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On the justification of the quasistationary approximation in the problem of motion of a viscous capillary drop

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We prove that the free boundary problem governing the motion of an isolated liquid mass in the case of a small Reynolds number ε has a unique solution in a certain time interval $(0, T_0)$ independent of ε and we show that the difference of the solution and of the quasistationary approximation to the solution has order $O(\varepsilon)$ for $t \in (t_0, T_0)$ with arbitrary positive t_0.

1. Introduction

The problem considered in the present paper consists of the determination of a bounded domain Ω_t, $t > 0$, which is given at the initial moment of time $t = 0$, of the vector field of velocity $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and of a scalar pressure $p(x, t)$ satisfying in Ω_t the Navier–Stokes equations (in a dimensionless form)

$$\varepsilon(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\, \mathbf{v}) - \nabla^2 \mathbf{v} + \nabla p = 0, \quad \nabla \cdot \mathbf{v} = 0 \qquad (x \in \Omega_t, \ \ t > 0), \tag{1.1}$$

initial conditions

$$\mathbf{v}(x, 0) = \mathbf{v}_0(x) \qquad (x \in \Omega_0), \tag{1.2}$$

and conditions at the free (unknown) boundary $\Gamma_t = \partial \Omega_t$

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma_t} = V_n, \quad T(\mathbf{v}, p)\mathbf{n} - H\mathbf{n}|_{\Gamma_t} = 0. \tag{1.3}$$

Here V_n is the velocity of the motion of the free surface Γ_t in the direction of the exterior (with respect to Ω_t) normal $\mathbf{n}$, H is the doubled mean curvature of Γ_t, negative for convex surfaces, $T(\mathbf{v}, p) = -pI + S(\mathbf{v})$ is the stress tensor, $S(\mathbf{v}) = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ is the doubled rate-of-strain tensor, I is a unit matrix, and ε is a small positive parameter.

The first kinematic boundary condition (1.3) means that at every point of the surface Γ_t the velocity of evolution of this surface in the direction $\mathbf{n}$ coincides with the normal component of the velocity of the liquid. The second (dynamic) condition says that the tangential stresses at the free boundary vanish, and the normal stress is equilibreated with a capillary force. According to (1.1), there are no external forces acting on the drop Ω_t.

For a fixed positive ε, problem (1.1)–(1.3) has been studied in [12–15]. It has been shown that the solution exists in a certain finite time interval, however, if Ω_0 is close to a ball and $\mathbf{v}_0(x)$ is small, then it is defined for all positive values of time, and a limiting regime as $t \to \infty$, depending on initial data, is rotation of the liquid as a rigid body about a certain axis moving with a constant speed. It has been also proved that for $t \to \infty$, Ω_t tends to a circle in the case of two spacial variables and to a certain equilibrium figure in the three-dimensional case.

In [5] an important special case of a two-dimensional problem (1.1)–(1.3) is studied, namely, the problem of motion of a ring filled with a viscous capillary liquid. In this case it turned out possible to make a complete analysis of the behaviour of the solution for $t > 0$, without any smallness assumptions on the data.

In a recent paper [4] by M. Günther and G. Prokert the same problem (1.1)–(1.3) is considered in a quasistationary approximation. This approximate model consists in the elimination of the terms $\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}$ in (1.1) by setting $\varepsilon = 0$ and of initial conditions (1.2) for $\mathbf{v}(x, t)$. Thus, the problem reduces to the solution of a linear elliptic (for $\epsilon = 0$) system (1.1) in a time-dependent domain Ω_t whose evolution is determined completely by conditions (1.3). It has been proved in [4] that the solution of this problem is determined in a certain finite time interval, or for all $t > 0$, if Ω_0 is close to a ball. Another proof of the local solvability of the same problem in the Hölder spaces was given in [17]. The discussion of the range of applicability of quasistationary approximation is contained in [9].

In the present paper we prove that the problem (1.1)–(1.3) is solvable in a certain finite time interval independent of $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \ll 1$. This is done by comparing $(\mathbf{v}, p)$ with a quasistationary approximation $(\mathbf{w}, r)$ and by making a better analysis of the corresponding linear problem than has been done in [7, 8, 16]. In comparison with these papers, our arguments here are much simpler. For the differences $\mathbf{v} - \mathbf{w}$, $\quad p - r$ written as functions of the Lagrangean coordinates we obtain representation formulas (1.18) and estimates (1.14), (4.20), (1.20) which show that these differences have order $O(\varepsilon)$ for $t > t_0$, $\forall t_0 > 0$. This provides justification of the quasistationary approximation.

The relation between the Eulerian coordinates $x = (x_1, x_2, x_3) \in \Omega_t$ and the Lagrangean coordinates $\xi = (\xi_1, \xi_2, \xi_3) \in \Omega_0 \equiv \Omega$ is given by the formula

$$\mathbf{x} = \boldsymbol{\xi} + \int_0^t \mathbf{u}(\xi, \tau)\, \mathrm{d}\tau \equiv \mathbf{X}_u(\xi, t), \qquad \xi \in \Omega, \tag{1.4}$$

where $\mathbf{x}$ and $\boldsymbol{\xi}$ are radii-vectors corresponding to the points x and ξ, and $\mathbf{u}(\xi, t)$ is the velocity vector field written as a function of the Lagrangean coordinates. This formula determines a mapping $x = X_u(\xi, t)$ of Ω onto Ω_t and of $\Gamma_0 \equiv \Gamma$ onto Γ_t (sometimes, in order to avoid confusion, we denote Γ_t by $\Gamma_u(t)$ and the normal $\mathbf{n}$ to Γ_t by $\mathbf{n}_u$). By virtue of $\nabla \cdot \mathbf{v} = 0$, this mapping is invertible, and the determinant of its Jacobi matrix equals 1. It tranforms (1.1)–(1.3) into

$$\begin{gathered} \varepsilon\, \mathbf{u}_t - \nabla_u^2 \mathbf{u} + \nabla_u q = 0, \quad \nabla_u \cdot \mathbf{u} = 0, \qquad \xi \in \Omega, \ \ t > 0, \\ \mathbf{u}(\xi, 0) = \mathbf{v}_0(\xi), \\ T_u(\mathbf{u}, q)\mathbf{n} - H\mathbf{n}|_\Gamma = 0 \end{gathered} \tag{1.5}$$

(condition $\mathbf{v} \cdot \mathbf{n} = V_n$ is equivalent to $X_u(\Omega) = \Omega_t$). Here

$$\nabla_u = \mathcal{A}\nabla = \left(\sum_{j=1}^3 A_{ij} \frac{\partial}{\partial \xi_j} \right)_{i=1,2,3},$$

A_{ij} is a cofactor of the element $a_{ij}(\xi, t) = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial \xi_j}\, \mathrm{d}\tau$ of the Jacobi matrix of the transformation X_u, $q(\xi, t) = p(X(\xi, t), t)$, $\ T_u(\mathbf{u}, q) = -qI + S_u(\mathbf{u})$, and

$$S_u(\mathbf{u}) = \left(\sum_{k=1}^3 \left(A_{ik} \frac{\partial u_j}{\partial \xi_k} + A_{jk} \frac{\partial u_i}{\partial \xi_k} \right) \right)_{i,j=1,2,3}.$$

It is convenient to modify (1.5) by introducing some additional terms into the Navier–Stokes equations and by using the well known formula $H\mathbf{n} = \Delta_u(t)\mathbf{x} = \Delta_u(t)\mathbf{X}_u(\xi, t)$ where $\Delta_u(t)$ is the Laplace–Beltrami operator on Γ_t. Instead of (1.5), we consider the problem

$$\begin{aligned} &\varepsilon\,\mathbf{u}_t + \ell_u(\mathbf{u}) - \nabla_u^2\mathbf{u} + \nabla_u q = \sum_{k=1}^{6} \mu_k\,\boldsymbol{\varphi}_k(X_u(\xi, t)),\\ &\nabla_u \cdot \mathbf{u} = 0, \qquad (\xi \in \Omega, \quad t > 0),\\ &\mathbf{u}(\xi, 0) = \mathbf{v}_0(\xi),\\ &T_u(\mathbf{u}, q)\,\mathbf{n} - \Delta_u(t)\,\mathbf{X}_u|_\Gamma = 0 \end{aligned} \tag{1.6}$$

where $\boldsymbol{\varphi}_k(x)$, $k = 1, \ldots, 6$, are linearly independent vector fields in the space of vectors $\boldsymbol{\varphi}(x) = \mathbf{a} + \mathbf{b} \times \mathbf{x}$ of rigid displacement, for instance, $\boldsymbol{\varphi}_1 = (1, 0, 0) = \mathbf{e}_1$, $\boldsymbol{\varphi}_2 = (0, 1, 0) = \mathbf{e}_2$, $\boldsymbol{\varphi}_3 = (0, 0, 1) = \mathbf{e}_3$, $\boldsymbol{\varphi}_4 = (0, -x_3, x_2)$, $\boldsymbol{\varphi}_5 = (x_3, 0, -x_1)$, $\boldsymbol{\varphi}_6 = (-x_2, x_1, 0)$,

$$\begin{aligned} &\ell_u(\mathbf{u}) = \sum_{k=1}^{6} \left(\int_\Omega \mathbf{u}(\xi', t) \cdot \boldsymbol{\varphi}_k(X_u(\xi', t))\,\mathrm{d}\xi' \right) \boldsymbol{\varphi}_k(X_u(\xi, t)),\\ &\mu_k = \int_\Omega \mathbf{v}_0(x) \cdot \boldsymbol{\varphi}_k(x)\,\mathrm{d}x. \end{aligned} \tag{1.7}$$

Multiplying (1.6_1) by $\boldsymbol{\varphi}_m(X_u(\xi, t))$ and integrating over Ω, we obtain (see details in [12])

$$\varepsilon\,\frac{\mathrm{d}\lambda_m}{\mathrm{d}t} + \sum_{k=1}^{6} \lambda_k(t) \int_{\Omega_t} \boldsymbol{\varphi}_k \cdot \boldsymbol{\varphi}_m\,\mathrm{d}x = 0$$

where

$$\lambda_k(t) = \int_{\Omega_t} \mathbf{v}(y, t) \cdot \boldsymbol{\varphi}_k(y)\,\mathrm{d}y - \mu_k = \int_\Omega \mathbf{u}(\xi, t) \cdot \boldsymbol{\varphi}_k(X_u(\xi, t))\,\mathrm{d}\xi - \mu_k,$$

and, as a consequence,

$$\begin{aligned} &\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m=1}^{6} \lambda_m^2(t) + c \sum_{m=1}^{6} \lambda_m^2(t)\\ &\leqslant \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{m=1}^{6} \lambda_m^2(t) + \sum_{k,m=1}^{6} \lambda_k(t)\,\lambda_m(t) \int_{\Omega_t} \boldsymbol{\varphi}_k \cdot \boldsymbol{\varphi}_m\,\mathrm{d}x = 0. \end{aligned}$$

Since $\lambda_m(0) = 0$, we conclude that $\lambda_m(t) = 0$, i.e.

$$\int_\Omega \mathbf{u}(\xi, t) \cdot \boldsymbol{\varphi}_m(X_u(\xi, t))\,\mathrm{d}\xi = \mu_m$$

for all $t > 0$, so (1.6) is equivalent to (1.5).

We prove the solvability of this problem by comparing $(\mathbf{u}, q)$ with the solution $(\mathbf{w}, r)$ of the same problem in a quasistationary approximation [4,17]. In the Lagrangean coordinates, the latter problem can be written as follows:

$$-\nabla_w^2\mathbf{w} + \nabla_w r + \ell_w(\mathbf{w}) = \sum_{k=1}^{6} \mu_k\,\boldsymbol{\varphi}(X_w(\xi, t)),$$

$$\nabla_w \cdot \mathbf{w} = 0, \qquad \xi \in \Omega,\ t > 0, \tag{1.8}$$
$$T_w(\mathbf{w}, q)\,\mathbf{n}_w - \Delta_w(t)\,\mathbf{X}_w(\xi, t)|_\Gamma = 0.$$

Here $\mathbf{X}_w(\xi, t) = \boldsymbol{\xi} + \int_0^t \mathbf{w}(\xi, t)\,\mathrm{d}t$, $\Delta_w(t)$ is the Laplace–Beltrami operator on $\Gamma_w(t) = X_w(\Gamma)$, $\mathbf{n}_w$ is a unit exterior normal to $\Gamma_w(t)$, μ_k are the same numbers as in (1.7) and

$$\ell_w(\mathbf{v}) = \sum_{k=1}^{6} \left(\int_\Omega \mathbf{v}(\xi', t) \cdot \boldsymbol{\varphi}_k(X_w(\xi', t))\,\mathrm{d}\xi' \right) \boldsymbol{\varphi}_k(X_w(\xi, t)).$$

The solvability of problem (1.8) on a certain finite time interval in the Hölder spaces of functions was established in [17] where the following theorem was proved.

THEOREM 1.1 Let $\Gamma \in C^{3+l}$ where l is an arbitrary fixed positive non-integral number. There exists such $T_0 > 0$ that problem (1.8) is uniquely solvable on the time interval $(0, T_0)$. The solution satisfies the conditions

$$\int_\Omega \mathbf{w}(\xi, t) \cdot \boldsymbol{\varphi}_k(X_w(\xi, t))\,\mathrm{d}\xi = \mu_k, \quad k = 1, ..., 6, \tag{1.9}$$

and the inequality

$$\sup_{t<T_0} |\mathbf{w}(\cdot, t)|_{C^{2+l}(\Omega)} + \sup_{t<T_0} |r(\cdot, t)|_{C^{1+l}(\Omega)} \leqslant c\Big(|H_0|_{C^{l+1}(\Gamma)} + \sum_{k=1}^{6} |\mu_k|\Big) \tag{1.10}$$

where H_0 is the doubled mean curvature of Γ.

Here $C^l(\Omega)$ means a standard Hölder space of functions (or vector fields) $u(x)$, $x \in \Omega$, with the norm

$$|u|_{C^l(\Omega)} = \sum_{|j|<l} \sup_\Omega |D^j u(x)| + [u]_\Omega^{(l)},$$

$$[u]_\Omega^{(l)} = \sum_{|j|=[l]} \sup_{x,y\in\Omega} |x - y|^{-l+[l]}\,|D^j u(x) - D^j u(y)|,$$

and $C^l(\Gamma)$ is defined, as usual, with the help of partition of unity on Γ and of local maps. The norm in the space $C^{l,l/2}(\Omega \times (0, T))$ may be defined by the formula

$$|u|_{C^{l,l/2}(\Omega\times(0,T))} = \sup_{t\in(0,T)} |u(\cdot, t)|_{C^l(\Omega)} + \sup_\Omega [u(x, \cdot)]_{(0,T)}^{(l/2)}$$

but most often another equivalent norm in this space is used; it contains maxima moduli and Hölder constants of the mixed derivatives $D_t^k D_x^j u(x, t)$, $2k+|j| < l$. Finally, by $\tilde{C}(0, T; C^l(\Omega))$ we mean the space of functions (or vector fields) continuous with respect to $(x, t) \in \Omega \times (0, T)$ and having a finite norm

$$\sup_{t\in(0,T)} |u(\cdot, t)|_{C^l(\Omega))}$$

(unfortunately, in [17] a misleading notation $C(0, T; C^l(\Omega))$ for this space was used).

We also consider two linear evolution problems:

$$\begin{aligned}
&\varepsilon \mathbf{V}_t - \nabla_w^2 \mathbf{V} + \ell_w(\mathbf{V}) + \nabla_w P = 0, \\
&\nabla_w \cdot \mathbf{V} = 0, \qquad \xi \in \Omega, \quad t < T_0, \\
&\mathbf{V}|_{t=0} = \mathbf{v}_0(\xi) - \mathbf{u}_0(\xi) \equiv \mathbf{V}_0(\xi), \\
&T_w(\mathbf{V}, P)\mathbf{n}_w|_\Gamma = 0
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
&\varepsilon \boldsymbol{\eta}_t - \nabla_w^2 \boldsymbol{\eta} + \ell_w(\boldsymbol{\eta}) + \nabla_w \pi = \mathbf{f}(\xi, t), \\
&\nabla_w \cdot \boldsymbol{\eta} = g(\xi, t), \qquad \xi \in \Omega, \quad t < T_0, \\
&\boldsymbol{\eta}|_{t=0} = 0, \\
&\Pi_w S_w(\boldsymbol{\eta})\, \mathbf{n}_w|_\Gamma = \mathbf{b}, \\
&\mathbf{n}_w \cdot T_w(\boldsymbol{\eta}, \pi)\mathbf{n}_w - \mathbf{n}_w \cdot \Delta_w(t) \int_0^t \boldsymbol{\eta}\, \mathrm{d}\tau \Big|_\Gamma = b + \int_0^t B(\xi, \tau)\, \mathrm{d}\tau
\end{aligned} \tag{1.12}$$

where Π_w is a projector onto the tangent plane to $\Gamma_w(t)$ at the point $X_w(\xi, t)$, i.e.

$$\Pi_w \mathbf{f} = \mathbf{f} - \mathbf{n}_w(\mathbf{n}_w \cdot \mathbf{f}).$$

By Π_0 we mean a projector onto the tangent plane to Γ at the point ξ, i.e.

$$\Pi_0 \mathbf{f} = \mathbf{f} - \mathbf{n}_0(\mathbf{n}_0 \cdot \mathbf{f}),$$

$\mathbf{n}_0$ being the exterior normal to Γ.

We prove the following theorems.

THEOREM 1.2 If $\Gamma \in C^{3+\alpha}$, $\mathbf{v}_0 \in C^{2+\alpha}(\Omega)$ $(\alpha \in (0, 1))$ and $\nabla \cdot \mathbf{V}_0(\xi) = 0$, $\Pi_0 S(\mathbf{V}_0)\mathbf{n}_0|_\Gamma = 0$, then problem (1.11) has a unique solution in the interval $(0, T_0)$, and the solution satisfies the inequalities

$$\varepsilon \sup_{\tau<t} |\mathbf{V}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\mathbf{V}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t} |P(\cdot, \tau)|_{C^{l+1}(\Omega)} \leqslant c|\mathbf{V}_0|_{C^{2+\alpha}(\Omega)}, \tag{1.13}$$

$$\|\mathbf{V}(\cdot, t)\|_{L_2(\Omega)} \leqslant c\|\mathbf{V}_0|_{L_2(\Omega)} \exp(-bt\varepsilon^{-1}), \qquad b > 0, \quad t \in (0, T_0), \tag{1.14}$$

$$\int_0^{T_0} |\mathbf{V}|_{C^{2+\alpha}(\Omega)}\, \mathrm{d}t \leqslant c\,\varepsilon\, |\mathbf{V}_0|_{C^{2+\alpha}(\Omega)}. \tag{1.15}$$

If $\Gamma \in C^{4+\alpha}$, then also

$$\int_0^{T_0} |\mathbf{V}|_{C^{3+\alpha}(\Gamma)}\, \mathrm{d}t \leqslant c\,\varepsilon\, |\mathbf{V}_0|_{C^{2+\alpha}(\Omega)}. \tag{1.16}$$

The constants in (1.13)–(1.16) are independent of ε.

THEOREM 1.3 Assume that $\Gamma \in C^{3+\alpha}$, $\mathbf{f} \in \tilde{C}(0,T_0; C^{\alpha}(\Omega))$, $g \in \tilde{C}(0,T_0;\ C^{1+\alpha}(\Omega))$, $g(\xi,t) = \nabla\cdot\mathbf{h}(\xi,t) + h_0(\xi,t)$, $\mathbf{h}_t \in \tilde{C}(0,T_0; C^{\alpha}(\Omega))$, $h_{0t} \in C(0,T_0; C(\Omega))$, $\mathbf{b} \in C^{1+\alpha,(1+\alpha)/2}(\Gamma\times(0,T_0))$, $b \in \tilde{C}(0,T_0; C^{1+\alpha}(\Gamma))$, $B \in \tilde{C}(0,T_0; C^{\alpha}(\Gamma))$, and $\mathbf{b}$, g satisfy the conditions

$$\mathbf{b}\cdot\mathbf{n}_0 = 0, \qquad \mathbf{b}(\xi,0) = 0, \qquad g(\xi,0) = 0.$$

Then problem (1.12) has a unique solution $\boldsymbol{\eta} \in \tilde{C}(0,T_0; C^{2+\alpha}(\Omega))$, $\pi \in \tilde{C}(0,T_0; C^{1+\alpha}(\Omega))$ such that $\boldsymbol{\eta}_t \in \tilde{C}(0,T_0; C^{\alpha}(\Omega))$, and

$$\begin{aligned}
&\varepsilon \sup_{\tau<t}|\boldsymbol{\eta}_\tau(\cdot,\tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<t}|\boldsymbol{\eta}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t}|\pi(\cdot,\tau)|_{C^{1+\alpha}(\Omega)} \leqslant \\
&\quad\leqslant c\Bigg(\sup_{\tau<t}|\mathbf{f}(\cdot,\tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<t}|g(\cdot,\tau)|_{C^{1+\alpha}(\Omega)} + \varepsilon\sup_{\tau<t}|\mathbf{h}_\tau(\cdot,\tau)|_{C^{\alpha}(\Omega)} \\
&\quad + \varepsilon \sup_{\Omega\times(0,t)}|h_{0\tau}(x,\tau)| + \sup_{\tau<t}|\mathbf{b}(\cdot,\tau)|_{C^{1+\alpha}(\Gamma)} + \sup_{\tau<t}|b(\cdot,\tau)|_{C^{1+\alpha}(\Gamma)} \\
&\quad + \varepsilon^{(1+\alpha)/2}\sup_{\Gamma}[\mathbf{b}(\xi,\cdot)]^{((1+\alpha)/2)}_{(0,t)} + \sup_{\tau<t}|B(\cdot,\tau)|_{C^{\alpha}(\Gamma)}\Bigg)
\end{aligned} \tag{1.17}$$

for arbitrary $t \in (0,T_0)$.

To prove the solvability of non-linear problem (1.6), we introduce new unknown functions

$$\boldsymbol{\eta} = \mathbf{u} - \mathbf{w} - \mathbf{V}, \qquad \pi = q - r - P. \tag{1.18}$$

It is easy to see that (1.6) is equivalent to the problem

$$\begin{aligned}
&\varepsilon\boldsymbol{\eta}_t - \nabla_w^2\boldsymbol{\eta} + \ell_w(\boldsymbol{\eta}) + \nabla_w\pi = -\varepsilon\mathbf{w}_t + \mathcal{L}_1(\boldsymbol{\eta},\pi), \\
&\nabla_w\cdot\boldsymbol{\eta} = \mathcal{L}_2(\boldsymbol{\eta}), \qquad \xi\in\Omega,\quad t<T_0, \\
&\boldsymbol{\eta}|_{t=0} = 0, \\
&\Pi_w S_w(\boldsymbol{\eta})\mathbf{n}_w|_\Gamma = \mathcal{L}_3(\boldsymbol{\eta}), \\
&\mathbf{n}_w\cdot T_w(\boldsymbol{\eta},\pi)\mathbf{n}_w - \mathbf{n}_w\cdot\Delta_w(t)(\mathbf{X}_u - \mathbf{X}_w)\big|_\Gamma = \mathcal{L}(\boldsymbol{\eta},\pi)
\end{aligned} \tag{1.19}$$

where

$$\begin{aligned}
\mathcal{L}_1(\boldsymbol{\eta},\pi) &= (\nabla_u^2 - \nabla_w^2)\mathbf{u} - (\nabla_u - \nabla_w)q \\
&\quad + \ell_w(\mathbf{u}) - \ell_u(\mathbf{u}) + \sum_{k=1}^{6}\mu_k(\boldsymbol{\varphi}_k(X_u(\xi,t)) - \boldsymbol{\varphi}_k(X_w(\xi,t))), \\
\mathcal{L}_2(\boldsymbol{\eta}) &= (\nabla_w - \nabla_u)\cdot\mathbf{u}, \\
\mathcal{L}_3(\boldsymbol{\eta}) &= \Pi_w\Big(\Pi_w S_w(\mathbf{u})\mathbf{n}_w - \Pi_u S_u(\mathbf{u})\mathbf{n}_u\Big), \\
\mathcal{L}(\boldsymbol{\eta},\pi) &= \mathbf{n}_w\cdot\Big(T_w(\mathbf{u},q)\mathbf{n}_w - T_u(\mathbf{u},q)\mathbf{n}_u\Big) + \mathbf{n}_w\cdot(\Delta_u(t) - \Delta_w(t))\,\mathbf{X}_u(\xi,t)
\end{aligned}$$

and $\mathbf{u} = \mathbf{w} + \mathbf{V} + \boldsymbol{\eta}$, $q = r + P + \pi$.

Applying Theorem 1.2 and using estimates of the coefficients of the operators $\nabla_u - \nabla_w$ and $\Delta_u(t) - \Delta_w(t)$, we prove the following main result of the paper.

THEOREM 1.4 If $\Gamma \in C^{4+\alpha}$, then problem (1.19) has a unique solution $\boldsymbol{\eta}, \pi$ in a certain time interval $t \in (0, T_1)$, $T_1 \leqslant T_0$, and the solution satisfies the inequality

$$\varepsilon \sup_{\tau<t} |\boldsymbol{\eta}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\boldsymbol{\eta}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t} |\pi(\cdot, \tau)|_{C^{1+\alpha}(\Omega)} \leqslant c\varepsilon, \tag{1.20}$$

with arbitrary $t \in (0, T_1)$ and with the constant c independent of ε.

This theorem guarantees the existence of solution $\mathbf{u} = \boldsymbol{\eta} + \mathbf{w} + \mathbf{V}$, $q = \pi + r + P$ of problem (1.6) for $t \in (0, T_1)$, moreover, the difference $\mathbf{u} - \mathbf{w}$ is represented as the sum of a boundary layer type function $\mathbf{V}$ (at $t = 0$) and of the vector field $\boldsymbol{\eta}$ for which uniform estimate (1.20) is obtained. Hence, for $t > t_0 > 0$ this difference does not exceed $c(t_0)\varepsilon$.

Finally, we prove that the solution of problem (1.6) can be extended into the whole interval $t \in (0, T_0)$.

