Existence for an Allen–Cahn/Cahn–Hilliard system with degenerate mobility

Roberta Dal Passo[†]

Dipartimento di Matematica, Università di Roma "Tor Vergata", 00133 Roma, Italy

Lorenzo Giacomelli[‡]

Dipartimento di Matematica, Università di Roma "La Sapienza", 00185 Roma, Italy

AND

Amy Novick-Cohen[§]

Department of Mathematics, Technion-IIT, Haifa, Israel 32000

We prove existence in one space dimension of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equation with singular lower-order terms. This system arises in the description of phase separation and ordering in binary alloys.

1. Introduction

Our paper is concerned with the Allen–Cahn/Cahn–Hilliard system

$$
(\mathbf{P}) \begin{cases}
u_t = [\omega_1 Q(u, v)(\mathcal{F}_u(u, v) - \varepsilon u_{xx})_x]_x & (x, t) \in \Omega_T \\
v_t = -\omega_2 Q(u, v)(\mathcal{F}_v(u, v) - \varepsilon v_{xx}) & (x, t) \in \Omega_T \\
u_x = v_x = \omega_1 Q(u, v)(\mathcal{F}_u(u, v) - \varepsilon u_{xx})_x = 0 & (x, t) \in S_T \\
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & x \in \Omega \\
(u, v) \in \overline{B} & (x, t) \in \Omega_T
\end{cases}
$$

where $\Omega \subset \mathbb{R}$ is a bounded open interval, $T > 0$, $\Omega_T = \Omega \times (0, T)$ and $S_T = \partial\Omega \times (0, T)$. The dimensionless homogeneous free energy $\mathcal{F}$ is assumed here to have the form

$$
\mathcal{F}(u, v) = F(u + v) + F(u - v) + F(1 - (u + v)) + F(1 - (u - v)) + \alpha u(1 - u) - \beta v^2,
$$

where $F(s) = \frac{\Theta}{2} s \ln s$ and Θ denotes the (dimensionless) absolute temperature; the function $\mathcal{F}$ is defined in the square

$$
B = \{(u, v) \in \mathbb{R}^2 : 0 < u + v < 1, \ 0 < u - v < 1\},
$$

and $Q(u, v)$, the mobility, is non-negative. Here ε is a gradient energy coefficient, and ε, α, β, ω_1 and ω_2 are positive constants. The system (**P**) with constant mobility was introduced by Cahn

[†]Email: dalpasso@mat.uniroma2.it
[‡]Email: giacomel@mat.uniroma1.it
[§]Email: amync@techunix.technion.ac.il

& Novick-Cohen [13] in the description of simultaneous phase separation and order–disorder transition in a BCC Fe–Al binary alloy: u and v represent a conserved (typically an average concentration) and a non-conserved order parameter, respectively. See [13] for further relevant references. In the context of their derivation ω_1 and ω_2 should assume the values $4\varepsilon^2$ and $\frac{1}{4}$ respectively. For the sake of simplicity in our analysis we let $\omega_1 = \omega_2 = 1$, though our conclusions are valid for arbitrary positive values of ω_1 and ω_2. The derivation in [13] can be extended to allow a non-constant mobility, and in the present paper we shall make the further assumption that the mobility Q is given by

$$Q(u, v) = u(1 - u)(\frac{1}{4} - v^2) =: Q_1(u)Q_2(v);$$

such a form satisfies the physical notion that the mobility should vanish at the pure phases and also turns out to be analytically convenient to consider.

It is known that Allen–Cahn and Cahn–Hilliard equations can serve as diffuse interface models for limiting sharp interface motion. The Allen–Cahn equation serves as a diffuse interface model for antiphase grain boundary coarsening in the sense that a singular limit of the equation ([1],[28],[22]) yields a geometric problem in which a sharp interface separating two phase variants evolves according to motion by mean curvature ($V = M\kappa$). The Cahn–Hilliard equation was constructed to describe mass conservative phase separation, and by considering an appropriate singular limit can describe the motion of interphase boundaries separating two phases of differing composition during the later stages of coarsening. For the constant mobility case with a non-singular free energy, this limiting motion ([27], [16]) is given by a free boundary problem known as a Mullins–Sekerka problem. For the Cahn–Hilliard equation with a degenerate mobility and with a free energy with logarithmic terms (as in (**P**) setting $v = 0$), it has been shown [12] that when $\Theta = \mathcal{O}(\varepsilon^{1/2})$, the limiting geometric boundary motion is predicted to be given by $V = -M\triangle_s\kappa$; i.e. surface diffusion. Rigorous justification has yet to be given for the surface diffusion description; however, the numerical results of Blowey & Elliott [8], Barrett & Blowey [2], and Barrett, Blowey, & Garcke [3] lend credibility to these results. Similarly, a singular limit for a system of Allen–Cahn equations can yield geometric motion in which an array of antiphase boundaries move by motion by mean curvature and couple together at triple junctions according to Young's law [11], and singular limits for a system of Cahn–Hilliard equations can lead to an array of interphase boundaries governed by a generalized Mullins–Sekerka problem [10] or by motion by minus the surface Laplacian [23].

As remarked above, by its construction, the Allen–Cahn/Cahn–Hilliard system was designed to describe simultaneous order–disorder and phase separation. By examining the free energy prescribed in Problem (**P**), it is clear that at low temperatures three phases—a disordered phase given by $(u, v) \approx (0, 0)$ or $(1, 0)$, and two ordered phase variants given by $(u, v) \approx (\frac{1}{2}, \pm\frac{1}{2})$— should roughly dominate the system at late times when it is nearly equilibrated. A description of the motion of both antiphase boundaries (APBs) separating ordered phase variants and interphase boundaries (IPBs) separating ordered variants from regions of disordered phase should be provided by the Allen–Cahn/Cahn–Hilliard system (**P**) in an appropriate singular limit. In the proximity of the deep quench limit ($\mathcal{O}(\varepsilon^{1/2})$), formal asymptotic analysis for Problem (**P**) (Cahn & Novick-Cohen [14], Novick-Cohen [25]) has predicted geometric motion in which APBs are governed by motion by mean curvature, IPBs move by motion by minus the surface Laplacian of the curvature, and there is quasi-static diffusion of the chemical potential along APBs. Under the assumption that the curvature of APBs is $\mathcal{O}(\varepsilon^{-1/2})$ and the curvature of IPBs is $\mathcal{O}(\varepsilon^{3/2})$, the two motions occur on the same time scale, and as in the analysis for a system of degenerate Cahn–Hilliard equations

[23], triple junction motion is governed by balance of mass flux, Young's law, and continuity of the chemical potential. To leading order, complete wetting is predicted; i.e. two IPBs connecting up with an APB at a triple junction meet with an angle of zero between them. However, the previously mentioned analysis allows the possibility of dynamic inhibition of complete wetting; i.e. the angle between two IPBs meeting at a triple junction may be small but non-vanishing. We remark that recently, Cahn & Novick-Cohen [15] studied limiting motion for an Allen–Cahn/Cahn–Hilliard system with non-constant mobility and a polynomial free energy designed to model the neighborhood of a tricritical point. By varying the values of the free energy polynomial parameters, they were able to mimic various levels of adsorption, prewetting, and impurity drag. Also, as discussed in [25], the Allen–Cahn/Cahn–Hilliard system can be viewed as a diffuse interface model for sintering of small grains and grain boundary grooving of polycrystalline films.

