our goal is to establish differentiability and local Hölder continuity for the gradient of viscosity solutions for such a class of equations. We also aim at a comprehensive regularity theory which

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allows Φ to impel degenerate and singular characters on the diffusibility of the governing operator. To be more precise, we now state the main assumptions to be imposed on equation (1.3) throughout the paper.

(A1) The operator $F : \mathcal{S}(n) \to \mathbb{R}$ in (1.3) is uniformly (λ, Λ)-elliptic in the sense that

$$\lambda \operatorname{tr}(N) \leqslant F(M) - F(M+N) \leqslant \Lambda \operatorname{tr}(N)$$

holds with some constants $0 < \lambda \leqslant \Lambda$, whenever $M, N \in \mathcal{S}(n)$ with $N \geqslant 0$; and $F(0) = 0$.

(A2) The function $\Phi : B_1 \times [0,\infty) \to [0,\infty)$ in (1.3) is a continuous map satisfying the following properties:

1. There exist constants $s(\Phi) \geqslant i(\Phi) > -1$ such that the map $t \mapsto \frac{\Phi(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing with a constant $L \geqslant 1$ in $(0,\infty)$ and the map $t \mapsto \frac{\Phi(x,t)}{t^{s(\Phi)}}$ is almost non-increasing with a constant $L \geqslant 1$ in $(0,\infty)$ for all $x \in B_1$.
2. There exist constants $0 < \nu_0 \leqslant \nu_1$ such that $\nu_0 \leqslant \Phi(x,1) \leqslant \nu_1$ for all $x \in B_1$.

(A3) The term f on the right-hand side of (1.3) belongs to $C(B_1) \cap L^\infty(B_1)$.

Equation (1.3) features an inhomogeneous degenerate or singular term modeled on the integrand of the functional with Uhlenbeck-type structure; namely,

$$v \mapsto \int_{B_1} \varphi(x, |Dv|)\, dx \tag{1.4}$$

for an integral density $\varphi : B_1 \times [0,\infty) \to [0,\infty)$. From a variational point of view, the functional in (1.4) is a highly general non-autonomous functional including the following significant model functionals for the regularity theory:

1. p-growth: $\varphi(x,t) = t^p$ for $p > 1$; see, for instance, [46, 49, 59, 61, 62, 70–72].
2. $p(x)$-growth: $\varphi(x,t) = t^{p(x)}$ for $p(\cdot) > 1$; see, for instance, [1, 2, 39, 40]
3. Orlicz growth: $\varphi(x,t) = \phi(t)$; see, for, instance [7, 22, 41, 42, 60].
4. (p,q)-double phase: $\varphi(x,t) = t^p + a(x)t^q$ for $1 < p \leqslant q$; see, for instance, [12, 14, 28–30, 37].
5. Variable double phase: $\varphi(x,t) = t^{p(x)} + a(x)t^{q(x)}$ for $1 < p(\cdot) \leqslant q(\cdot)$; see, for instance, [23, 27, 69].
6. Borderline case of double phase: $\varphi(x,t) = t^p + a(x)t^p \log(1+t)$ for $1 < p$; see, for instance, [13, 24].
7. Multi-phase: $\varphi(x,t) = t^p + a(x)t^q + b(x)t^s$ for $1 < p \leqslant q, s$; see, for instance, [10, 38].
8. Orlicz multi-phase: $\varphi(x,t) = \phi(t) + a(x)\psi_a(t) + b(x)\psi_b(t)$; see, for instance, [6, 8, 10, 25].

Over the last several decades, a systematic analysis of the aforementioned functionals has been an object of intensive studies for the regularity theory. All the examples of the above-mentioned functionals fall within a realm of functionals with non-standard growth treated first in a series of papers (see [63–65]); and problems with non-standard growth conditions have been studied extensively over last decades—see, for instance, [43–45, 48] and references therein. Hölder continuity for the gradient of local minima of functional (1.4) under suitable optimal assumptions has been investigated in [50], where fundamental assumptions on the integral density function φ in (1.4) are that there exist constants $1 < p, q$ such that the map $t \mapsto \frac{\varphi(x,t)}{t^p}$ is almost non-decreasing and the map $t \mapsto \frac{\varphi(x,t)}{t^q}$ is almost non-increasing for all $x \in B_1$; see [50, Definition 3.1]. In this regard, the assumptions on Φ in (1.3) described in (A2) are absolutely reasonable.

On the other hand, the non-variational counterparts of the models described above could be cast as singular or degenerate fully non-linear equations. A first important special case of equation (1.3) is derived by replacing $\Phi(x,t) = t^p$ with $i(\Phi) = s(\Phi) = p > -1$ in (A2), whose most celebrated type is

$$|Du|^p F(D^2u) = f(x) \quad \text{in } B_1. \tag{1.5}$$

Structures of this type often occur in the theory of stochastic games [5, 11] in the setting of viscosity solutions. The fairly comprehensive investigation of these kinds of fully non-linear elliptic equations has been carried out. Birindelli and Demengel [15] proved the comparison principle and Liouville-type theorems in singular case ($-1 < p \leqslant 0$), and showed the regularity and uniqueness of the first eigenfunction in [16]. Alexandrov–Bakelman–Pucci estimates and the Harnack inequality have been also obtained in the papers [34, 35, 51]. In particular, Imbert and Silvestre [52] proved local Hölder continuity for the gradient of viscosity solutions of (1.5) in degenerate cases ($p \geqslant 0$). Later, Araújo, Ricarte, and Teixeira [4] proved the optimality of Hölder regularity for the gradient of viscosity solutions for the same problem in [52] by showing that viscosity solutions are $C^{1,\beta}_{\text{loc}}$ with $\beta = \min\{\hat{\alpha}, 1/(p+1)\}$, where $\hat{\alpha} \in (0,1)$ is the Hölder exponent coming from the Krylov–Safonov regularity for equation (1.11). It is worth mentioning that in the recent paper [3], the authors considered a degenerate fully non-linear equation of the type

$$\sigma(|Du|)F(D^2u) = f(x) \quad \text{in } B_1 \tag{1.6}$$

and proved local C^1 regularity for viscosity solutions of (1.6) under the condition that $\sigma : (0,\infty) \to (0,\infty)$ is a map whose inverse has a Dini-continuous modulus of continuity near the origin. This equation does not fall into the class of fully non-linear equations under consideration in general. However, it would be interesting to find an optimal condition on σ which leads to C^1 or $C^{1,\beta}$ regularity of viscosity solutions of (1.6). Next, we mention that De Fillippis [36] introduced the double phase-type degeneracies to the fully non-linear equation

$$(|Du|^p + a(x)|Du|^q)F(D^2u) = f(x) \quad \text{in } B_1, \quad 0 < p \leqslant q \tag{1.7}$$

and proved local Hölder continuity for the gradient of viscosity solutions to equation (1.7). Moreover, in this degenerate case, the sharpness of the local $C^{1,\beta}$-regularity estimates for bounded viscosity solutions is shown in [32]. Note that equation (1.7) can be derived from (1.3) by setting $\Phi(x,t) = t^p + a(x)t^q$ with $s(\Phi) = \max\{p,q\} \geqslant i(\Phi) = \min\{p,q\} > -1$ in (A2) and $0 \leqslant a(\cdot) \in C(B_1)$. Meanwhile, under rather general conditions, in [21], it has been shown that viscosity solutions to the fully non-linear elliptic equations having variable exponent degeneracies,

$$|Du|^{p(x)} F(D^2u) = f(x) \quad \text{in } B_1, \tag{1.8}$$

are locally of class $C^{1,\beta}$ for a universal constant $\beta \in (0,1)$ under the key assumption that $p(\cdot) : B_1 \to \mathbb{R}$ is a continuous function satisfying $\inf_{x\in B_1} p(x) > -1$. Again, we note that equation (1.8) corresponds to the choice of $\Phi(x,t) = t^{p(x)}$ in (1.3) with $i(\Phi) = \inf_{x\in B_1} p(x) > -1$ and $s(\Phi) = \sup_{x\in B_1} p(x)$ in (A2). In this paper, we provide a novel way to prove Hölder continuity for the gradient of viscosity solutions of (1.3) for both degenerate/singular cases in the full generality. The fully non-linear equation with double phase type degeneracies having variable exponents

$$(|Du|^{p(x)} + a(x)|Du|^{q(x)})F(D^2u) = f(x) \text{ in } B_1, \quad 0 \leqslant p(\cdot) \leqslant q(\cdot) \tag{1.9}$$

has been studied in [47] as a combination of equations (1.7) and (1.8), and local Hölder continuity for the gradient of viscosity solutions to (1.9) is also proved. It is clear to see that equation (1.9) can be generated from (1.3) by choosing $\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)}$ with functions $0 \leqslant a(\cdot) \in C(B_1)$ and $-1 < p(\cdot), q(\cdot)$ in $C(B_1)$, where the constants in (A2) are defined by $i(\Phi) = \inf_{x\in B_1}\{p(x), q(x)\}$ and $s(\Phi) = \sup_{x\in B_1}\{p(x), q(x)\}$. Finally, we point out another recent paper [57] dealing with viscosity solutions to the equation

$$|Du|^{\beta(x,u,Du)} F(D^2u) = f(x) \quad \text{in } B_1, \tag{1.10}$$

where $\beta : B_1 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a map satisfying $0 < \beta_m \leqslant \beta(\cdot) \leqslant \beta_M$ for some positive constants β_m and β_M. In [57], as a follow-up of the arguments introduced in [68], local Hölder continuity for the gradient of viscosity solutions of (1.10) is obtained, but the singular case is not treated due to the methods employed there and equation (1.3) cannot be represented as (1.10) generally.