The paper is organized as follows. In Sections 2 and 3 we study a linear problem

$$\begin{gathered}\varepsilon \mathbf{v}_t - \nabla^2 \mathbf{v} + \ell(\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = g = \nabla \cdot \mathbf{h}, \qquad x \in \Omega, \quad t \in (0, T), \\ \mathbf{v}|_{t=0} = 0,\end{gathered} \tag{1.21}$$

$$\begin{gathered}\Pi_0 S(\mathbf{v})\mathbf{n}_0|_\Gamma = \mathbf{b}, \\ \mathbf{n}_0 \cdot T(\mathbf{v}, p)\mathbf{n}_0 - \mathbf{n}_0 \cdot \Delta_0 \int_0^t \mathbf{v}\,\mathrm{d}\tau\Big|_\Gamma = b + \int_0^t B\,\mathrm{d}\tau\end{gathered} \tag{1.22}$$

where $\ell(\mathbf{v}) = \sum_{k=1}^{6} \left(\int_\Omega \mathbf{v}(\xi', t) \cdot \boldsymbol{\varphi}_k(\xi', t)\,\mathrm{d}\xi'\right) \boldsymbol{\varphi}_k(\xi, t)$, Δ_0 is the Laplace–Beltrami operator on Γ, and the corresponding model problem in the half-space $\mathbb{R}^3_+ = \{x_3 > 0\}$. In the case of a fixed $\varepsilon > 0$, this problem was considered earlier in [7, 8, 16], in particular, in anisotropic Hölder spaces $C^{2+\alpha,1+\alpha/2}(\Omega \times (0, T))$, and one of the most difficult technical points was the estimate of the Hölder constants of the time derivative of $\mathbf{v}$ with respect to t. For our purposes here a more modest estimate of the norm

$$Y_t(\mathbf{v}, p) = \varepsilon \sup_{\tau<t} |\mathbf{v}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\mathbf{v}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t} |p(\cdot, \tau)|_{C^{1+\alpha}(\Omega)}$$

is sufficient, and it is proved by simpler arguments. For instance, we do not use the theory of Fourier multipliers in the Hölder spaces which is replaced here by Proposition 2.2 on the pointwise estimates of the kernels (2.27) made in the spirit of [1]. The final results of Sections 2 and 3 are contained in the following theorem.

THEOREM 1.5 Let $\Gamma \in C^{2+\alpha}$, $\mathbf{f} \in \tilde{C}(0, T; C^\alpha(\Omega))$, $g \in \tilde{C}(0, T;\ C^{\alpha+1}(\Omega))$, $\mathbf{h}_t \in \tilde{C}(0, T; C^\alpha(\Omega))$, $h_{0t} \in C(\Omega \times (0, T))$, $\mathbf{b} \in C^{1+\alpha,(1+\alpha)/2}(\Gamma \times (0, T))$, $b \in \tilde{C}(0, T; C^{1+\alpha}(\Gamma))$, $B \in \tilde{C}(0, T; C^\alpha(\Gamma))$, and let $\mathbf{b}$, g satisfy the compatibility conditions

$$\mathbf{b} \cdot \mathbf{n}_0 = 0, \qquad \mathbf{b}(x, 0) = 0, \qquad g(x, 0) = 0.$$

Then problem (1.21)–(1.22) has a unique solution $\mathbf{v} \in \tilde{C}(0, T; C^{2+\alpha}(\Omega))$, $p \in \tilde{C}(0, T; C^{1+\alpha}(\Omega))$ with $\mathbf{v}_t \in \tilde{C}(0, T; C^{\alpha}(\Omega))$, and the solution satisfies the inequality

$$\begin{aligned} Y_t(\mathbf{v}, p) \leqslant c \Bigg(&\sup_{\tau<t} |\mathbf{f}(\cdot, \tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<t} |g(\cdot, \tau)|_{C^{1+\alpha}(\Omega)} \\ &+ \sup_{\tau<t} |\mathbf{h}_{\tau}(\cdot, \tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<t} \sup_{\Omega} |h_{0\tau}(\xi, \tau)| + \sup_{\tau<t} |\mathbf{b}(\cdot, \tau)|_{C^{1+\alpha}(\Gamma)} \\ &+ \varepsilon^{(1+\alpha)/2} \sup_{\Gamma} [\mathbf{b}(\xi, \cdot)]_{(0,t)}^{((1+\alpha)/2)} + \sup_{\tau<t} |b(\cdot, \tau)|_{C^{1+\alpha}(\Gamma)} + \sup_{\tau<t} |B(\cdot, \tau)|_{C^{\alpha}(\Gamma)} \Bigg). \end{aligned} \tag{1.23}$$

We also consider the initial-boundary value problem for (1.21) with initial and boundary conditions

$$\begin{aligned} &\mathbf{v}|_{t=0} = \mathbf{v}_0(\xi), \\ &T(\mathbf{v}, p)\mathbf{n}_0|_{\Gamma} = \mathbf{a}. \end{aligned} \tag{1.24}$$

THEOREM 1.6 Let $\Gamma \in C^{2+\alpha}$, $\mathbf{f} \in \tilde{C}(0, T; C^{\alpha}(\Omega))$, $g \in \tilde{C}(0, T; C^{\alpha+1}(\Omega))$, $\mathbf{h}_t \in \tilde{C}(0, T; C^{\alpha}(\Omega))$, $h_{0t} \in C(\Omega \times (0, T))$, $\mathbf{v}_0 \in C^{2+\alpha}(\Omega)$, $\Pi_0\mathbf{a} \in C^{1+\alpha,(1+\alpha)/2}(\Gamma \times (0, T))$, $\mathbf{a} \cdot \mathbf{n}_0 \in \tilde{C}(0, T; C^{1+\alpha}(\Gamma))$, and let the compatibility conditions

$$\nabla \cdot \mathbf{v}_0(x) = g(x, 0), \quad \Pi_0\mathbf{v}_0(x)|_{\Gamma} = \Pi_0\mathbf{a}(x, 0)$$

be satisfied. Then problem (1.21)–(1.24) has a unique solution $\mathbf{v} \in \tilde{C}(0, T; C^{2+\alpha}(\Omega))$, $p \in \tilde{C}(0, T; C^{1+\alpha}(\Omega))$ with $\mathbf{v}_t \in \tilde{C}(0, T; C^{\alpha}(\Omega))$, and the solution satisfies the inequality

$$\begin{aligned} Y_t(\mathbf{v}, p) \leqslant c \Bigg(&\sup_{\tau<t} |\mathbf{f}(\cdot, \tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<t} |g(\cdot, \tau)|_{C^{1+\alpha}(\Omega)} \\ &+ \sup_{\tau<t} |\mathbf{h}_{\tau}(\cdot, \tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<t} \sup_{\Omega} |h_{0\tau}(\xi, \tau)| + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)} \\ &+ \sup_{\tau<t} |\mathbf{a}(\cdot, \tau)|_{C^{1+\alpha}(\Gamma)} + \varepsilon^{(1+\alpha)/2} \sup_{\Gamma} [\Pi_0\mathbf{a}(\xi, \cdot)]_{(0,t)}^{((1+\alpha)/2)} \Bigg). \end{aligned} \tag{1.25}$$

For a fixed $\varepsilon > 0$, problem (1.21)–(1.24) is considered in [10,11] and estimate (1.25) is proved in [18]. A uniform (with respect to $\varepsilon > 0$) estimate (1.25) is obtained in the same way as the inequality (1.23), and we omit the details.

In Section 4 we consider the problem (1.8), estimate the time derivative $\mathbf{w}_t(\xi, t)$ and prove theorems 1.2 and 1.3. Finally, in Section 5 we prove Theorem 1.4.

2. Model problem in the half-space

2.1 *Construction of the solution*

In this section we consider the model initial-boundary value problem in the half-space $\mathbb{R}^3_+$ $(x_3 > 0)$:

$$\begin{aligned} &\varepsilon\, \mathbf{v}_t - \nabla^2\mathbf{v} + \nabla p = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad (x \in \mathbb{R}^3_+, \quad t > 0), \\ &\mathbf{v}\Big|_{t=0} = 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\frac{\partial v_j}{\partial x_3} + \frac{\partial v_3}{\partial x_j}\Big|_{x_3=0} = b_j(x', t), \quad j = 1, 2, \\ &- p + 2\,\frac{\partial v_3}{\partial x_3} + \Delta' \int_0^t v_3(x, t)\,\mathrm{d}\tau\Big|_{x_3=0} = b_3'(x', t) \quad (x' = (x_1, x_2) \in \mathbb{R}^2,\ \ t > 0), \end{aligned} \tag{2.2}$$

where $\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. We assume that b_1, b_2, b_3' are smooth functions decaying at infinity sufficiently rapidly. Making the Fourier transform with respect to x' and the Laplace transform with respect to t defined by the formula

$$FLh \equiv \widetilde{h}(\xi, x_3, t) = \int_0^\infty \int_{\mathbb{R}^2} e^{-ix'\xi}\, e^{-st}\, h(x', x_3, t)\,\mathrm{d}x'\,\mathrm{d}t \tag{2.3}$$

we reduce (2.1)–(2.2) to the boundary value problem for the system of ordinary differential equations

$$\begin{aligned} &-\frac{\mathrm{d}^2\widetilde{v}_j}{\mathrm{d}x_3^2} + r_\varepsilon^2\,\widetilde{v}_j + i\,\xi_j\,\widetilde{p} = 0, \quad j = 1, 2, \\ &-\frac{\mathrm{d}^2\widetilde{v}_3}{\mathrm{d}x_3^2} + r_\varepsilon^2\,\widetilde{v}_3 + \frac{\mathrm{d}\widetilde{p}}{\mathrm{d}x_3} = 0, \\ &i\,\xi_1\,\widetilde{v}_1 + i\,\xi_2\,\widetilde{v}_2 + \frac{\mathrm{d}\widetilde{v}_3}{\mathrm{d}x_3} = 0, \quad x_3 > 0, \\ &\widetilde{v}_k(\xi, x_3, s) \to 0,\ \ \widetilde{p}(\xi, x_3, s) \to 0 \quad (x_3 \to +\infty), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\frac{\mathrm{d}\widetilde{v}_j}{\mathrm{d}x_3} + i\,\xi_j\,\widetilde{v}_3\Big|_{x_3=0} = \widetilde{b}_j(\xi, s), \quad j = 1, 2, \\ &-\widetilde{p} + 2\,\frac{\mathrm{d}\widetilde{v}_3}{\mathrm{d}x_3} - \frac{|\xi|^2}{s}\widetilde{v}_3\Big|_{x_3=0} = \widetilde{b}_3'(\xi, s), \end{aligned} \tag{2.5}$$

where $r_\varepsilon = \sqrt{\varepsilon\, s + |\xi|^2}$, $\mathrm{Re}\, r_\varepsilon \geqslant 0$ for $\mathrm{Re}\, s \geqslant 0$.

We also consider the initial-boundary value problem for the Stokes equations (2.1) with boundary conditions (2.2) and (2.5) replaced by

$$\begin{aligned} &\frac{\partial v_j}{\partial x_3} + \frac{\partial v_3}{\partial x_j}\Big|_{x_3=0} = a_j(x', t), \quad j = 1, 2, \\ &- p + 2\,\frac{\partial v_3}{\partial x_3}\Big|_{x_3=0} = a_3(x', t), \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\frac{\mathrm{d}\widetilde{v}_j}{\mathrm{d}x_3} + i\,\xi_j\,\widetilde{v}_3\Big|_{x_3=0} = \widetilde{a}_j(\xi, s), \quad j = 1, 2, \\ &-\widetilde{p} + 2\,\frac{\mathrm{d}\widetilde{v}_3}{\mathrm{d}x_3}\Big|_{x_3=0} = \widetilde{a}_3(\xi, s). \end{aligned} \tag{2.7}$$

As shown in [10], problem (2.4)–(2.7) can be solved explicitly in the form

$$\begin{aligned}\widetilde{v}_i(\xi, x_3, s) &= \frac{(1-\delta_{3i})\,\widetilde{a}_i}{r_\varepsilon}\, e^{-r_\varepsilon x_3} + \sum_{j=1}^{3} \frac{R_{ij}(r_\varepsilon, \xi)}{r_\varepsilon\, Q(r_\varepsilon, |\xi|)}\, \widetilde{a}_j\, e^{-r_\varepsilon x_3} \\ &\quad + \sum_{j=1}^{3} \frac{S_{ij}(r_\varepsilon, \xi)}{Q(r_\varepsilon, |\xi|)} \widetilde{a}_j\, \frac{e^{-r_\varepsilon x_3} - e^{-|\xi| x_3}}{r_\varepsilon - |\xi|},\end{aligned} \tag{2.8}$$

$$\widetilde{p}(\xi, x_3, s) = -\widetilde{a}_3\, e^{-|\xi| x_3} + \sum_{j=1}^{3} P_j(r_\varepsilon, \xi)\, \widetilde{a}_j\, e^{-|\xi| x_3}, \tag{2.9}$$

where

$$\begin{aligned} &P_j(r_\varepsilon, \xi) = 2i\, \xi_j\, \frac{r_\varepsilon(r_\varepsilon + |\xi|)}{Q(r_\varepsilon, |\xi|)}, \quad j = 1, 2, \\ &P_3(r_\varepsilon, s) = 2\,|\xi|^2\, \frac{r_\varepsilon - |\xi|}{Q(r_\varepsilon, |\xi|)}, \\ &Q(r_\varepsilon, |\xi|) = r_\varepsilon^3 + |\xi|\, r_\varepsilon^2 + 3\, r_\varepsilon\, |\xi|^2 - |\xi|^3 = (r_\varepsilon + |\xi|) M(\varepsilon s, |\xi|), \\ &M(\varepsilon s, |\xi|) = \varepsilon s + 4|\xi|^2 \frac{r_\varepsilon}{r_\varepsilon + |\xi|} \end{aligned} \tag{2.10}$$

and R_{ij}, S_{ij} are elements of the matrices

$$\mathcal{R} = \begin{pmatrix} \xi_1^2(3r_\varepsilon - |\xi|) & \xi_1\,\xi_2(3r_\varepsilon - |\xi|) & i\,\xi_1\, r_\varepsilon(r_\varepsilon - |\xi|) \\ \xi_1\,\xi_2(3r_\varepsilon - |\xi|) & \xi_2^2(3r_\varepsilon - |\xi|) & i\,\xi_2\, r_\varepsilon(r_\varepsilon - |\xi|) \\ -i\,\xi_1\, r_\varepsilon(r_\varepsilon - |\xi|) & -\,i\,\xi_2\, r_\varepsilon(r_\varepsilon - |\xi|) & -\,|\xi|\, r_\varepsilon(r_\varepsilon + |\xi|) \end{pmatrix},$$

$$\mathcal{S} = \begin{pmatrix} -2\, r_\varepsilon \xi_1^2 & -\,2\, r_\varepsilon \xi_1\, \xi_2 & -\,i\,\xi_1(r_\varepsilon^2 + |\xi|^2) \\ -2\, r_\varepsilon \xi_1\, \xi_2 & -\,2\, r_\varepsilon \xi_2^2 & -\,i\,\xi_2(r_\varepsilon^2 + |\xi|^2) \\ -2\,i\,\xi_1\,|\xi|\, r_\varepsilon & -\,2\,i\,\xi_2\,|\xi|\, r_\varepsilon & |\xi|(r_\varepsilon^2 + |\xi|^2) \end{pmatrix}.$$

Hence,

$$\begin{aligned} -\widetilde{p} + 2\,\frac{d\widetilde{v}_3}{dx_3} - \frac{|\xi|^2}{s}\,\widetilde{v}_3\Big|_{x_3=0} &= \widetilde{a}_3 - \frac{|\xi|^2}{s} \sum_{j=1}^{3} \frac{R_{3j}(r_\varepsilon, |\xi|)}{r_\varepsilon\, Q(r_\varepsilon, |\xi|)} \widetilde{a}_j \\ &= \widetilde{a}_3\left(1 + \frac{|\xi|^3}{s\, M(\varepsilon s, |\xi|)}\right) + \frac{|\xi|^2}{s\, M(\varepsilon s, |\xi|)} \frac{r_\varepsilon - |\xi|}{r_\varepsilon + |\xi|} \sum_{j=1}^{2} i\,\xi_j\, \widetilde{a}_j. \end{aligned}$$

It follows that the solution of problem (2.4)–(2.5) may be considered as a solution of problem (2.4)–(2.7) with $\widetilde{a}_1 = \widetilde{b}_1$, $\widetilde{a}_2 = \widetilde{b}_2$ and

$$\begin{aligned} \widetilde{a}_3 &= \frac{s\, M(\varepsilon s, |\xi|)}{P_\varepsilon(s, |\xi|)}\, \widetilde{b}_3' - \frac{|\xi|^2}{P_\varepsilon(s, |\xi|)} \frac{r_\varepsilon - |\xi|}{r_\varepsilon + |\xi|} \sum_{j=1}^{2} i\,\xi_j \widetilde{b}_j \\ &= \widetilde{b}_3'\left(1 - \frac{|\xi|^3}{P_\varepsilon(s, |\xi|)}\right) - \frac{|\xi|^2}{P_\varepsilon(s, |\xi|)} \frac{r_\varepsilon - |\xi|}{r_\varepsilon + |\xi|} \sum_{j=1}^{2} i\,\xi_j \widetilde{b}_j, \end{aligned} \tag{2.11}$$

$$P_\varepsilon(s, |\xi|) = s\, M(\varepsilon s, |\xi|) + |\xi|^3. \tag{2.12}$$

Making the inverse Fourier–Laplace transformation

$$F^{-1} L^{-1} \widetilde{h} \equiv h(x', x_3, t) = \frac{1}{(2\pi)^2\, 2\pi\, i} \int_{\mathbb{R}^2} e^{i x\cdot\xi}\, \mathrm{d}\xi \int_{\mathrm{Re}\, s=a\geqslant 0} e^{st}\, \widetilde{h}(\xi, x_3, s)\, \mathrm{d}s \tag{2.13}$$

we find that (2.11) is equivalent to

$$a_3(x'.t) = \sum_{j=1}^{2} \left(V * \frac{\partial b_j}{\partial x_j} \right) + b_3' + \sum_{j=1}^{2} \left(V_j * \frac{\partial b_3'}{\partial x_j} \right) \tag{2.14}$$

where

$$\widetilde{V} = -\frac{|\xi|^2}{P_\varepsilon} \frac{r_\varepsilon - |\xi|}{r_\varepsilon + |\xi|}, \qquad \widetilde{V}_j = \frac{i\, \xi_j |\xi|}{P_\varepsilon}$$

and the convolution is taken with respect to x' and t.

The initial-boundary value problem for the non-homogeneous Stokes equations

$$\begin{aligned}
&\varepsilon\, \mathbf{v}_t - \nabla^2 \mathbf{v} + \nabla p = \mathbf{f}(x, t), \;\; \nabla \cdot \mathbf{v} = g(x, t) \quad (x \in \mathbb{R}^3_+, \;\; t \in (0, T)),\\
&\mathbf{v}\Big|_{t=0} = \mathbf{v}_0(x),\\
&\frac{\partial v_j}{\partial x_3} + \frac{\partial v_3}{\partial x_j}\bigg|_{x_3=0} = b_j(x', t), \quad j = 1, 2,\\
&-p + 2\frac{\partial v_3}{\partial x_3} + \Delta' \int_0^t v_3(x, t)\, \mathrm{d}\tau \bigg|_{x_3=0} = b_3'(x', t) \quad (x' = (x_1, x_2) \in \mathbb{R}^2, \;\; t \in (0, T)),
\end{aligned} \tag{2.15}$$

can be reduced to problem (2.1)–(2.2) by construction of auxiliary functions

$$\begin{aligned}
\mathbf{v}^{(1)}(x, t) &= \int_0^t \mathrm{d}\tau \int_{\mathbb{R}^3} \Gamma_\varepsilon(x - y, t - \tau)\, \mathbf{f}^*(y, \tau)\, \mathrm{d}y + \varepsilon \int_{\mathbb{R}^3} \Gamma_\varepsilon(x - y, t)\, \mathbf{v}_0^*(y)\, \mathrm{d}y,\\
\mathbf{v}^{(2)}(x, t) &= \nabla \int_{\mathbb{R}^3_+} \Big(E(x - y) - E(x - y^*)\Big) \Big(g(y, t) - \nabla \cdot \mathbf{v}^{(1)}(y, t)\Big)\, \mathrm{d}y,\\
p^{(2)} &= g(x, t) - \nabla \cdot \mathbf{v}^{(1)}(x, t) - \varepsilon \int_{\mathbb{R}^3_+} \Big(E(x - y) - E(x - y^*)\Big) \Big(g_t - \nabla \cdot \mathbf{v}_t^{(1)}\Big)\, \mathrm{d}y.
\end{aligned} \tag{2.16}$$

Here $\mathbf{f}^*$ and $\mathbf{v}_0^*$ are extensions of the functions $\mathbf{f}$ and $\mathbf{v}_0$, respectively, into the domain $\{x_3 < 0\}$, $\Gamma_\varepsilon(x, t) = \varepsilon^{-1}(4\pi\, t\, \varepsilon^{-1})^{-3/2} \exp(-\varepsilon |x|^2/4t)$ and $E(x) = -\frac{1}{4\pi|x|}$ are fundamental solutions of the heat and Laplace equations, respectively, and $y^* = (y_1, y_2, -y_3)$. It can be easily seen that

$$\varepsilon \mathbf{v}_t^{(1)} - \nabla^2 \mathbf{v}^{(1)} = \mathbf{f}^*(x, t), \qquad \mathbf{v}^{(1)}(x, 0) = \mathbf{v}_0^*(x),$$

$$\nabla \cdot \mathbf{v}^{(2)} = g - \nabla \cdot \mathbf{v}^{(1)},$$

$$\varepsilon \mathbf{v}_t^{(2)} - \nabla^2 \mathbf{v}^{(2)} + \nabla p^{(2)}(x, t) = 0, \qquad v_1^{(2)}|_{x_3=0} = v_2^{(2)}|_{x_3=0} = 0,$$

and, if the compatibility condition $g(x,0) = \nabla\cdot\mathbf{v}_0(x) = \nabla\cdot\mathbf{v}^{(1)}(x,0)$ is satisfied, then $\mathbf{v}^{(2)}|_{t=0} = 0$. Hence, for $\mathbf{v}^{(3)} = \mathbf{v} - \mathbf{v}^{(1)} - \mathbf{v}^{(2)}$, $\quad p^{(3)} = p - p^{(2)}$ we obtain the problem of the type (2.1)–(2.2):

$$\varepsilon\,\mathbf{v}_t^{(3)} - \nabla^2\mathbf{v}^{(3)} + \nabla p^{(3)} = 0, \qquad \nabla\cdot\mathbf{v}^{(3)} = 0 \quad (x\in\mathbb{R}^3,\ \ t\in(0,T)),$$

$$\mathbf{v}^{(3)}\Big|_{t=0} = 0, \tag{2.17}$$

$$\frac{\partial v_j^{(3)}}{\partial x_3} + \frac{\partial v_3^{(3)}}{\partial x_j}\Big|_{x_3=0} = b_j(x',t) - \left(\frac{\partial v_j^{(1)}}{\partial x_3} + \frac{\partial v_3^{(1)}}{\partial x_j}\right)\Big|_{x_3=0} - \left(\frac{\partial v_j^{(1)}}{\partial x_3} + \frac{\partial v_3^{(1)}}{\partial x_j}\right)\Big|_{x_3=0}, \quad j=1,2,$$

$$-p^{(3)} + 2\frac{\partial v_3^{(3)}}{\partial x_3} + \Delta'\int_0^t v_3^{(3)}(x,\tau)\,d\tau\Big|_{x_3=0} =$$

$$= b_3'(x',t) + \left(p^{(2)} - 2\frac{\partial v_3^{(1)}}{\partial x_3} - 2\frac{\partial v_3^{(2)}}{\partial x_3}\right)\Big|_{x_3=0} - \Delta'\int_0^t v_3^{(1)}(x,\tau)\,d\tau\Big|_{x_3=0}.$$

If $\mathbf{f}$ g have compact supports, then $\mathbf{v}^{(1)}$ decays at infinity exponentially and $\mathbf{v}^{(2)}$, $p^{(2)}$ decay like power functions, at least, like $|x'|^{-1}$.

2.2 *Auxiliary propositions*

We start with the proof of some auxiliary inequalities which are necessary for the estimate of convolution integrals in (2.14). We consider at first the function (2.12). We observe that the functions

$$r(s,\xi) = \sqrt{s+|\xi|^2}, \qquad \frac{r(s,\xi)}{r(s,\xi)+|\xi|}, \qquad M(s,|\xi|) = s + 4|\xi|^2\,\frac{r(s,\xi)}{r(s,\xi)+|\xi|}$$

satisfy the inequalities

$$c_1\Big(|s|+|\xi|^2\Big)^{1/2} \leqslant \operatorname{Re} r(s,\xi) \leqslant |r(s,\xi)| \leqslant c_2\Big(|s|+|\xi|^2\Big)^{1/2}, \tag{2.18}$$

$$c_3 \leqslant \operatorname{Re}\frac{r(s,\xi)}{r(s,\xi)+|\xi|} \leqslant c_4, \tag{2.19}$$

$$c_5\Big(|s|+|\xi|^2\Big) \leqslant |M(s,\xi)| \leqslant c_6\Big(|s|+|\xi|^2\Big) \tag{2.20}$$

for arbitrary $\xi\in\mathbb{R}^2$ and $s\in\mathbb{C}$ such that

$$\operatorname{Re}s + \kappa\,|\operatorname{Im}s| > -\delta\,|\xi|^2$$

where δ and κ are small positive numbers. Moreover, these inequalities (maybe, with other constants) hold true if ξ_j is replaced with $\zeta_j = \xi_j + i\,\eta_j$, $\ \xi_j,\eta_j\in\mathbb{R}$, $\ j=1,2$, and

$$|\eta| \leqslant \delta_1|\xi| \tag{2.21}$$

where $\delta_1 > 0$ is a small positive number. With the help of (2.18)–(2.20) we can evaluate the function

$$P^{(\tau)}(s,\langle\zeta\rangle) = sM(s,\langle\zeta\rangle) + \tau\langle\zeta\rangle^3$$

where $\tau > 0$, $\ \langle\zeta\rangle = (\zeta_1^2+\zeta_2^2)^{1/2}$, $\ \zeta_j = \xi_j + i\,\eta_j$.