The analytical study of the Cahn–Hilliard equation in the case of constant mobility and logarithmic free energy is due to Elliott & Luckhaus [21] who demonstrated existence and uniqueness. By passing to the limit $\Theta \to 0$, they obtained existence and uniqueness also in the deep quench limit; i.e. with a free energy of the form

$$\mathcal{F}(u) = \begin{cases} u(1 - u) & 0 \leqslant u \leqslant 1 \\ +\infty & \text{elsewhere,} \end{cases}$$

as had been proven earlier by Blowey & Elliott [7]. The Cahn–Hilliard equation with non-singular lower-order terms was considered by Novick-Cohen in [26], where the effect of a concentration dependent non-degenerate mobility on stability was studied. The degenerate case with possibly singular free energy was studied by Elliott & Garcke [19] who proved existence under general assumptions on Q and $\mathcal{F}$ which include the case

$$Q(u) = u(1 - u),$$
$$\mathcal{F}(u) = F(u) + F(1 - u) + \alpha u(1 - u);$$

existence for non-degenerate mobilities with a possibly singular free energy is a byproduct of [19]. In the case of concentration dependent non-degenerate mobilities with a logarithmic free energy, uniqueness was proven by Barrett & Blowey [2] for sufficiently regular initial data. Recently, Elliott & Garcke [20] and Garcke & Novick-Cohen [23] have obtained existence results for respectively systems of Cahn–Hilliard equations with degenerate mobilities and the limiting geometric motions to which such systems give raise in the long time, low temperature limit. We mention that systems of Allen–Cahn/Cahn–Hilliard type with non-singular lower-order terms and non-degenerate mobilities have been considered by Brochet, Hilhorst, & Novick-Cohen in [9], where global existence as well as existence of a global attractor and inertial sets was proven.

The problem of existence of solutions for system (**P**) is delicate in several respects. In fact, this kind of operator combines together various non-trivial features. For example, it is known (see [4], [17], [18], [6]) that the degeneracy of the principal part leads, even when considering each equation in the system separately, several open problems on well-posedness. Below we shall describe some of the problematics in detail (see also Remarks 1.2 and 1.3).

In considering Problem (**P**), one has to deal with the constraint $(u, v) \in \overline{B}$; in addition, note that the partial derivatives of $\mathcal{F}$ can be extended to be continuous functions with values in $\mathbb{R} \cup \{\pm\infty\}$ only in $\overline{B} \setminus \{(0, 0), (1, 0), (\frac{1}{2}, \pm\frac{1}{2})\}$, and not in $\overline{B}$. This difficulty can be handled in the case of

positive mobilities via a-priori estimates in $L^1(\Omega_T)$ on the terms

$$F'(u + v), \ldots, F'(1 - (u - v)) \tag{1.1}$$

which are logarithmic. This estimate implies that $(u, v) \in B$ almost everywhere in Ω_T in the case of positive mobilities. Another peculiarity in the structure of the operator, which turns out to be crucial in our analysis, is the fact that the mobility degenerates only at the vertices of the box B, whereas the derivatives of $\mathcal{F}$ are singular over the whole boundary ∂B. Therefore there is no obvious way to bound a-priori the product of mobilities and derivatives of $\mathcal{F}$. For a single degenerate C–H equation such difficulties do not arise as the locus of the zeroes of the mobility and the singularities of the logarithmic terms appearing in the free energy is identical. Thus, this represents a main difference between the system which we consider here and the case of a single degenerate C–H equation [19]. On the other hand, the assumed form of Q allows the derivation of an estimate for v, namely

$$\int_\Omega \frac{1}{\left(\frac{1}{4} - v^2(t)\right)^{\beta-1}} \leqslant c + \int_\Omega \frac{1}{\left(\frac{1}{4} - v_0^2\right)^{\beta-1}} \quad \text{if } \beta \in (1, 4]. \tag{1.2}$$

This estimate permits control of the degeneracy of the system with respect to the variable v. It is worth noting that the estimate (1.2) only holds if one knows a-priori that the range of the solution is contained in the closure of the square. For this reason we first prove an existence result for the case of non-degenerate mobilities with a logarithmic free energy, whose solutions can be shown to satisfy $(u, v) \in \overline{B}$ by virtue of the estimates on terms of the type given in (1.1). Afterwards we derive an estimate in $L^1(\Omega_T)$ for the terms

$$u(1 - u)vF'(u + v), \ldots, u(1 - u)vF'(1 - (u - v)) \tag{1.3}$$

which holds for arbitrarily positive mobilities and implies that the constraint $(u, v) \in B \cup \{(0, 0), (1, 0)\}$ is satisfied almost everywhere. Indeed, we are not able to exclude the possibility that the sets in Ω_T where $(u, v) = (0, 0)$ or $(u, v) = (1, 0)$ have positive measure—a possibility which has also not been excluded in the case of a single degenerate C–H equation with free energy and mobility of analogous form.

Throughout the paper we make the following assumption on the initial data:

$$u_0 \in H^1(\Omega), \quad v_0 \in H^1(\Omega),$$
$$(u_0, v_0) \in \overline{B}, \quad v_0 \in (-1/2, 1/2), \quad \overline{u_0} \in (0, 1), \tag{1.4}$$

where $\overline{f}$ denotes the mean value of f in Ω of a given $f \in L^1(\Omega)$. Let us introduce some notation which we shall use in the sequel. Given $u, v \in C(\overline{\Omega}_T)$, we define the sets

$$\mathcal{D}_\eta(u) = \{(x, t) \in \overline{\Omega}_T : \eta < u(x, t) < 1 - \eta\}, \quad 0 \leqslant \eta < \tfrac{1}{2},$$
$$\mathcal{B}(u, v) = \{(x, t) \in \overline{\Omega}_T : (u(x, t), v(x, t)) \in B\},$$

with the convention that $\mathcal{D}(u) = \mathcal{D}_0(u)$. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.

Let us define our concept of a weak solution for Problem (**P**) as follows.