Lastly, let us recall a consequence of the classical Krylov–Safonov–Harnack inequality (see [26])—namely, that viscosity solutions to the homogeneous equation

$$F(D^2h) = 0 \quad \text{in } B_1, \tag{1.11}$$

under the assumption that $F : \mathcal{S}(n) \to \mathbb{R}$ satisfies (A1), are locally of $C^{1,\widehat{\alpha}}(B_1)$ with the estimate

$$\|h\|_{C^{1,\widehat{\alpha}}(\overline{B}_{1/2})} \leqslant c\,\|h\|_{L^\infty(B_1)} \tag{1.12}$$

for universal constants $\widehat{\alpha} \equiv \widehat{\alpha}(n,\lambda,\Lambda) \in (0,1)$ and $c \equiv c(n,\lambda,\Lambda)$. The main results of this paper read as follows:

Theorem 1.1. *Let $u \in C(B_1)$ be a viscosity solution of equation* (1.3) *under assumptions* (A1)–(A3). *Then, $u \in C^{1,\beta}_{\mathrm{loc}}(B_1)$ for all $\beta > 0$ satisfying*

$$\beta \in \begin{cases} (0,\hat{\alpha}) \cap \left(0, \frac{1}{1+s(\Phi)}\right] & \text{if } i(\Phi) \geqslant 0, \\ (0,\hat{\alpha}) \cap \left(0, \frac{1}{1+s(\Phi)-i(\Phi)}\right] & \text{if } -1 < i(\Phi) < 0, \end{cases} \tag{1.13}$$

where $\hat{\alpha}$ is given in (1.12). *Moreover, for every β in* (1.13), *there exists a constant $c \equiv c(n,\lambda,\Lambda,i(\Phi),L,\beta)$ such that*

$$\|Du\|_{L^\infty(B_{1/2})} + \sup_{x \neq y \in B_{1/2}} \frac{|Du(x)-Du(y)|}{|x-y|^\beta} \leqslant c\left(1 + \|u\|_{L^\infty(B_1)} + \left\|\frac{f}{\nu_0}\right\|_{L^\infty(B_1)}^{\frac{1}{1+i(\Phi)}}\right). \tag{1.14}$$

The results of Theorem 1.1 are sharp in view of an example given in [52, Example 1]. As we have discussed above, the results of Theorem 1.1 cover the main results of the papers [21,36,47,52] for fully non-linear equations having both degenerate/singular terms in a unified way. Moreover, the results of Theorem 1.1 cover another important examples, for instance,

1. $\Phi(x,t) = t^p + a(x)t^p \log(1+t)$ with $-1 < p$ and $0 \leqslant a(\cdot) \in C(B_1)$, where the constants in (A2) can be defined by $i(\Phi) = p$ and $s(\Phi) = p + \varepsilon$ for any $\varepsilon > 0$.
2. $\Phi(x,t) = \frac{\phi(t)}{t^2} + a(x)\frac{\psi(t)}{t^2}$ for suitable N-functions ϕ, ψ, and $0 \leqslant a(\cdot) \in C(B_1)$.

As we conclude this introduction, let us briefly explain the techniques used in the proof of Theorem 1.1. While a main idea for getting $C^{1,\beta}$ regularity of the viscosity solution u of (1.3) is showing that the graph of the function u can be approximated by affine functions with an error bounded by $CR^{1+\beta}$ in any ball of radius R, as shown in previous papers [21, 36, 47, 52], there are several difficulties to overcome by means of new ideas and tools. In fact, for the degenerate case ($0 \leqslant i(\Phi)$), we show that a viscosity solution v of the equation

$$\Phi(x, |\xi + Dv|)F(D^2v) = f(x) \quad \text{in } B_1, \quad \xi \in \mathbb{R}^n \text{ is any vector},$$

is locally Hölder continuous with an exponent β, which is independent of the size of ξ, using the method introduced by Ishii and Lions [56] in the case of large scopes and the Harnack inequality approach of Caffarelli and Cabré [26] employed in [54] for small scopes. This information is an important step towards the proof of Theorem 1.1 based on iterative and compactness arguments, in which we show that the graph of u is better approximated by affine functions in smaller balls. For the singular case ($-1 < i(\Phi) < 0$), the main result of Theorem 1.1 is an outcome of the degenerate case after showing that the viscosity solution u of equation (1.3) is Lipschitz continuous; see Lemma 3.1 below.

Finally, we outline the organization of the paper. In the next section, we provide basic notations and assumptions to be used, and also smallness regime and basic regularity results. In Section 3, we prove basic regularity properties of viscosity solutions of (2.5) depending on the sign of $i(\Phi)$ and the size of quantity $|\xi|$. Section 4 is devoted to the approximation procedure for viscosity solutions of (1.3). Finally, in last section we provide the proof of Theorem 1.1.

2. Preliminaries

2.1. Notation

Throughout the paper, we denote by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the open ball of $\mathbb{R}^n$ with $n \geqslant 2$ centered at x_0 with positive radius r. If the center is clear in the context, we shall omit the center point by writing $B_r \equiv B_r(x_0)$. Also, $B_1 \equiv B_1(0) \subset \mathbb{R}^n$ denote the unit ball. We shall always denote by c a generic positive constant, possibly varying line to line, having dependencies on parameters using brackets; for example, $c \equiv c(n, i(\Phi), \nu_0)$ means that c is a positive constant depending only on $n, i(\Phi)$, and ν_0. For a measurable map $g : \mathcal{B} \subset B_1 \to \mathbb{R}^N$ $(N \geqslant 1)$ with $\beta \in (0, 1]$ being a given number, we shall use the notation

$$[g]_{C^{0,\beta}(\mathcal{B})} := \sup_{x \neq y \in \mathcal{B}} \frac{|g(x) - g(y)|}{|x - y|^\beta}, \quad [g]_{C^{0,\beta}} := [g]_{C^{0,\beta}(B_1)}.$$

The Pucci extremal operators $P^{\pm}_{\lambda,\Lambda} : \mathcal{S}(n) \to \mathbb{R}$ are defined as

$$P^{+}_{\lambda,\Lambda}(M) := -\lambda \sum_{\lambda_k > 0} \lambda_k - \Lambda \sum_{\lambda_k < 0} \lambda_k \tag{2.1}$$

and

$$P^{-}_{\lambda,\Lambda}(M) := -\Lambda \sum_{\lambda_k > 0} \lambda_k - \lambda \sum_{\lambda_k < 0} \lambda_k, \tag{2.2}$$

where $\{\lambda_k\}_{k=1}^n$ are the eigenvalues of the matrix M. The (λ, Λ)-ellipticity of the operator F via the Pucci extremal operators can be formulated as

$$P^{-}_{\lambda,\Lambda}(N) \leqslant F(M + N) - F(M) \leqslant P^{+}_{\lambda,\Lambda}(N) \tag{2.3}$$

for all $M, N \in \mathcal{S}(n)$; see [26, Chapter 2] for detailed discussions. In what follows, for any vector $\xi \in \mathbb{R}^n$, we define maps $G_\xi : B_1 \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ by

$$G_\xi(x, p, M) := \Phi(x, |\xi + p|)F(M) - f(x) \tag{2.4}$$

under assumptions (A1)–(A3). Note that the map G_ξ in (2.4) satisfies ellipticity condition (1.2) (monotonicity property), that is,

$$G_\xi(x, z, M) \leqslant G_\xi(x, z, N) \quad \text{whenever } M, N \in \mathcal{S}(n) \text{ satisfy } M \geqslant N.$$

Then, we shall focus on viscosity solutions of the equation

$$G(x, Du, D^2u) := \Phi(x, |Du|)F(D^2u) - f(x) = 0 \quad \text{in } B_1 \tag{2.5}$$

or

$$G_\xi(x, Du, D^2u) = 0 \quad \text{in } B_1, \tag{2.6}$$

where ξ is any vector in $\mathbb{R}^n$.

For the reader's convenience, we give the definition of a viscosity solution u of equation (2.5) as defined in [16].