PROPOSITION 2.1 Assume that

$$\operatorname{Re} s + \kappa |\operatorname{Im} s| \geqslant a \gg \tau^2$$

(see inequality (2.24)) and that condition (2.21) is satisfied. Then

$$\begin{aligned} c_7\Big(|s|\Big(|s| + |\xi|^2\Big) + \tau\,|\xi|^3\Big) &\leqslant |P^{(\tau)}(s, \langle\zeta\rangle)| \\ &\leqslant c_8\Big(|s|\Big(|s| + |\xi|^2\Big) + \tau\,|\xi|^3\Big). \end{aligned} \tag{2.22}$$

Moreover, if δ_2/τ and $\delta_2/\sqrt{a}$ are small enough, then

$$\begin{aligned} c_9\Big(|s|\Big(|s| + |\xi|^2\Big) + \tau\,|\xi|^3\Big) &\leqslant |P^{(\tau)}(s - \delta_2\langle\zeta\rangle, \langle\zeta\rangle)| \\ &\leqslant c_{10}\Big(|s|\Big(|s| + |\xi|^2\Big) + \tau\,|\xi|^3\Big). \end{aligned} \tag{2.23}$$

The constants $c_7 - c_{10}$ are independent of τ.

Proof. We start with the estimate of $P^{(\tau)}(s, |\xi|)$, $\xi \in \mathbb{R}^2$, from below (the estimate from above is evident). We consider the sum

$$\operatorname{Re}\frac{P^{(\tau)}(s, |\xi|)}{M(s, |\xi|)} + \kappa_1\left|\operatorname{Im}\frac{P^{(\tau)}(s, |\xi|)}{M(s, |\xi|)}\right| = \operatorname{Re}(s + T\bar{M}) + \kappa_1|\operatorname{Im}(s + T\bar{M})|$$

where $T = \tau|\xi|^3/|M|^2$, $\kappa_1 > \kappa$, and κ_1 is small. We have

$$T \leqslant \frac{\tau\sqrt{|s|}|\xi|^3}{\min\sqrt{|s|}|M|^2} \leqslant \frac{(1+\kappa^2)^{1/4}}{c_5^2\sqrt{a}}\frac{\tau\sqrt{|s|}|\xi|^3}{(|s| + |\xi|^2)^2} \leqslant \frac{(1+\kappa^2)^{1/4}}{c_5^2}\frac{\tau}{\sqrt{a}} \leqslant \frac{1}{2},$$

if

$$\frac{\tau}{\sqrt{a}} \leqslant \frac{c_5^2}{2(1+\kappa^2)^{1/4}}, \tag{2.24}$$

and, under this condition,

$$\begin{aligned} \operatorname{Re}\frac{P^{(\tau)}(s, |\xi|)}{M(s, |\xi|)} &+ \kappa_1\left|\operatorname{Im}\frac{P^{(\tau)}(s, |\xi|)}{M(s, |\xi|)}\right| \geqslant \\ &\geqslant (1+T)\operatorname{Re} s + \kappa_1(1-T)|\operatorname{Im} s| + 4|\xi|^2 T\left(\operatorname{Re}\frac{r}{r+|\xi|} - \kappa_1|\operatorname{Im}\frac{r}{r+|\xi|}|\right) \\ &\geqslant \frac{1}{2}(\operatorname{Re} s + \kappa_1|\operatorname{Im} s|) + 4|\xi|^2 T(c_3 - \kappa_1 c_4) \\ &\geqslant c\Big(|s| + 4|\xi|^2 T\Big) \geqslant \frac{2}{3}c\Big(|s| + T(|s| + 4|\xi|^2)\Big), \end{aligned}$$

provided that $\kappa_1 \leqslant c_3/2c_4$. Hence,

$$\mathrm{Re}\,\frac{P^{(\tau)}(s,|\xi|)}{M(s,|\xi|)} + \kappa_1\left|\mathrm{Im}\,\frac{P^{(\tau)}(s,|\xi|)}{M(s,|\xi|)}\right| \geqslant c\left(|s| + \frac{\tau|\xi|^3}{M|}\right),$$

which implies

$$\left|\frac{P^{(\tau)}(s,|\xi|)}{M(s,|\xi|)}\right| \geqslant \frac{c}{\sqrt{1+\kappa_1^2}}\left(|s| + \frac{\tau|\xi|^3}{|M|}\right)$$

and gives a necessary estimate

$$|P^{(\tau)}(s,|\xi|)| \geqslant c(|s|(|s|+|\xi|^2) + \tau|\xi|^3). \tag{2.25}$$

Inequality (2.22) follows from (2.25) and from

$$|P^{(\tau)}(s,\langle\zeta\rangle) - P^{(\tau)}(s,|\xi|)| \leqslant c|\eta|(|s||\xi| + \tau|\xi|^2) \leqslant c\delta_1(|s||\xi|^2 + \tau|\xi|^3),$$

if δ_1 is small.

Finally, to prove (2.23), we evaluate the difference

$$P^{(\tau)}(s-\delta_2\langle\zeta\rangle,\langle\zeta\rangle) - P^{(\tau)}(s,\langle\zeta\rangle) = -\delta_2\langle\zeta\rangle\int_0^1 \frac{\partial}{\partial s}P^{(\tau)}(s-\delta_2\lambda\langle\zeta\rangle,\langle\zeta\rangle)\,\mathrm{d}\lambda$$

$$= -\delta_2\langle\zeta\rangle\int_0^1\left[M(s-\delta_2\lambda\langle\zeta\rangle,\langle\zeta\rangle) + (s-\delta_2\lambda\langle\zeta\rangle)\frac{\partial M(s-\delta_2\lambda\langle\zeta\rangle,\langle\zeta\rangle)}{\partial s}\right]\mathrm{d}\lambda.$$

Since

$$\mathrm{Re}(s-\delta_2\lambda\langle\zeta\rangle) + \kappa|\,\mathrm{Im}(s-\delta_2\lambda\langle\zeta\rangle)| \geqslant a - c\delta_2|\xi| \geqslant -\delta|\xi|^2 + a - \frac{c^2\delta_2^2}{4\delta} \geqslant -\delta|\xi|^2,$$

if $\delta_2/\sqrt{a} \leqslant 2\sqrt{\delta}/c$, the functions $r(s-\delta_2\lambda\langle\zeta\rangle,\langle\zeta\rangle)$ and $M(s-\delta_2\lambda\langle\zeta\rangle,\langle\zeta\rangle)$ satisfy the inequalities (2.18) and (2.20). Hence,

$$|P^{(\tau)}(s-\delta_2\langle\zeta\rangle,\langle\zeta\rangle) - P^{(\tau)}(s,\langle\zeta\rangle)| \leqslant c\delta_2|\xi|(|s|+|\xi|^2)$$

$$\leqslant c\delta_2\frac{(1+\kappa^2)^{1/4}}{\sqrt{a}}|s|^{3/2}|\xi| + c\frac{\delta_2}{\tau}\tau|\xi|^3 \leqslant c(1+\kappa^2)^{1/4}\max\left(\frac{\delta_2}{\sqrt{a}},\frac{\delta_2}{\tau}\right)(|s|(|s|+|\xi|^2)+\tau|\xi|^3). \tag{2.26}$$

This implies (2.23), if $\delta_2/\sqrt{a}$ and δ_2/τ are small. The proposition is proved. □

PROPOSITION 2.2 The functions

$$V_1(x',t) = F^{-1}L^{-1}\frac{p_k(\xi,|\xi|)}{P_\varepsilon(s,|\xi|)}, \quad k>0,$$

$$V_2 = F^{-1}L^{-1}\frac{p_{k+1}(\xi,|\xi|)}{P_\varepsilon(s,|\xi|)\,(r_\varepsilon+|\xi|)}, \quad k>0, \tag{2.27}$$

$$V_3(x', t) = F^{-1} L^{-1} \frac{M(\varepsilon s, |\xi|)\, p_{k-2}(\xi, |\xi|)}{P_\varepsilon(s, |\xi|)}, \quad k \geqslant 2,$$

where p_k is a homogeneous polynomial of order k with respect to ξ_j, $|\xi|$, satisfy the inequalities

$$|D^j_{x'} V_i(x', t)| \leqslant \frac{c(t, j)}{\Big(|x'| + t\Big)^{k+|j|}} \tag{2.28}$$

in which $c(t, j)$ are increasing functions of t independent of ε.

Proof. Since the number k can be arbitrarily large, it suffices to prove (2.28) for $j = 0$. We consider the function $V_1(x', t)$, $t > 0$. Using the Cauchy theorem, we change the contour of integration with respect to s in the formula (2.13) for V_1 replacing the line $\operatorname{Re} s = a > 0$ with

$$\ell(a) = \{\operatorname{Re} s = a - \kappa |\operatorname{Im} s|\}$$

and we introduce new variables $\xi' = \sqrt{\frac{t}{\varepsilon}}\xi$, $s' = ts$. This gives

$$V_1(x', t) = \frac{1}{(2\pi)^2\, 2\pi\, i} \left(\frac{\varepsilon}{t}\right)^{\frac{k}{2}} \int_{\mathbb{R}^2} e^{iy'\cdot\xi'}\, p_k(\xi', |\xi'|)\, \mathrm{d}\xi' \int_{\ell(a)} e^{s'} \frac{\mathrm{d}s'}{P^{(\tau)}(s', |\xi'|)}$$

where $y' = x'(\frac{\varepsilon}{t})^{1/2}$, $\tau = \sqrt{\varepsilon\, t}$. The number $a > 0$ is arbitrary because the parallel shift of the contour $\ell(a)$ does not change the value of the integral. We set

$$a = 1 + K\varepsilon t \equiv a(t), \qquad K \gg 1,$$

and, taking account of Proposition 2.2, shift the contour $\ell(a(t))$ to the left by $\delta_3\tau|\xi|$, $\delta_3 \ll 1$, to obtain

$$V_1(x,' t) = \frac{1}{(2\pi)^3\, i} \left(\frac{\varepsilon}{t}\right)^{\frac{k}{2}} \int_{\mathbb{R}^2} e^{iy'\cdot\xi'}\, p_k(\xi, |\xi|)\, e^{-\delta_3\,\tau\,|\xi'|}\, \mathrm{d}\xi \int_{\ell(a(t))} \frac{e^s\, \mathrm{d}s}{P^{(\tau)}(s - \delta_3\tau|\xi'|, |\xi'|)}.$$

Finally, assuming that $y_1 = y_2 = |y|/\sqrt{2}$ (which can be achieved by rotation of coordinate axes), we deform the contours of integration with respect to ξ_1 and ξ_2 (i.e. $\mathbb{R}$) into the contours $\zeta_j = \xi_j + i\delta_1|\xi_j|$, $\quad j = 1, 2$, with a small δ_1. This is possible due to the Jordan lemma, because the integrand decays at infinity and

$$\operatorname{Re}(i\, y_j\zeta_j) = -\frac{|y|}{\sqrt{2}}\eta_j < 0 \qquad \text{for} \quad \eta_j > 0,$$

and this leads to

$$V_1(x', t) = \frac{1 + \delta_1^2}{(2\pi)^3\, i} \left(\frac{\varepsilon}{t}\right)^{\frac{k}{2}} \int_{\mathbb{R}^2} e^{iy'\cdot\xi' - \frac{\delta_1|y'|}{\sqrt{2}} \sum_j |\xi_j| - \delta_3\,\tau\,\langle\zeta\rangle}\, p_k(\zeta, \langle\zeta\rangle)\, \mathrm{d}\xi \int_{\ell(a(t))} \frac{e^s\, \mathrm{d}s}{P^{(\tau)}(s - \delta_3\,\tau\,\langle\zeta\rangle, \langle\zeta\rangle)} \tag{2.29}$$

(we observe that this device was used in [1]). Since, in virtue of (2.23),

$$|P^{(\tau)}(s - \delta_3\,\tau\,\langle\zeta\rangle, \langle\zeta\rangle)| \geqslant c\Big(|s|(|s| + |\xi|^2) + \tau|\xi|^3\Big) \geqslant c|\xi|^2,$$

we have

$$|V_1(x',t)| \leqslant \frac{c\,\varepsilon^{\frac{k}{2}}}{t^{\frac{k}{2}}} \int_{\mathbb{R}^2} e^{-c(|y'|+\sqrt{\varepsilon t})\,|\xi|}\,|\xi|^{k-2}\,\mathrm{d}\xi \int_0^\infty e^{a(t)-\kappa r}\,\mathrm{d}r$$

$$\leqslant \frac{c\,\varepsilon^{\frac{k}{2}}}{t^{\frac{k}{2}}\left(|y'|+\sqrt{\varepsilon\, t}\right)^k} \leqslant \frac{c}{\left(|x'|+t\right)^k}.$$

Inequality (2.28) with $j=0$ for V_1 is proved. For V_2 it is obtained in exactly the same way. For V_3, instead of (2.29), we have

$$V_3(x',t) = \frac{1+\delta_1^2}{(2\pi)^3\, i}\left(\frac{\varepsilon}{t}\right)^{\frac{k}{2}} \int_{\mathbb{R}^2} e^{iy'\cdot\xi'-\frac{\delta_1|y'|}{\sqrt{2}}\sum_j|\xi_j|-\delta_3\,\tau\,\langle\zeta\rangle}\, p_{k-2}(\zeta,\langle\zeta\rangle)\,\mathrm{d}\xi\,\cdot$$
$$\int_{\ell(a(t))} \frac{M(s-\delta_3\,\tau\,\langle\zeta\rangle,\langle\zeta\rangle)e^s\,\mathrm{d}s}{P^{(\tau)}(s-\delta_3\,\tau\,\langle\zeta\rangle,\langle\zeta\rangle)},$$

so, using the inequality $|M|/|P^{(\tau)}| \leqslant c|s|^{-1} \leqslant c$, we arrive at the same estimate (2.28).

If $t<0$, then, by the Jordan lemma,

$$\int_{\mathrm{Re}\, s=a} e^{st}\frac{\mathrm{d}s}{P_\varepsilon(s,|\xi|)} = 0, \qquad \int_{\mathrm{Re}\, s=a} e^{st}\frac{M(s,|\xi|)\,\mathrm{d}s}{P_\varepsilon(s,|\xi|)} = 0,$$

hence, $V_1(x',t)=V_2(x',t)=V_3(x',t)=0$. The proposition is proved. □

COROLLARY 2.1 If $k=2$, then the kernels V_i satisfy the inequalities

$$\int_0^t \mathrm{d}\tau \int_{\mathbb{R}^2} |\varDelta_j^m(h)V_i(x',\tau)| \leqslant c(t)h, \qquad m>1, \tag{2.30}$$

where

$$\varDelta_j^m(h)V_i(x',\tau) = \sum_{k=0}^m (-1)^{m-k}C_m^k V_i(x'+e_jkh,\tau)$$

is a finite difference of V_i with respect to x_j of order m, $e_1=(1.0)$, $e_2=(0,1)$.

Indeed, we have

$$\int_0^t \mathrm{d}\tau \int_{\mathbb{R}^2} |\varDelta_j^m(h)V_i(x',\tau)|\,\mathrm{d}x' \leqslant \sum_{k=0}^m C_m^k \int_0^t \mathrm{d}\tau \int_{|x'|<2mh} |V_i(x'+e_jkh,\tau)|\,\mathrm{d}x'$$

$$+\int_0^t \mathrm{d}\tau \int_{|x'|>2mh} \left|\int_0^h \ldots \int_0^h \frac{\partial^m V_i(x+e_j(t_1+\ldots t_m))}{\partial x_j^m}\,\mathrm{d}t_1\ldots\mathrm{d}t_m\right|$$

$$\leqslant c\sum_{k=0}^m \int_0^t \mathrm{d}\tau \int_{|x'|<2mh} \frac{\mathrm{d}x'}{(|x'+e_jkh|+\tau)^2} + c\int_0^t \mathrm{d}\tau \int_{|x'|>2mh} \frac{h^m\,\mathrm{d}x'}{(|x'|+\tau)^{2+m}} \leqslant ch.$$

2.3 *Estimates of solutions of Problems (2.1), (2.2), and (2.15)*

THEOREM 2.1 Assume that b_1, b_2 are smooth functions decaying like power functions as $|x'| \to \infty$ together with their derivatives and satisfying the conditions

$$b_1(x', 0) = b_2(x', 0) = 0.$$

Further, let

$$b_3'(x', t) = b_3(x', t) + \int_0^t B(x', \tau)\,d\tau$$

with $b_3,\ B$ possessing the same properties. Then the solution of problem (2.1)–(2.2) constructed in Subsection 2.1 satisfies the inequality

$$\begin{aligned}
&\varepsilon \sup_{\tau<t}[\mathbf{v}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\mathbf{v}(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[p(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+} \leqslant \\
&\qquad \leqslant c\Big(\sum_{k=1}^m \sup_{\tau<t}[b_k(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2} + \varepsilon^{\frac{1+\alpha}{2}} \sum_{j=1}^2 \sup_{\mathbb{R}^2}[b_j(x',\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)} + \sup_{\tau<t}[B(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^2}\Big). \qquad (2.31)
\end{aligned}$$

Proof. We consider the solution of problem (2.1)–(2.2) as the solutions of (2.1)–(2.6) with $a_1 = b_1,\ a_2 = b_2$ and with a_3 given by formulas (2.11) and (2.14). Under our hypotheses, these formulas take the form

$$\widetilde{a}_3 = \widetilde{b}_3 + \sum_{j=1}^2 \frac{|\xi|\,i\,\xi_j}{P_\varepsilon(s,|\xi|)}\,i\,\xi_j\,\widetilde{b}_3 + \frac{M(\varepsilon s,|\xi|)}{P_\varepsilon(s,|\xi|)}\,\widetilde{B} - \left(\frac{|\xi|^2}{P_\varepsilon(s,|\xi|)} - \frac{2\,|\xi|^3}{P_\varepsilon(s,|\xi|)\,(r_\varepsilon+|\xi|)}\right)\sum_{j=1}^2 i\,\xi_j\,\widetilde{b}_j,$$

or

$$\begin{aligned}
a_3(x', t) = &\sum_{j=1}^2 \int_0^t d\tau \int_{\mathbb{R}^2} W(x'-y', t-\tau)\,\frac{\partial b_j(y',\tau)}{\partial y_j}\,dy' \\
&+ \sum_{j=1}^2 \int_0^t d\tau \int_{\mathbb{R}^2} W_j(x'-y', t-\tau)\,\frac{\partial b_3(y',\tau)}{\partial y_j}\,dy' + b_3(x', t) \qquad (2.32)\\
&+ \int_0^t d\tau \int_{\mathbb{R}^2} V(x'-y', t-\tau)\,B(y',\tau)\,dy'
\end{aligned}$$

with the kernels $W,\ W_j,\ V$ defined by

$$W = -F^{-1}L^{-1}\left(\frac{|\xi|^2}{P_\varepsilon(s,|\xi|)} - \frac{2\,|\xi|^3}{P_\varepsilon(s,|\xi|)(r_\varepsilon+|\xi|)}\right),$$

$$W_j = F^{-1}L^{-1}\,\frac{i\,\xi_j\,|\xi|}{P_\varepsilon(s,|\xi|)}, \quad V = F^{-1}L^{-1}\,\frac{M(\varepsilon s,|\xi|)}{P_\varepsilon(s,|\xi|)}.$$

All these kernels satisfy inequalities (2.28) and (2.30), so the integrals in (2.32) are convergent, if ∇b_k, B decay at infinity like power functions. Moreover, from (2.30) and from a general result due to K. K. Golovkin (see [2, 3]) it follows that

$$\sup_{\tau<t}[a_3(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2} \leqslant c\Big(\sum_{k=1}^3 \sup_{\tau<t}[b_k(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2} + \sup_{\tau<t}[B(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^2}\Big). \qquad (2.33)$$

Solutions of the initial-boundary value problem for the Stokes equations with boundary conditions (2.6) were estimated in the paper [18]; in particular, Theorem 4.1 of this paper yields the following inequality for the solution of Problem (2.1) and (2.6):

$$\varepsilon \sup_{\tau<t}[\mathbf{v}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\mathbf{v}(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[p(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+}$$
$$\leqslant c\left(\sum_{k=1}^{3}\sup_{\tau<t}[a_k(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2} + \varepsilon^{\frac{1+\alpha}{2}}\sum_{j=1}^{2}\sup_{\mathbb{R}^2}[a_j(x',\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)}\right). \tag{2.34}$$

Estimate (2.31) follows from (2.33) and (2.34). The theorem is proved. □

THEOREM 2.2 Let $\mathbf{f}$, g, $\mathbf{v}_0$, b_1, b_2 be smooth functions with compact supports defined for $t \in (0,T)$ and satisfying the compatibility conditions

$$\nabla\cdot\mathbf{v}_0(x) = g(x,0), \qquad \frac{\partial v_{0j}}{\partial x_3} + \frac{\partial v_{03}}{\partial x_j}\Big|_{x_3=0} = b_j(x',0), \qquad j=1,2,$$

moreover, let

$$b_3'(x',t) = b_3(x',t) + \int_0^t B(x',\tau)\,\mathrm{d}\tau \qquad g(x,t) = \nabla\cdot\mathbf{h}(x,t) + h_0(x,t)$$

where b_3, B, $\mathbf{h}$, h_0 are also smooth and have compact supports. Then Problem (2.15) has a unique solution in a class of bounded continuous functions with a finite norm

$$Y_T(\mathbf{v},p) = \varepsilon\sup_{t<T}[\mathbf{v}_t(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T}[\mathbf{v}_t(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{t<T}[p(\cdot,t)]^{(1+\alpha)}_{\mathbb{R}^3_+}, \tag{2.35}$$

and for arbitrary $t < T$

$$Y_t(\mathbf{v},p) \leqslant c\Bigg(\sup_{\tau<t}[\mathbf{f}(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[g(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+} + [\mathbf{v}_0]^{(2+\alpha)}_{\mathbb{R}^3_+} + \varepsilon\sup_{\tau<t}[\mathbf{h}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} \tag{2.36}$$
$$+\varepsilon D^{1-\alpha}\sup_{\tau<t}\sup_{\mathbb{R}^3_+}|h_{0\tau}(x,\tau)| + \sum_{k=1}^{3}\sup_{\tau<t}[b_k(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2}$$
$$+\varepsilon^{\frac{1+\alpha}{2}}\sum_{j=1}^{2}\sup_{\mathbb{R}^2}[b_j(x',\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)} + \sup_{\tau<t}[B(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^2}\Bigg) \equiv F_t$$

where $D = \sup_{\tau<T}(\operatorname{diam}\operatorname{supp} h_0(x,\tau))$.

Proof. We obtain the estimate (2.36) for the solution of Problem (2.15) constructed above. If the extension of $\mathbf{f}$ and $\mathbf{v}_0$ into the domain $x_3 < 0$ in (2.16) is made in such a way that

$$[\mathbf{f}^*(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3} \leqslant c[\mathbf{f}(\cdot,t)]^{(\alpha)}_{\mathbb{R}^3_+}, \qquad [\mathbf{v}_0^*]^{(2+\alpha)}_{\mathbb{R}^3} \leqslant c[\mathbf{v}_0]^{(2+\alpha)}_{\mathbb{R}^3_+},$$

then, as it follows from the estimates (4.9) and (4.11) in [18],

$$\sum_{j=1}^{2}\left(\varepsilon\sup_{\tau<t}[\mathbf{v}^{(j)}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\mathbf{v}^{(j)}(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+}\right) + \sup_{\tau<t}[p^{(2)}(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+}$$

$$\leqslant c\Big(\sup_{\tau<t}[\mathbf{f}(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+}+\sup_{\tau<t}[g(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+}+[\mathbf{v}_0]^{(2+\alpha)}_{\mathbb{R}^3_+} \tag{2.37}$$

$$+\varepsilon\sup_{\tau<t}[\mathbf{h}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+}+\varepsilon D^{1-\alpha}\sup_{\tau<t}\sup_{\mathbb{R}^3_+}|h_{0\tau}(x,\tau)|\Big).$$

Finally, applying Theorem 2.1 to Problem (2.17), we obtain

$$Y_t(\mathbf{v}^{(3)},p^{(3)})\leqslant c\Big(\sum_{k=1}^{3}\sup_{\tau<t}[b_k(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2}+\varepsilon^{\frac{1+\alpha}{2}}\sum_{j=1}^{2}\sup_{\mathbb{R}^2}[b_j(x',\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)}+\sup_{\tau<t}[B(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^2}$$

$$+\sum_{j=1}^{2}\sup_{\tau<t}[\mathbf{v}^{(j)}_\tau(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+}+\sup_{\tau<t}[p^{(2)}(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+}+\varepsilon^{\frac{1+\alpha}{2}}\sum_{j=1}^{2}\sup_{\mathbb{R}^2}[\nabla\mathbf{v}^{(j)}(x',\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)}\Big).$$

But the last term, in virtue of interpolation inequality

$$\varepsilon^{\frac{1+\alpha}{2}}\sup_{\mathbb{R}^2}[\nabla u(x',\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)}\leqslant c\Big(\varepsilon\sup_{\tau<t}[u^{(j)}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+}+\sup_{\tau<t}[u(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+}\Big) \tag{2.38}$$

(see [6, 18]) can be estimated by the left-hand side of (2.37). Hence, putting all the estimates together, we arrive at (2.36).

The uniqueness of the solution of Problem (2.15) can be proved in the same way as in [18]. Let $(\mathbf{w},s)$ be a bounded continuous solution of a homogeneous problem with a finite norm (2.35) and let $\zeta\in C_0^\infty(\mathbb{R}^3)$ be a function equal to one for $|x|<1$ and to zero for $|x|>2$. The functions $\mathbf{W}_R=\zeta_R\mathbf{w}$, $S_R=\zeta_R s$ where $\zeta_R(x)=\zeta(x/R)$ satisfy the equations (2.15) with

$$\mathbf{f}=-2\nabla\mathbf{w}\nabla\zeta_R-\mathbf{w}\nabla^2\zeta_R+s\nabla\zeta_R,\qquad g=\nabla\zeta_R\cdot\mathbf{w}\equiv h_0,$$

$$b_j=w_3\frac{\partial\zeta_R}{\partial x_j}+w_j\frac{\partial\zeta_R}{\partial x_3}\Big|_{x_3=0},\qquad b_3=2w_3\frac{\partial\zeta_R}{\partial x_3}\Big|_{x_3=0},\qquad B=(\nabla'w_3\cdot\nabla'\zeta_R+w_3\nabla'^2\zeta_R)\Big|_{x_3=0}.$$

Clearly, $\mathbf{w}_R, s_R$ are expressed in terms of these functions as indicated above. Since $\mathbf{w}, s$ and their derivatives are bounded, estimate (2.36) gives

$$Y_t(\mathbf{W}_R,S_R)\leqslant cR^{-\alpha}\to 0,\qquad \text{as}\ \ R\to\infty.$$

Hence, $\mathbf{w}=0$, $s=0$, q.e.d. The theorem is proved. □

3. Proof of Theorem 5

We start with the proof of the estimate (1.23) by a standard method of Schauder; the proof is very close to that of inequality (1.3) in [18]. We estimate the solution in a small neighbourhood of an arbitrary point $x_0\in\overline{\Omega}$ assuming (without loss of generality) that $x_0=0$ and that Ox_3-axis is directed along the interior normal $-\mathbf{n}_0(x_0)$.