DEFINITION 1   A triplet (u, v, w) is called a weak solution of (**P**) if:

(i) $u \in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T) \cap L^\infty(0, T; H^1(\Omega))$,

$v \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T) \cap L^\infty(0, T; H^1(\Omega))$,

$w \in L^2(\Omega_T)$;

(ii) $u_{xx} \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$, $v_{xx} \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$;

(iii) $u_t \in L^2(0, T; (H^1(\Omega))')$, $v_t \in L^2(\Omega_T)$;

(iv) $(u, v) \in \overline{B}$, $v \in (-1/2, 1/2)$;

(v) $\mathcal{F}_u(u, v) \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$, $\mathcal{F}_v(u, v) \in L^2_{\mathrm{loc}}(\mathcal{D}(u))$;

(vi) $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$;

(vii) $u_x|_{S_T \cap \mathcal{D}(u)} = v_x|_{S_T \cap \mathcal{D}(u)} = 0$ in $L^2_{\mathrm{loc}}(S_T \cap \mathcal{D}(u))$;

(viii) (u, v, w) solves the system (**P**) in the following sense:

$$\int_0^T \langle u_t, \phi \rangle \, \mathrm{d}t = -\iint_{\Omega_T} [Q(u, v)]^{1/2} w \phi_x \quad \forall \phi \in L^2(0, T; H^1(\Omega)), \tag{1.5}$$

$$\iint_{\Omega_T} v_t \psi = -\iint_{\Omega_T} Q(u, v)(\mathcal{F}_v(u, v) - \varepsilon v_{xx})\psi \quad \forall \psi \in L^2(\Omega_T) : \ \mathrm{supp}\,\psi \subset \mathcal{D}(u), \tag{1.6}$$

where $w = Q^{1/2}(u, v)(\mathcal{F}_u(u, v) - \varepsilon u_{xx})_x$ in the sense that

$$\iint_{\Omega_T} w\phi = -\iint_{\Omega_T} (\mathcal{F}_u(u, v) - \varepsilon u_{xx}) \left([Q(u, v)]^{1/2}\phi\right)_x \tag{1.7}$$

for all $\phi \in L^2(0, T; H^1_0(\Omega))$ with $\mathrm{supp}\,\phi \subset \mathcal{D}(u)$.

REMARK 1.1 The terms in the Definition are all well defined. In particular:

1. The traces of u_x, v_x in $S_T \cap \mathcal{D}(u)$ can be seen to be well defined by the following argument: Let $K \subset S_T \cap \mathcal{D}(u)$ compact; then $K \subseteq S_T \cap \mathcal{D}_\eta(u)$ for some $\eta > 0$. Since $u \in C(\overline{\Omega}_T)$, we can select $\zeta \in C^\infty(\overline{\Omega}_T)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $\mathcal{D}_\eta(u)$ and $\zeta = 0$ in $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$. Then $\zeta u \in L^2(0, T; H^2(\Omega))$ and therefore $(\zeta u)_x|_{S_T} \in L^2(0, T; L^2(\partial\Omega))$. It follows that

$$u_x|_{S_T \cap K} \in L^2(S_T \cap K).$$

The argument for v_x is the same.

2. Since (iv) implies that $Q_2(v)$ is strictly positive, from (i) it then follows that the function $\left([Q(u, v)]^{1/2}\phi\right)_x$ has compact support in $\mathcal{D}(u)$ and belongs to $L^2(\Omega_T)$ for any $\phi \in L^2(0, T; H^1(\Omega))$ with $\mathrm{supp}\,\phi \subset \mathcal{D}(u)$. Therefore the integral at the right hand side of (1.7) makes sense.

REMARK 1.2 *Comparison with other definitions of solution.*
The choice of test functions which are compactly supported in $\mathcal{D}(u)$, as well as the regularity properties given by *(ii)* and *(v)*, differ from (and actually weaken) the definition of solution given in [19] in the case of a single degenerate C–H equation. This reflects the lack of control on the set

$\{u = 0, 1\}$, which is essentially due to the difference between the locus of the zeroes of Q and that of the singularities of $\mathcal{F}_u, \mathcal{F}_v$.

Note that our choice of the unknowns—which were taken in [19] to be (u, v, J) with $J = Q(u, v)(\mathcal{F}_u - \varepsilon u_{xx})_x$—is not weaker: indeed, choosing $\phi = \sqrt{Q}\varphi$ in (1.7), one can identify the flux J in the sense that

$$\int_0^T \langle u_t, \phi \rangle \, \mathrm{d}t = -\iint_{\Omega_T} J\phi_x \quad \forall \phi \in L^2(0, T; H^1(\Omega)),$$

where

$$\iint_{\Omega_T} J\varphi = -\iint_{\Omega_T} (\mathcal{F}_u - \varepsilon u_{xx})(Q\varphi)_x$$

for all $\varphi \in L^2(0, T; H_0^1(\Omega))$ with supp $\varphi \subset \mathcal{D}(u)$.

REMARK 1.3   *Uniqueness.*

According to our definition, problem (**P**) admits more than one solution. To demonstrate this, let us take $v \equiv 0$; then (**P**) formally reduces to a single C–H equation to be solved in $\mathcal{D}(u)$. Hence, this is a free boundary problem (the free boundary being given by $\partial \mathcal{D}(u)$), but our definition does not specify any free boundary condition. Note that the definition of solution given in [19], and in particular the regularity property $u \in L^2(0, T; H^2(\Omega))$, does contain in one space dimension the free boundary condition: $u_x|_{\partial \mathcal{D}(u)} = 0$ almost everywhere. On this basis, it is strongly believed that solutions with this property should be unique; in fact, this is probably the major open problem in the present theory of fourth-order degenerate parabolic equations with principal part of the form $-\mathrm{div}(m(u)\nabla \Delta u), m \geqslant 0$.

We can actually construct an example of non-uniqueness according to our definition. Let $u(x)$ be the unique solution of the initial value problem

$$\begin{cases} \varepsilon u'' = 2\log \frac{u}{1-u} + \alpha(1 - 2u), & x > 0 \\ u(0) = \overline{u}_0 > 0 \\ u'(0) = 0. \end{cases}$$

It is easy to verify that for $\overline{u}_0 = \overline{u}_0(\varepsilon, \alpha)$ sufficiently small there exists $x_0 > 0$ such that $u(x) \to 0$ as $x \nearrow x_0$; in addition

$$u'(x_0) = -c < 0.$$

Define

$$u_0(x) = \begin{cases} u(|x|) & |x| < x_0 \\ 0 & \text{elsewhere.} \end{cases}$$

The pair $(u_0, 0)$ is, according to Definition 1, a steady solution of Problem (**P**) in $\Omega = (-2x_0, 2x_0)$. Note that u_0 does not belong to $H^2((-2x_0, 2x_0))$. On the other hand, let $\tilde{u}$ denote a solution of the C–H equation in the sense of [19] with initial data u_0. The pair $(\tilde{u}, 0)$ is also a solution of (**P**) in the sense of Definition 1, and, since $\tilde{u} \in L^2(0, T; H^2(\Omega))$, we conclude that $\tilde{u}(t) \neq u_0$.

The non-degeneracy of the system with respect to the unknown v given by (iv) should be required, as far as v is concerned, to characterize uniquely a solution. This would not be the case if

degeneracy with respect to v was allowed: indeed, Dal Passo & Luckhaus have proved ([18], see in particular Remark 1 and Section 3) that the Dirichlet problem for

$$z_t = z\Delta z - z \ln z$$

admits more than one solution for a suitable class of initial data.

The main result of the paper is the following

THEOREM 1   For all u_0, v_0 satisfying (1.4) there exists a weak solution of Problem (**P**) in the sense of Definition 1; in addition, the set

$$\mathcal{D}(u) \setminus \mathcal{B}(u, v) = \left\{ (x, t) \in \overline{\Omega}_T : (u(x, t), v(x, t)) \in \partial B \setminus \{(0, 0), (1, 0)\} \right\}$$

has measure zero.