Definition 2.1. A lower semicontinuous function v is called a viscosity supersolution of (2.5) if for any $x_0 \in B_1$, either there exists $\delta > 0$ such that u is constant in $B_\delta(x_0)$ and $f(x) \leqslant 0$ for all $x \in B_\delta(x_0)$; or, for all $\varphi \in C^2(B_1)$ such that $v - \varphi$ has a local minimum at x_0 and $D\varphi(x_0) \neq 0$,

$$G(x_0, D\varphi(x_0), D^2\varphi(x_0)) \geqslant 0$$

holds. An upper semicontinuous function w is called a viscosity subsolution of (2.5) if for all $x_0 \in B_1$, either there exists $\delta > 0$ such that u is constant in $B_\delta(x_0)$ and $f(x) \geqslant 0$ for all $x \in B_\delta(x_0)$; or, for all $\varphi \in C^2(B_1)$ such that $w - \varphi$ has a local maximum at x_0 and $D\varphi(x_0) \neq 0$,

$$G(x_0, D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0$$

holds. We say that $u \in C(B_1)$ is a viscosity solution of (2.5) if u is a viscosity supersolution and a subsolution simultaneously.

Remark 2.1. We remark that the above definition is necessary only for the case $-1 < i(\Phi) < 0$, due to the fact that the operator appearing in equation (2.5) may not be defined when the gradient is zero. In the case $i(\Phi) \geqslant 0$, the classical definition of a viscosity solution is equivalent to Definition 2.1; see [31]. We also note that a viscosity solution of equation (2.6) can be understood as a viscosity solution of equation (2.5), by replacing $u(x) = v(x) + \langle \xi, x \rangle$.

Following the proof of [19, Proposition 1.2], one can prove the following assertion:

Proposition 2.1. *Suppose that for $-1 < i(\Phi) < 0$, u is a viscosity solution of (2.5) in the sense of Definition 2.1 under assumptions (A1)–(A3). Then, u is a classical viscosity solution of*

$$|Du|^{-i(\Phi)}\Phi(x, |Du|)F(D^2u) = f(x)|Du|^{-i(\Phi)}. \tag{2.7}$$

Note that the map $\Psi : B_1 \times [0, \infty) \to [0, \infty)$, given by $\Psi(x, t) = t^{-i(\Phi)}\Phi(x, t)$, satisfies condition (A2) for numbers $i(\Psi) = 0$ and $s(\Psi) = s(\Phi) - i(\Phi)$. Moreover, Proposition 2.1 means that if $-1 < i(\Phi) < 0$ in (A2), then any viscosity solution u of equation (2.5) in the sense of Definition 2.1 can be understood as a classical viscosity solution of equation (2.7).

2.2. Small regime

Here we verify that, for a viscosity solution u of (2.6), we are able to assume that

$$\operatorname{osc}_{B_1} u \leqslant 1 \quad \text{and} \quad \|f\|_{L^\infty(B_1)} \leqslant \varepsilon_0 \tag{2.8}$$

for some constant $0 < \varepsilon_0 < 1$ small enough, and also that $\nu_0 = \nu_1 = 1$, without loss of generality. In order to consider the problem in a small regime as in (2.8), for a fixed ball $B_R(x_0) \subset B_1$, we define $\hat{u} : B_1 \to \mathbb{R}$ by

$$\hat{u}(x) := \frac{u(x_0 + Rx)}{K} \tag{2.9}$$

for positive constants $K \geqslant 1 \geqslant R$ which will be determined in a few lines. It can be seen that $\hat{u}$ is a viscosity solution of

$$\widehat{G}_{\hat{\xi}}(x, D\hat{u}, D^2\hat{u}) := \widehat{\Phi}(x, |\hat{\xi} + D\hat{u}|)\widehat{F}(D^2\hat{u}) - \hat{f}(x) = 0, \tag{2.10}$$

where

$$\begin{aligned} \widehat{\Phi}(x,t) &:= \frac{\Phi\left(x_0 + Rx, \frac{K}{R}t\right)}{\Phi\left(x_0 + Rx, \frac{K}{R}\right)}, \\ \widehat{F}(M) &:= \frac{R^2}{K} F\left(\frac{K}{R^2} M\right), \\ \hat{f}(x) &:= \frac{R^2}{\Phi\left(x_0 + Rx, \frac{K}{R}\right)K} f(x_0 + Rx) \quad \text{and} \quad \hat{\xi} := \frac{R}{K}\xi. \end{aligned}$$

Note that $\widehat{F}$ is still a uniformly (λ, Λ)-elliptic operator, the map $t \mapsto \frac{\widehat{\Phi}(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing, and the map $t \mapsto \frac{\widehat{\Phi}(x,t)}{t^{s(\Phi)}}$ is almost non-increasing with the same constant $L \geqslant 1$ as in assumption (A2); and $\widehat{\Phi}(x, 1) = 1$ for all $x \in B_1$. Moreover, (A2) implies

$$\|\hat{f}\|_{L^\infty(B_1)} \leqslant \frac{LR^{2+i(\Phi)}}{\nu_0 K^{1+i(\Phi)}} \|f\|_{L^\infty(B_1)} \leqslant \frac{L}{\nu_0 K^{1+i(\Phi)}} \|f\|_{L^\infty(B_1)}.$$

By recalling $i(\Phi) > -1$ and setting

$$K := 2\left(\varepsilon + \|u\|_{L^\infty(B_1)} + \left[\frac{L}{\varepsilon_0 \nu_0} \|f\|_{L^\infty(B_1)}\right]^{\frac{1}{1+i(\Phi)}}\right) \quad \text{for any } \varepsilon \geqslant 1,$$

we see that $\hat{u}$ solves equation (2.10) in the same class as does (2.6) under the small regime described in (2.8).

2.3. Basic regularity results

In this subsection, we discuss some basic regularity results for (2.6). First, we recall important notions of superjets and subjets introduced in [31].

Definition 2.2. Let $v : \Omega \to \mathbb{R}$ be an upper semicontinuous function and $w : \Omega \to \mathbb{R}$ be a lower semicontinuous function. For every $x_0 \in \Omega$, we define the *second-order superjet* of v at x_0 by

$$J^{2,+}_{\Omega} v(x_0) := \Big\{(p, M) \in \mathbb{R}^n \times \mathcal{S}(n) : v(x) \leq v(x_0) + \langle p, x - x_0\rangle \\ + \frac{1}{2}\langle M(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \text{ as } x \to x_0\Big\}$$

and the *second-order subjet* of w at x_0 by

$$J^{2,-}_{\Omega} w(x_0) := \Big\{(p, M) \in \mathbb{R}^n \times \mathcal{S}(n) : w(x) \geq w(x_0) + \langle p, x - x_0\rangle \\ + \frac{1}{2}\langle M(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \text{ as } x \to x_0\Big\}.$$

We further define a *limiting* superjet of v and subjet of w, respectively:

(1) A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a *limiting superjet* of v at $x_0 \in \Omega$ if there exists a sequence $\{x_k, p_k, M_k\} \to \{x, p, M\}$ as $k \to \infty$ in such a way that $(p_k, M_k) \in J^{2,+}_{\Omega} v(x_k)$ and $\lim_{k\to\infty} v(x_k) = v(x_0)$. We denote the set of all limiting superjets of v at $x_0 \in \Omega$ by $\overline{J}^{2,+}_{\Omega} v(x_0)$.

(2) A couple $(p, M) \in \mathbb{R}^n \times \mathcal{S}(n)$ is a *limiting subjet* of w at $x_0 \in \Omega$ if there exists a sequence $\{x_k, p_k, M_k\} \to \{x, p, M\}$ as $k \to \infty$ in such a way that $(p_k, M_k) \in J^{2,-}_{\Omega} w(x_k)$ and $\lim_{k\to\infty} w(x_k) = w(x_0)$. We denote the set of all limiting subjets of w at $x_0 \in \Omega$ by $\overline{J}^{2,-}_{\Omega} w(x_0)$.

Using the above concept of superjet and subjet, the notions of viscosity supersolutions, subsolutions, and solutions of (1.1) are defined as follows (see [31, Definition 2.2]):

Definition 2.3. Let $\mathcal{F} : B_1 \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ be a map satisfying (1.2). An upper semicontinuous function $v : B_1 \to \mathbb{R}$ is called a viscosity subsolution of equation (1.1) (equivalently, a viscosity solution of $\mathcal{F} \leqslant 0$) if

$$\mathcal{F}(x, p, M) \leqslant 0 \tag{2.11}$$

holds true for all $x \in B_1$ and $(p, M) \in J^{2,+}_{B_1} v(x)$. Similarly, a lower semicontinuous function $w : B_1 \to \mathbb{R}$ is called a viscosity subsolution of equation (1.1) (equivalently, a viscosity solution of $\mathcal{F} \geqslant 0$) if

$$\mathcal{F}(x, p, M) \geqslant 0 \tag{2.12}$$

holds true for all $x \in B_1$ and $(p, M) \in J^{2,-}_{B_1} w(x)$. Finally, a continuous function $u : B_1 \to \mathbb{R}$ is a viscosity solution of equation (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1).