Let

$$x_3=\phi(x'),\quad x'\in\mathcal{B}_d=\{|x'|<d\},\ \ \phi\in C^{2+\alpha}(\mathcal{B}_d),$$

be the equation of the surface Γ in the neighbourhood of the point $x_0 = 0$. Then the normal and the Laplace–Beltrami operator on Γ are defined by

$$\mathbf{n}_0(x) = \left(\frac{\phi_{x_1}}{\sqrt{1+|\nabla\phi(x')|^2}},\ \frac{\phi_{x_2}}{\sqrt{1+|\nabla\phi(x')|^2}},\ -\frac{1}{\sqrt{1+|\nabla\phi(x')|^2}} \right)$$

and

$$\Delta_0 f = \frac{1}{\sqrt{g^{(0)}}} \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha} \left(\frac{\widehat{g}^{(0)}_{\alpha\beta}}{\sqrt{g^{(0)}}} \frac{\partial f}{\partial x_\beta} \right)$$

where $g^{(0)} = \det(g^{(0)}_{\alpha\beta})_{\alpha,\beta=1,2}$,

$$g^{(0)}_{\alpha\beta} = \frac{\partial \mathbf{r}}{\partial x_\alpha} \cdot \frac{\partial \mathbf{r}}{\partial x_\beta} = \delta_{\alpha\beta} + \phi_{x_\alpha}\phi_{x_\beta}, \quad \mathbf{r} = (x_1, x_2, \phi(x')),$$

and $\widehat{g}^{(0)}_{\alpha\beta}$ are elements of associated matrix to $(g^{(0)}_{\alpha\beta})_{\alpha,\beta=1,2}$, i.e.

$$\widehat{g}^{(0)}_{11} = g^{(0)}_{22}, \quad \widehat{g}^{(0)}_{22} = g^{(0)}_{11}, \quad \widehat{g}^{(0)}_{12} = \widehat{g}^{(0)}_{21} = -g^{(0)}_{12}.$$

We 'rectify' Γ near the origin by the transformation $y = Z(x)$:

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 - \phi(x'), \ \ x' \in K'_d,$$

and we set

$$\mathbf{u} = \zeta_r \mathbf{v}, \quad q = \zeta_r\, p$$

where $\zeta_r(y) = \zeta(y/r)$ and $\zeta(y)$ is the same function as in Theorem 2.2. By virtue of (1.21) and (1.22),

$$\begin{aligned}
&\varepsilon\, \mathbf{u}_t - \nabla^2 \mathbf{u} + \nabla q = \zeta_r\, \mathbf{f}_1 + \mathbf{f}' + \mathbf{m}_1(\mathbf{u}, q), \\
&\nabla \cdot \mathbf{u} = g\, \zeta_r + g' + m_2(\mathbf{u}), \quad \mathbf{u}\Big|_{t=0} = 0, \\
&S_{i3}(\mathbf{u})\Big|_{y_3=0} = -(\zeta_r\, b_i + b'_i) + m_{3i}(\mathbf{u}), \quad i = 1, 2, \\
&T_{33}(\mathbf{u}, q) + \int_0^t \Delta' u_3(y', \tau)\, \mathrm{d}\tau\Big|_{y_3=0} = b\, \zeta_r + b' + m_4(\mathbf{u}) + \int_0^t \Big(B\, \zeta_r + B' + m_5(\mathbf{u}) \Big)\, \mathrm{d}\tau,
\end{aligned} \tag{3.1}$$

where $\mathbf{f}_1 = \mathbf{f} - \ell(\mathbf{v})$,

$$\mathbf{f}' = -2\, \widehat{\nabla} \mathbf{v}\, \widehat{\nabla} \zeta_r - \mathbf{v}\, \widehat{\nabla}^2 \zeta_r + p\, \widehat{\nabla} \zeta_r,$$

$$g' = \mathbf{v} \cdot \widehat{\nabla} \zeta_r,$$

$$b'_i = \left(\frac{\partial \zeta_r}{\partial y_i} - \phi_{y_i} \frac{\partial \zeta_r}{\partial y_3} \right) (\mathbf{v} \cdot \mathbf{n}_0) + (\widehat{\nabla} \zeta_r \cdot \mathbf{n}_0)\, v_i - 2\, n_i (\widehat{\nabla} \zeta_r \cdot \mathbf{n}_0)\, (\mathbf{v} \cdot \mathbf{n}_0)\Big|_{y_3=0}, \quad i = 1, 2,$$

$$b' = 2(\widehat{\nabla} \zeta_r \cdot \mathbf{n}_0)\, (\mathbf{v} \cdot \mathbf{n}_0)\Big|_{y_3=0},$$

$$B' = \mathbf{n}_0 \cdot \Big(\Delta(\zeta_r\, \mathbf{v}) - \zeta_r\, \Delta \mathbf{v} \Big)\Big|_{y_3=0},$$

$$
\begin{aligned}
&\mathbf{m}_1(\mathbf{u}, q) = -\nu(\nabla^2 - \widehat{\nabla}^2)\,\mathbf{u} + (\nabla - \widehat{\nabla})\,q,\\
&m_2(\mathbf{u}) = (\nabla - \widehat{\nabla}) \cdot \mathbf{u},\\
&m_{3i}(\mathbf{u}) = \Big(S_{i3}(\mathbf{u}) + (\widehat{S}(\mathbf{u})\,\mathbf{n}_0)_i\Big) - n_{0i}\Big(\mathbf{n}_0 \cdot \widehat{S}(\mathbf{u})\,\mathbf{n}_0\Big)\Big|_{y_3=0},\\
&m_4(\mathbf{u}) = 2\Big(S_{33}(\mathbf{u}) - \mathbf{n}_0 \cdot \widehat{S}(\mathbf{u})\,\mathbf{n}_0\Big)\Big|_{y_3=0},\\
&m_5(\mathbf{u}) = (\Delta' u_3 + \mathbf{n}_0 \cdot \Delta\mathbf{u})\Big|_{y_3=0}.
\end{aligned} \tag{3.2}
$$

We observe that all the functions in (3.1) vanish for $|y| > 2r$, and the leading coefficients in the expressions (3.2) vanish at the origin. Hence, inequality (2.36) implies

$$
\begin{aligned}
&\varepsilon \sup_{\tau<t}[\mathbf{u}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\mathbf{u}(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[q(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+}\\
&\leqslant c\,r\Big(\varepsilon \sup_{\tau<t}[\mathbf{u}_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\mathbf{u}(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[q(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+}\Big)\\
&\quad + c\Bigg(\sup_{\tau<t}[\zeta_r\,\mathbf{f}_1(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \sup_{\tau<t}[\zeta_r\,g(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^3_+} + \varepsilon\sup_{\tau<t}[\zeta_r\,\mathbf{h}_t(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^3_+} + \varepsilon\sup_{\tau<t}\sup_{\mathbb{R}^3_+}|\zeta_r\,h_{0t}(y,\tau)|\\
&\quad + \sum_{i=1}^{2}\sup_{\tau<t}[\zeta_r\,b_i(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2} + \sup_{\tau<t}[\zeta_r\,b(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}^2} + \varepsilon^{\frac{1+\alpha}{2}}\sum_{i=1}^{2}\sup_{\mathbb{R}^2}[\zeta_r\,b_i]^{(\frac{1+\alpha}{2})}_{(0,t)} + \sup_{\tau<t}[\zeta_r\,B(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}^2}\Bigg)\\
&\quad + c(r)\Bigg(\sum_{0\leqslant|j|\leqslant2}\sup_{\tau<t}\sup_{\Omega}|\mathcal{D}^j\mathbf{v}| + \sup_{\tau<t}\sup_{\Omega}|\nabla p(x,\tau)| + \sup_{\tau<t}\sup_{\Omega}|p(x,t)|\\
&\quad + \varepsilon\sup_{\tau<t}\sup_{\Omega}|\mathbf{v}_\tau(x,\tau)| + \varepsilon^{\frac{1+\alpha}{2}}\sup_{\Gamma}[\mathbf{v}(x,\cdot)]^{(\frac{1+\alpha}{2})}_{(0,t)}\Bigg).
\end{aligned}
$$

Choosing r so small that $c\,r \leqslant \frac{1}{2}$, we eliminate the first three terms in the right-hand side. A similar estimate can be obtained in the case $\operatorname{dist}(x_0, \Gamma) > r$. Combination of these two cases gives the inequality

$$
\begin{aligned}
Y(t) &\equiv \varepsilon\sup_{\tau<t}|\mathbf{v}_\tau(\cdot,\tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t}|\mathbf{v}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t}|p(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}\\
&\leqslant c(N(t) + Y_0(t)),
\end{aligned} \tag{3.3}
$$

where $N(t)$ is the sum of norms in the right-hand side of (1.23) and

$$
Y_0(t) = \sum_{0\leqslant|j|\leqslant2}\sup_{\tau<t}\sup_{\Omega}|D^j_x\mathbf{v}(x,t)| + \sup_{\tau<t}\sup_{\Omega}|\nabla p(x,\tau)| + \sup_{\tau<t}\sup_{\Omega}|p(x,t)|.
$$

By virtue of interpolation inequalities,

$$
Y_0(t) \leqslant \varepsilon_1 Y(t) + c(\varepsilon_1)\left(\sup_{\tau<t}\|\mathbf{v}\|_{L_2(\Omega)} + \sup_{\tau<t}\sup_{\Omega}|p(x,t)|\right) \tag{3.4}
$$

where ε_1 is an arbitrarily small positive number. The function $p(x,t)$ can be considered as a solution of the Dirichlet problem

$$\nabla^2 p = \nabla\cdot\mathbf{f} + \nabla^2 g - \varepsilon(\nabla\cdot\mathbf{h}_t + h_{0t}), \quad x\in\Omega,$$
$$p\Big|_\Gamma = \Big(2\,\mathbf{n}_0\cdot\frac{\partial\mathbf{v}}{\partial n_0} - \mathbf{n}_0\cdot\int_0^t \Delta_0\mathbf{v}\,\mathrm{d}\tau\Big)\Big|_\Gamma - b - \int_0^t B\,\mathrm{d}\tau,$$

and it satisfies the inequality

$$\begin{aligned}\sup_\Omega|p(x,t)| \leqslant c\Big(&\sup_\Omega|\mathbf{f}(x,t)| + \sup_\Omega|\nabla g(x,t)| + \varepsilon(\sup_\Omega|\mathbf{h}_t(x,t)| + \sup_\Omega|h_{0t}(x,t)|)\\ &+ \sup_\Gamma|b(x,t)| + \int_0^t\sup_\Gamma|B(x,\tau)|\,\mathrm{d}\tau + \sup_\Omega|\nabla\mathbf{v}(x,t)| + \int_0^t\sup_\Gamma|\Delta_0\mathbf{v}(x,\tau)|\,\mathrm{d}\tau\Big)\end{aligned} \tag{3.5}$$

(see, for instance, [10], Lemma 5.1). Finally, we have the energy relation

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2_{L_2(\Omega)} + \frac{1}{2}\|S(\mathbf{v})\|^2_{L_2(\Omega)} + \sum_{k=1}^6\Big|\int_\Omega \mathbf{v}\cdot\boldsymbol{\varphi}_k\,\mathrm{d}x\Big|^2\\ &\quad= \int_\Omega(\mathbf{f}-\nabla g)\cdot\mathbf{v}\,\mathrm{d}x + \int_\Gamma\Big(\mathbf{b}\cdot\Pi_0\mathbf{v} + (b + \int_0^t B\,\mathrm{d}\tau)(\mathbf{v}\cdot\mathbf{n}_0)\Big)\,\mathrm{d}S\\ &\qquad+ \int_\Gamma\Big(\mathbf{v}(x,t)\cdot\mathbf{n}_0(x)\Big)\,\mathrm{d}S\int_0^t \mathbf{n}_0(x)\,\Delta_0\mathbf{v}(x,\tau)\,\mathrm{d}\tau + \int_\Omega p\,g\,\mathrm{d}x.\end{aligned} \tag{3.6}$$

By virtue of the Korn inequality (in the form presented in [13]),

$$\frac{1}{2}\|S(\mathbf{v})\|^2_{L_2(\Omega)} + \sum_{k=1}^6\Big|\int_\Omega\mathbf{v}\cdot\boldsymbol{\varphi}_k\,\mathrm{d}x\Big|^2 \geqslant \frac{c}{2}\|\mathbf{v}\|^2_{W_2^1(\Omega)},$$

so (3.6) implies

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2 + c\,\|\mathbf{v}\|^2 \leqslant c_1\Big(N(t) + \int_0^t Y_0(\tau)\,\mathrm{d}\tau\Big)^2$$

and

$$\begin{aligned}\|\mathbf{v}(\cdot,t)\|^2 &\leqslant \frac{c_1}{\varepsilon}\int_0^t e^{-\frac{c}{\varepsilon}(t-\tau)}\Big(N(\tau) + \int_0^\tau Y_0(\tau')\,\mathrm{d}\tau'\Big)^2\mathrm{d}\tau\\ &\leqslant \frac{c_1}{c}\Big(\sup_{\tau<t}N(\tau) + \int_0^t Y_0(\tau)\,\mathrm{d}\tau\Big)^2.\end{aligned}$$

From this inequality and from (3.4)–(3.6) it follows that

$$Y_0(t) \leqslant c\Big(N(t) + \int_0^t Y_0(\tau)\,\mathrm{d}\tau\Big),$$

$Y_0(t) \leqslant cN(t)$, and finally, taking (3.3) into account we arrive at (1.23).

As for the solvability of problem (1.21)–(1.22), we prove it making use of Theorem 6. This gives us the possibility to restrict ourselves to the case $\mathbf{f} = 0$, $g = 0$, $\mathbf{b} = 0$, $b = 0$ and to consider the problem

$$\begin{aligned} &\varepsilon \mathbf{v}_t - \nabla^2\mathbf{v} + \ell(\mathbf{v}) + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0, \quad x \in \Omega, \quad t \in (0,T), \\ &\mathbf{v}\Big|_{t=0} = 0, \\ &\Pi_0\, S(\mathbf{v})\,\mathbf{n}_0\Big|_{\Gamma} = 0, \\ &\mathbf{n}_0\cdot T(\mathbf{v},p)\mathbf{n}_0 - \mathbf{n}_0\cdot\Delta_0\int_0^t \mathbf{v}\,\mathrm{d}\tau = \int_0^t B\,\mathrm{d}\tau, \end{aligned} \tag{3.7}$$

with $B \in \tilde{C}(0,T;C^{\alpha}(\Gamma))$.

For a fixed $\varepsilon > 0$, the solvability of this problem in anisotropic Sobolev spaces was already established in [16], Theorem 5.1. Using similar scheme of the proof, we construct for arbitrary $B \in \tilde{C}(0,T;C^{\alpha}(\Gamma))$ a couple of functions $(\mathbf{u}, q)$ ($\mathbf{u} \in \tilde{C}(0,T;C^{2+\alpha}(\Omega))$, $\mathbf{u}_t \in \tilde{C}(0,T;C^{\alpha}(\Omega))$) $q \in \tilde{C}(0,T;C^{1+\alpha}(\Omega))$, satisfying (3.7) with B replaced by

$$B_1 = B + \mathcal{L}\,B,$$

where $\mathcal{L}$ is a linear operator in the space $\tilde{C}(0,T;C^{\alpha}(\Gamma))$ such that $I + \mathcal{L}$ has a bounded inverse. This immediately implies the solvability of the problem (3.7).

For the construction of $\mathbf{u}$, q, we introduce on Γ and in $\Omega_{\delta/2} = \{x \in \Omega:\ \mathrm{dist}(x,\Gamma) < \delta/2\}$ a smooth partition of unity $\{\zeta_j(x)\}_{j=1,\ldots,N}$ subordinated to the covering of $\Omega_{\delta/2}$ with the balls $K_j = \{|x-\xi^j| < \delta\}$, $\xi^j \in \Gamma$, and we assume that there exist smooth functions η_j, also with $\mathrm{supp}\,\eta_j \subset K_j$, such that $\eta_j\,\zeta_j = \zeta_j$. Let T_j be a tangent plane to Γ at the point ξ_j and let (y_1^j, y_2^j, y_3^j) be a Cartesian coordinate system with the origin in ξ^j and with y_3^j- axis directed along the interior normal $-\mathbf{n}_0(\xi^j)$. In the d-neighbourhood of the origin the surface Γ can be given by the equation

$$y_3^j = \phi^j(y_1^j, y_2^j), \quad \phi^j \in C^{3+\alpha}(\mathcal{B}_d),$$

where $\mathcal{B}_d = \{\sqrt{y_1^{j2} + y_2^{j2}} < d\}$. Coordinates (x_1, x_2, x_3) and (y_1^j, y_2^j, y_3^j) are related to each other by the transformation

$$y^j = U_j(x - \xi^j) \equiv U^j(x),$$

where U_j is an orthogonal matrix. The mapping

$$x = Y^j(z) = (U^j)^{-1}\circ Z_j \circ U^j(z), \tag{3.8}$$

where $Z_j(y^j) \equiv (y_1^j, y_2^j, y_3^j + \phi^j(y_1^j, y_2^j))$ transforms the semi-ball $|z-\xi^j| < d$, $(\mathbf{z}-\boldsymbol{\xi}^j)\cdot\mathbf{n}(\xi^j) < 0$, into a subdomain $\omega_d \subset \Omega$ containing $K_j \cap \Omega$. Since $\nabla\phi^j = 0$ at the point $y_1^j = 0$, $y_2^j = 0$, the Jacobi matrix J_j of the transformation (3.8) satisfies the condition

$$J_j(\xi^j) = I.$$

In the half-space $\mathbb{R}_j = \{(\mathbf{z} - \boldsymbol{\xi}^j) \cdot \mathbf{n}(\xi^j) < 0\}$ we consider the problem

$$\begin{aligned}
&\varepsilon\, \mathbf{v}^j_t - \nabla^2 \mathbf{v}^j + \nabla p^j = 0, \quad \nabla \cdot \mathbf{v}^j = 0, \quad z \in \mathbb{R}_j, \quad t \in (0, T),\\
&\mathbf{v}^j\Big|_{t=0} = 0,\\
&\Pi^j S(\mathbf{v}^j)\, \mathbf{n}_0(\xi^j)\Big|_{T_j} = 0,\\
&\mathbf{n}_0(\xi^j) \cdot T(\mathbf{v}^j, p^j)\, \mathbf{n}_0(\xi^j) - \mathbf{n}_0(\xi^j) \cdot \Delta^j \int_0^t \mathbf{v}^j \,\mathrm{d}\tau\Big|_{T_j} = \int_0^t B^j(z, \tau)\,\mathrm{d}\tau,
\end{aligned} \tag{3.9}$$

where Π^j is a projector onto T_j, Δ^j is the Laplacean on T_j, and

$$B^j(z, \tau) = B(x, \tau)\, \zeta_j(x)\Big|_{z=(Y^j)^{-1}(x)}, \quad x \in K_j \cap \Gamma.$$

Further we set

$$\mathbf{u}^j(x, t) = \mathbf{v}^j \circ (Y^j)^{-1}, \quad q^j(x, t) = p^j \circ (Y^j)^{-1},$$
$$\mathbf{u}'(x, t) = \sum_j \eta_j(x)\, \mathbf{u}^j(x, t), \quad q'(x, t) = \sum_j \eta_j(x)\, q^j(x, t),$$

and we define $(\mathbf{u}'', q'')$ as the solution of the problem

$$\begin{aligned}
&\varepsilon\, \mathbf{u}''_t - \nabla^2 \mathbf{u}'' + \ell(\mathbf{u}'') + \nabla q'' = -\varepsilon\, \mathbf{u}'_t + \nabla^2 \mathbf{u}' - \ell(\mathbf{u}') + \nabla q' \equiv \mathbf{f},\\
&\nabla \cdot \mathbf{u}'' = -\nabla \cdot \mathbf{u}' \equiv g, \quad x \in \Omega, \quad t \in (0, T),\\
&\mathbf{u}''\Big|_{t=0} = 0,\\
&\Pi_0\, S(\mathbf{u}'')\, \mathbf{n}_0\Big|_{\Gamma} = -\Pi_0\, S(\mathbf{u}')\, \mathbf{n}_0\Big|_{\Gamma} \equiv \mathbf{b},\\
&\mathbf{n}_0 \cdot T(\mathbf{u}'', q'')\, \mathbf{n}_0\Big|_{\Gamma} = -\mathbf{n}_0 \cdot T(\mathbf{u}', q')\, \mathbf{n}_0 + \sum_j \eta_j(x) \Big(\mathbf{n}_0(\xi^j) \cdot T(\mathbf{v}^j, p^j)\, \mathbf{n}_0(\xi^j)\Big)\Big|_{z=Y_j^{-1}(x)} \equiv b.
\end{aligned}$$

Then $\mathbf{u} = \mathbf{u}' + \mathbf{u}''$, $q = q' + q''$ is a solution of (3.7) with

$$B_1 = B - \sum_j \Big(\mathbf{n}_0(x) \cdot \Delta_0(\eta_j\, \mathbf{u}^j) - \mathbf{n}_0(\xi^j)\eta_j \cdot (\Delta^j \mathbf{v}^j) \circ (Y^j)^{-1})\Big) + \mathbf{n}_0(x) \cdot \Delta_0 \mathbf{u}''$$

instead of B. Hence,

$$\mathcal{L}\, B = -\sum_j \Big(\mathbf{n}_0(x) \cdot \Delta_0(\eta_j\, \mathbf{u}^j) - \mathbf{n}_0(\xi^j)\eta_j \cdot (\Delta^j \mathbf{v}^j \circ (Y^j)^{-1})\Big) + \mathbf{n}_0(x) \cdot \Delta_0 \mathbf{u}''.$$

Our main objective now is to prove that there holds the estimate

$$\sup_{\tau<t} |\mathcal{L}\, B|_{C^\alpha(\Gamma)} \leqslant \varepsilon_1' \sup_{\tau<t} |B(\cdot, \tau)|_{C^\alpha(\Gamma)} + c(\varepsilon_1', \varepsilon) \int_0^t |B(\cdot, \tau)|_{C^\alpha(\Gamma)}\,\mathrm{d}\tau \tag{3.10}$$

with arbitrarily small $\varepsilon_1' > 0$. Then the existence of the bounded inverse operator $(I + \mathcal{L})^{-1}$ is evident.