In Section 2 we obtain (by means of a Galerkin approximation) an existence result for suitably regularized systems; i.e. for systems with the form of Problem (**P**) but which have been made to be uniformly parabolic by assuming sufficient additional regularity properties for Q and $\mathcal{F}$. The proof of Theorem 1 is contained in Section 3, where by rescaling, without loss of generality we have assumed $\Theta = 2$: in Subsection 3.1 we define a two-parameter family of approximating regular systems and derive an energy inequality yielding uniform estimates for the corresponding solutions; in Subsection 3.2 we obtain intermediate results for non-degenerate systems based on a free energy which is now of the original logarithmic form, $\mathcal{F}(u, v)$; in Subsection 3.3 we use these approximating solutions to construct a weak solution of Problem (**P**).

We observe that most of the results contained in Subsections 3.1 and 3.2 remain valid in higher space dimensions. Actually it is the continuity of the solution that enables us to control the boundary of the box B uniformly with respect to the approximating procedure; however, the need for continuity forces our results to be restricted, up to now, to space dimension one.

2. Regularized systems

In this section we prove an existence theorem for the Allen–Cahn/Cahn–Hilliard system under additional regularity and positivity assumptions on the mobility and the homogeneous free energy. This result will be used later to construct approximating solutions to Problem (**P**).

Consider the following system:

$$(\mathbf{P}') \begin{cases} u_t = [q_1(u, v)(f_1(u, v) - \varepsilon u_{xx})_x]_x & (x, t) \in \Omega_T \\ v_t = -q_2(u, v)(f_2(u, v) - \varepsilon v_{xx}) & (x, t) \in \Omega_T \\ v_x = u_x = u_{xxx} = 0 & (x, t) \in S_T \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & x \in \Omega \end{cases}$$

where q_i, f_i satisfy:
(H1) $q_i \in C(\mathbb{R}^2; \mathbb{R}^+)$, with $q_{min} \leqslant q_i \leqslant q_{max}$ for some $0 < q_{min} \leqslant q_{max}$;
(H2) $f_1 \in C^1(\mathbb{R}^2; \mathbb{R})$ and $f_2 \in C(\mathbb{R}^2, \mathbb{R})$, with $\|f_1\|_{C^1} + \|f_2\|_{C^0} \leqslant F_0$ for some $F_0 > 0$.

THEOREM 2.1   Assuming (H1), (H2) and $u_0, v_0 \in H^1(\Omega)$, there exists a pair of functions (u, v) such that:

(i) $u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap C([0, T]; H^\lambda(\Omega)), \lambda < 1$;

(ii) $v \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^\lambda(\Omega)), \lambda < 1$;

(iii) $u_t \in L^2(0, T; (H^1(\Omega))'), v_t \in L^2(\Omega_T)$;

(iv) $u(0) = u_0$ and $v(0) = v_0$ in $L^2(\Omega)$;

(v) $u_x|_{S_T} = v_x|_{S_T} = 0$ in $L^2(S_T)$;

(vi) (u, v) solves $(\mathbf{P}')$ in the following sense:

$$\int_0^T \langle u_t, \phi \rangle \, dt = -\iint_{\Omega_T} q_1(u, v)(f_1(u, v) - \varepsilon u_{xx})_x \phi_x \quad \forall \phi \in L^2(0, T; H^1(\Omega))$$

$$\iint_{\Omega_T} v_t \psi = -\iint_{\Omega_T} q_2(u, v)(f_2(u, v) - \varepsilon v_{xx}) \psi \quad \forall \psi \in L^2(\Omega_T).$$

Proof. We apply a Galerkin approximation. Let us denote by $\psi_i, i \in \mathbf{N}$, the eigenfunctions of minus the Laplacian with Neumann boundary conditions:

$$\begin{cases} -\psi_i'' = \lambda_i \psi_i & x \in \Omega \\ \psi_i' = 0 & x \in \partial\Omega. \end{cases}$$

Without loss of generality, we assume that the eigenfunctions ψ_i are orthogonal in $H^1(\Omega)$, orthonormal in $L^2(\Omega)$, and that the first eigenvalue is zero ($0 = \lambda_1 < \lambda_2 \ldots$). The Galerkin ansatz for $(\mathbf{P}')$

$$u^N(x, t) = \sum_{i=1}^N a_i^N(t) \psi_i(x), \quad v^N(x, t) = \sum_{i=1}^N b_i^N(t) \psi_i(x)$$

yields the following initial value problem:

$$\frac{da_j^N}{dt} = \int_\Omega u_t^N \psi_j = -\int_\Omega q_1(u^N, v^N)(f_1(u^N, v^N) - \varepsilon u_{xx}^N)_x \psi_{jx} \tag{2.1}$$

$$\frac{db_j^N}{dt} = \int_\Omega v_t^N \psi_j = -\int_\Omega q_2(u^N, v^N)(f_2(u^N, v^N) - \varepsilon v_{xx}^N) \psi_j \qquad j = 1, \ldots, N \tag{2.2}$$

$$a_j^N(0) = (u_0, \psi_j)_{L^2(\Omega)}, \quad b_j^N(0) = (v_0, \psi_j)_{L^2(\Omega)}. \tag{2.3}$$

The problem has a local solution because the right hand side is continuous with respect to $(a_1^N, \ldots, a_N^N, b_1^N, \ldots, b_N^N)$. We now derive an a-priori estimate which allows us to infer global existence. Multiplying each equation of (2.1) by $-\lambda_j a_j(t)$ and summing (that is, using u_{xx}^N as a test function), we get

$$\frac{d}{dt} \frac{1}{2} \int_\Omega (u_x^N)^2 + \varepsilon \int_\Omega q_1 (u_{xxx}^N)^2 = \int_\Omega q_1 f_{1u} u_x^N u_{xxx}^N + \int_\Omega q_1 f_{1v} v_x^N u_{xxx}^N;$$

using v_{xx}^N as a test function in (2.2) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\Omega}(v_x^N)^2 + \varepsilon\int_{\Omega}q_2(v_{xx}^N)^2 = \int_{\Omega}q_2 f_2 v_{xx}^N.$$

Summing the two identities, using (H1)–(H2), and applying the Cauchy–Schwarz and Young's inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega}(u_x^N)^2 + \int_{\Omega}(v_x^N)^2\right] + \int_{\Omega}(u_{xxx}^N)^2 + \int_{\Omega}(v_{xx}^N)^2 \leqslant c_1 + c_2\left[\int_{\Omega}(u_x^N)^2 + \int_{\Omega}(v_x^N)^2\right],$$

from which we obtain by a Gronwall argument

$$\int_{\Omega}(u_x^N)^2 + \int_{\Omega}(v_x^N)^2 + \iint_{\Omega_T}(u_{xxx}^N)^2 + \iint_{\Omega_T}(v_{xx}^N)^2 \leqslant c_3(T). \tag{2.4}$$