Remark 2.2. We remark that in the above definition, if the map $\mathcal{F} : B_1 \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ is continuous, then $J^{2,+}_{B_1}$ can be replaced by $\overline{J}^{2,+}_{B_1}$ for a viscosity subsolution and $J^{2,-}_{B_1}$ can be replaced by $\overline{J}^{2,-}_{B_1}$ for a viscosity supersolution, respectively; see [31, Remark 2.4].

There are some examples and properties of superjets and subjet described in [31, 55], among which is the following result, which we need as it is useful to our case (see also [58, Lemma 2.1]):

Lemma 2.1. *Let $v : \Omega \to \mathbb{R}$ be an upper semicontinuous function and $w : \Omega \to \mathbb{R}$ be a lower semicontinuous function. Then, for any point $x_0 \in \Omega$, we have*

$$\begin{aligned} J^{2,+}_{\Omega} v(x_0) = \big\{ &(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega),\ \varphi(x_0) = v(x_0), \\ &\varphi \geqslant v \text{ in a neighborhood of } x_0 \big\}, \\ J^{2,-}_{\Omega} w(x_0) = \big\{ &(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2(\Omega),\ \varphi(x_0) = w(x_0), \\ &\varphi \leqslant v \text{ in a neighborhood of } x_0 \big\}. \end{aligned}$$

Remark 2.3. Essentially, Lemma 2.1 tells us that Definitions 2.3 and 2.1 are equivalent for equation (1.3). Next, we discuss a variation of the celebrated Ishii–Jensen lemma (see [31, Theorem 3.2]), which will be used afterwards.

Lemma 2.2. *Let u be a viscosity solution of (2.6) under assumptions (A1)–(A3), where $\xi \in \mathbb{R}^n$ is any vector. Suppose that $\mathcal{B} \Subset B_1$ is an open subset and $\psi \in C^2(\mathcal{B} \times \mathcal{B})$. Define a map $v : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ as*

$$v(x, y) := u(x) - u(y).$$

Suppose further $(\hat{x}, \hat{y}) \in \mathcal{B} \times \mathcal{B}$ is a local maximum point of $v - \psi$ in $\mathcal{B} \times \mathcal{B}$. Then, for each $\delta > 0$, there exist matrices $X_\delta, Y_\delta \in \mathcal{S}(n)$ such that

$$G_\xi(\hat{x}, D_x\psi(\hat{x}, \hat{y}), X_\delta) \leqslant 0 \leqslant G_\xi(\hat{y}, -D_y\psi(\hat{x}, \hat{y}), Y_\delta) \tag{2.13}$$

and

$$-\Big(\frac{1}{\delta} + \|A\|\Big) I \leqslant \begin{pmatrix} X_\delta & 0 \\ 0 & -Y_\delta \end{pmatrix} \leqslant A + \delta A^2 \tag{2.14}$$

with $A := D^2\psi(\hat{x}, \hat{y})$.

Proof. We are able to apply [31, Theorem 3.2] in our setting. In turn, for each $\delta > 0$, there exist $X_\delta, Y_\delta \in \mathcal{S}(n)$ such that

$$(D_x\psi(\hat{x}, \hat{y}), X_\delta) \in \overline{J}^{2,+}_{\mathcal{B}} u(\hat{x}) \quad \text{and} \quad (D_x\psi(\hat{x}, \hat{y}), -Y_\delta) \in \overline{J}^{2,+}_{\mathcal{B}} (-u)(\hat{y}), \tag{2.15}$$

and the block diagonal matrix with entries X_δ and $-Y_\delta$ satisfy (2.14). Since u is a viscosity solution of (2.6), the validity of (2.13) is immediate by Remarks 2.2 and 2.3. ∎

3. Hölder continuity

In this section, we provide Hölder regularity for solutions of (2.6), where ξ is any vector, under the small regime.

Lemma 3.1 (Hölder continuity). *Let u be a viscosity solution of (2.6) under assumptions (A1)–(A3) with $\operatorname{osc}_{B_1} u \leqslant 1$, $\|f\|_{L^\infty(B_1)} \leqslant \varepsilon_0 < 1$, and $\nu_0 = \nu_1 = 1$. Let $B_R \equiv B_R(x_0) \subset B_1$ be any ball. Then, we have the following results:*

(R1) *If $-1 < i(\Phi) < 0$ and $|\xi| = 0$, then u is Lipschitz continuous in $B_{R/2}$ with the estimate*

$$[u]_{C^{0,1}(B_{R/2})} \leqslant C_{sl} \tag{3.1}$$

for some constant $C_{sl} \equiv C_{sl}(n, \lambda, \Lambda, i(\Phi), L, R)$.

(R2) *If $i(\Phi) \geqslant 0$ and $|\xi| > A_0$ with $A_0 \equiv A_0(n, \lambda, \Lambda, i(\Phi), L, R) \geqslant 1$, then u is Lipschitz continuous in $B_{R/2}$ with the estimate*

$$[u]_{C^{0,1}(B_{R/2})} \leqslant C_{dl} \tag{3.2}$$

for some constant $C_{dl} \equiv C_{dl}(n, \lambda, \Lambda, i(\Phi), L, R)$.

(R3) *If $i(\Phi) \geqslant 0$ and $|\xi| \leqslant A_0$, then $u \in C^{0,\beta}(B_{R/2})$ with the estimate*

$$[u]_{C^{0,\beta}(B_{R/2})} \leqslant C_{ds}, \tag{3.3}$$

where $\beta \equiv \beta(n, \lambda, \Lambda, i(\Phi), L, R, A_0) \in (0, 1)$ and $C_{ds} \equiv C_{ds}(n, \lambda, \Lambda, i(\Phi), L, R, A_0)$.

Proof. For the proof of (R1) and (R2), it suffices to show that there exist positive constants L_1 and L_2 such that

$$\mathcal{L} := \sup_{x,y \in B_R} (u(x) - u(y) - L_1\omega(|x-y|) - L_2(|x-z_0|^2 + |y-z_0|^2)) \leqslant 0 \tag{3.4}$$

for every $z_0 \in B_{R/2}$, where

$$\omega(t) = \begin{cases} t - \omega_0 t^{\frac{3}{2}} & \text{if } t \leqslant t_0 := \left(\frac{2}{3\omega_0}\right)^2, \\ \omega(t_0) & \text{if } t \geqslant t_0. \end{cases} \tag{3.5}$$

We choose $\omega_0 \in (0, 2/3)$ in such a way that $t_0 \geqslant 1$. For instance, we take any constant $\omega_0 \leqslant 1/3$. By contradiction, suppose that there are no such positive constants L_1 and L_2 satisfying (3.4) for every $z_0 \in B_{R/2}$. Then, there exists a point $z_0 \in B_{R/2}$ so that $\mathcal{L} > 0$ for all numbers $L_1 > 0$ and $L_2 > 0$. Now we define two auxiliary functions $\phi, \psi : \overline{B_R} \times \overline{B_R} \to \mathbb{R}$ given by

$$\psi(x, y) := L_1\omega(|x-y|) + L_2(|x-z_0|^2 + |y-z_0|^2) \tag{3.6}$$

and

$$\phi(x, y) := u(x) - u(y) - \psi(x, y). \tag{3.7}$$

Let $(\hat{x}, \hat{y}) \in \overline{B_R} \times \overline{B_R}$ be a maximum point for ϕ. Then, we have

$$\phi(\hat{x}, \hat{y}) = \mathcal{L} > 0$$

and

$$L_1\omega(|\hat{x}-\hat{y}|)+L_2(|\hat{x}-z_0|^2+|\hat{y}-z_0|^2)\leqslant \operatorname{osc}_{B_1} u\leqslant 1.$$

Now we select

$$L_2:=\frac{64}{R^2}.$$

This choice of L_2 ensures

$$|\hat{x}-z_0|+|\hat{y}-z_0|\leqslant \frac{R}{4}\quad\text{and}\quad |\hat{x}-\hat{y}|\leqslant\frac{R}{4}. \tag{3.8}$$

This means that the points $\hat{x}$ and $\hat{y}$ belong to the open ball B_R, and we are able to assume that $\hat{x}\neq\hat{y}$, otherwise $\mathcal{L}\leqslant 0$ is clear. The rest of the proof is divided into several steps for simplicity of the presentation.