At first we estimate $\mathbf{u}''$, q''. By virtue of (3.9), we have

$$\mathbf{f}(x,t) = -\ell(\mathbf{u}') + \sum_j q_j \nabla\eta_j + \sum_j \eta_j(\nabla - \nabla_j)\, q^j + 2\sum_j \nabla\mathbf{u}^j\,\nabla\eta_j$$

$$+ \sum_j \mathbf{u}^j\,\nabla^2\eta_j + \sum_j \eta_j(\nabla^2 - \nabla_j^2)\,\mathbf{u}^j,$$

$$g = -\sum_j \mathbf{u}^j\cdot\nabla\eta_j - \sum_j \eta_j(\nabla - \nabla_j)\cdot\mathbf{u}^j,$$

$$\mathbf{b} = -\sum_j (\Pi_0\,\nabla\eta_j)\,(\mathbf{u}^j\cdot\mathbf{n}) - \sum_j \frac{\partial\eta_j}{\partial n}(\Pi_0\,\mathbf{u}^j)$$

$$+ \sum_j \eta_j\Big(\Pi^j\, S_j(\mathbf{u}^j)\,\mathbf{n}_0(\xi^j) - \Pi_0\, S(\mathbf{u}^j)\,\mathbf{n}_0(x)\Big)\Big|_\Gamma,$$

$$b = -2\sum_j \frac{\partial\eta_j}{\partial n}(\mathbf{u}^j\cdot\mathbf{n}_0)|_\Gamma + \sum_j \eta_j\Big(\mathbf{n}_0(\xi^j)\cdot S_j(\mathbf{u}^j)\,\mathbf{n}_0(\xi^j) - \mathbf{n}_0(x)\cdot S(\mathbf{u}^j)\,\mathbf{n}_0(x)\Big)\Big|_\Gamma$$

where $\nabla_j = J_j^T\nabla$ and

$$S_j(\mathbf{u}) = (\nabla_j\mathbf{u}) + (\nabla_j\mathbf{u})^T = \sum_{m=1}^{2}\left((J_j)_{mk}\,\frac{\partial u_q}{\partial x_m} + (J_j)_{mq}\,\frac{\partial u_k}{\partial x_m}\right)_{k,q=1,2,3.}$$

We observe also that, since $\det J_j^T = 1$, $g(x,t)$ can be written in the form

$$g = \nabla\cdot\mathbf{h} + h_0,$$

$$\mathbf{h} = \sum_j \eta_j(J_j - I)\,\mathbf{u}^j, \quad h_0 = -\sum_j \nabla\eta_j(J_j - I)\,\mathbf{u}^j - \sum_j \mathbf{u}^j\cdot\nabla\eta_j = -\sum_j \nabla\eta_j\cdot J_j\,\mathbf{u}^j.$$

Making use of the fact that $\nabla_j\Big|_{z=\xi^j} = \nabla$, of Theorem 6 and of interpolation inequalities (applied to $\mathbf{v}^j$, p^j), we obtain the estimate for $(\mathbf{u}'', q'')$

$$\varepsilon \sup_{\tau<t}|\mathbf{u}''_\tau(\cdot,\tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t}|\mathbf{u}''(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t}|q''(\cdot,\tau)|_{C^{1+\alpha}(\Omega)} \leqslant$$

$$\leqslant c(\delta+\varepsilon_1)\max_j\Big(\varepsilon\sup_{\tau<t}[\mathbf{v}^j_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j} + \sup_{\tau<t}[\mathbf{v}^j(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}_j} + \sup_{\tau<t}[p^j(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}_j}\Big)$$

$$+ c(\delta,\varepsilon_1)\max_j\Big(\sup_{\tau<t}\sup_{\mathbb{R}_j}|\mathbf{v}^j(z,\tau)| + \sup_{\tau<t}\sup_{\mathbb{R}_j}|p^j(z',\tau)|\Big),$$

and, since the principle part of Δ_0 at the point ξ^j coincides with Δ^j, a similar inequality holds for $\mathcal{L}B$:

$$\sup_{\tau<t}|\mathcal{L}B|_{C^\alpha(\Gamma)} \leqslant c(\delta+\varepsilon_1)\max_j\Big(\varepsilon\sup_{\tau<t}[\mathbf{v}^j_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j} + \sup_{\tau<t}[\mathbf{v}^j(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}_j} + \sup_{\tau<t}[p^j(\cdot,\tau)]^{(1+\alpha)}_{\mathbb{R}_j}\Big)$$

$$+c(\delta,\varepsilon_1)\max_j\Big(\sup_{\tau<t}\sup_{\mathbb{R}_j}|\mathbf{v}^j(z,\tau)|+\sup_{\tau<t}\sup_{\mathbb{R}_j}|p^j(z',\tau)|\Big).$$

The first term in the right-hand side can be estimated by the inequality (2.31) for the problem (3.9). The function $p^j(z,t)$ is harmonic in $\mathbb{R}_j$ and satisfies the boundary conditions

$$p^j\Big|_{T_j}=\left(2\mathbf{n}(\xi^j)\cdot\frac{\partial\mathbf{v}^j}{\partial n^j}-\mathbf{n}(\xi^j)\cdot\Delta^j\int_0^t\mathbf{v}^j\,\mathrm{d}\tau\right)\Big|_{T_j}-\int_0^t B^j\,\mathrm{d}\tau'\equiv p_0^j(z,t)$$

hence,

$$\sup_{\mathbb{R}_j}|p^j(z,t)|\leqslant\sup_{T_j}|p_0^j(z,t)|$$

$$\leqslant\varepsilon_2[\mathbf{v}^j(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}_j}+c(\varepsilon_2)[\mathbf{v}^j(\cdot,t)]^{(\alpha)}_{\mathbb{R}_j}+\int_0^t\Big([\mathbf{v}(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}_j}+[\mathbf{v}(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j}\Big)\,\mathrm{d}\tau+\int_0^t\sup_{T_j}|B^j(z,\tau)|\,\mathrm{d}\tau.$$

Making use of the elementary estimate

$$[\mathbf{v}^j(\cdot,t)]^{(\alpha)}_{\mathbb{R}_j}\leqslant\int_0^t[\mathbf{v}^j_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j}\,\mathrm{d}\tau=\varepsilon^{-1}\int_0^t\varepsilon[\mathbf{v}^j_\tau(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j}\,\mathrm{d}\tau$$

and of (2.31), we obtain

$$\sup_{\mathbb{R}_j}|p^j(z,t)|\leqslant\varepsilon_2[\mathbf{v}^j(\cdot,t)]^{(2+\alpha)}_{\mathbb{R}_j}+c(\varepsilon_2,\varepsilon)\int_0^t\Big([B^j(\cdot,\tau)]^{(\alpha)}_{T_j}+\sup_{T_j}|B^j(z,\tau)|\Big)\,\mathrm{d}\tau,$$

$$\int_0^t\sup_{\mathbb{R}_j}|p^j(z,\tau)|\,\mathrm{d}\tau\leqslant c(\varepsilon_2,\varepsilon)\int_0^t\Big([B^j(\cdot,\tau)]^{(\alpha)}_{T_j}+\sup_{T_j}|B^j(z,\tau|\Big)\,\mathrm{d}\tau,$$

and, finally,

$$\sup_{\mathbb{R}_j}|\mathbf{v}^j(z,t)|\leqslant\varepsilon^{-1}\int_0^t\varepsilon\sup_{\mathbb{R}_j}|\mathbf{v}^j_\tau(z,\tau|\,\mathrm{d}\tau\leqslant\varepsilon^{-1}\int_0^t\Big(\sup_{\mathbb{R}_j}|\nabla^2\mathbf{v}^j(z,\tau)|+\sup_{\mathbb{R}_j}|\nabla p^j(z,\tau)|\Big)\,\mathrm{d}\tau$$

$$\leqslant c\varepsilon^{-1}\int_0^t\Big([\mathbf{v}^j(\cdot,\tau)]^{(2+\alpha)}_{\mathbb{R}_j}+[\mathbf{v}^j(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j}\Big)\,\mathrm{d}\tau+c\varepsilon^{-1}\int_0^t\Big([\nabla p^j(\cdot,\tau)]^{(\alpha)}_{\mathbb{R}_j}+\sup_{\mathbb{R}_j}|p^j(z,\tau)|\Big)\,\mathrm{d}\tau$$

$$\leqslant c(\varepsilon)\int_0^t\Big([B^j(\cdot,\tau)]^{(\alpha)}_{T_j}+\sup_{T_j}|B^j(z,\tau)|\Big)\,\mathrm{d}\tau.$$

If we introduce in $C^\alpha(\Gamma)$ the norm

$$\|B\|_{C^\alpha(\Gamma)}=\max_j|B\,\zeta_j|_{C^\alpha(T_j)}$$

and put all the above estimates together, we arrive at the inequality (3.10) for the norm $\|\mathcal{L}B\|_{C^\alpha(\Gamma)}$.

More careful arguments give for $\|\mathcal{L}\|$ a bound independent of ε.

The proof of Theorem 5 is now complete.

4. Proof of Theorem 2

4.1 *Estimates of coefficients of operators ∇_u and $\Delta_u(t)$*

As mentioned in Section 1, the mapping $x = X_u(\xi, t)$ where X_u is defined in (1.4) transforms the operator $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ into $\nabla_u = \mathcal{A}\nabla = (\sum_{k=1}^{3} A_{kj} \frac{\partial}{\partial \xi_k})_{j=1,2,3}$, if $\mathbf{u}$ satisfies the equation $\nabla_u \cdot \mathbf{u} = 0$. The cofactor A_{kj} of $a_{kj} = \delta_{kj} + \int_0^t \frac{\partial u_k}{\partial \xi_j}\,\mathrm{d}\tau$ has the form

$$A_{kj} = \delta_{kj} + B_{kj},$$

where B_{kj} is the sum of homogeneous linear and quadratic functions of $b_{im} = \int_0^t \frac{\partial u_i}{\partial \xi_m}\,\mathrm{d}\tau$. Sometimes, in order to avoid confusion, we denote A_{kj} and B_{kj} by $A_{kj}^{(u)}$ and $B_{kj}^{(u)}$, respectively.

There holds the following evident proposition.

PROPOSITION 4.1 If $\mathbf{u}, \mathbf{w} \in \tilde{C}(0, T, C^{k+1+\alpha}(\Omega))$, $k = 0, 1, \ldots$, then

$$\sup_{\tau<t} |A_{kj}^{(u)}(\cdot, \tau) - A_{kj}^{(w)}(\cdot, \tau)|_{C^{k+\alpha}(\Omega)} \leqslant c \int_0^t |\mathbf{u} - \mathbf{w}|_{C^{k+1+\alpha}(\Omega)}\,\mathrm{d}\tau \Big(1 + \int_0^t |\mathbf{u}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)}\,\mathrm{d}\tau + \int_0^t |\mathbf{w}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)}\,\mathrm{d}\tau\Big).$$

In particular,

$$\sup_{\tau<t} |A_{kj}^{(u)}(\cdot, \tau) - \delta_{kj}|_{C^{k+\alpha}(\Omega)} \leqslant c \int_0^t |\mathbf{u}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)}\,\mathrm{d}\tau \Big(1 + \int_0^t |\mathbf{u}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)}\,\mathrm{d}t\Big).$$

Moreover,

$$\left|\frac{\partial}{\partial t} A_{kj}^{(u)}(\cdot, t)\right|_{C^{k+\alpha}(\Omega)} \leqslant c\, |\mathbf{u}(\cdot, t)|_{C^{k+1+\alpha}(\Omega)} \Big(1 + \int_0^t |\mathbf{u}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)}\,\mathrm{d}\tau\Big).$$

Let $\Gamma_u(t) = X_u(\Gamma)$ and let (η_1, η_2) be local coordinates on the submanifold $\Gamma' \subset \Gamma$. The normal $\mathbf{n}_u$ to $\Gamma'_u(t) = X_u(\Gamma')$ and the Laplace–Beltrami operator $\Delta_u(t)$ on $\Gamma'_u(t)$ are defined by

$$\mathbf{n}_u = \pm \frac{\mathbf{r}_{\eta_1} \times \mathbf{r}_{\eta_2}}{|\mathbf{r}_{\eta_1} \times \mathbf{r}_{\eta_2}|} \tag{4.1}$$

and

$$\Delta_u(t) f = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial \eta_\alpha} \frac{\widehat{g}_{\alpha\beta}}{\sqrt{g}} \frac{\partial f}{\partial \eta_\beta} \equiv \sum_{\alpha,\beta=1}^{2} h_{\alpha\beta} \frac{\partial^2 f}{\partial \xi_\alpha\, \partial \xi_\beta} + \sum_{\beta=1}^{2} h_\beta \frac{\partial f}{\partial \xi_\beta},$$

where

$$\mathbf{r}(\eta_1, \eta_2, t) = \mathbf{X}_u(\xi, t)\Big|_{\xi=\xi(\eta_1,\eta_2)} = \boldsymbol{\xi}(\eta_1, \eta_2) + \int_0^t \boldsymbol{\omega}(\eta_1, \eta_2, \tau)\,\mathrm{d}\tau, \tag{4.2}$$

$$\boldsymbol{\omega}(\eta_1, \eta_2, t) = \mathbf{u}(\xi(\eta_1, \eta_2), t),$$

$g = \det(g_{\alpha\beta})_{\alpha,\beta=1,2}$, $g_{\alpha\beta} \equiv g^{(u)}_{\alpha\beta} = \frac{\partial \mathbf{r}}{\partial \eta_\alpha} \cdot \frac{\partial \mathbf{r}}{\partial \eta_\beta}$, $\widehat{g}_{\alpha\beta}$ are elements of the associated matrix to $(g_{\alpha\beta})_{\alpha,\beta=1,2}$, i.e.

$$\widehat{g}_{11} = g_{22}, \quad \widehat{g}_{22} = g_{11}, \quad \widehat{g}_{12} = \widehat{g}_{21} = -g_{12},$$

$$h_{\alpha\beta} = \frac{\widehat{g}_{\alpha\beta}}{g}, \quad h_\beta = \frac{1}{\sqrt{g}} \sum_{\alpha=1}^{2} \frac{\partial}{\partial \xi_\alpha} \frac{\hat{g}^{\alpha\beta}}{\sqrt{g}}.$$

We set $\dot{g}_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial t}$, $\dot{h}_{\alpha\beta} = \frac{\partial h_{\alpha\beta}}{\partial t}$, $\dot{h}_\beta = \frac{\partial h_\beta}{\partial t}$ and

$$\dot{\Delta}(t) = \sum_{\alpha,\beta=1}^{2} \dot{h}_{\alpha\beta} \frac{\partial^2}{\partial \xi_\alpha \, \partial \xi_\beta} + \sum_{\beta=1}^{2} \dot{h}_\beta \frac{\partial}{\partial \xi_\beta}.$$

It follows from (4.2) that

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \frac{\partial \mathbf{r}_0}{\partial \eta_\alpha} \cdot \int_0^t \frac{\partial \boldsymbol{\omega}}{\partial \eta_\beta} \, \mathrm{d}\tau + \frac{\partial \mathbf{r}_\alpha}{\partial \eta_\beta} \cdot \int_0^t \frac{\partial \boldsymbol{\omega}}{\partial \eta_\alpha} \, \mathrm{d}\tau + \int_0^t \frac{\partial \boldsymbol{\omega}}{\partial \eta_\alpha} \, \mathrm{d}\tau \cdot \int_0^t \frac{\partial \boldsymbol{\omega}}{\partial \eta_\beta} \, \mathrm{d}\tau, \tag{4.3}$$

$$\dot{g}_{\alpha\beta} = \frac{\partial \mathbf{r}_0(\eta)}{\partial \eta_\alpha} \cdot \frac{\partial \boldsymbol{\omega}(\eta, t)}{\partial \eta_\beta} + \frac{\partial \mathbf{r}_0}{\partial \eta_\beta} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \eta_\alpha} + \frac{\partial \boldsymbol{\omega}}{\partial \eta_\alpha} \cdot \int_0^t \frac{\partial \boldsymbol{\omega}}{\partial \eta_\beta} \, \mathrm{d}\tau + \frac{\partial \boldsymbol{\omega}}{\partial \eta_\beta} \cdot \int_0^t \frac{\partial \boldsymbol{\omega}}{\partial \eta_\alpha} \, \mathrm{d}\tau,$$

$$g^{(0)}_{\alpha\beta} = \frac{\partial \mathbf{r}_0}{\partial \eta_\alpha} \cdot \frac{\partial \mathbf{r}_0}{\partial \eta_\beta}, \quad \mathbf{r}_0 = \boldsymbol{\xi}(\eta_1, \eta_2). \tag{4.4}$$

The following proposition is an easy consequence of formulas (4.1)–(4.4).

PROPOSITION 4.2 Assume that

$$\int_0^t |\mathbf{u}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)} \, \mathrm{d}\tau + \int_0^t |\mathbf{w}(\cdot, \tau)|_{C^{k+1+\alpha}(\Omega)} \, \mathrm{d}\tau \leqslant \theta, \tag{4.5}$$

where $\theta > 0$ is so small that $|\mathbf{r}_{\eta_1} \times \mathbf{r}_{\eta_2}|$ and $g(\eta, t)$ are strictly positive. Then

$$\left|\mathbf{n}_u(\cdot, t) - \mathbf{n}_w(\cdot, t)\right|_{C^{k+\alpha}(\Gamma')} + \left|g^{(u)}_{\alpha\beta}(\cdot, t) - g^{(w)}_{\alpha\beta}(\cdot, t)\right|_{C^{k+\alpha}(\Gamma')} + \left|h^{(u)}_{\alpha\beta}(\cdot, t) - h^{(w)}_{\alpha\beta}(\cdot, t)\right|_{C^{k+\alpha}(\Gamma')} \leqslant c \int_0^t |\mathbf{u} - \mathbf{w}|_{C^{k+1+\alpha}(\Gamma')} \, \mathrm{d}\tau$$

and if $k \geqslant 1$, then also

$$\left|h^{(u)}_\beta - h^{(w)}_\beta\right|_{C^{k-1+\alpha}(\Gamma')} \leqslant c \int_0^t \left|\mathbf{u}(\xi(\cdot), \tau) - \mathbf{w}(\xi(\cdot), \tau)\right|_{C^{k+1+\alpha}(\Gamma')} \mathrm{d}\tau.$$

Moreover,

$$\left|\dot{g}^{(u)}_{\alpha\beta}(\cdot, t) - \dot{g}^{(w)}_{\alpha\beta}(\cdot, t)\right|_{C^{k+\alpha}(\Gamma')} + \left|\dot{h}^{(u)}_{\alpha\beta} - \dot{h}^{(w)}_{\alpha\beta}\right|_{C^{k+\alpha}(\Gamma')} \leqslant c \left|\mathbf{u}(\xi(\cdot), t) - \mathbf{w}(\xi(\cdot), t)\right|_{C^{k+1+\alpha}(\Gamma')},$$

$$\left|\dot{h}^{(u)}_\beta - \dot{h}^{(u)}_\beta\right|_{C^{k-1+\alpha}(\Gamma')} \leqslant c \left|\mathbf{u}(\xi(\cdot), t) - \mathbf{w}(\xi(\cdot), t)\right|_{C^{k+1+\alpha}(\Gamma')}, \qquad k \geqslant 1,$$

$$\left|\frac{\partial}{\partial t} \mathbf{n}_w\right|_{C^{k+\alpha}(\Gamma')} \leqslant c \left|\mathbf{w}(\xi(\cdot), t)\right|_{C^{k+1+\alpha}(\Gamma')}.$$

COROLLARY 4.1 If $\mathbf{u}$, $\mathbf{w}$ satisfy the condition (4.5) with $k = 1$, then

$$\begin{aligned}
&\sup_{\tau<t}\Big|(\nabla_u^2 - \nabla_w^2)\,\mathbf{v}(\cdot,\tau)\Big|_{C^\alpha(\Omega)} + \sup_{\tau<t}\Big|\Pi_u\, S_u(\mathbf{v})\,\mathbf{n}_u - \Pi_w\, S_w(\mathbf{v})\,\mathbf{n}_w\Big|_{C^{1+\alpha}(\Gamma)} \\
&\qquad + \sup_{\tau<t}\Big|\mathbf{n}_u\cdot S_u(\mathbf{v})\,\mathbf{n}_u - \mathbf{n}_w\cdot S_w(\mathbf{v})\,\mathbf{n}_w\Big|_{C^{1+\alpha}(\Gamma)} + \sup_{\tau<t}\Big|\Delta_u\mathbf{v} - \Delta_w\mathbf{v}\Big|_{C^\alpha(\Gamma)} e \\
&\leqslant c\sup_{\tau<t}|\mathbf{v}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\int_0^t |\mathbf{u}(\cdot,\tau) - \mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau,
\end{aligned} \tag{4.6}$$

$$\sup_{\tau<t}|(\nabla_u - \nabla_w)\,p(\cdot,\tau)|_{C^\alpha(\Omega)} \leqslant c\sup_{\tau<t}|p(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}\int_0^t |\mathbf{u}(\cdot,\tau) - \mathbf{w}(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}\,\mathrm{d}\tau. \tag{4.7}$$

PROPOSITION 4.3 There holds the estimate

$$\begin{aligned}
\varepsilon^{\frac{1+\alpha}{2}}\sup_\Gamma[f\,\nabla u]_{(0,T)}^{(\frac{1+\alpha}{2})} &\leqslant \Big(\varepsilon\sup_{G_T}|f_t'| + 2\sup_{G_T}|f|\Big)\sup_{G_T}|\nabla u| \\
&\quad + c\sup_{G_T}|f|\,\Big(\varepsilon\sup_{t<T}|u_t(\cdot,t)|_{C^\alpha(\Omega)} + \sup_{G_T}|u(\cdot,t)|_{C^{2+\alpha}(\Omega)}\Big)
\end{aligned} \tag{4.8}$$

where $G_T = \Gamma \times (0,T)$.

Proof. Indeed, the left-hand side of (4.8) does not exceed

$$\varepsilon^{\frac{1+\alpha}{2}}\sup_\Gamma[f]_{(0,T)}^{(\frac{1+\alpha}{2})}\sup_{G_T}|\nabla u| + \sup_{G_T}|f|\,\varepsilon^{\frac{1+\alpha}{2}}\sup_\Gamma[\nabla u]_{(0,T)}^{(\frac{1+\alpha}{2})}.$$

Evaluating the Hölder constant of ∇u with respect to t by (2.38) and taking into account that

$$\Big|f(\xi,t+h) - f(\xi,t)\Big| \leqslant \Big(\int_t^{t+h}|f_\tau'(\xi,\tau)|\,\mathrm{d}\tau\Big)^{\frac{1+\alpha}{2}}\Big(2\sup_{G_T}|f|\Big)^{\frac{1-\alpha}{2}}$$

and, as a consequence,

$$\varepsilon^{\frac{1+\alpha}{2}}\sup_\Gamma[f]_{(0,T)}^{(\frac{1+\alpha}{2})} \leqslant \Big(\varepsilon\sup_{G_T}|f_t'|\Big)^{\frac{1+\alpha}{2}}\Big(2\sup_{G_T}|f|\Big)^{\frac{1-\alpha}{2}},$$

we easily obtain (4.8). The proposition is proved. □

Inequality (4.8) may be used for the estimate of the Hölder constant of $\Pi_u S_u(\mathbf{v})\,\mathbf{n}_u - \Pi_w S_w(\mathbf{v})\,\mathbf{n}_w$ with respect to t. We have:

$$\begin{aligned}
&\varepsilon^{\frac{1+\alpha}{2}}\sup_\Gamma\Big[\Pi_u\, S_u(\mathbf{v})\,\mathbf{n}_u - \Pi_w\, S_w(\mathbf{v})\,\mathbf{n}_w\Big]_{(0,t)}^{(\frac{1+\alpha}{2})} \\
&\qquad\leqslant c\Big(\int_0^t\sup_\Gamma\Big|\nabla(\mathbf{u}(\xi,\tau) - \mathbf{w}(\xi,\tau))\Big|\,\mathrm{d}\tau + \varepsilon\sup_{\tau<t}\sup_{G_t}\Big|\nabla(\mathbf{u}(\xi,\tau) - \mathbf{w}(\xi,\tau))\Big|\Big) \\
&\qquad\times\Big(\varepsilon\sup_{\tau<t}|\mathbf{v}_\tau(\cdot,\tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t}|\mathbf{v}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\Big).
\end{aligned} \tag{4.9}$$

4.2 *Quasistationary approximation*

Assume that $\Gamma \in C^{4+\alpha}$. Then, according to Theorem 1, there exists a unique solution of the problem (1.8) defined on a certain finite time interval $(0, T_0)$ and satisfying conditions (1.9) and inequality (1.10) for $\ell = 1 + \alpha$. It follows that

$$\int_0^T |\mathbf{w}(\cdot,t)|_{C^{3+\alpha}(\Omega)}\,\mathrm{d}t \leqslant T \sup_{t\in(0,T)} |\mathbf{w}(\cdot,t)|_{C^{3+\alpha}(\Omega)} \leqslant c\,T\Big(|H_0|_{C^{2+\alpha}(\Gamma)} + \sum_{k=1}^{6} |\mu_k|\Big), \quad T \leqslant T_0.$$

Let $\tau \in (0, T_0)$, $t \in (\tau, T_0)$ and

$$\mathbf{w}^{(\tau)}(x,t) = \mathbf{w}\Big(X_w^{-1}(x,\tau), t\Big), \quad x \in \Omega_\tau.$$

Clearly,

$$\frac{\partial \mathbf{w}^{(\tau)}}{\partial x_k} = \sum_{j=1}^{3} A_{jk}(\xi,\tau)\,\frac{\partial \mathbf{w}(\xi,t)}{\partial \xi_j}\bigg|_{\xi = X_w^{-1}(x,\tau)}.$$

Differentiating this formula we can easily show that $\mathbf{w}^{(\tau)} \in C^{3+\alpha}(\Omega_T)$ and that

$$\sup_{t\in(\tau,T_0)} |\mathbf{w}^{(\tau)}(\cdot,t)|_{C^{3+\alpha}(\Omega_T)} \leqslant c, \tag{4.10}$$

where c depends on T_0 and on $|H_0|_{C^{2+\alpha}(\Gamma)} + \sum_{k=1}^{6} |\mu_k|$.

We need also the estimate of the time derivative $\mathbf{w}_t(\xi, t)$.

PROPOSITION 4.4 If $\Gamma \in C^{4+\alpha}$, then $\mathbf{w}_t \in C^{1+\alpha}(\Omega)$ and

$$|\mathbf{w}_t(\cdot,t)|_{C^{\alpha}(\Omega)} \leqslant c_1, \tag{4.11}$$

where c_1 depends on the sum of norms $\sup_{\tau<t} |\mathbf{w}(\cdot,\tau)|_{C^{3+\alpha}(\Omega)} + |r(\cdot,t)|_{C^{1+\alpha}(\Omega)}$.

Proof. Differentiation of (1.8) and (1.9) with respect to t gives

$$\begin{aligned}
&-\nabla_w^2 \mathbf{w}_t + \nabla_w r_t = (\nabla_w\cdot\dot{\nabla}_w + \dot{\nabla}_w\cdot\nabla_w)\,\mathbf{w} - \dot{\nabla}_w r \equiv \mathbf{f}(\xi,t),\\
&\nabla_w\cdot\mathbf{w}_t = -\dot{\nabla}_w\cdot\mathbf{w} \equiv g(\xi,t),\\
&T_w(\mathbf{w}_t, r_t)\,\mathbf{n}_w\Big|_\Gamma = -\dot{S}_w(\mathbf{w},r)\,\mathbf{n}_w - T_w(\mathbf{w},r)\,\dot{\mathbf{n}}_w + \Big(\Delta_w(t)\,\mathbf{w} + \dot{\Delta}_w(t)\,\mathbf{X}_w\Big) \equiv \mathbf{a}(\xi,t),\\
&\int_\Omega \mathbf{w}_t(\xi,t)\cdot\boldsymbol{\varphi}_k(X_w(\xi,t))\,\mathrm{d}\xi = -\int_\Omega \mathbf{w}(\xi,t)\cdot\frac{\partial}{\partial t}\boldsymbol{\varphi}_k(X_w(\xi,t))\,\mathrm{d}\xi \equiv \nu_k,
\end{aligned} \tag{4.12}$$

where

$$\dot{\nabla}_w = \left(\sum_{k=1}^{3} \frac{\partial A_{kj}}{\partial t} \frac{\partial}{\partial \xi_j}\right)_{j=1,2,3},$$

$$\dot{\mathbf{n}}_w = \frac{\partial}{\partial t}\mathbf{n}_w(\xi, t),$$

$$\dot{S}_w(\mathbf{v}) = \left(\sum_{k=1}^{3}\left(\frac{\partial A_{kj}}{\partial t}\frac{\partial v_i}{\partial \xi_j} + \frac{\partial A_{ij}}{\partial t}\frac{\partial v_k}{\partial \xi_j}\right)\right)_{k,i=1,2,3}.$$

Clearly, $\mathbf{f} \in C^{1+\alpha}(\Omega)$, $g \in C^{2+\alpha}(\Omega)$, $\mathbf{a} \in C^{1+\alpha}(\Gamma)$. In the Eulerian coordinates, (4.12) takes the form

$$\begin{aligned} &-\nabla^2\mathbf{w}' + \nabla r' = \mathbf{f}', \quad \nabla\cdot\mathbf{w}' = g', \quad \text{in } \Omega_w(t) = X_w(\Omega),\\ &T(\mathbf{w}', r')\,\mathbf{n} = \mathbf{a}' \quad \text{on } \Gamma_w(t) = \partial\Omega_w(t),\\ &\int_{\Omega_w} \mathbf{w}'\cdot\boldsymbol{\varphi}_k(x)\,\mathrm{d}x = \nu_k, \qquad k = 1, ..., 6, \end{aligned} \tag{4.13}$$

where

$$f'(x,t) = f\left(X_w^{-1}(x,t), t\right), \quad x \in \Omega_w(t).$$

Inequality (4.11) follows from (4.10) and from the Schauder estimate for the problem (4.13). The proposition is proved. □

4.3 *Solvability of the problem (1.11)*

We proceed to the proof of Theorem 2 and consider at first a more general problem

$$\begin{aligned} &\varepsilon\,\mathbf{V}_t - \nabla_w^2\mathbf{V} + \ell_w(\mathbf{V}) + \nabla_w P = \mathbf{f},\\ &\nabla_w\cdot\mathbf{V} = g,\\ &\mathbf{V}\Big|_{t=0} = \mathbf{V}_0(\xi), \quad \xi \in \Omega,\\ &T_w(\mathbf{V}, P)\,\mathbf{n}_w\Big|_{\Gamma} = \mathbf{a}, \end{aligned} \tag{4.14}$$

on a certain small time interval $(0, t_1)$.