Since $\psi_1' \equiv 0$, it follows from (2.1) that $\int_{\Omega}u_t^N = 0$; hence, by Poincaré inequality

$$\|u^N\|_{L^\infty(0,T;\,H^1(\Omega))} \leqslant c_4(T). \tag{2.5}$$

The relation

$$\iint_{\Omega_T}(v_t^N)^2 = -\iint_{\Omega_T}v_t^N q_2(f_2 - \varepsilon v_{xx}^N) \leqslant \left(\iint_{\Omega_T}(v_t^N)^2\right)^{1/2}\left(\iint_{\Omega_T}q_2^2(f_2 - \varepsilon v_{xx}^N)^2\right)^{1/2}$$

coupled with (2.4) gives

$$\|v_t^N\|_{L^2(\Omega_T)} \leqslant c_5(T), \tag{2.6}$$

which in turn implies a bound on $b_1^N(t)$. Therefore, using the Poincaré inequality and (2.4) we have

$$\|v^N\|_{L^\infty(0,T;\,H^1(\Omega))} \leqslant c_6(T), \tag{2.7}$$

$$\|v^N\|_{L^2(0,T;\,H^2(\Omega))} \leqslant c_6(T). \tag{2.8}$$

From (2.5) and (2.7), it follows that $(a_1^N, \ldots, a_N^N, b_1^N, \ldots, b_N^N)$ are uniformly bounded, and therefore there exists a solution of (2.1)–(2.3) in $(0, T)$.

In order to derive an estimate for u_t^N, we introduce the projection Π_N of $L^2(\Omega)$ onto $span\{\psi_1, \ldots, \psi_N\}$; for any $\psi \in L^2(0, T;\, H^1(\Omega))$ we obtain

$$\left|\iint_{\Omega_T}u_t^N\psi\right| = \left|\iint_{\Omega_T}q_1(f_1 - \varepsilon u_{xx}^N)_x(\Pi_N\psi)_x\right| \leqslant c_7(T)\|\psi\|_{L^2(0,T;H^1(\Omega))}$$

(where (2.4) has been used); it follows that

$$\|u_t^N\|_{L^2(0,T;(H^1(\Omega))')} \leqslant c_7(T). \tag{2.9}$$

Finally, since

$$\int_\Omega u^N_{xx} = 0, \tag{2.10}$$

the estimate (2.4), Poincaré's inequality and (2.5) yield

$$\|u^N\|_{L^2(0,T;\,H^3(\Omega))} \leqslant c_8(T). \tag{2.11}$$

From the estimates (2.5)–(2.9) and (2.11), by well-known compactness results (see also [29]) it is possible to select a subsequence (still denoted by (u^N, v^N)) such that

$$
\begin{aligned}
u^N, v^N &\overset{*}{\rightharpoonup} & u, v & \quad\text{in} & L^\infty(0, T;\, H^1(\Omega)), \\
u^N &\longrightarrow & u & \quad\text{in} & L^2(0, T;\, H^3(\Omega)), \\
v^N &\longrightarrow & v & \quad\text{in} & L^2(0, T;\, H^2(\Omega)), \\
u^N_t &\longrightarrow & u_t & \quad\text{in} & L^2(0, T;\, (H^1(\Omega))'), \\
v^N_t &\longrightarrow & v_t & \quad\text{in} & L^2(\Omega_T), \\
u^N, v^N &\longrightarrow & u, v & \quad\text{in} & C([0, T];\, H^\lambda(\Omega)), \quad \lambda < 1, \\
u^N &\longrightarrow & u & \quad\text{in} & L^2(0, T;\, H^\lambda(\Omega)), \quad \lambda < 3, \\
v^N &\longrightarrow & v & \quad\text{in} & L^2(0, T;\, H^\lambda(\Omega)), \quad \lambda < 2.
\end{aligned}
$$

In particular, the strong convergence in $C([0, T];\, L^2(\Omega))$ implies (iv), and the last two convergence statements imply (v) in view of the continuous linear mapping

$$L^2(0, T;\, H^{1-\lambda}(\Omega)) \to L^2(0, T;\, H^{\frac{1}{2}-\lambda}(\partial\Omega)), \quad \lambda < \frac{1}{2}$$

applied to u^N_x, v^N_x with $\lambda > 0$.   □

REMARK 2.2 Note that, using the same arguments as above with some minor modifications, Theorem 2.1 can be proved in any space dimension.

3. Proof of Theorem 1

3.1 *Approximating systems*

We shall approximate Problem (**P**) by regularized systems; i.e. systems with positive mobility and smooth bulk energy. For $\delta > 0$ we introduce positive mobilities $Q_\delta(u, v)$ by defining (see Fig. 1)

$$
Q_{2\delta}(v) =
\begin{cases}
\left(\frac{1}{2} + \delta\right)^2 - v^2 & \text{if} \quad v \in \left[0, \frac{1}{2} + \frac{\delta}{2}\right], \\[2ex]
\frac{\delta^2}{4}\left[1 + (1+\delta)\left(v - \frac{1}{2}\right)^{-1}\right] & \text{if} \quad v > \frac{1}{2} + \frac{\delta}{2}, \\[2ex]
Q_{2\delta}(-v) & \text{if} \quad v < 0,
\end{cases}
$$

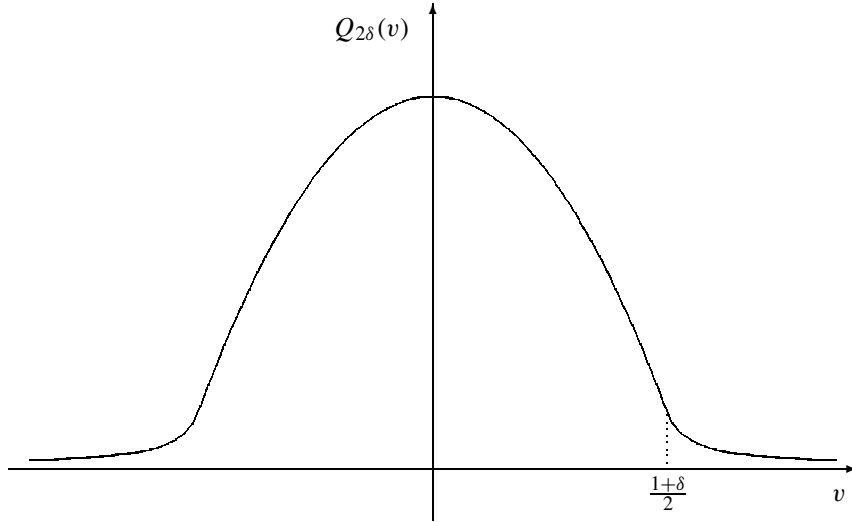$$Q_{1\delta}(u) = Q_{2\delta}(u - \tfrac{1}{2}),$$

FIG. 1. The function $Q_{2\delta}(v)$.