Step 1. We are in a position to apply Lemma 2.2 in order to ensure the existence of a limiting subjet $(\xi_{\hat{x}}, X_\delta)$ of u at $\hat{x}$ and a limiting superjet $(\xi_{\hat{y}}, Y_\delta)$ of u at $\hat{y}$, where

$$\xi_{\hat{x}}:=D_x\psi(\hat{x},\hat{y})=L_1\omega'(|\hat{x}-\hat{y}|)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}+2L_2(\hat{x}-z_0)$$

and

$$\xi_{\hat{y}}:=-D_y\psi(\hat{x},\hat{y})=L_1\omega'(|\hat{x}-\hat{y}|)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}-2L_2(\hat{y}-z_0),$$

such that the matrices X_δ and Y_δ satisfy the matrix inequality

$$\begin{pmatrix} X_\delta & 0\\ 0 & -Y_\delta\end{pmatrix}\leqslant\begin{pmatrix} Z & -Z\\ -Z & Z\end{pmatrix}+(2L_2+\delta)I, \tag{3.9}$$

where

$$\begin{aligned} Z&:=L_1D^2(\omega(|\cdot|))(\hat{x}-\hat{y})\\ &=L_1\Big[\frac{\omega'(|\hat{x}-\hat{y}|)}{|\hat{x}-\hat{y}|}I+\Big(\omega''(|\hat{x}-\hat{y}|)-\frac{\omega'(|\hat{x}-\hat{y}|)}{|\hat{x}-\hat{y}|}\Big)\frac{(\hat{x}-\hat{y})\otimes(\hat{x}-\hat{y})}{|\hat{x}-\hat{y}|^2}\Big]\end{aligned}$$

and the constant $\delta>0$ only depends on the norm of Z, which can be selected to be sufficiently small. Applying inequality (3.9) for vectors of the form $(z,z)\in\mathbb{R}^{2n}$, we find

$$\langle (X_\delta-Y_\delta)z,z\rangle\leqslant(4L_2+2\delta)|z|^2.$$

The last inequality yields that all the eigenvalues of the matrix $(X_\delta-Y_\delta)$ are not larger than $4L_2+2\delta$. On the other hand, applying again (3.9) for the vector $\hat{z}:=(\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|},\frac{\hat{y}-\hat{x}}{|\hat{x}-\hat{y}|})$, we have

$$\begin{aligned}\langle (X_\delta-Y_\delta)\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|},\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\rangle&\leqslant\Big(4L_2+2\delta+4L_1\omega''(|\hat{x}-\hat{y}|)\Big)\Big|\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\Big|^2\\ &=\Big(4L_2+2\delta-\frac{6\omega_0L_1}{|\hat{x}-\hat{y}|^{1/2}}\Big)\Big|\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\Big|^2\\ &\leqslant(4L_2+2\delta-6\omega_0L_1)\Big|\frac{\hat{x}-\hat{y}}{|\hat{x}-\hat{y}|}\Big|^2,\end{aligned}$$

where we have used the definition of ω in (3.5) together with $|\hat{x}-\hat{y}|\leqslant 1/4$ in (3.8). As a consequence of the last display, at least one eigenvalue of $(X_\delta - Y_\delta)$ is not larger than $4L_2+2\delta-6\omega_0 L_1$, where this quantity can be negative for large values of L_1. By the definition of the extremal Pucci operator in (2.1)–(2.2), we see that

$$\begin{aligned} P^-_{\lambda,\Lambda}(X_\delta - Y_\delta) &\geqslant -\lambda(4L_2+2\delta-6\omega_0 L_1) - \Lambda(n-1)(4L_2+2\delta) \\ &\geqslant -(\lambda+(n-1)\Lambda)(4L_2+2\delta)+6\omega_0\lambda L_1. \end{aligned}$$

From the two viscosity inequalities and (2.3), we have

$$\Phi(\hat{x},|\xi+\xi_{\hat{x}}|)F(X_\delta)\leqslant f(\hat{x}),\quad \Phi(\hat{y},|\xi+\xi_{\hat{y}}|)F(Y_\delta)\geqslant f(\hat{y})$$

and

$$F(X_\delta)\geqslant F(Y_\delta)+P^-_{\lambda,\Lambda}(X_\delta-Y_\delta).$$

Combining last three displays, we have

$$6\omega_0\lambda L_1 \leqslant (\lambda+(n-1)\Lambda)(4L_2+2\delta)+\frac{f(\hat{x})}{\Phi(\hat{x},|\xi+\xi_{\hat{x}}|)}-\frac{f(\hat{y})}{\Phi(\hat{y},|\xi+\xi_{\hat{y}}|)}. \tag{3.10}$$

At this stage, we separate the remaining part of the proof into several cases depending on the quantity of $|\xi|$ and the positiveness of $i(\Phi)$.

Step 2: Proof of (R1). Suppose $-1<i(\Phi)<0$ and $\xi=0$. By triangle inequality $(3.8)_2$, we observe that

$$|\xi_{\hat{x}}|\leqslant L_1\Big(1+\frac{3}{2}\omega_0\Big)+2L_2\leqslant \frac{7}{4}L_1 \tag{3.11}$$

and

$$|\xi_{\hat{x}}|\geqslant L_1\Big(1-\frac{3\omega_0}{2}|\hat{x}-\hat{y}|^{\frac{1}{2}}\Big)-3L_2\geqslant \frac{3L_1}{4}-3L_2\geqslant 3L_2 \tag{3.12}$$

for all $L_1\geqslant 8L_2$. In exactly the same way, we see

$$|\xi_{\hat{y}}|\leqslant \frac{7}{4}L_1 \quad\text{and}\quad |\xi_{\hat{y}}|\geqslant 2L_2 \tag{3.13}$$

for all $L_1\geqslant 8L_2$. Using (A2) and last two displays, we have

$$\frac{f(\hat{x})}{\Phi(\hat{x},|\xi_{\hat{x}}|)}\leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi_{\hat{x}}|^{i(\Phi)}}\leqslant \frac{c}{L_1^{i(\Phi)}} \tag{3.14}$$

and

$$\frac{-f(\hat{y})}{\Phi(\hat{y},|\xi_{\hat{y}}|)}\leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi_{\hat{y}}|^{i(\Phi)}}\leqslant \frac{c}{L_1^{i(\Phi)}} \tag{3.15}$$

for a constant $c\equiv c(i(\Phi),L)$. Using the last two displays in (3.10), we obtain

$$6\omega_0\lambda L_1\leqslant(\lambda+(n-1)\Lambda)(4L_2+2\delta)+\frac{c}{L_1^{i(\Phi)}}$$

for a constant $c\equiv c(n,\lambda,\Lambda,i(\Phi),L,R)$. Recalling $-1<i(\Phi)<0$ and taking L_1 large enough, depending only on $n,\lambda,\Lambda,i(\Phi),L$ and R, we get a contradiction. Thus, the first part of Lemma 3.1 is proved.

Step 3: Proof of (R2). We suppose that $i(\Phi) \geqslant 0$ and $|\xi| > A_0$ with a constant A_0 defined by

$$A_0 := \frac{35L_1}{2} \tag{3.16}$$

for $L_1 > 1$ to be selected soon. This choice of A_0 together with (3.11) and (3.14) leads to

$$|\xi + \xi_{\hat{x}}| \geqslant A_0 - \frac{A_0}{10} = \frac{9A_0}{10} \quad \text{and} \quad |\xi + \xi_{\hat{y}}| \geqslant \frac{9A_0}{10}.$$

Therefore, using the last display and (A2), we have

$$\frac{f(\hat{x})}{\Phi(\hat{x}, |\xi + \xi_{\hat{x}}|)} \leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi + \xi_{\hat{x}}|^{i(\Phi)}} \leqslant \frac{c}{A_0^{i(\Phi)}}$$

and

$$\frac{-f(\hat{y})}{\Phi(\hat{y}, |\xi + \xi_{\hat{y}}|)} \leqslant c\frac{\|f\|_{L^\infty(B_1)}}{|\xi + \xi_{\hat{y}}|^{i(\Phi)}} \leqslant \frac{c}{A_0^{i(\Phi)}}$$

for a constant $c \equiv c(i(\Phi), L)$. Again using the last two displays in (3.10), we obtain

$$6\omega_0\lambda L_1 \leqslant (\lambda + (n-1)\Lambda)(4L_2 + 2\delta) + \frac{c}{L_1^{i(\Phi)}}$$

for a constant $c \equiv c(n, \lambda, \Lambda, i(\Phi), L, R)$. By choosing L_1 large enough, depending only on $n, \lambda, \Lambda, i(\Phi), L$ and R, we have again a contradiction. Thus, we have proved the second part of Lemma 3.1.