PROPOSITION 4.5 Assume that $\Gamma \in C^{3+\alpha}$, $\alpha \in (0,1)$, $\mathbf{f} \in \tilde{C}(0,T; C^{\alpha}(\Omega))$, $g \in \tilde{C}(0,T; C^{1+\alpha}(\Omega))$, $\mathbf{V}_0 \in C^{2+\alpha}(\Omega)$, $\mathbf{a} \in \tilde{C}(0,T; C^{1+\alpha}(\Gamma))$, $\Pi_w\mathbf{a} \in C^{1+\alpha,(1+\alpha)/2}(\Gamma\times(0,T))$, and that there are satisfied the conditions

$$\nabla\cdot\mathbf{V}_0(\xi) = g(\xi, 0), \quad \Pi_0\,\mathbf{V}_0(\xi)\Big|_{\Gamma} = \Pi_0\,\mathbf{a}(\xi, 0)$$

and

$$g_t = \nabla\cdot\mathbf{h}(\xi,t) + h_0(\xi,t),$$

where $\mathbf{h}$, h_0 have the time derivatives $\mathbf{h}_t \in \tilde{C}(0,T,C^{\alpha}(\Omega))$, $h_{0t} \in C(0,T,C(\Omega))$. There exists such $t_1 > 0$ depending on $\mathbf{w}$ (see condition (4.17)) that problem (4.14) has a unique solution

$\mathbf{V} \in \tilde{C}(0, t_1, C^{\alpha}(\Omega))$, $P \in \tilde{C}(0, t_1, C^{1+\alpha}(\Omega))$ such that $\mathbf{V}_t \in \tilde{C}(0, t_1, C^{\alpha}(\Omega))$, and it satisfies the inequality

$$Y_t(\mathbf{V}, P) \leqslant c\, N(t), \quad t \in (0, t_1), \tag{4.15}$$

where

$$Y_t(\mathbf{V}, P) = \varepsilon \sup_{\tau<t} |\mathbf{V}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\mathbf{V}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t} |P(\cdot, \tau)|_{C^\alpha(\Omega)},$$

$$N(t) = \sup_{\tau<t} |\mathbf{f}(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |g(\cdot, \tau)|_{C^{1+\alpha}(\Omega)} + \sup_{\tau<t} |\mathbf{h}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)}$$
$$+ \sup_{\tau<t} \sup_{\Omega} |h_{0\tau}(\xi, t)| + \sup_{\tau<t} |\mathbf{a}(\cdot, \tau)|_{C^{1+\alpha}(\Gamma)} + \varepsilon^{\frac{1+\alpha}{2}} \sup_{\Gamma} \Big[\Pi_w\, \mathbf{a}\Big]^{(\frac{1+\alpha}{2})}_{(0,t)}.$$

Proof. If $\mathbf{n}_w \cdot \mathbf{n}_0 > 0$ (which is the case for small t), then we can write (4.14) in the form

$$\varepsilon\, \mathbf{V}_t - \nabla^2 \mathbf{V} + \ell(\mathbf{V}) + \nabla P = M_1(\mathbf{V}, P) + \mathbf{f},$$
$$\nabla \cdot \mathbf{V} = M_2(\mathbf{V}) + g,$$
$$\mathbf{V}\Big|_{t=0} = \mathbf{V}_0,$$
$$\Pi_0\, S(\mathbf{V})\, \mathbf{n}_0|_\Gamma = M_3(\mathbf{V}) + \Pi_0\, \Pi_w\, \mathbf{a},$$
$$\mathbf{n}_0 \cdot T(\mathbf{V}, P)\mathbf{n}_0|_\Gamma = M_4(\mathbf{V}) + \mathbf{a} \cdot \mathbf{n}_w,$$

where

$$\begin{aligned} M_1(\mathbf{V}, P) &= (\nabla_w^2 - \nabla^2)\, \mathbf{V} + (\nabla - \nabla_w)\, P + \sum_{k=1}^{6} \int_\Omega \Big[\mathbf{V}(\xi', t) \cdot \boldsymbol{\varphi}_k(\xi')\, \boldsymbol{\varphi}_k(\xi) \\ &\quad - \mathbf{V}(\xi', t) \cdot \boldsymbol{\varphi}_k(X_w(\xi', t))\, \boldsymbol{\varphi}_k(X_w(\xi, t))\Big]\, d\xi', \\ M_2(\mathbf{V}) &= (\nabla - \nabla_w) \cdot \mathbf{V} = \nabla H, \quad H = (I - \mathcal{A}^{(w)})^T\, \mathbf{V}, \\ M_3(\mathbf{V}) &= \Pi_0\Big(\Pi_0\, S(\mathbf{V})\, \mathbf{n}_0 - \Pi_w\, S_w(\mathbf{V})\, \mathbf{n}_w\Big), \\ M_4(\mathbf{V}) &= \mathbf{n}_0 \cdot S(\mathbf{V})\, \mathbf{n}_0 - \mathbf{n}_w \cdot S_w(\mathbf{V})\, \mathbf{n}_w. \end{aligned} \tag{4.16}$$

and $\mathcal{A}^T$ means a transposed matrix. In virtue of (4.6), (4.7), and (4.9),

$$\sup_{\tau<t} |M_1(\mathbf{V}, P)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |M_2(\mathbf{V})|_{C^{1+\alpha}(\Omega)} + \sup_{\tau<t} |M_3(\mathbf{V})|_{C^{1+\alpha}(\Omega)} + \sup_{\tau<t} |M_4(\mathbf{V})|_{C^{1+\alpha}(\Gamma)}$$
$$+ \varepsilon^{\frac{1+\alpha}{2}} \sup_\Gamma \Big[\Pi_0\, M_3(\mathbf{V})\Big]^{(\frac{1+\alpha}{2})}_{(0,t)} + \varepsilon \sup |H_t(\mathbf{V})|_{C^\alpha(\Omega)}$$
$$\leqslant c\Big(\int_0^t |\mathbf{w}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)}\, \mathrm{d}\tau + \varepsilon \sup_{\tau<t} |\mathbf{w}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)}\Big)\, Y_t(\mathbf{V}, P).$$

Therefore, if t_1 is so small that

$$c\,c'\Big(\int_0^{t_1} |\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau + \varepsilon \sup_{\tau<t_1} |\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\Big) \leqslant \frac{1}{2}, \tag{4.17}$$

where c' is the constant in the inequality (1.25), then the solvability of the problem (4.14) and the estimate (4.15) follow from Theorem 6 and from the contraction mapping principle. The proposition is proved. □

We have constructed the solution for $t \in (0, t_1)$. To extend it to a larger time interval, we consider the same problem for $t > t_1$, taking $\mathbf{V}(\xi, t_1)$ as an initial function and introducing a new independent variable

$$\xi' = X_w(\xi, t_1) \in \Omega' \equiv X_w(\Omega, t_1).$$

Then this problem takes the form

$$\begin{aligned}
&\varepsilon\,\mathbf{V}'_t - \nabla'^2_w \mathbf{V}' + \ell'_w(\mathbf{V}') + \nabla'_w P' = \mathbf{f}',\\
&\nabla'_w \cdot \mathbf{V}' = g' \quad (\xi' \in \Omega',\ t > t_1),\\
&\mathbf{V}'\Big|_{t=t_1} = \mathbf{V}_1(\xi') \equiv \mathbf{V}(X_w^{-1}(\xi', t_1)),\\
&T'_w(\mathbf{V}', P')\,\mathbf{n}'_w\Big|_{\Gamma'} = \mathbf{a}',
\end{aligned} \tag{4.18}$$

where $F'(\xi', t) = F(X_w^{-1}(\xi', t_1), t)$ $(t \geqslant t_1)$, $\Gamma' = \partial\Omega'$, $\mathbf{n}'_w$ is the exterior normal to the surface $\Gamma_w(t)$ which is now considered as $X'_w(\Gamma', t)$, and

$$\mathbf{X}'_w(\xi', t) = \boldsymbol{\xi}' + \int_{t_1}^{t} \mathbf{w}'(\xi', \tau)\,\mathrm{d}\tau, \quad \mathbf{w}'(\xi', \tau) = \mathbf{w}(X_w^{-1}(\xi', t_1), \tau).$$

The operators ∇'_w, T'_w, ℓ'_w are expressed in terms of the transformation X'_w in the same way as ∇_w, T_w, ℓ_w are expressed in terms of X_w. Finally, it is easy to see that

$$g' = \nabla \cdot \mathbf{h}'' + h_0'$$

where

$$\mathbf{h}''_j = \sum_{j=1}^{3}\Big(\delta_{jk} + \int_0^{t_1} \frac{\partial w'_j(\xi,\tau)}{\partial \xi_k}\,\mathrm{d}\tau\Big)\Big|_{\xi=X_w^{-1}(\xi',t_1)} h'_k(\xi', t).$$

Due to uniform boundedness of the norm of $\mathbf{w}^{(\tau)}(x, t)$ (see (4.10)), problem (4.18) can be solved by the same method on the time interval $t \in (t_1, 2t_1)$, and the solution can be estimated by the norms of the data $\mathbf{f}'$, g', $\mathbf{h}''$, h'_0, $\mathbf{V}_1$, $\mathbf{a}'$ exactly as above (see (4.15)). This proves the solvability of Problem (4.14) and the estimate (4.15) for $t \in (0, 2t_1)$. Repeating this argument we can extend the solution into the whole interval $(0, T_0)$ and prove inequality (4.15) for $t \in (0, T_0)$.

(1.13) is just a particular case of (4.15).

4.4 *Proof of estimates (1.14)–(1.16)*

Problem (1.11) can be written in the Eulerian coordinates $x = X_w(\xi, t)$ in the form

$$\varepsilon\Big(\mathbf{v}_t + (\mathbf{w}\cdot\nabla)\,\mathbf{v}\Big) - \nabla^2\mathbf{v} + \ell(\mathbf{v}) + \nabla p = 0, \quad \nabla\cdot\mathbf{v} = 0, \;\; x \in \Omega_w(t),$$

$$\mathbf{v}\Big|_{t=0} = \mathbf{V}_0(x), \quad x \in \Omega_0,$$

$$T(\mathbf{v}, p)\,\mathbf{n}\Big|_{\Gamma_w(t)} = 0.$$

In order to prove (1.14), we multiply the first equation by $\mathbf{v}$ and integrate over $\Omega_w(t)$ which gives

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}(\cdot,t)\|^2_{L_2(\Omega_w(t))} + \frac{1}{2}\,\|S(\mathbf{v})\|^2_{L_2(\Omega_w(t))} + \sum_{k=1}^{6}\Big(\int_{\Omega_w(t)}\mathbf{v}\cdot\boldsymbol{\varphi}_k\,\mathrm{d}x\Big)^2 = 0. \tag{4.19}$$

By the Korn inequality,

$$\frac{1}{2}\,\|S(\mathbf{v})\|^2_{L^2(\Omega_w(t))} + \sum_{k=1}^{6}\Big(\int_{\Omega_w(t)}\mathbf{v}\cdot\boldsymbol{\varphi}_k\,\mathrm{d}x\Big)^2 \geqslant c\,|\mathbf{v}\|^2_{L_2(\Omega_w(t))}$$

with a certain positive constant c independent of t. Hence, (4.19) yields

$$\|\mathbf{v}(\cdot,t)\|^2_{L_2(\Omega_w(t))} \leqslant e^{-\frac{2c}{\varepsilon}}\,\|\mathbf{V}_0\|^2_{L_2(\Omega)}$$

which is equivalent to (1.14).

The proof of (1.15) and (1.16) is based on (1.14) and on the following two propositions.

PROPOSITION 4.6 If $\Gamma \in C^{3+\alpha}$, then the solution of Problem (1.11) satisfies the inequality

$$\varepsilon \sup_{t-\delta<\tau<t}|\mathbf{V}_\tau(\cdot,\tau)|_{C^\alpha(\Omega)} + \sup_{t-\delta<\tau<t}|\mathbf{V}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau-\delta<\tau<t}|\mathbf{P}(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}$$
$$\leqslant c\varepsilon\delta^{-1}(1+\varepsilon\,\delta^{-1})^{\frac{2\alpha+3}{4}}\sup_{t-2\delta<\tau<t}\|\mathbf{V}(\cdot,\tau)\|_{L_2(\Omega)}, \tag{4.20}$$

where $t \in (\varepsilon, T_0)$, $\delta = \min(t/4, t_1/4)$, t_1 is the same as in Proposition 4.5.

Proof. We fix $t_0 \in (0, T_0)$, $\delta_0 = \min(\frac{t_0}{4}, \frac{t_1}{4})$ and introduce the function $\omega_\lambda(t)$, $t > 0$, which is smooth, monotone, equals zero for $t < t_0 - 2\delta_0 + \frac{\lambda}{2}$, equals one for $t > t_0 - 2\delta_0 + \lambda$ ($\lambda \in (0, \delta_0)$) and satisfies the estimate

$$|\omega'_\lambda(t)| \leqslant c\,\lambda^{-1}.$$

The functions $\mathbf{U} = \mathbf{V}\omega_\lambda$, $Q = P\omega_\lambda$ satisfy the equations

$$\varepsilon\,\mathbf{U}_t + \ell_w(\mathbf{U}) - \nabla^2_w\mathbf{U} + \nabla_w Q = \varepsilon\,\mathbf{V}\,\omega'_\lambda(t),$$

$$\nabla_w\cdot\mathbf{U} = 0,$$

$$\mathbf{U}\Big|_{t=0} = 0,$$

$$T_w(\mathbf{U}, Q)\,\mathbf{n}_w\Big|_{\Gamma} = 0,$$

hence, by virtue of (4.15),

$$Y_{t_0}(\mathbf{U}, Q) \leqslant c_0\, \varepsilon\, \lambda^{-1} \sup_{t-2\delta_0+\lambda/2<\tau<t} |\mathbf{V}(\cdot, \tau)|_{C^\alpha(\Omega)}. \tag{4.21}$$

We estimate the right-hand side by the interpolation inequality

$$|\mathbf{V}(\cdot, \tau)|_{C^\alpha(\Omega)} \leqslant \mu\, |\mathbf{V}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + c\, \mu^{-\frac{2\alpha+3}{4}} \|\mathbf{V}(\cdot, \tau)\|_{L_2(\Omega)}$$

which is true for arbitrary $\mu < 1$, set

$$\mu = \frac{\chi}{1+\varepsilon\, \lambda^{-1}}, \quad \chi \in (0, 1),$$

and multiply (4.21) by $\lambda(1+\varepsilon\lambda^{-1})^{-(3/4+\alpha/2)}$. This gives

$$f(\lambda) \leqslant 2c_0\chi\, \frac{\varepsilon\, \lambda^{-1}}{1+\varepsilon\, \lambda^{-1}} \left(\frac{1+2\,\varepsilon\, \lambda^{-1}}{1+\varepsilon\, \lambda^{-1}}\right)^{\frac{3}{4}+\frac{\alpha}{2}} f\left(\frac{\lambda}{2}\right) + c\varepsilon\chi^{-\frac{\alpha}{2}-\frac{3}{4}} \sup_{t_0-2\delta_0<t<t_0} \|\mathbf{V}(\cdot, \tau)\|_{L_2(\Omega)}$$

where

$$f(\lambda) = \lambda(1+\varepsilon\, \lambda^{-1})^{-(\frac{3}{4}+\frac{\alpha}{2})}\Bigg(\varepsilon \sup_{t_0-2\delta_0+\lambda<\varepsilon<t_0} |\mathbf{V}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)}$$
$$+ \sup_{t_0-2\delta_0+\lambda<\varepsilon<t_0} |\mathbf{V}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{t_0-2\delta_0+\lambda<\varepsilon<t_0} |P(\cdot, \tau)|_{C^{1+\alpha}(\Omega)}\Bigg).$$

If

$$2^{\frac{11}{4}+\frac{\alpha}{2}} c_0\chi < 1,$$

then the last inequality implies

$$f(\lambda) \leqslant \frac{1}{2}\, f\left(\frac{\lambda}{2}\right) + \varepsilon c_0\chi^{-\frac{\alpha}{2}-\frac{3}{4}} \sup_{t_0-2\delta_0<\tau<t_0} \|\mathbf{V}(\cdot, \tau)\|_{L_2(\Omega)},$$

and, since $f(0) = 0$,

$$f(\lambda) \leqslant 2c_0\varepsilon\chi^{-\frac{\alpha}{2}-\frac{3}{4}} \sup_{t_0-2\delta_0<\tau<t_0} \|\mathbf{V}(\cdot, \tau)\|_{L_2(\Omega)},$$

or

$$\varepsilon \sup_{t_0-2\delta_0+\lambda<\tau<t_0} |\mathbf{V}_t(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{t_0-2\delta_0+\lambda<\tau<t_0} |\mathbf{V}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} +$$
$$+ \sup_{t_0-2\delta_0+\lambda<\tau<t_0} |P(\cdot, \tau)|_{C^{1+\alpha}(\Omega)}$$
$$\leqslant 2c_0\varepsilon\lambda^{-1}(1+\varepsilon\, \lambda^{-1})^{\frac{2\alpha+3}{4}} \sup_{t_0-2\delta_0<\tau<t_0} \|\mathbf{V}(\cdot, \tau)\|_{L_2(\Omega)}.$$

Setting here $\lambda = \delta_0$, we arrive at (4.20) for arbitrary $t_0 < T_0$. The proposition is proved. □

PROPOSITION 4.7 If $\Gamma \in C^{4+\alpha}$, $\mathbf{V}_0 \in C^{3+\alpha}(\Omega)$ and conditions

$$\nabla \cdot \mathbf{V}_0 = 0, \quad \Pi_0\, S(\mathbf{V}_0)\mathbf{n}_0\Big|_\Gamma = 0$$

hold, then the solution of the problem (1.11) satisfies the inequalities

$$\sup_{\tau<t} |\mathbf{V}(\cdot,\tau)|_{C^{3+\alpha}(\Gamma)} \leqslant c\, |\mathbf{V}_0|_{C^{3+\alpha}(\Omega)}, \tag{4.22}$$

$$\sup_{t-\delta<\tau<t} |\mathbf{V}(\cdot,\tau)|_{C^{3+\alpha}(\Gamma)} \leqslant c\varepsilon\delta^{-1}(1+\varepsilon\,\delta^{-1})^{\frac{2\alpha+7}{4}} \sup_{t-2\delta<\tau<t} \|\mathbf{V}(\cdot,\tau)\|_{L_2(\Omega)}, \tag{4.23}$$

where $\delta(t)$ is the same as in the Proposition 4.6: $\delta = \min(\frac{t}{4}, \frac{t_1}{4})$.

Proof. We obtain stronger inequalities than (4.22) and (4.23), namely, we estimate in the whole domain Ω the second derivatives of the functions

$$\mathbf{V}^i = \partial_i \mathbf{V}, \quad P^i = \partial_i P,$$

where

$$\partial_i = \frac{\partial}{\partial \xi_i} - N_i(\xi) \sum_{k=1}^{3} N_k(\xi)\, \frac{\partial}{\partial \xi_k}$$

and $N_i \in C^{3+\alpha}(\Omega)$ is an extension of $n_{0_i} \in C^{3+\alpha}(\Gamma)$ from Γ into Ω. For $\xi \in \Gamma$, ∂_i are the tangential components of the gradient, therefore $\partial_i f\Big|_\Gamma$ depend only on $f\Big|_\Gamma$ and the operators ∂_i can be applied to functions defined on Γ.

Applying ∂_i to the equations (1.11) we see that $\mathbf{V}^i$, P^i are solutions to the problems

$$\begin{aligned}
&\varepsilon\, \mathbf{V}^i_t - \nabla^2_w \mathbf{V}^i + \ell_w(\mathbf{V}^i) + \nabla_w P^i = Q^i_1(\mathbf{V}, P),\\
&\nabla_w \cdot \mathbf{V}^i = Q^i_2(\mathbf{V}),\\
&\mathbf{V}^i\Big|_{t=0} = \mathbf{V}^i_0,\\
&\Pi_w\, S_w(\mathbf{V}^i)\, \mathbf{n}_w = Q^i_3(\mathbf{V}),\\
&\mathbf{n}_w \cdot T_w(\mathbf{V}^i)\, \mathbf{n}_w = Q^i_4(\mathbf{V}),
\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
&Q^i_1(\mathbf{V}, P) = (\ell_w(\mathbf{V}^i) - \partial_i \ell_w(\mathbf{V})) + [\partial_i, \nabla^2_w]\, \mathbf{V} - [\partial_i, \nabla_w]\, P,\\
&Q^i_2(\mathbf{V}) = -[\partial_i, \nabla_w] \cdot \mathbf{V},\\
&Q^i_3(\mathbf{V}) = \Pi_w\Big(\Pi_w\, S_w(\partial_i \mathbf{V})\, \mathbf{n}_w - \partial_i(\Pi_w\, S_w(\mathbf{V})\, \mathbf{n}_w)\Big),\\
&Q^i_4(\mathbf{V}) = \mathbf{n}_w \cdot S_w(\partial_i \mathbf{V})\, \mathbf{n}_w - \partial_i(\mathbf{n}_w \cdot S_w(\mathbf{V})\, \mathbf{n}_w).
\end{aligned}$$

and by $[\partial_i, \nabla_w]$ we mean the commutator

$$\partial_i \nabla_w - \nabla_w \partial_i = (\partial_i A^{(w)})\, \nabla + \sum_{k=1}^{3} \nabla_w(N_i\, N_k)\, \frac{\partial}{\partial \xi_k} \equiv \left(\sum_{k=1}^{3} b^i_{mk}\, \frac{\partial}{\partial \xi_k}\right)_{m=1,2,3}.$$

Moreover, we have

$$Q_2^i(\mathbf{V}) = \nabla \cdot \mathbf{R}^i + R_0^i,$$

$$R_k^i = \sum_{m=1}^{3} b_{mk}^i V_m, \quad R_0^i = -\sum_{m=1}^{3} \frac{\partial b_{mk}^i}{\partial \xi_k} V_m.$$

It is easy to see that the coefficients of Q_j^i have the same differentiability properties as the coefficients of M_j and that

$$\begin{aligned} &\sup_{\tau<t} |Q_1^i(\mathbf{V}, P)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |Q_2^i(\mathbf{V})|_{C^{1+\alpha}(\Omega)} + \varepsilon \sup_{\tau<t} |\mathbf{R}_t^i(\cdot, \tau)|_{C^\alpha(\Omega)} \\ &\quad + \varepsilon \sup_{\tau<t} \sup_{\Omega} |R_{0t}^i(\xi, \tau)| + \sup_{\tau<t} |Q_3^i(\mathbf{V})|_{C^{1+\alpha}(\Gamma)} + \sup_{\tau<t} |Q_4^i(\mathbf{V})|_{C^{1+\alpha}(\Gamma)} + \varepsilon^{\frac{1+\alpha}{2}} \sup_{\Gamma} [Q_3^i(\mathbf{V})]_{(0,t)}^{(\frac{1+\alpha}{2})} \\ &\quad \leqslant c\, Y_t(\mathbf{V}, P). \end{aligned} \tag{4.25}$$

Compatibility conditions $Q_2^i(\mathbf{V}_0^i)\Big|_{t=0} = \nabla \cdot \mathbf{V}_0^i$, $Q_3^i(\mathbf{V}_0^i)\Big|_{t=0} = \Pi_0 S(\mathbf{V}_0^i)\, \mathbf{n}_0\Big|_{\Gamma}$ are easily verified. Hence, in virtue of inequality (4.15),

$$Y_t(\mathbf{V}^i, P^i) \leqslant c\Big(Y_t(\mathbf{V}, P) + |\mathbf{V}_0^i|_{C^{2+\alpha}(\Omega)}\Big) \leqslant c|\mathbf{V}_0|_{C^{3+\alpha}(\Omega)}.$$

This implies (4.22):

$$\sup_{\tau<t} |\mathbf{V}(\cdot, \tau)|_{C^{3+\alpha}(\Gamma)} \leqslant c \sum_{i=1}^{3} \sup_{\tau<t} |\mathbf{V}^i(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} \leqslant c\, |\mathbf{V}_0|_{C^{3+\alpha}(\Omega)}.$$

In order to obtain (4.23), we proceed as in the proof of Proposition 5.6. We introduce the functions $\mathbf{U}^i = V^i \omega_\delta$, $Q^i = P^i \omega_\delta$ which satisfy the equations

$$\begin{aligned} &\varepsilon \mathbf{U}_t^i + \ell_w(\mathbf{U}^i) - \nabla_w^2 \mathbf{U}^i + \nabla_w Q^i = \varepsilon \mathbf{V}^i \omega_\delta' + Q_1^i(\mathbf{V}\, \omega_\delta, P\, \omega_\delta), \\ &\nabla_w \cdot \mathbf{U}^i = Q_2^i(\mathbf{V}\, \omega_\delta), \\ &\mathbf{U}^i\Big|_{t=0} = 0, \\ &\Pi_w\, S_w(\mathbf{U}^i)\, \mathbf{n}_w = Q_3^i(\mathbf{V}\, \omega_\delta), \\ &\mathbf{n}_w \cdot T(\mathbf{U}^i)\, \mathbf{n}_w = Q_4^i(\mathbf{V}\, \omega_\delta). \end{aligned}$$

By virtue of (4.15) and (4.25),

$$\begin{aligned} Y_{t_0}(\mathbf{U}^i, Q^i) &\leqslant c\Big(Y_{t_0}(\mathbf{V}\, \omega_\delta, P\, \omega_\delta) + \varepsilon\, \delta^{-1} \sup_{t_0-\delta<t<t_0} |\mathbf{V}^i(\xi, t)|_{C^\alpha(\Omega)}\Big) \\ &\leqslant c\Big(\varepsilon \sup_{t_0-3\delta/2<\tau<t_0} |\mathbf{V}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{t_0-3\delta/2<\tau<t_0} |\mathbf{V}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{t_0-3\delta/2<\tau<t_0} |P(\cdot, \tau)|_{C^{1+\alpha}(\Omega)}\Big) \\ &\quad + c\, \varepsilon\, \delta^{-1} \sup_{t_0-3\delta/2<\tau<t_0} |\mathbf{V}(\xi, t)|_{C^{1+\alpha}(\Omega)}. \end{aligned}$$

The right-hand side may be estimated by inequality (4.20) which gives

$$Y_{t_0}(\mathbf{U}^i, Q^i) \leqslant c\varepsilon\delta^{-1}(1+\varepsilon\,\delta^{-1})^{\frac{2\alpha+7}{4}} \sup_{t_0-2\delta<\tau<t} \|\mathbf{V}(\cdot,\tau)\|_{L_2(\Omega)}.$$

The proposition is proved. □

COROLLARY 4.2 If $\Gamma \in C^{3+\alpha}$, $\mathbf{V}_0 \in C^{2+\alpha}(\Omega)$, then

$$\int_0^{T_0} |\mathbf{V}|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}t + \int_0^{T_0} |P|_{C^{1+\alpha}(\Omega)}\,\mathrm{d}t \leqslant c\,\varepsilon. \tag{4.26}$$

If $\Gamma \in C^{4+\alpha}$, $\mathbf{V}_0 \in C^{3+\alpha}(\Omega)$, then

$$\int_0^{T_0} |\mathbf{V}|_{C^{3+\alpha}(\Gamma)}\,\mathrm{d}t \leqslant c\,\varepsilon. \tag{4.27}$$

Indeed, making use of (1.13) and (4.20) for $t < \varepsilon$ and $t > \varepsilon$, respectively, we obtain

$$\begin{aligned}
&\int_0^{\varepsilon}\Big(|\mathbf{V}|_{C^{2+\alpha}(\Omega)} + |P|_{C^{1+\alpha}(\Omega)}\Big)\,\mathrm{d}t + \int_{\varepsilon}^{T_0}\Big(|\mathbf{V}|_{C^{2+\alpha}(\Omega)} + |P|_{C^{1+\alpha}(\Omega)}\Big)\,\mathrm{d}t \\
&\qquad \leqslant c\bigg(\varepsilon|\mathbf{V}_0|_{C^{2+\alpha}(\Omega)} + \|\mathbf{V}_0\|_{L_2(\Omega)}\int_{\varepsilon}^{T_0} \varepsilon\delta^{-1}(t)(1+\varepsilon\,\delta^{-1}(t))^{\frac{2\alpha+3}{4}}\,e^{-\frac{b\,t}{\varepsilon}}\,\mathrm{d}t\bigg) \\
&\qquad \leqslant c\,\varepsilon\,|\mathbf{V}_0|_{C^{2+\alpha}(\Omega)}.
\end{aligned}$$

The proof of (4.27) is similar.