and setting

$$Q_\delta(u, v) = Q_{1\delta}(u) Q_{2\delta}(v).$$

With this choice we have $Q''_{i\delta} \in L^\infty(\mathbb{R})$ and

$$\frac{\delta^2}{4} < Q_{i\delta} \leqslant \left(\frac{1}{2} + \delta\right)^2,$$

$$|Q'_{i\delta}| \leqslant 1 + \delta,$$

$$Q_{i\delta} \longrightarrow Q_i \quad \text{uniformly in} \quad C^1(\text{supp}\,[Q_i]_+).$$

For $\sigma \in (0, 1/e)$ we choose $F'_\sigma(s)$ such that (see Fig. 2)

$$F'_\sigma(s) = \begin{cases} \left[1 - \frac{s-\sigma}{\sigma}\right]^{-1} + \ln\sigma, & \text{if} \quad s < \sigma, \\ \ln s + 1, & \text{if} \quad \sigma \leqslant s \leqslant 1 - \sigma, \\ f_\sigma(s), & \text{if} \quad 1 - \sigma < s < 2, \\ 1, & \text{if} \quad s \geqslant 2, \end{cases}$$

where $f_\sigma \in C^1([1 - \sigma, 2])$ has the following properties:

$$f_\sigma \leqslant F', \quad f'_\sigma \geqslant 0,$$

$$f_\sigma(1 - \sigma) = F'(1 - \sigma), \quad f_\sigma(2) = 1,$$

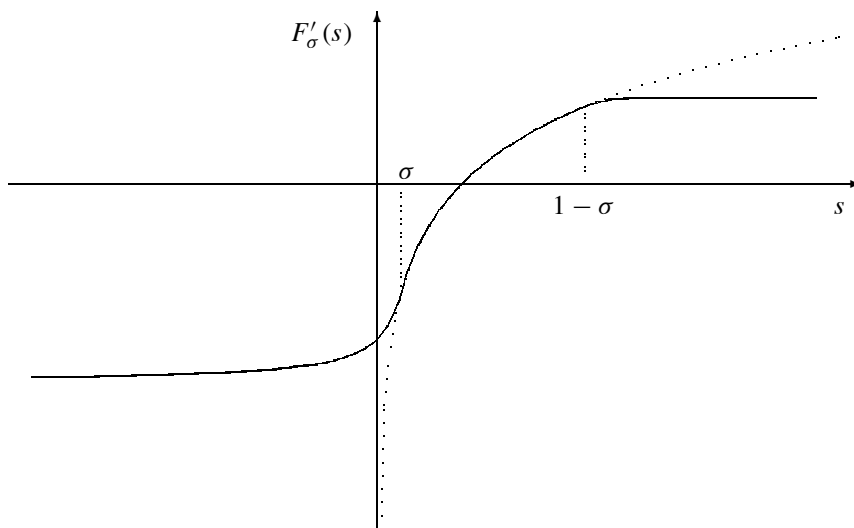$$f'_\sigma(1 - \sigma) = F''(1 - \sigma), \quad f'_\sigma(2) = 0.$$

FIG. 2. The function $F'_\sigma(s)$ (the dotted curve is $F'(s)$).

Defining

$$F_\sigma(s) = -\frac{1}{e} + \int_{\frac{1}{e}}^{s} F'_\sigma(\xi)\,d\xi,$$

we have

$$F_\sigma \in C^2(\mathbb{R}), \qquad F_\sigma(s) \leqslant F(s) \ \text{ if } \ 0 < s,$$
$$F''_\sigma \geqslant 0, \qquad F_\sigma(s) = F(s) \ \text{ if } \ \sigma \leqslant s \leqslant 1 - \sigma.$$

Note that F_σ is defined for all real values of s, whereas F is defined for $s > 0$ only. We also introduce U, V such that

$$\|U\|_{C^2(\mathbb{R})} \leqslant U_0, \qquad \|V\|_{C^2(\mathbb{R})} \leqslant V_0,$$
$$U(u) = \alpha u(1 - u) \ \text{ if } \ 0 \leqslant u \leqslant 1, \qquad V(v) = -\beta v^2 \ \text{ if } \ -\tfrac{1}{2} \leqslant v \leqslant \tfrac{1}{2},$$

and we define the approximating homogeneous free energies as follows:

$$\mathcal{F}_\sigma(u, v) = F_\sigma(u + v) + F_\sigma(u - v) + F_\sigma(1 - (u + v)) + F_\sigma(1 - (u - v)) + U(u) + V(v).$$

We observe that

$$-U_0 - V_0 - \frac{4}{e} \leqslant \mathcal{F}_\sigma(u, v) \ \text{ in } \ \mathbb{R}^2, \quad \text{and} \quad \mathcal{F}_\sigma(u, v) < U(u) + V(v) \ \text{ in } \ \overline{B}. \tag{3.1}$$

Applying Theorem 2.1, for each $\delta, \sigma > 0$ there exists a solution $(u_{\delta\sigma}, v_{\delta\sigma})$ of

$$(\mathbf{P}_{\delta\sigma}) \begin{cases} u_t = [Q_\delta(u, v)(\mathcal{F}_{\sigma u}(u, v) - \varepsilon u_{xx})_x]_x & (x, t) \in \Omega_T \\ v_t = -Q_\delta(u, v)(\mathcal{F}_{\sigma v}(u, v) - \varepsilon v_{xx}) & (x, t) \in \Omega_T \\ u_x = v_x = Q_\delta(u, v)(\mathcal{F}_{\sigma u}(u, v) - u_{xx})_x = 0 & (x, t) \in S_T \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & x \in \Omega. \end{cases}$$

We shall hereafter denote by $(\mathbf{P}_\delta)$, Problem $(\mathbf{P}_{\delta\sigma})$ where $\mathcal{F}_\sigma$ is replaced by $\mathcal{F}$, and we shall denote by δ_0, σ_0 generic positive constants.
First of all we observe that

$$\overline{u_{\delta\sigma}(t)} = \overline{u_0} \in (0, 1). \tag{3.2}$$

LEMMA 3.1   There exists a constant C_1 independent of δ, σ such that the following estimates hold for all $\delta < \delta_0, \sigma < \sigma_0$:

(i) $\|u_{\delta\sigma}\|_{L^\infty(0,T; H^1(\Omega))} \leqslant C_1$,

(ii) $\|v_{\delta\sigma}\|_{L^\infty(0,T; H^1(\Omega))} \leqslant C_1$,

(iii) $\|Q_\delta^{1/2}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x\|_{L^2(\Omega_T)} \leqslant C_1$,

(iv) $\|Q_\delta^{1/2}(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})\|_{L^2(\Omega_T)} \leqslant C_1$,

(v) $\|u_{\delta\sigma t}\|_{L^2(0,T;(H^1(\Omega))')} \leqslant C_1$,

(vi) $\|Q_\delta^{-1/2} v_{\delta\sigma t}\|_{L^2(\Omega_T)} \leqslant C_1$,

(vii) $\|\mathcal{F}_\sigma(u_{\delta\sigma}, v_{\delta\sigma})\|_{L^\infty(0,T; L^1(\Omega))} \leqslant C_1$.