Step 4: Proof of (R3). Finally, we turn our attention to proving (R3). Suppose $|\xi| \leqslant A_0$, where A_0 has been defined in (3.16). We look at an operator $\mathcal{G} : B_1 \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ given by

$$\mathcal{G}(x, p, M) := \Phi(x, |\xi + p|)F(M) - f(x).$$

In fact, for any $(x, p, M) \in B_1 \times \mathbb{R}^n \times \mathcal{S}(n)$ with $|p| > 2A_0$ and $\mathcal{P}^-_{\lambda,\Lambda}(M) \geqslant 0$, we see that

$$\begin{aligned} \mathcal{G}(x, p, M) &\geqslant \Phi(x, |\xi + p|)\mathcal{P}^-_{\lambda,\Lambda}(M) - \|f\|_{L^\infty(B_1)} \\ &\geqslant \frac{1}{L}|\xi + p|^{i(\Phi)}\mathcal{P}^-_{\lambda,\Lambda}(M) - 1 \\ &\geqslant \frac{A_0^{i(\Phi)}}{L}\mathcal{P}^-_{\lambda,\Lambda}(M) - 1, \end{aligned} \tag{3.17}$$

where we have used (2.3) and (A2) together with the facts that $i(\Phi) \geqslant 1$ and $|\xi + p| \geqslant |p| - |\xi| \geqslant A_0 \geqslant 1$. In turn, we see that u is a viscosity subsolution of the equation

$$\mathcal{M}^-(Du, D^2u) \leqslant 1, \tag{3.18}$$

where the operator $\mathcal{M}^- : \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\mathcal{M}^-(p, M) := \begin{cases} \frac{A_0^{i(\Phi)}}{L}\mathcal{P}^-_{\lambda,\Lambda}(M) & \text{if } |p| > 2A_0, \\ -\infty & \text{otherwise.} \end{cases} \tag{3.19}$$

Similarly, we show that u is a viscosity supersolution of the equation

$$\mathcal{M}^+(Du, D^2u) \geqslant -1, \tag{3.20}$$

where the operator $\mathcal{M}^+ : \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{M}^+(p, M) := \begin{cases} \frac{A_0^{i(\Phi)}}{L} \mathcal{P}^+_{\lambda,\Lambda}(M) & \text{if } |p| > 2A_0, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.21}$$

Therefore, we are in an exact position to apply [53, Theorem 1.1] (see also [67] for degenerate/singular elliptic equations with unbounded drift) so that we find an exponent $\beta \equiv \beta(n, \lambda, \Lambda, L, R, A_0) \in (0, 1)$ satisfying (3.3). The proof is complete. ■

4. Approximation

Now we prove a key approximation lemma, which plays a crucial role in later arguments.

Lemma 4.1. *Let $u \in C(B_1)$ be a viscosity solution of equation (2.6) with $\operatorname{osc}_{B_1} \leqslant 1$, where $\xi \in \mathbb{R}^n$ is arbitrarily given. Suppose assumptions (A1)–(A3) hold true for $i(\Phi) \geqslant 0$ and $\nu_0 = \nu_1 = 1$. Then, for any $\mu > 0$, there exists a constant $\delta \equiv \delta(n, \lambda, \Lambda, i(\Phi), L, \mu)$ such that if*

$$\|f\|_{L^\infty(B_1)} \leqslant \delta, \tag{4.1}$$

then one can find $h \in C^{1,\hat{\alpha}}(\overline{B}_{3/4})$, with the estimate $\|h\|_{C^{1,\hat{\alpha}}(\overline{B}_{3/4})} \leqslant c \equiv c(n, \lambda, \Lambda)$ for some $\hat{\alpha} \in (0, 1)$ as in (1.12), satisfying

$$\|u - h\|_{L^\infty(B_{1/2})} \leqslant \mu. \tag{4.2}$$

Proof. We point out that the proof is inspired by the proof of [52, Lemma 6]. By contradiction, we suppose the conclusion of the lemma fails. Then, there exist $\mu_0 > 0$ and sequences of $\{F_k\}_{k=1}^\infty$, $\{\Phi_k\}_{k=1}^\infty$, $\{f_k\}_{k=1}^\infty$, and $\{u_k\}_{k=1}^\infty$ and a sequence of vectors $\{\xi_k\}_{k=1}^\infty$ such that

- (C1) $F_k \in C(\mathcal{S}(n), \mathbb{R})$ is uniformly (λ, Λ)-elliptic;
- (C2) $\Phi_k \in C(B_1 \times [0, \infty), [0, \infty))$ such that the map $t \mapsto \frac{\Phi_k(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing and the map $t \mapsto \frac{\Phi_k(x,t)}{t^{s(\Phi)}}$ is almost non-increasing with constant $L \geqslant 1$, and $\Phi_k(x, 1) = 1$ for all $x \in B_1$;
- (C3) $f_k \in C(B_1)$ with $\|f_k\|_{L^\infty(B_1)} \leqslant 1/k$;
- (C4) $u_k \in C(B_1)$ with $\operatorname{osc}_{B_1} u_k \leqslant 1$ solves the equation

$$\Phi_k(x, |\xi_k + Du_k|) F_k(D^2u_k) = f_k(x), \tag{4.3}$$

but

$$\sup_{x\in B_{1/2}} |u_k(x) - h(x)| > \mu_0 \tag{4.4}$$

for all $h \in C^{1,\hat{\alpha}}(\overline{B}_{3/4})$ and every $0 < \hat{\alpha} < 1$.

Condition (C1) implies that F_k converges to some uniformly (λ, Λ)-elliptic operator $F_\infty \in C(\mathcal{S}(n), \mathbb{R})$. Since u_k is a viscosity solution of (4.3) and the map Φ_k satisfies (C2), we are able to apply Lemma 3.1 to find that $u_k \in C^{0,\beta}_{\mathrm{loc}}(B_1) \cap C(B_1)$ for some $\beta \in (0, 1)$. Using estimates (3.2) and (3.3) and the Arzelà–Ascoli theorem, we have that the sequence $\{u_k\}_{k=1}^\infty$ converges to a function u_∞, locally uniformly in B_1. In particular, it holds that

$$u_\infty \in C(B_1) \quad \text{and} \quad \operatorname{osc}_{B_1} u_\infty \leqslant 1. \tag{4.5}$$

Now we prove that the limiting function u_∞ is a viscosity solution of the homogeneous equation

$$F_\infty(D^2u_\infty) = 0 \quad \text{in } B_{3/4}. \tag{4.6}$$

For this, we first verify that u_∞ is a viscosity supersolution. Let

$$p(x) := \frac{1}{2}\langle M(x-y), x-y\rangle + \langle b, x-y\rangle + u_\infty(y)$$

be a quadratic polynomial touching u_∞ from below at a point $y \in B_{3/4}$. Note that there is no need to assume that $Dp(y) \neq 0$, since $i(\Phi) \geqslant 0$ and we only look at a classical viscosity solution (see Remark 2.1). Without loss of generality, let us assume $|y| = u_\infty(y) = 0$. Then, there exists a sequence $x_k \to 0$ as $k \to \infty$ such that $u_k - \varphi$ has a local minimum at x_k. Observe that $D\varphi(x_k) \to b$ and $D^2\varphi(x_k) \to M$. Since u_k is a viscosity solution of (4.3), we have

$$\Phi_k(x_k, |\xi_k + D\varphi(x_k)|)F_k(D^2\varphi(x_k)) \geqslant f_k(x_k). \tag{4.7}$$

For ease of presentation, from now on we shall consider several cases depending on the boundedness of the sequence $\{\xi_k\}_{k=1}^\infty$.

Case 1: The sequence $\{\xi_k\}_{k=1}^\infty$ is unbounded. In this case, we can assume $|\xi_k| \to \infty$ (up to a subsequence). As a consequence, we can show (up to a subsequence) that

$$|\xi_k + D\varphi(x_k)| \geqslant |\xi_k| - |D\varphi(x_k)| \geqslant |\xi_k| - (|b| + 1) \geqslant 1, \tag{4.8}$$

where we have used the triangle inequality and the convergence $D\varphi(x_k) \to b$, which implies that

$$\begin{aligned} F_\infty(M) = \lim_{k\to\infty} F_k(D^2\varphi(x_k)) &\geqslant \lim_{k\to\infty} \frac{f_k(x_k)}{\Phi_k(x_k, |\xi_k + D\varphi(x_k)|)} \\ &\geqslant -\lim_{k\to\infty} \frac{L}{k|\xi_k + D\varphi(x_k)|^{i(\Phi)}} = 0, \end{aligned}$$

where we have used condition (C2) and (4.7).

Case 2: The sequence $\{\xi_k\}_{k=1}^{\infty}$ is bounded. In this case, we may assume $\xi_k \to \xi_\infty$ (up to a subsequence). Therefore, for the case $|\xi_\infty + b| \neq 0$, in exactly the same way as in (4.8), we infer that $F_\infty(M) \geqslant 0$. Then, we focus on the case $|\xi_\infty + b| = 0$. There are two possibilities, as $|b| = |\xi_\infty| = 0$ or $b = -\xi_\infty$ with $|b|, |\xi_\infty| > 0$. In those scenarios, we prove that $F_\infty(M) \geqslant 0$. By contradiction, suppose

$$F_\infty(M) < 0. \tag{4.9}$$

Hence, the matrix M has at least one positive eigenvalue, by the uniformly ellipticity condition of F_∞. Let $\mathbb{R}^n = E \oplus Q$ be an orthogonal sum, where $E = \text{span}\{e_1, \dots, e_m\}$ is the space consisting of those eigenvectors corresponding to positive eigenvalues of M and $Q = \{v \in \mathbb{R}^n : \langle v, w\rangle = 0 \text{ for all } w \in E\}$.