5. Proof of Theorems 3 and 4

5.1 *Proof of Theorem 3*

At first we consider the problem (1.12) with a modified boundary condition, namely,

$$\begin{aligned}
&\varepsilon\,\boldsymbol{\eta}_t - \nabla_w^2\boldsymbol{\eta} + \ell_w(\boldsymbol{\eta}) + \nabla_w\pi = \mathbf{f}, \\
&\nabla_w\cdot\boldsymbol{\eta} = g, \\
&\boldsymbol{\eta}\Big|_{t=0} = 0, \\
&\Pi_w\,S_w(\boldsymbol{\eta})\,\mathbf{n}_w\Big|_{\Gamma} = \mathbf{b}, \\
&\mathbf{n}_w\cdot T_w(\boldsymbol{\eta},\pi)\,\mathbf{n}_w - \int_0^t \mathbf{n}_w(\xi,\tau)\cdot\Delta_w(\tau)\,\boldsymbol{\eta}\,\mathrm{d}\tau\Big|_{\Gamma} = b + \int_0^t B\,\mathrm{d}\tau
\end{aligned} \tag{5.1}$$

and prove its solvability on a certain small but fixed time interval $(0, t_2)$. We write (5.1) in the form

$$\begin{aligned}
&\varepsilon\,\boldsymbol{\eta}_t - \nabla^2\boldsymbol{\eta} + \ell(\boldsymbol{\eta}) + \nabla\pi = \mathbf{f} + M_1(\boldsymbol{\eta}, \pi),\\
&\nabla\cdot\boldsymbol{\eta} = g + M_2(\boldsymbol{\eta}),\\
&\boldsymbol{\eta}\Big|_{t=0} = 0,\\
&\Pi_0\, S(\boldsymbol{\eta})\,\mathbf{n}_0\Big|_\Gamma = \Pi_0\,\mathbf{b} + M_3(\boldsymbol{\eta}),\\
&\mathbf{n}_0\cdot T(\boldsymbol{\eta}, \pi)\,\mathbf{n}_0 - \mathbf{n}_0\cdot\Delta_0\int_0^t \boldsymbol{\eta}\,d\boldsymbol{\eta}\Big|_\Gamma = b + \int_0^t B\,\mathrm{d}\tau + M_4(\boldsymbol{\eta}) + \int_0^t M_5(\boldsymbol{\eta})\,\mathrm{d}\tau,
\end{aligned}\tag{5.2}$$

where $M_1 - M_4$ are defined above in (4.16), and

$$M_5(\boldsymbol{\eta}) = (\mathbf{n}_0\cdot\Delta_0 - \mathbf{n}_w\cdot\Delta_w(t))\,\boldsymbol{\eta}.$$

Problems (5.1) and (5.2) are equivalent if $\mathbf{n}_w\cdot\mathbf{n}_0 > 0$ which holds for small t.

By virtue of (4.6)–(4.8), there holds the estimate

$$\begin{aligned}
&\sup_{\tau<t}|M_1(\boldsymbol{\eta},\pi)|_{C^\alpha(\Omega)} + \sup_{\tau<t}|M_2(\boldsymbol{\eta})|_{C^{1+\alpha}(\Omega)} + \varepsilon\sup_{\tau<t}|H_t(\boldsymbol{\eta})|_{C^\alpha(\Omega)} + \sup_{\tau<t}|M_3(\boldsymbol{\eta})|_{C^{1+\alpha}(\Gamma)}\\
&\qquad + \varepsilon^{\frac{1+\alpha}{2}}\sup_\Gamma[M_3(\boldsymbol{\eta})]^{(\frac{1+\alpha}{2})}_{(0,t)} + \sup_{\tau<t}|M_4(\boldsymbol{\eta})|_{C^{1+\alpha}(\Gamma)} + \sup_{\tau<t}|M_5(\boldsymbol{\eta})|_{C^\alpha(\Gamma)}\\
&\leqslant c_1\Big(\int_0^t|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau + \varepsilon\sup_{\tau<t}|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\Big)\Big(\varepsilon\sup_{\tau<t}|\boldsymbol{\eta}_\tau(\cdot,\tau)|_{C^\alpha(\Omega)}\\
&\qquad + \sup_{\tau<t}|\boldsymbol{\eta}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t}|\pi(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}\Big).
\end{aligned}$$

Therefore, if t_2 is so small that

$$c_1\,c_2\Big(\int_0^{t_2}|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau + \varepsilon\sup_{\tau<t_2}|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\Big) \leqslant \frac{1}{2},\tag{5.3}$$

where c_2 is a constant in the inequality (1.23), then the solvability ot the problem (5.1) can be established by the method of successive approximations or deduced from the contraction mapping principle. Inequality (1.23) and condition (5.3) imply (1.17) for $t < t_2$.

To extend the solution outside the interval $(0, t_2)$ we introduce a smooth monotone function $\chi_1(t)$ such that $\chi_1(t) = 1$ for $t < t_2/3$, $\chi_1(t) = 0$ for $t_1 > 2\,t_2/3$, and we set $\chi_2(t) = 1 - \chi_1(t)$. It can be easily verified that $\boldsymbol{\eta}\,\chi_i \equiv \boldsymbol{\eta}_i$, $\pi\,\chi_i \equiv \pi_i$ satisfy the equations

$$\varepsilon\,\boldsymbol{\eta}_{it} - \nabla_w^2\boldsymbol{\eta}_i + \ell_w(\boldsymbol{\eta}_i) + \nabla_w\pi_i = \mathbf{f}\,\chi_i + \varepsilon\,\boldsymbol{\eta}\,\chi_i',\tag{5.4}$$

$$\nabla_w\cdot\boldsymbol{\eta}_i = g\,\chi_i,\tag{5.5}$$

$$\boldsymbol{\eta}_i\Big|_{t=0} = 0,$$

$$\Pi_w\,S_w(\boldsymbol{\eta}_i)\,\mathbf{n}_w\Big|_\Gamma = \mathbf{b}\,\chi_i,\tag{5.6}$$

$$\mathbf{n}_w \cdot T_w(\boldsymbol{\eta}_i, \pi_i)\, \mathbf{n}_w - \int_0^t \mathbf{n}_w \cdot \Delta_w(\tau)\, \boldsymbol{\eta}_i \, \mathrm{d}\tau\Big|_\Gamma$$
$$= \int_0^t \Big(\mathbf{n}_w \cdot T_w(\boldsymbol{\eta}, \pi)\, \mathbf{n}_w\Big|_\Gamma - b\Big)\, \chi_i' \, \mathrm{d}\tau + \int_0^t B\, \chi_i \, \mathrm{d}\tau + b\, \chi_i. \tag{5.7}$$

The functions $\boldsymbol{\eta}_2, \pi_2$ vanish for $t \in (0, t_2/3)$, so $\boldsymbol{\eta}_2$ satisfies the condition

$$\boldsymbol{\eta}_2\Big|_{t=t_2/3} = 0 \tag{5.8}$$

hence, integration in (5.7) in the case $i = 2$ is carried out in the limits $\tau \in (t_2/3, t)$ (in the first integral of the right-hand side, in the limits $\tau \in (t_2/3, \min(t, 2t_2/3))$). In order to show that the solution $(\boldsymbol{\eta}_2, \pi_2)$ of problem (5.4)–(5.8), $i = 2$, is defined for $t \in (t_2/3, 4t_2/3)$, we make the change of variables

$$\xi' = X_w(\xi, t_2/3) \equiv X_w(\xi)$$

and write this problem in the form

$$\begin{aligned}
&\varepsilon\, \boldsymbol{\eta}_{2t}' - \nabla_w'^2 \boldsymbol{\eta}_2' + \ell_w'(\boldsymbol{\eta}_2') + \nabla_w' \pi_2' = \mathbf{f}_2,\\
&\nabla_w' \cdot \boldsymbol{\eta}_2' = g_2, \quad \xi' \in \Omega_{t_1/3} \equiv \Omega', \quad t > t_1/3,\\
&\boldsymbol{\eta}_2'\Big|_{t=t_1/3} = 0,\\
&\Pi_w'\, S_w'(\boldsymbol{\eta}_2')\, \mathbf{n}_w'\Big|_{\Gamma'} = \mathbf{b}_2,\\
&\mathbf{n}_w' \cdot T_w'(\boldsymbol{\eta}_2', \pi_2')\, \mathbf{n}_w' - \int_{t_1/3}^t \mathbf{n}_w' \cdot \Delta_w'(\tau)\, \boldsymbol{\eta}_2' \, \mathrm{d}\tau\Big|_{\Gamma'} = b_2 + \int_{t_1/3}^t B_2 \, \mathrm{d}\tau,
\end{aligned} \tag{5.9}$$

(cf. (4.18)) where

$$\Gamma' = \partial\Omega', \quad \boldsymbol{\eta}_2' = \boldsymbol{\eta}_2 \circ X_w^{-1}, \quad \pi_2' = \pi_2 \circ X_w^{-1}, \quad \mathbf{f}_2 = (\mathbf{f}\, \chi_2 + \varepsilon\, \boldsymbol{\eta}\, \chi_2') \circ X_w^{-1},$$
$$g_2 = g\, \chi_2 \circ X_w^{-1}, \quad \mathbf{b}_2 = \mathbf{b}\, \chi_2 \circ X_w^{-1}, \quad B_2 = B\, \chi_2 \circ X_w^{-1},$$
$$b_2 = \Big[\int_{t_1/3}^t \Big(\mathbf{n}_w \cdot T_w(\boldsymbol{\eta}, \pi)\, \mathbf{n}_w\Big|_\Gamma - b\Big)\, \chi_2' \, \mathrm{d}\tau + b\, \chi_2\Big] \circ X_w^{-1}.$$

Problem (5.9) has exactly the same form as (5.1), so by virtue of uniform estimate (4.10) it is solvable in the interval $t \in (t_2/3, 4t_2/3)$ of the length t_2 and the solution satisfies the estimate of the type (1.17):

$$\varepsilon \sup_{\tau\in(t_2/3,t)} |\boldsymbol{\eta}_{2\tau}'(\cdot, \tau)|_{C^\alpha(\Omega')} + \sup_{\tau\in(t_2/3,t)} |\boldsymbol{\eta}_2'(\cdot, \tau)|_{C^{2+\alpha}(\Omega')} + \sup_{\tau\in(t_2/3,t)} |\pi_2'(\cdot, \tau)|_{C^{1+\alpha}(\Omega')} \leqslant$$
$$c\Big(\sup_{\tau\in(t_2/3,t)} |\mathbf{f}_2(\cdot, \tau)|_{C^\alpha(\Omega')} + \sup_{\tau\in(t_2/3,t)} |g_2(\cdot, \tau)|_{C^{1+\alpha}(\Omega')} + \sup_{\tau\in(t_2/3,t)} |\mathbf{h}_{2\tau}(\cdot, \tau)|_{C^\alpha(\Omega')}$$
$$+ \sup_{\tau\in(t_2/3,t)} \sup_{\Omega'} |h_{20\tau}(\xi', \tau)| + \sup_{\tau\in(t_2/3,t)} |\mathbf{b}_2(\cdot, \tau)|_{C^{1+\alpha}(\Gamma')} + \varepsilon^{\frac{1+\alpha}{2}} \sup_{\Gamma'} [\mathbf{b}_2]_{(t_2/3,t)}^{(\frac{1+\alpha}{2})}$$

$$+ \sup_{\tau\in(t_2/3),t} |b_2(\cdot,\tau)|_{C^{1+\alpha}(\Gamma')} + \sup_{\tau\in(t_2/3,t)} |B(\cdot,\tau)|_{C^{\alpha}(\Gamma')}\Big), \qquad t \in (t_2/3, 4t_2/3), \tag{5.10}$$

where $\mathbf{h}_2 = \mathbf{h}\,\chi_2 \circ X_w^{-1}$, $h_{02} = h_0\,\chi_2 \circ X_w^{-1}$. Clearly, $\boldsymbol{\eta} = \boldsymbol{\eta}\,\chi_1 + \boldsymbol{\eta}_2$, $\pi = \pi\,\chi_1 + \pi_2$ is a solution of (5.1) in the interval $(0, 4t_2/3)$.

Estimate (1.17) for $t \in (0, 4t_2/3)$ follows from (5.10) and from (1.17), already proved for $t < t_2$. This argument can be repeated till the solution of (5.1) is extended into the interval $t \in (0, T_0)$.

Now, we consider the problem (1.12). The last boundary condition can be written in the form

$$\mathbf{n}_w \cdot T_w(\boldsymbol{\eta},\pi)\,\mathbf{n}_w - \int_0^t \mathbf{n}_w(\xi,\tau)\cdot \Delta_w(\tau)\boldsymbol{\eta}(\xi,\tau)\,\mathrm{d}\tau\Big|_{\Gamma} = b + \int_0^t B\,\mathrm{d}\tau + \int_0^t Q(\boldsymbol{\eta})\,\mathrm{d}\tau,$$

where

$$Q(\boldsymbol{\eta}) = \Big[\dot{\mathbf{n}}_w(\xi,t)\cdot \Delta_w(t) + \mathbf{n}_w(\xi,t)\cdot \dot{\Delta}_w(t)\Big]\int_0^t \boldsymbol{\eta}(\xi,\tau)\,\mathrm{d}\tau.$$

Since

$$\sup_{\tau<t} |Q(\boldsymbol{\eta})|_{C^{\alpha}(\Gamma)} \leqslant c\int_0^t |\boldsymbol{\eta}(\cdot,\tau)|_{C^{2+\alpha}(\Gamma)}\,\mathrm{d}\tau,$$

we can use the theorem on the solvability of problem (5.1) which has been just proved to obtain the solution of (1.12) by the method of successive approximations.

This completes the proof of Theorem 5.

5.2 *Proof of Theorem 4*

We find the solution of problem (1.19) by the method of successive approximations. At zero approximation, we take $\boldsymbol{\eta}_0 = 0$, $\pi_0 = 0$ and we define $\boldsymbol{\eta}_{m+1}$, π_{m+1} $m = 0, 1, \ldots$ as solutions to the problems

$$\begin{aligned}
&\varepsilon(\boldsymbol{\eta}_{m+1})_t - \nabla_w^2\boldsymbol{\eta}_{m+1} + \ell_w(\boldsymbol{\eta}_{m+1}) + \nabla_w\pi_{m+1} = -\varepsilon\mathbf{w}_t + \mathcal{L}_1(\boldsymbol{\eta}_m,\pi_m),\\
&\nabla_w\cdot\boldsymbol{\eta}_{m+1} = \mathcal{L}_2(\boldsymbol{\eta}_m),\\
&\boldsymbol{\eta}_{m+1}\Big|_{t=0} = 0,\\
&\Pi_w\,S_w(\boldsymbol{\eta}_{m+1})\,\mathbf{n}_w\Big|_{\Gamma} = \mathcal{L}_3(\boldsymbol{\eta}_m),\\
&\mathbf{n}_w\cdot T_w(\boldsymbol{\eta}_{m+1},\pi_{m+1})\,\mathbf{n}_w - \mathbf{n}_w\cdot\Delta_w(X_{m+1}-X_w)\Big|_{\Gamma} = \mathcal{L}(\boldsymbol{\eta}_m,\pi_m),
\end{aligned}$$

where $\mathcal{L}_i$, $\mathcal{L}$ are defined in Section 1, $X_m \equiv X_{u_m}$. For $m=0$ we have

$$\varepsilon\,\boldsymbol{\eta}_{1t} - \nabla_w^2\boldsymbol{\eta}_1 + \ell_w(\boldsymbol{\eta}_1) + \nabla_w\pi_1 = -\varepsilon\mathbf{w}_t + (\nabla_{u_0}^2 - \nabla_w^2)\,\mathbf{u}_0 - (\nabla_{u_0} - \nabla_w)\,q_0$$
$$+\,\ell_w(\mathbf{u}_0) - \ell_{u_0}(\mathbf{u}_0) + \sum_{k=1}^{6}\mu_k\Big(\boldsymbol{\varphi}_k(X_{u_0}(\xi,t)) - \boldsymbol{\varphi}_k(X_w(\xi,t))\Big) \equiv \mathbf{f}_0(\xi,t),$$

$$\nabla_w\cdot\boldsymbol{\eta}_1 = (\nabla_w - \nabla_{u_0})\cdot\mathbf{u}_0 \equiv g_0(\xi,t),$$

$$\boldsymbol{\eta}_1|_{t=0} = 0,$$

$$\varPi_w\,S(\boldsymbol{\eta}_1)\,\mathbf{n}_w\Big|_\varGamma = \varPi_w\Big(\varPi_w\,S_w(\mathbf{u}_0)\,\mathbf{n}_w - \varPi_{u_0}\,S_{u_0}(\mathbf{u}_0)\mathbf{n}_{u_0}\Big)\Big|_\varGamma \equiv \mathbf{b}_0(\xi,t),$$

$$\mathbf{n}_w\cdot T_w(\boldsymbol{\eta}_1,\pi_1)\,\mathbf{n}_w - \mathbf{n}_w\cdot\varDelta_w(t)\int_0^t\boldsymbol{\eta}_1\,\mathrm{d}\tau\Big|_\varGamma$$
$$= \mathbf{n}_w\cdot\Big(T_w(\mathbf{u}_0,q_0)\,\mathbf{n}_w - T_{u_0}(\mathbf{u}_0,q_0)\,\mathbf{n}_{u_0}\Big)\Big|_\varGamma + \mathbf{n}_w\cdot\Big(\varDelta_{u_0}(t) - \varDelta_w(t)\Big)\mathbf{X}_{u_0}$$
$$+\,\mathbf{n}_w\cdot\varDelta_w(t)\int_0^t\mathbf{V}(\xi,t)\,\mathrm{d}\tau\Big|_\varGamma \equiv b_0(\xi,t)$$

where $\mathbf{u}_0 = \mathbf{w} + \mathbf{V}$, $q_0 = r + P$.

We observe that

$$g_0 = \nabla\cdot\mathbf{h}_0,\quad \mathbf{h}_0 = (\mathcal{A}^{(w)} - \mathcal{A}^{(u_0)})^T\,\mathbf{u}_0.$$

By virtue of (4.6)–(4.8), we have

$$\sup_{t<T_0}|\mathbf{f}_0(\cdot,t)|_{C^\alpha(\varOmega)} + \sup_{t<T_0}|g_0(\cdot,t)|_{C^{1+\alpha}(\varOmega)} + \sup_{t<T_0}|\mathbf{h}_{0t}(\cdot,t)|_{C^\alpha(\varOmega)}$$
$$+ \sup_{\tau<T_0}|\mathbf{b}_0(\cdot,t)|_{C^{1+\alpha}(\varGamma)} + \sup_{\tau<T_0}|b_0(\cdot,t)|_{C^{1+\alpha}(\varGamma)} + \varepsilon^{\frac{1+\alpha}{2}}\sup_\varGamma[\mathbf{b}_0]^{(\frac{1+\alpha}{2})}_{(0,T_0)} \leqslant c\,\varepsilon,$$

hence, making use of the inequality (1.17), we obtain

$$\varepsilon\sup_{t<T_0}|\boldsymbol{\eta}_{1t}(\cdot,t)|_{C^\alpha(\varOmega)} + \sup_{t<T_0}|\boldsymbol{\eta}_1(\cdot,t)|_{C^{2+\alpha}(\varOmega)} + \sup_{t<T_0}|\pi_1(\cdot,t)|_{C^{1+\alpha}(\varOmega)} \leqslant c\,\varepsilon.$$

The functions $\boldsymbol{\psi}_{m+1} = \boldsymbol{\eta}_{m+1} - \boldsymbol{\eta}_m$, $\rho_{m+1} = \pi_{m+1} - \pi_m$ satisfy the equations

$$\begin{aligned}
&\varepsilon(\boldsymbol{\psi}_{m+1})_t - \nabla_w^2\boldsymbol{\psi}_{m+1} + \ell_w(\boldsymbol{\psi}_{m+1}) + \nabla_w\rho_{m+1} = \mathcal{L}_1(\boldsymbol{\eta}_m,\pi_m) - \mathcal{L}_1(\boldsymbol{\eta}_{m-1},\pi_{m-1}),\\
&\nabla_w\cdot\boldsymbol{\psi}_{m+1} = \mathcal{L}_2(\boldsymbol{\eta}_m) - \mathcal{L}_2(\boldsymbol{\eta}_{m-1}),\\
&\boldsymbol{\psi}_{m+1}\Big|_{t=0} = 0,\\
&\varPi_w\,S_w(\boldsymbol{\psi}_{m+1})\,\mathbf{n}_w\Big|_\varGamma = \mathcal{L}_3(\boldsymbol{\eta}_m) - \mathcal{L}_3(\boldsymbol{\eta}_{m-1}),\\
&\mathbf{n}_w\cdot T_w(\boldsymbol{\psi}_{m+1},\rho_m)\,\mathbf{n}_w - \mathbf{n}_w\cdot\varDelta_w(t)\int_0^t\boldsymbol{\psi}_{m+1}\,\mathrm{d}\tau\Big|_\varGamma\\
&\qquad\qquad = \mathcal{L}(\boldsymbol{\eta}_m,\pi_m) - \mathcal{L}(\boldsymbol{\eta}_{m-1},\pi_{m-1}).
\end{aligned}\tag{5.11}$$