Proof. Choosing $\phi = (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})$ and $\psi = (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})$ as test functions in the equations for $u_{\delta\sigma}$ and $v_{\delta\sigma}$ respectively, we obtain

$$\frac{\varepsilon}{2}\int_\Omega [u_{\delta\sigma x}(t)]^2 + \frac{\varepsilon}{2}\int_\Omega [v_{\delta\sigma x}(t)]^2 + \int_\Omega \mathcal{F}_\sigma(u_{\delta\sigma}(t), v_{\delta\sigma}(t))$$

$$+ \iint_{\Omega_t} Q_\delta [(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x]^2 + \iint_{\Omega_t} Q_\delta(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2$$

$$= \frac{\varepsilon}{2}\int_\Omega (u_{0x})^2 + \frac{\varepsilon}{2}\int_\Omega (v_{0x})^2 + \int_\Omega \mathcal{F}_\sigma(u_0, v_0)$$

for almost every $t \in (0, T]$ (see [19] for a detailed proof). Using (1.4) and (3.1) we obtain

$$\frac{\varepsilon}{2}\int_\Omega [u_{\delta\sigma x}(t)]^2 + \frac{\varepsilon}{2}\int_\Omega [v_{\delta\sigma x}(t)]^2 + \int_\Omega \mathcal{F}_\sigma(u_{\delta\sigma}(t), v_{\delta\sigma}(t))$$

$$+ \iint_{\Omega_t} Q_\delta [(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x]^2 + \iint_{\Omega_t} Q_\delta(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2 \leqslant c_9 \tag{3.3}$$

which implies (iii), (iv) and (vii) since $-U_0 - V_0 - \frac{4}{e} \leqslant \mathcal{F}_\sigma$. Using the Poincaré inequality and (3.2), (i) is also verified. To prove (vi), we choose $\psi = v_{\delta\sigma t}/Q_\delta(u_{\delta\sigma}, v_{\delta\sigma})$ as a test function in the

second equation of ($\mathbf{P}_{\delta\sigma}$), which yields

$$\iint\limits_{\Omega_T} \frac{(v_{\delta\sigma t})^2}{Q_\delta(u_{\delta\sigma}, v_{\delta\sigma})} = -\iint\limits_{\Omega_T} (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})v_{\delta\sigma t} \leqslant \left(\iint\limits_{\Omega_T} Q_\delta(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx})^2\right)^{1/2} \left(\iint\limits_{\Omega_T} \frac{(v_{\delta\sigma t})^2}{Q_\delta}\right)^{1/2}.$$

Since $Q_\delta \leqslant 1$ for δ small enough, in particular it holds that

$$\int\limits_\Omega v_{\delta\sigma}^2(t) \leqslant 2\int\limits_\Omega v_0^2 + 2t\iint\limits_{\Omega_t} (v_{\delta\sigma t})^2 \leqslant c_{10}$$

which together with (3.3) implies (ii). Finally, (v) follows since

$$\left|\iint\limits_{\Omega_T} u_{\delta\sigma t}\phi\right| \leqslant \left(\iint\limits_{\Omega_T} Q_\delta \left[(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x\right]^2\right)^{1/2} \left(\iint\limits_{\Omega_T} (\phi_x)^2\right)^{1/2}$$

for all $\phi \in L^2(0, T; H^1(\Omega))$. $\qquad\qquad\square$

Arguing in a standard fashion (see [5] for a proof) we have

COROLLARY 3.2   There exists a constant C_2 independent of $0 < \sigma < \sigma_0, 0 < \delta < \delta_0$ such that

$$\begin{aligned}
\|u_{\delta\sigma}\|_{C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T)} &\leqslant C_2, \\
\|v_{\delta\sigma}\|_{C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T)} &\leqslant C_2.
\end{aligned} \tag{3.4}$$

3.2   *Systems with positive mobility*

Throughout this subsection we fix $\delta > 0$, and assume the mobility to be given by the form $Q_\delta(u, v)$ introduced in Subsection 3.1. However, we remark that the results in this subsection are independent of the particular form of the mobility, and hold for any $\tilde{Q}(u, v) \in C(\mathbb{R}^2; \mathbb{R})$ with $0 < Q_{\min} \leqslant \tilde{Q} \leqslant Q_{\max}$.

By Corollary 3.2 we can extract a subsequence (still denoted by $(u_{\delta\sigma}, v_{\delta\sigma})$) such that

$$\begin{aligned}
(u_{\delta\sigma}, v_{\delta\sigma}) &\to (u_\delta, v_\delta) \quad \text{uniformly in } \overline{\Omega}_T \text{ as } \sigma \to 0 \\
u_\delta &\in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T), \quad v_\delta \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T).
\end{aligned} \tag{3.5}$$

We now demonstrate that the limit (u_δ, v_δ) lies within the square B.

LEMMA 3.3   For all $0 < \delta < \delta_0$,

$$|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| = 0.$$

Proof. Let N denote minus the inverse of the Laplacian with zero Neumann boundary conditions: given $f \in (H^1(\Omega))'_{\text{null}} := \{f \in (H^1(\Omega))' : \langle f, 1\rangle = 0\}$, we define $Nf \in H^1(\Omega)$ as the unique

solution of

$$\int_{\Omega} (Nf)'\psi' = \langle f, \psi \rangle \quad \forall \psi \in H^1(\Omega)$$

$$\int_{\Omega} Nf = 0.$$

By (3.2) and Lemma 3.1 (i), $N(u_{\delta\sigma}(t) - \overline{u_{\delta\sigma}})$ is well defined. Choosing $\phi = Q_\delta^{-1} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}})$ as a test function in the equation for $u_{\delta\sigma}$, we have

$$\int_0^T \langle u_{\delta\sigma t}, Q_\delta^{-1} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}) \rangle \, dt =$$

$$= -\iint_{\Omega_T} (\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x (N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}))_x$$

$$+ \iint_{\Omega_T} Q_\delta^{-1}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x Q_{\delta x} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}})$$

$$= -\iint_{\Omega_T} (u_{\delta\sigma} - \overline{u_{\delta\sigma}})\mathcal{F}_{\sigma u} - \varepsilon \iint_{\Omega_T} (u_{\delta\sigma x})^2$$

$$+ \iint_{\Omega_T} Q_\delta^{-1}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x Q_{\delta x} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}).$$

Choosing $\psi = Q_\delta^{-1} v_{\delta\sigma}$ as a test function in the equation for $v_{\delta\sigma}$,

$$\iint_{\Omega_T} Q_\delta^{-1} v_{\delta\sigma} v_{\delta\sigma t} = -\iint_{\Omega_T} v_{\delta\sigma} \mathcal{F}_{\sigma v} - \varepsilon \iint_{\Omega_T} (v_{\delta\sigma x})^2.$$