Case 3: $b = -\xi_\infty$ with $|b|, |\xi_\infty| > 0$. Let $\gamma > 0$ and set

$$p_\gamma(x) := p(x) + \gamma|P_E(x)| = \frac{1}{2}\langle Mx, x\rangle + \langle b, x\rangle + \gamma|P_E(x)|,$$

where P_E stands for the orthogonal projection on E. Since $u_k \to u_\infty$ locally uniformly in B_1 and $p(x)$ touches $u_\infty(x)$ from below at the origin, for γ small enough, $p_\gamma(x)$ touches $u_k(x)$ from below at a point $x_k^\gamma \in B_r$ (B_r is a small neighborhood of the origin). Moreover, it holds that $x_k^\gamma \to x_\infty^\gamma$ for some x_∞^γ as $k \to \infty$. At this point we consider two scenarios: $P_E(x_k^\gamma) = 0$ for all $k \in \mathbb{N}$ (up to a subsequence), or $P_E(x_k^\gamma) \neq 0$ for all $k \in \mathbb{N}$ (up to a subsequence).

Scenario 1: $P_E(x_k^\gamma) = 0$ for all $k \in \mathbb{N}$ (up to a subsequence). In this scenario, we first note that

$$\widehat{p}_\gamma(x) := \frac{1}{2}\langle Mx, x\rangle + \langle b, x\rangle + \gamma\langle e, P_E(x)\rangle$$

touches u_k from below at x_k^γ for every $e \in \mathbb{S}^{n-1}$. A straightforward computation gives us

$$D\widehat{p}_\gamma(x_k^\gamma) = Mx_k^\gamma + b + \gamma P_E(e) \quad \text{and} \quad D^2\widehat{p}_\gamma(x_k^\gamma) = M.$$

Now we select $e \in E \cap \mathbb{S}^{n-1}$ such that $P_E(e) = e$. Therefore, by u_k being a viscosity solution of (4.3), we see

$$\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma e|)F_k(M) \geqslant f_k(x_k^\gamma).$$

We also notice that if $Mx_\infty^\gamma = 0$, then for k large enough, we have

$$|\xi_k + Mx_k^\gamma + b| \leqslant \gamma/2 \quad \text{and} \quad 3\gamma/2 \geqslant |\xi_k + Mx_k^\gamma + b + \gamma e| \geqslant \gamma/2.$$

Therefore, combining the last two displays and using (C2) together with $\gamma \ll 1$, we have

$$\begin{aligned} F_k(M) &\geqslant \frac{f_k(x_k^\gamma)}{\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma e|)} \\ &\geqslant \frac{-L|f_k(x_k^\gamma)|}{|\xi_k + Mx_k^\gamma + b + \gamma e|^{s(\Phi)}} \geqslant -\frac{L}{k}\left(\frac{2}{\gamma}\right)^{s(\Phi)}. \end{aligned}$$

Letting $k \to \infty$ in the last display, we obtain $F_\infty(M) \geqslant 0$. In the situation $|Mx_\infty^\gamma| > 0$, we first look at the subcase $E \equiv \mathbb{R}^n$ and choose $e \in \mathbb{S}^{n-1}$ such that

$$|Mx_\infty^\gamma + \gamma P_E(e)| = |Mx_\infty^\gamma + \gamma e| > 0.$$

Therefore, for k large enough, we have

$$|Mx_k^\gamma + \gamma e| \geqslant \frac{1}{2}|Mx_\infty^\gamma + \gamma e| > 0 \quad \text{and} \quad |\xi_k + b| \leqslant \frac{1}{8}|Mx_\infty^\gamma + \gamma e|. \tag{4.10}$$

On the other hand, if $E \not\equiv \mathbb{R}^n$, then we can find $e \in \mathbb{S}^{n-1} \cap E^\perp$ so that

$$|Mx_\infty^\gamma + \gamma P_E(e)| = |Mx_\infty^\gamma| > 0.$$

Again for k large enough, we have

$$|Mx_k^\gamma| \geqslant \frac{1}{2}|Mx_\infty^\gamma| \quad \text{and} \quad |\xi_k + b| \leqslant \frac{1}{8}|Mx_\infty^\gamma|. \tag{4.11}$$

As a consequence, using either (4.10) or (4.11), we see

$$|\xi_k + Mx_k^\gamma + b + \gamma P_E(e)| > \frac{1}{4}|Mx_\infty^\gamma + \gamma P_E(e)| > 0.$$

Again applying (C2) and taking into account the last display, we have

$$\begin{aligned} F_k(M) &\geqslant \frac{f_k(x_k^\gamma)}{\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma P_E(e)|)} \\ &\geqslant -\Big(\frac{L}{|\xi_k + Mx_k^\gamma + b + \gamma P_E(e)|^{i(\Phi)}} \\ &\qquad + \frac{L}{|\xi_k + Mx_k^\gamma + b + \gamma P_E(e)|^{s(\Phi)}}\Big)|f_k(x_k^\gamma)| \\ &\geqslant \frac{-L4^{s(\Phi)}}{k}\Big(\frac{1}{|Mx_\infty^\gamma + \gamma P_E(e)|^{i(\Phi)}} + \frac{1}{|Mx_\infty^\gamma + \gamma P_E(e)|^{s(\Phi)}}\Big). \end{aligned}$$

Again letting $k \to \infty$ in the last display, we again arrive at $F_\infty(M) \geqslant 0$.

Scenario 2: $P_E(x_k^\gamma) \neq 0$ for all $k \in \mathbb{N}$ (up to a subsequence). In this scenario, we note that $P_E(x)$ is smooth and convex in a small neighborhood of x_k^γ. Let us denote

$$\zeta_k^\gamma := \frac{P_E(x_k^\gamma)}{|P_E(x_k^\gamma)|}.$$

A direct computation yields

$$D(|P_E(\cdot)|)(x_k^\gamma) = \zeta_k^\gamma \quad \text{and} \quad D^2(P_E(|\cdot|))(x_k^\gamma) = \frac{1}{|P_E(x_k^\gamma)|}(I - \zeta_k^\gamma \otimes \zeta_k^\gamma).$$

Hence, with u_k being a viscosity solution of equation (4.3), we have the following viscosity inequality:

$$\Phi_k(x_k^\gamma, |\xi_k + Mx_k^\gamma + b + \gamma\zeta_k^\gamma|)F_k\Big(M + \frac{1}{|P_E(x_k^\gamma)|}(I - \zeta_k^\gamma \otimes \zeta_k^\gamma)\Big) \geqslant f_k(x_k^\gamma).$$

Observing that $|\zeta_k^\gamma| = 1$ and letting $e := \zeta_k^\gamma$, we can perform the same procedure as in the first scenario of $P_E(x_k^\gamma) = 0$ by considering the cases of $Mx_\infty^\gamma = 0$ and $Mx_\infty^\gamma \neq 0$. Finally, we conclude that $F_\infty(M) \geqslant 0$ when $b = -\xi_\infty \neq 0$, which contradicts (4.9).

Case 4: $b = \xi_\infty = 0$. This case is much easier to handle. Since $\frac{1}{2}\langle Mx, x\rangle$ touches $u_\infty(x)$ from below at the origin and $u_k \to u_\infty$ locally uniformly, the function

$$\hat{p}_\gamma(x) := \frac{1}{2}\langle Mx, x\rangle + \gamma|P_E(x)|$$

touches u_k from below at a point $\hat{x}_k^\gamma \in B_r$ (B_r is a small neighborhood of the origin) for $\gamma > 0$ sufficiently small. Again, the sequence $\{\hat{x}_k^\gamma\}$ is uniformly bounded. As in Case 3, we analyze those two scenarios: $P_E(\hat{x}_k^\gamma) = 0$ for all $k \in \mathbb{N}$ (up to a subsequence), and $P_E(\hat{x}_k^\gamma) \neq 0$ for all $k \in \mathbb{N}$ (up to a subsequence). In this case, we conclude $F_\infty(M) \geqslant 0$.

Finally, taking into account all the cases we have analyzed above, we have shown that u_∞ is a viscosity supersolution of (4.6). In order to prove that u_∞ is a viscosity subsolution of (4.6), we show that $-u_\infty$ is a viscosity supersolution of $\widehat{F}_\infty(D^2h) = 0$, where $\widehat{F}_\infty(M) = -F_\infty(-M)$ is a uniformly (λ, Λ)-elliptic operator as well. Therefore, u_∞ is a viscosity solution of (4.6). From the regularity results of [26, Chap. 5], we see $u_\infty \in C^{1,\hat{\alpha}}_{\mathrm{loc}}(B_{3/4})$ for some $\hat{\alpha} \in (0, 1)$. Moreover, $\|u_\infty\|_{C^{1,\hat{\alpha}}(\overline{B}_{1/2})} \leqslant c \equiv c(n, \lambda, \Lambda)$ via (4.5). So, choosing $h := u_\infty$ in (4.4), we get a contradiction. The proof is complete. ■

5. Proof of Theorem 1.1

Now we provide the proof of Theorem 1.1. Let $u \in C(B_1)$ be a viscosity solution with $\operatorname{osc}_{B_1} u \leqslant 1$, $\|f\|_{L^\infty(B_1)} \leqslant \delta \ll 1$ for a constant $\delta \equiv \delta(n, \lambda, \Lambda, i(\Phi), L)$ to be determined in a moment and $\nu_0 = \nu_1 = 1$. The proof is divided into two main parts, where in the first part we shall deal with the case $i(\Phi) \geqslant 0$ and the remaining case $-1 < i(\Phi) < 0$ will be investigated in the second part.