Setting $\nabla_{u_m} \equiv \nabla_m$, $X_{u_m} \equiv X_m$, $\Pi_{u_m} \equiv \Pi_m$, $S_{u_m} \equiv S_m$, $\mathbf{n}_{u_m} \equiv \mathbf{n}_m$ we easily show that

$$\mathcal{L}_1(\boldsymbol{\eta}_m, \pi_m) - \mathcal{L}_1(\boldsymbol{\eta}_{m-1}, \pi_{m-1}) = (\nabla_m^2 - \nabla_w^2)\,\boldsymbol{\psi}_m - (\nabla_m - \nabla_w)\,\rho_m$$
$$+ (\ell_w(\boldsymbol{\psi}_m) - \ell_{u_m}(\boldsymbol{\psi}_m)) + (\ell_{u_{m-1}}(\mathbf{u}_{m-1}) - \ell_{u_m}(\mathbf{u}_{m-1}))$$
$$+ (\nabla_m^2 - \nabla_{m-1}^2)\,\mathbf{u}_{m-1} - (\nabla_m - \nabla_{m-1})\,q_{m-1} + \sum_{k=1}^{6} \mu_k \Big(\varphi_k(\mathbf{X}_m) - \varphi_k(\mathbf{X}_{m-1})\Big),$$
$$\mathcal{L}_2(\boldsymbol{\eta}_m) - \mathcal{L}_2(\boldsymbol{\eta}_{m-1}) = (\nabla_w - \nabla_m)\cdot\boldsymbol{\psi}_m + (\nabla_{m-1} - \nabla_m)\cdot\mathbf{u}_{m-1} = \nabla\cdot\vec{\mathcal{H}}_m,$$
$$\vec{\mathcal{H}}_m = (\mathcal{A}^{(w)} - \mathcal{A}^{(m)})^T\,\boldsymbol{\psi}_m + (\mathcal{A}^{(m-1)} - \mathcal{A}^{(m)})^T\,\mathbf{u}_{m-1},$$
$$\mathcal{L}_3(\boldsymbol{\eta}_m) - \mathcal{L}_3(\boldsymbol{\eta}_{m-1}) = \Pi_w\Big(\Pi_w\,S_w(\boldsymbol{\psi}_m)\,\mathbf{n}_w - \Pi_m\,S_m(\boldsymbol{\psi}_m)\,\mathbf{n}_m\Big)$$
$$+ \Pi_w\Big(\Pi_{m-1}\,S_{m-1}(\mathbf{u}_{m-1})\,\mathbf{n}_{m-1} - \Pi_m\,S_m(\mathbf{u}_{m-1})\,\mathbf{n}_m\Big),$$
$$\mathcal{L}(\boldsymbol{\eta}_m, \pi_m) - \mathcal{L}(\boldsymbol{\eta}_{m-1}, \pi_{m-1}) = \mathcal{P}_m + \int_0^t \mathcal{Q}_m'\,\mathrm{d}\tau + \int_0^t \mathcal{Q}_m''\,\mathrm{d}\tau,$$

where

$$\begin{aligned}\mathcal{P}_m &= \mathbf{n}_w\cdot\Big(T_w(\boldsymbol{\psi}_m, \rho_m)\,\mathbf{n}_w - T_m(\boldsymbol{\psi}_m, \rho_m)\,\mathbf{n}_m\Big)\\ &+ \mathbf{n}_w\cdot\Big(T_{m-1}(\mathbf{u}_{m-1}, q_{m-1})\,\mathbf{n}_{m-1} - T_m(\mathbf{u}_{m-1}, q_{m-1})\,\mathbf{n}_m\Big)\\ &+ \mathbf{n}_0\cdot(\Delta_m - \Delta_{m-1})\,\boldsymbol{\xi},\end{aligned} \tag{5.12}$$

$$\mathcal{Q}_m' = \Big[\dot{\mathbf{n}}_w\cdot(\Delta_m - \Delta_w) + \mathbf{n}_w\cdot(\dot{\Delta}_m - \dot{\Delta}_w)\Big]\int_0^t \boldsymbol{\psi}_m\,\mathrm{d}\tau + \mathbf{n}_w\cdot(\Delta_m - \Delta_w)\,\boldsymbol{\psi}_m$$
$$+\Big[\dot{\mathbf{n}}_w(\Delta_m - \Delta_{m-1})(\boldsymbol{\xi} + \int_0^t \mathbf{u}_{m-1}\,\mathrm{d}\tau) + \mathbf{n}_w(\Delta_m - \Delta_{m-1})\mathbf{u}_{m-1}\Big]$$
$$\mathcal{Q}_m'' = (\mathbf{n}_w - \mathbf{n}_0)\,(\dot{\Delta}_m - \dot{\Delta}_{m-1})\,\boldsymbol{\xi} + \mathbf{n}_w\cdot(\dot{\Delta}_m - \dot{\Delta}_{m-1})\int_0^t \mathbf{u}_{m-1}\,\mathrm{d}\tau.$$

Assume that for $t \leqslant T_1$

$$\varepsilon \sup_{\tau<t} |\mathbf{u}_{m\tau}(\cdot,\tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\mathbf{u}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + + \sup_{\tau<t} |q_m(\cdot,\tau)|_{C^{1+\alpha}(\Omega)} \leqslant c_0,$$

$$\int_0^t |\mathbf{u}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau \leqslant \theta, \qquad m = 1, ..., n, \tag{5.13}$$

where θ is the same number as in Proposition 4.2. Then all the problems (5.11), $m = 1, ..., n$, are solvable in the interval $(0, T_1)$. The norms

$$Y_m(t) = \varepsilon \sup_{\tau<t} |\boldsymbol{\psi}_{m\tau}(\cdot,\tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\boldsymbol{\psi}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t} |\rho_m(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}$$

can be estimated by inequality (1.17) as follows:

$$\begin{aligned}Y_{m+1}(t) \leqslant c\biggl(&\sup_{\tau<t}\Bigl|\mathcal{L}_1(\boldsymbol{\eta}_m,\pi_m)-\mathcal{L}_1(\boldsymbol{\eta}_{m-1},\pi_{m-1})\Bigr|_{C^{\alpha}(\Omega)}+\sup_{\tau<t}\Bigl|\mathcal{L}_2(\boldsymbol{\eta}_m)-\mathcal{L}_2(\boldsymbol{\eta}_{m-1})\Bigr|_{C^{1+\alpha}(\Omega)}\\&+\varepsilon\sup_{\tau<t}|\vec{\mathcal{H}}_{m\tau}|_{C^{\alpha}(\Omega)}+\sup_{\tau<t}\Bigl|\mathcal{L}_3(\boldsymbol{\eta}_m)-\mathcal{L}_3(\boldsymbol{\eta}_{m-1})\Bigr|_{C^{1+\alpha}(\Gamma)}\\&+\varepsilon^{\frac{1+\alpha}{2}}\sup_{\Gamma}\Bigl[\mathcal{L}_3(\boldsymbol{\eta}_m)-\mathcal{L}_3(\boldsymbol{\eta}_{m-1})\Bigr]_{(0,t)}^{(\frac{1+\alpha}{2})}+\sup_{\tau<t}|\mathcal{P}_m|_{C^{1+\alpha}(\Gamma)}+\sup_{\tau<t}|\mathcal{Q}'_m|_{C^{\alpha}(\Gamma)}\\&+\sup_{\tau<t}|\mathcal{Q}''_m|_{C^{\alpha}(\Gamma)}\biggr).\end{aligned} \tag{5.14}$$

Consider the last term in (5.12), using local coordinates (see Subsection 4.1). Since $\mathbf{n}_0\cdot\frac{\partial\boldsymbol{\xi}}{\partial\eta_\alpha}=0$, the terms with $h_\beta(\eta,t)$ drop out, and we have, by virtue of Proposition 4.2:

$$|\mathbf{n}_0\cdot(\Delta_m-\Delta_{m-1})\,\boldsymbol{\xi}|_{C^{1+\alpha}(\Gamma)}\leqslant c\int_0^t|\boldsymbol{\psi}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau\leqslant c\int_0^t Y_m(\tau)\,\mathrm{d}\tau.$$

Making use of (4.6), (4.7), (4.9), and (4.26), we obtain further

$$\sup_{\tau<t}|\mathcal{Q}''_m|_{C^{\alpha}(\Gamma)}\leqslant c\left(\int_0^t|\boldsymbol{\eta}_{m-1}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau+\int_0^t|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau+\varepsilon\right)|\boldsymbol{\psi}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}$$

Other terms in (5.11) can be easily estimated with the help of the same inequalities by

$$c\left(\int_0^t|\boldsymbol{\eta}_{m-1}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau+\varepsilon\right)Y_m(t)+c\int_0^t Y_m(\tau)\,\mathrm{d}\tau.$$

Putting all the estimates together we obtain

$$\begin{aligned}Y_{m+1}(t)\leqslant c_1&\left(\int_0^t\left(|\boldsymbol{\eta}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}+|\boldsymbol{\eta}_{m-1}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}+|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\right)\mathrm{d}\tau+\varepsilon\right)Y_m(t)\\&+c_1\int_0^t Y_m(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Let

$$\Sigma_n(t)=\sum_{m=1}^{n}Y_m(t).$$

Since $|\boldsymbol{\eta}_m(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\leqslant\Sigma_m(t)$, the last inequality implies

$$\begin{aligned}\Sigma_{n+1}(t)\leqslant c_1&\biggl(2\int_0^t\Sigma_n(\tau)\,\mathrm{d}\tau+\int_0^t|\mathbf{w}(\cdot,\tau)|_{C^{\alpha+2}(\Omega)}\,\mathrm{d}\tau+\varepsilon\biggr)\Sigma_n(t)\\&+c_1\int_0^t\Sigma_n(\tau)\,\mathrm{d}\tau+Y_1(t).\end{aligned}$$

Assume that T_1 and ε are so small that

$$c_1\Bigl(\int_0^{T_1}|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau+\varepsilon\Bigr)\leqslant\frac{1}{2}. \tag{5.15}$$

Then

$$\Sigma_{n+1}(t) \leqslant 4\,c_1\,\Sigma_n(t)\int_0^t \Sigma_n(\tau)\,\mathrm{d}\tau + 2\,c_1\int_0^t \Sigma_{n+1}(\tau)\,\mathrm{d}\tau + c\,\varepsilon, \quad t < T_1,$$

and, by the Gronwall lemma,

$$\begin{aligned}\Sigma_{n+1}(t) &\leqslant e^{2c_1T_1}\Big(4\,c_1\,\Sigma_n(t)\int_0^t \Sigma_n(\tau)\,\mathrm{d}\tau + c\,\varepsilon\Big)\\ &\equiv 2\,c_2\,\Sigma_n(t)\int_0^t \Sigma_n(\tau)\,\mathrm{d}\tau + c_3\,\varepsilon.\end{aligned} \tag{5.16}$$

Integrating this estimate and setting

$$R_n(t) = \int_0^t \Sigma_n(\tau)\,\mathrm{d}\tau$$

we obtain

$$R_{n+1}(t) \leqslant c_2\,R_n^2(t) + c_3\,\varepsilon\,T_1. \tag{5.17}$$

Now, let

$$4\,c_2\,c_3\,T_1\,\varepsilon < 1 \tag{5.18}$$

and let

$$x_0 = \frac{1}{2\,c_2} - \sqrt{\frac{1}{4\,c_2^2} - \frac{c_3\,\varepsilon\,T_1}{c_2}} = \frac{2\,c_3\,\varepsilon\,T_1}{1 + \sqrt{1 - 4\,c_2\,c_3\,T_1\,\varepsilon}}$$

be a minimal root of the quadratic equation

$$c_2\,X^2 - X + c_3\,\varepsilon\,T_1 = 0.$$

It follows from (5.17) that if $R_n(T_1) \leqslant x_0$, then also $R_{n+1}(T_1) \leqslant x_0$. Hence, $R_k(T_1) \leqslant x_0$ for $k = 1, ..., n+1$, further, in virtue of (5.16),

$$\Sigma_{n+1}(t) \leqslant 2\,c_2\,x_0\,\Sigma_n(t) + c_3\,\varepsilon = \frac{4\,c_2\,c_3\,\varepsilon\,T_1}{1 + \sqrt{1 - 4\,c_2\,c_3\,T_1\,\varepsilon}}\,\Sigma_{n+1} + c_3\,\varepsilon,$$

and

$$\Sigma_{n+1}(t) \leqslant \frac{c_3\,\varepsilon}{\sqrt{1 - 4\,c_2\,c_3\,T_1\,\varepsilon}}. \tag{5.19}$$

This shows that under the hypotheses (5.13) and (5.18), we have

$$\begin{aligned}&\varepsilon \sup_{\tau<T_1} |\mathbf{u}_{n+1}(\cdot,\tau)|_{C^{\alpha}(\Omega)} + \sup_{\tau<T_1} |\mathbf{u}_{n+1}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<T_1} |q_{n+1}(\cdot,\tau)|_{C^{1+\alpha}(\Omega)}\\ &\quad\leqslant \varepsilon \sup_{\tau<T_1} |\mathbf{w}_\tau + \mathbf{V}_\tau|_{C^{\alpha}(\Omega)} + \sup_{\tau<T_1} |\mathbf{w} + \mathbf{V}|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<T_1} |r + P|_{C^{1+\alpha}(\Omega)} + \frac{c_3\,\varepsilon}{\sqrt{1 - 4\,c_2\,c_3\,T_1\,\varepsilon}},\end{aligned}$$

and, by virtue of (4.20) and (4.26),

$$\int_0^{T_1} |\mathbf{u}_{n+1}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)}\,\mathrm{d}\tau \leqslant +c\,\varepsilon + \int_0^{T_1}\Big(|\mathbf{w}(\cdot,\tau)|_{C^{2+\alpha}(\Omega)} + |r|_{C^{1+\alpha}(\Omega)}\Big)\mathrm{d}\tau,$$

i.e. inequalities (5.13), $n = 1, 2, ...$, hold for small ε and small T_1 independent of ε. Then (5.19) also holds for arbitrary n, and the successive approximations $(\boldsymbol{\eta}_m, \pi_m)$ are convergent to the solution $(\boldsymbol{\eta}, \pi)$ of the problem (1.19). It follows from (5.16) and from the above estimate for $R_n(T_1)$ that

$$\varepsilon \sup_{\tau<T_1} |\boldsymbol{\eta}_\tau(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<T_1} |\boldsymbol{\eta}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<T_1} |\pi(\cdot, \tau)|_{C^{1+\alpha}(\Omega)} \leqslant \frac{c_3\,\varepsilon}{\sqrt{1 - 4\,c_2\,c_3\,\varepsilon}}, \quad (5.20)$$

which implies (1.20).

The solution which we have found is unique. The difference of two solutions $\boldsymbol{\psi} = \boldsymbol{\eta} - \boldsymbol{\eta}'$, $\rho = \pi - \pi'$ satisfies the equations

$$\begin{aligned}
&\varepsilon\, \boldsymbol{\psi}_t - \nabla_w^2 \boldsymbol{\psi} + \nabla\rho = \mathcal{L}_1(\boldsymbol{\eta}, \pi) - \mathcal{L}_1(\boldsymbol{\eta}', \pi'),\\
&\nabla_w \cdot \boldsymbol{\psi} = \mathcal{L}_2(\boldsymbol{\eta}) - \mathcal{L}_2(\boldsymbol{\eta}'),\\
&\boldsymbol{\psi}\Big|_{t=0} = 0,\\
&\Pi_w\, S(\boldsymbol{\psi})\, \mathbf{n}_w\Big|_\Gamma = \mathcal{L}_3(\boldsymbol{\eta}) - \mathcal{L}_3(\boldsymbol{\eta}),\\
&\mathbf{n}_w \cdot T_w(\boldsymbol{\psi}, \rho)\, \mathbf{n}_w - \sigma\, \mathbf{n}_w \cdot \Delta_w(t) \int_0^t \boldsymbol{\psi}\, \mathrm{d}\tau\Big|_\Gamma = \mathcal{L}(\boldsymbol{\eta}, \pi) - \mathcal{L}(\boldsymbol{\eta}', \pi').
\end{aligned}$$

Repeating the above arguments, we obtain for the norm

$$Y(t) = \varepsilon \sup_{\tau<t} |\boldsymbol{\psi}(\cdot, \tau)|_{C^\alpha(\Omega)} + \sup_{\tau<t} |\boldsymbol{\psi}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + \sup_{\tau<t} |\rho(\cdot, \tau)|_{C^{1+\alpha}(\Omega)}$$

the estimate similar to (5.14), i.e.

$$Y(t) \leqslant c_1 \left[\int_0^t \Big(|\boldsymbol{\eta}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{\eta}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega)} + |\mathbf{w}(\cdot, \tau)|_{C^{2+\alpha}(\Omega)}\Big)\, \mathrm{d}\tau + \varepsilon\right] Y(t) + c_1 \int_0^t Y(\tau)\, \mathrm{d}\tau.$$

Let us assume that $(\boldsymbol{\eta}', q')$ is the solution constructed above. Then (5.20) implies

$$Y(t) \leqslant 2\,c_1\, Y(t) \Big(\int_0^t Y(\tau)\, \mathrm{d}\tau + c\,\varepsilon\Big) + 2\,c_1 \int_0^t Y(\tau)\, \mathrm{d}\tau,$$

from which it follows that $Y(t) = 0$, if ε is sufficiently small. The theorem is proved.

5.3 *Extension of the solution of the problem (1.6) to the interval $t \in (0, T_0)$*

We extend $(\mathbf{u}, q)$ outside the interval $(0, T_1)$ by introducing new Lagrangean coordinates, as it has been already done above. Let $\xi' = X_w(\xi, T_1)$. In these coordinates, equations (1.8) can be written as follows:

$$\begin{gathered}
-\nabla_w'^2 \mathbf{w}' + \nabla_w' r' + \ell_w'(\mathbf{w}') = \sum_{k=1}^6 \mu_k \boldsymbol{\varphi}(X_w'(\xi', t)),\\
\nabla_w' \cdot \mathbf{w}' = 0, \qquad \xi' \in \Omega' \equiv X_w(\Omega, T_1),\ t > T_1,\\
T_w'(\mathbf{w}', q')\mathbf{n}_w' - \Delta_w'(t)\mathbf{X}_w'(\xi', t)|_{\Gamma'} = 0.
\end{gathered}$$

Here $\Gamma' = \partial\Omega'$, $\mathbf{w}'(\xi', t) = \mathbf{w}(X_w^{-1}(\xi', T_1), t)$, $r'(\xi', t) = r(X_w^{-1}(\xi', T_1), t)$, $\mathbf{X}'_w(\xi', t) = \boldsymbol{\xi}' + \int_{T_1}^t \mathbf{w}'(\xi', \tau)\,\mathrm{d}\tau$ $(t > T_1)$, $\nabla'_w = \mathcal{A}^{(w)'}\nabla_{\xi'}$, $\mathcal{A}^{(w)'}$ is the matrix of cofactors of $a'_{ij} = \delta_{ij} + \int_{T_1}^t \frac{\partial w'_i}{\partial \xi'_j}\,\mathrm{d}\tau$, $T'_w = -pI + S'_w$, $S'_w(\mathbf{w}') = \nabla'_w\mathbf{w} + (\nabla'_w\mathbf{w}')^T$, $\mathbf{n}'_w$ and Δ'_w are the normal vector and the Laplace–Beltrami operator on $\Gamma'_w(t) = X'_w(\Gamma')$, finally,

$$\ell'_w(\mathbf{w}') = \sum_{k=1}^{6}\int_{\Omega'} \mathbf{w}'(\xi'', t)\cdot\boldsymbol{\varphi}_k(X'_w(\xi'', t))\,\mathrm{d}\xi''\boldsymbol{\varphi}_k(X'_w(\xi', t)).$$

Equations (1.6) can be written in a similar way, namely,

$$\varepsilon\mathbf{u}'_t + \ell'_u(\mathbf{u}') - \nabla'^2_u\mathbf{u}' + \nabla'_u q' = \sum_{k=1}^{6}\mu_k\boldsymbol{\varphi}_k(X'_u(\xi', t)),$$

$$\nabla'_u\cdot\mathbf{u}' = 0, \qquad (\xi'\in\Omega',\quad t > T_1),$$

$$\mathbf{u}'(\xi', t)|_{t=T_1} = \mathbf{u}(X_w^{-1}(\xi', T_1), T_1),$$

$$T'_u(\mathbf{u}', q')\mathbf{n}' - \Delta'_u(t)\mathbf{X}'_u|_{\Gamma} = 0$$

where $\mathbf{u}'(\xi', t) = \mathbf{u}(X_w^{-1}(\xi', T_1), t)$, $q'(\xi', t) = q((X_w^{-1}(\xi', T_1), t)$,

$$\ell'_w(\mathbf{u}') = \sum_{k=1}^{6}\int_{\Omega'} \mathbf{u}'(\xi'', t)\cdot\boldsymbol{\varphi}_k(X'_u(\xi'', t)\,\mathrm{d}\xi''\boldsymbol{\varphi}_k(X'_u(\xi', t))$$

and $\mathbf{X}'_u$, ∇'_u are computed according to the formulas

$$\mathbf{X}'_u(\xi', t) = \mathbf{X}_u(\xi, t)|_{\xi = X_w^{-1}(\xi', T_1)} = \boldsymbol{\xi}' + \int_{T_1}^t \mathbf{u}'(\xi', \tau)\,\mathrm{d}\tau + \mathbf{R}(\xi'), \tag{5.21}$$

$$\mathbf{R}(\xi') = \int_0^{T_1}(\mathbf{u}(\xi, \tau) - \mathbf{w}(\xi, \tau))\,\mathrm{d}\tau|_{\xi = X_w^{-1}(\xi', T_1)},$$

$$\nabla'_u = \nabla_u|_{\xi = X_w^{-1}(\xi', T_1)} = \mathcal{A}^{(u)}(\xi, t)a^{(w)T}(\xi, T_1)|_{\xi = X_w^{-1}(\xi', T_1)}\nabla_{\xi'} \tag{5.22}$$

where

$$a^{(w)} = \left(\delta_{ij} + \int_0^{T_1}\frac{\partial w_i(\xi, \tau)}{\partial \xi_j}\,\mathrm{d}\tau\right)_{i,j=1,2,3}.$$

It follows from (5.21) and (5.22) that

$$\mathbf{X}'_u(\xi', t) - \mathbf{X}'_w(\xi', t) = \int_{T_1}^t [\mathbf{u}'(\xi', \tau) - \mathbf{w}'(\xi', \tau)]\,\mathrm{d}\tau + \mathbf{R}(\xi'),$$

$$\nabla'_u - \nabla'_w = \left((\mathcal{A}^{(u)}(\xi, t) - \mathcal{A}^{(w)}(\xi, t))a^{(w)T}(\xi, T_1)|_{\xi = X_w^{-1}(\xi', T_1)}\right)\nabla_{\xi'}. \tag{5.23}$$

By virtue of (1.15) and (5.20),

$$|\mathbf{R}|_{C^{2+\alpha}(\Omega')} \leqslant c\int_0^{T_1}|\mathbf{u} - \mathbf{w}|_{C^{2+\alpha}(\Omega')}\,\mathrm{d}\tau \leqslant c\varepsilon,$$

hence,

$$|X'_u(\cdot,t) - X'_w(\cdot,t)|_{C^{2+\alpha}(\Omega')} \leqslant c\left(\varepsilon + \int_{T_1}^t |\mathbf{u}' - \mathbf{w}'|_{C^{2+\alpha}(\Omega')}\,\mathrm{d}\tau\right)$$

and similar estimate holds for the coefficients of the operator (5.23). Estimates (4.6), (4.7), and (4.9) in new coordinates are conserved but $\int_0^t |\mathbf{u} - \mathbf{w}|_{C^{k+\alpha}(\Omega)}\,\mathrm{d}\tau$ should be replaced everywhere with $\int_{T_1}^t |\mathbf{u}' - \mathbf{w}'|_{C^{k+\alpha}(\Omega')}\,\mathrm{d}\tau + \varepsilon$.

Let $\mathbf{V}'$, P' be a solution of a linear problem

$$\begin{gathered}
\varepsilon\mathbf{V}'_t - \nabla'^2_w\mathbf{V}' + \ell'_w(\mathbf{V}') + \nabla_w P' = 0,\\
\nabla'_w\cdot\mathbf{V}' = 0, \qquad \xi\in\Omega',\quad t\in(T_1,T_0),\\
\mathbf{V}'|_{t=T_1} = \mathbf{u}'(\xi',T_1) - \mathbf{w}'(\xi',T_1),\\
T'_w(\mathbf{V}',P')\mathbf{n}'_w|_\Gamma = 0.
\end{gathered}$$

For the differences

$$\boldsymbol{\eta}' = \mathbf{u}' - \mathbf{w}' - \mathbf{V}', \qquad \pi' = q' - r' - P'$$

we obtain the problem similar to (1.19), namely,

$$\begin{gathered}
\varepsilon\boldsymbol{\eta}'_t - \nabla'^2_w\boldsymbol{\eta}' + \ell'_w(\boldsymbol{\eta}') + \nabla'_w\pi' = -\varepsilon\mathbf{w}'_t + \mathcal{L}'_1(\boldsymbol{\eta}',\pi'),\\
\nabla'_w\cdot\boldsymbol{\eta}' = \mathcal{L}'_2(\boldsymbol{\eta}'), \qquad \xi'\in\Omega',\quad t\in(T_1,T_0),\\
\boldsymbol{\eta}'|_{t=T_1} = 0,\\
\Pi'_w S'_w(\boldsymbol{\eta}')\mathbf{n}'_w|'_\Gamma = \mathcal{L}'_3(\boldsymbol{\eta}'),\\
\mathbf{n}'_w\cdot T'_w(\boldsymbol{\eta}',\pi')\mathbf{n}'_w - \mathbf{n}'_w\cdot\Delta'_w(t)(\mathbf{X}'_u - \mathbf{X}'_w)\big|'_\Gamma = \mathcal{L}'(\boldsymbol{\eta}',\pi')
\end{gathered}\tag{5.24}$$

where

$$\begin{gathered}
\mathcal{L}'_1(\boldsymbol{\eta}',\pi') = (\nabla'^2_u - \nabla'^2_w)\mathbf{u}' - (\nabla'_u - \nabla'_w)q'\\
+\ell'_w(\mathbf{u}') - \ell'_u(\mathbf{u}') + \sum_{k=1}^6 \mu_k(\boldsymbol{\varphi}_k(X'_u(\xi,t)) - \boldsymbol{\varphi}_k(X'_w(\xi,t)),\\
\mathcal{L}'_2(\boldsymbol{\eta}') = (\nabla'_w - \nabla'_u)\cdot\mathbf{u}',\\
\mathcal{L}'_3(\boldsymbol{\eta}') = \Pi'_w(\Pi'_w S'_w(\mathbf{u}')\mathbf{n}'_w - \Pi'_u S'_u(\mathbf{u}')\mathbf{n}'_u),\\
\mathcal{L}'(\boldsymbol{\eta}',\pi') = \mathbf{n}'_w\cdot(T'_w(\mathbf{u}',q')\mathbf{n}'_w - T'_u(\mathbf{u}',q')\mathbf{n}'_u) + \mathbf{n}'_w\cdot(\Delta'_u(t) - \Delta'_w(t))\mathbf{X}'_u(\xi,t)
\end{gathered}$$

and $\mathbf{u}' = \mathbf{w}' + \mathbf{V}' + \boldsymbol{\eta}'$, $q' = r' + P' + \pi'$. The presence of the extra term $c\varepsilon$ in (4.6)–(4.9) cannot be an obstacle for the construction of the solution of problem (5.24) in the interval $t\in(T_1,2T_1)$, as above, by a successive approximation procedure, since, by virtue of (1.14), (4.20), and (5.20), we have now a uniform estimate for $(\mathbf{V}',P')$:

$$\begin{gathered}
\varepsilon\sup_{T_1<\tau<t}|\mathbf{V}'_\tau(\cdot,\tau)|_{C^\alpha(\Omega')} + \sup_{T_1<\tau<t}|\mathbf{V}'(\cdot,\tau)|_{C^{2+\alpha}(\Omega')} + \sup_{T_1<\tau<t}|P'(\cdot,\tau)|_{C^{l+1}(\Omega)}\\
\leqslant c|\mathbf{u}'(\cdot,T_1) - \mathbf{w}'(\cdot,T_1)|_{C^{2+\alpha}(\Omega')} \leqslant c\varepsilon.
\end{gathered}$$

Repeating this procedure several times, we can extend $(\mathbf{u},q)$ onto the whole interval $(0,T_0)$.

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