Part 1: $i(\Phi) \geqslant 0$. Let us first fix a point $y \in B_{1/2}$ and an exponent $\beta \in (0, 1)$ such that

$$\beta \in (0, \hat{\alpha}) \cap \Big(0, \frac{1}{1 + s(\Phi)}\Big]. \tag{5.1}$$

We prove that there exist universal constants $0 < r < 1$, $C_0 > 1$ and a sequence of affine functions

$$l_k(x) := a_k + \langle b_k, x\rangle, \tag{5.2}$$

where $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and $\{b_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$, such that for every $k \in \mathbb{N}$:

(E1) $\sup_{x \in B_{r^k}(y)} |u(x) - l_k(x)| \leqslant r^{k(1+\beta)}$,

(E2) $|a_k - a_{k-1}| \leqslant C_0 r^{(k-1)(1+\beta)}$,

(E3) $|b_k - b_{k-1}| \leqslant C_0 r^{(k-1)\beta}$.

We show these estimates by mathematical induction. For ease of presentation, we divide the remaining part of the proof for Part 1 into several steps.

Step 1. Basis of induction. Without loss of generality, we can assume $y = 0$ by translating $x \mapsto y + \frac{1}{2}x$. Let us set

$$l_1(x) := h(0) + \langle Dh(0), x \rangle,$$

where h is the approximation function coming from Lemma 4.1 for a certain constant $\mu > 0$, to be determined in a few lines. Note that that there exist constants $\widehat{\alpha} \equiv \widehat{\alpha}(n, \lambda, \Lambda) \in (0, 1)$ and $C_0 \equiv C_0(n, \lambda, \Lambda) > 1$ such that

$$\|h\|_{C^{1,\widehat{\alpha}}(\overline{B}_{3/8})} \leqslant C_0 \quad \text{and} \quad \sup_{x \in B_r} |h(x) - l_1(x)| \leqslant C_0 r^{1+\widehat{\alpha}}$$

for every $r \leqslant 3/8$. The triangle inequality yields

$$\sup_{x \in B_r} |u(x) - l_1(x)| \leqslant \mu + C_0 r^{1+\widehat{\alpha}}.$$

First, we select a universal constant $0 < r < 1$ satisfying

$$r^{\beta} \leqslant \frac{1}{2}, \quad C_0 r^{1+\widehat{\alpha}} \leqslant \frac{1}{2} r^{1+\beta} \quad \text{and} \quad r^{1-\beta(1+s(\Phi))} \leqslant 1, \tag{5.3}$$

which is possible by (5.1). Later in the paper, we select a constant $\mu > 0$ as

$$\mu := \frac{1}{2} r^{1+\beta}, \tag{5.4}$$

which fixes an arbitrary constant $\mu > 0$ in Lemma 4.1. In turn, there exists a constant $\delta \equiv \delta(n, \lambda, \Lambda, i(\Phi), L, \beta)$ verifying the smallness assumption $\|f\|_{L^{\infty}(B_1)} \leqslant \delta$, but such a smallness assumption can be assumed without loss of generality. To conclude this step, we set

$$a_0 := 0, \quad a_1 := h(0), \quad b_0 = 0, \quad \text{and} \quad b_1 := Dh(0).$$

These choices with (5.3) and (5.4) verify that estimates (E1)–(E3) are satisfied for $k = 1$.

Step 2: Induction process. Now we suppose that hypotheses of the induction have been established for $k = 1, 2, \dots, m$ for $m \geqslant 1$. We show that estimates (E1)–(E3) hold true for $k = m + 1$. For this, we introduce an auxiliary function as

$$w_m(x) := \frac{u(r^m x) - l_m(r^m x)}{r^{m(1+\beta)}}.$$

We note that w_m solves the following equation in the viscosity sense:

$$\Phi_m(x, |r^{-m\beta} b_m + Dw_m|) F_m(D^2 w_m) = f_m(x),$$

where

$$F_m(M) := r^{m(1-\beta)} F(r^{(\beta-1)m} M)$$

is a uniformly (λ, Λ)-operator, and the function

$$\Phi_m(x,t) := \frac{\Phi(r^m x, r^{m\beta} t)}{\Phi(r^m x, r^{m\beta})} \quad (x \in B_1,\ t > 0)$$

still satisfies the properties that the map $t \mapsto \frac{\Phi_m(x,t)}{t^{i(\Phi)}}$ is almost non-decreasing, the map $t \mapsto \frac{\Phi_m(x,t)}{t^{s(\Phi)}}$ is almost non-increasing with the same constant $L \geqslant 1$ and $\Phi_m(x,1) = 1$ for all $x \in B_1$, and

$$f_m(x) := \frac{r^{m(1-\beta)} f(r^m x)}{\Phi(r^m x, r^{m\beta})}.$$

Using (A2) and (5.1), we notice that

$$\|f_m\|_{L^\infty(B_1)} \leqslant \frac{L r^{m(1-\beta)} \|f\|_{L^\infty(B_1)}}{r^{m\beta s(\Phi)}} \leqslant L\delta r^{m(1-(1+s(\Phi))\beta)} \leqslant L\delta.$$

Therefore, it is possible to apply Lemma 4.1 to w_m. In turn, there exists a function $\hat{h} \in C^{1,\hat{\alpha}}(\overline{B}_{3/4})$ such that

$$\sup_{x \in B_r} |w_m(x) - \hat{h}(x)| \leqslant \mu.$$

Arguing as in Step 1, we show that

$$\sup_{x \in B_r} |w_m(x) - \hat{l}(x)| \leqslant r^{1+\beta},$$

where

$$\hat{l}(x) := \hat{a} + \langle \hat{b}, x \rangle \quad \text{for some} \quad \hat{a} \in \mathbb{R} \text{ and } \hat{b} \in \mathbb{R}^n.$$

Denoting

$$l_{m+1} := l_m(x) + r^{m(1+\beta)} \hat{l}(r^{-m} x),$$

we see

$$\sup_{x \in B_{r^{m+1}}} |u(x) - l_{m+1}(x)| \leqslant r^{(m+1)(1+\beta)}$$

and

$$|a_{m+1} - a_m| + r^m |b_{m+1} - b_m| \leqslant C_0 r^{m(1+\beta)}.$$

Therefore, the $(m+1)$-th step of the induction is complete.

Step 3: Conclusion. Once we have the existence of universal constants $0 < r \ll 1$, $C_0 > 1$, and a sequence of affine functions in (5.2) verifying estimates (E1)–(E3), the remaining part of the proof is very standard; see, for instance, [36, 52]. Therefore, the proof of (1.14) is complete when $i(\Phi) \geqslant 0$.

Part 2: $-1 < i(\Phi) < 0$. Now we focus on the case when $-1 < i(\Phi) < 0$. Again, we fix a point $y \in B_{1/2}$. Without loss of generality, we may assume $y = 0$ by using the translation $x \mapsto y + \frac{1}{2}x$. Now we apply (R1) of Lemma 3.1 in order to ensure that

$$[u]_{C^{0,1}(\overline{B}_{3/4})} \leqslant C_{sl} \tag{5.5}$$

for a constant $C_{sl} \equiv C_{sl}(n, \lambda, \Lambda, i(\Phi), L)$. Therefore, using Proposition 2.1, we see that u is a classical viscosity solution of the equation

$$\widetilde{\Phi}(x, |Dv|)F(D^2v) = \widetilde{f}(x) \quad \text{in } B_{3/4},$$

where

$$\widetilde{\Phi}(x,t) := t^{-i(\Phi)}\Phi(x,t) \quad (x \in B_1, t > 0),$$

which satisfies the properties that the map $t \mapsto \widetilde{\Phi}(x,t)$ is almost non-decreasing, the map $t \mapsto \frac{\widetilde{\Phi}(x,t)}{t^{s(\Phi)-i(\Phi)}}$ is almost non-increasing with constant $L \geqslant 1$, $\widetilde{\Phi}(x,1) = 1$ for all $x \in B_1$, and

$$\widetilde{f}(x) = |Du(x)|^{-i(\Phi)} f(x).$$

Using estimate (5.5) together with $\|f\|_{L^\infty(B_1)} \leqslant \delta \ll 1$, we see

$$\|\widetilde{f}\|_{L^\infty(B_{3/4})} \leqslant C_{sl}^{-i(\Phi)}\delta.$$

Therefore, we are able to apply Part 1 of the proof in order to have (E1)–(E3). This means that we have estimate (1.14) for $-1 < i(\Phi) < 0$. The proof is therefore complete.

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