Summing the two above identities, we obtain

$$\iint_{\Omega_T} (u_{\delta\sigma} - \overline{u_{\delta\sigma}})\mathcal{F}_{\sigma u} + \iint_{\Omega_T} v_{\delta\sigma} \mathcal{F}_{\sigma v}$$

$$= -\int_0^T \langle u_{\delta\sigma t}, Q_\delta^{-1} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}) \rangle \, dt - \varepsilon \iint_{\Omega_T} \left[(u_{\delta\sigma x})^2 + (v_{\delta\sigma x})^2 \right]$$

$$+ \iint_{\Omega_T} Q_\delta^{-1}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x Q_{\delta x} N(u_{\delta\sigma} - \overline{u_{\delta\sigma}}) - \iint_{\Omega_T} Q_\delta^{-1} v_{\delta\sigma} v_{\delta\sigma t}.$$

The estimates in Lemma 3.1, the lower bound for Q_δ, the fact that $\|Q_\delta\|_{C^1} \leqslant 2$ for δ sufficiently small, and the definition of N imply

$$\iint_{\Omega_T} [(u_{\delta\sigma} - \overline{u_{\delta\sigma}})\mathcal{F}_{\sigma u} + v_{\delta\sigma} \mathcal{F}_{\sigma v}] \leqslant c_{11} \tag{3.6}$$

(where $c_{11} = c_{11}(\delta)$). We now exploit the sign property of the integrand at the left hand side. To this purpose, we observe that the following identity holds for any $c \in \mathbb{R}$:

$$
\begin{aligned}
(u - c)\mathcal{F}_{\sigma u}(u, v) + v\mathcal{F}_{\sigma v}(u, v) = &\left\{(u + v)\left[F'_\sigma(u + v) - 1\right] + (u - v)\left[F'_\sigma(u - v) - 1\right]\right. \\
&+(1 - (u + v))\left[F'_\sigma(1 - (u + v)) - 1\right] \\
&\left.+(1 - (u - v))\left[F'_\sigma(1 - (u - v)) - 1\right]\right\} \\
&-cF'_\sigma(u + v) - cF'_\sigma(u - v) - (1 - c)F'_\sigma(1 - (u + v)) \\
&-(1 - c)F'_\sigma(1 - (u - v)) + (u - c)U'(u) + vV'(v) + 2.
\end{aligned}
\tag{3.7}
$$

The terms inside the braces are bounded from below since

$$
\begin{aligned}
0 &\leqslant s\left[F'_\sigma(s) - 1\right], & s &\leqslant 0, \\
-\tfrac{1}{e} - \tfrac{1}{2} &\leqslant \sigma\left[\ln \sigma - 1\right] \leqslant s\left[F'_\sigma(s) - 1\right] \leqslant 0, & 0 &< s \leqslant \sigma, \\
-\tfrac{1}{e} &\leqslant s\ln s = s\left[F'_\sigma(s) - 1\right] \leqslant 0, & \sigma &\leqslant s \leqslant 1 - \sigma, \\
-2 &\leqslant s\left[F'_\sigma(s) - 1\right] \leqslant 0, & 1 - \sigma &\leqslant s \leqslant 2, \\
0 &= s\left[F'_\sigma(s) - 1\right], & s &\geqslant 2
\end{aligned}
$$

for $\sigma \in (0, 1/2)$. Since U', V' are uniformly bounded, setting $c = \overline{u_{\delta\sigma}} \equiv \overline{u_0} \in (0, 1)$ in (3.7) and noting that $F'_\sigma \leqslant 1$, it follows from (3.6) that

$$
\begin{aligned}
-\iint\limits_{\Omega_T} &\left[F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) + F'_\sigma(u_{\delta\sigma} - v_{\delta\sigma})\right. \\
&\left.+F'_\sigma(1 - (u_{\delta\sigma} + v_{\delta\sigma})) + F'_\sigma(1 - (u_{\delta\sigma} - v_{\delta\sigma}))\right] \leqslant c_{12}.
\end{aligned}
\tag{3.8}
$$

To complete the proof, suppose by contradiction $|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| > 0$. Since the argument can be repeated for each side of the box B, we can assume without loss of generality that the set

$$
A = \{(x, t) \in \Omega_T : u_\delta + v_\delta \leqslant 0\}
$$

has positive measure. Since $F'_\sigma \leqslant 1$, the estimate (3.8) gives

$$
-\iint_A F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) \leqslant c_{13}.
$$

Note, however, that the uniform convergence of $u_{\delta\sigma}$ and $v_{\delta\sigma}$ implies that

$$
\forall \lambda > 0 \ \exists \sigma_\lambda : u_{\delta\sigma} + v_{\delta\sigma} \leqslant \lambda \quad \forall x \in A, \ \sigma < \sigma_\lambda;
$$

therefore, due to the convexity of F_σ, $F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) \leqslant F'_\sigma(\lambda)$. Hence

$$
c_{13} \geqslant -\lim_{\sigma \to 0} \iint_A F'_\sigma(\lambda) = -|A|(\ln \lambda + 1),
$$

which leads to a contradiction for λ sufficiently small. $\qquad\square$

In the next Lemma we derive additional estimates which will allow us to pass to the limit as $\sigma \to 0$ in Problem ($\mathbf{P}_{\delta\sigma}$). To simplify the notation we define the function

$$
\varphi_\sigma(s) = F''_\sigma(s) + F''_\sigma(1 - s),
$$

which is positive in view of the definition of F'_σ.

LEMMA 3.4   Let $\delta > 0$ be fixed. Then there exists a constant C_3 which is independent of σ such that

(i) $\|\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma})\|_{L^2(\Omega_T)} \leqslant C_3$,

(ii) $\|u_{\delta\sigma xx}\|_{L^2(\Omega_T)} \leqslant C_3$,

(iii) $\|v_{\delta\sigma xx}\|_{L^2(\Omega_T)} \leqslant C_3$,

(iv) $\iint\limits_{\Omega_T} \varphi_\sigma(u_{\delta\sigma} + v_{\delta\sigma})\, [(u_{\delta\sigma} + v_{\delta\sigma})_x]^2 + \iint\limits_{\Omega_T} \varphi_\sigma(u_{\delta\sigma} - v_{\delta\sigma})\, [(u_{\delta\sigma} - v_{\delta\sigma})_x]^2 \leqslant C_3.$

Proof. From Lemma 3.1 (iv) we have

$$\iint\limits_{\Omega_T} (\mathcal{F}_{\sigma v})^2 + 2\varepsilon \iint\limits_{\Omega_T} (\mathcal{F}_{\sigma v})_x v_{\delta\sigma x} + \varepsilon^2 \iint\limits_{\Omega_T} (v_{\delta\sigma xx})^2 \leqslant c_{14}. \tag{3.9}$$

Defining $H_{\delta\sigma} = \mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx}$, since $u_{\delta\sigma x}|_{S_T} = 0$

$$\int\limits_\Omega H_{\delta\sigma}(t) = \int\limits_\Omega \mathcal{F}_{\sigma u}(u_{\delta\sigma}(t), v_{\delta\sigma}(t)),$$

and from Lemma 3.1 (iii),

$$\iint\limits_{\Omega_T} (H_{\delta\sigma x})^2 \leqslant c_{15}.$$

Hence we can write

$$\iint\limits_{\Omega_T} (\mathcal{F}_{\sigma u})^2 + 2\varepsilon \iint\limits_{\Omega_T} (\mathcal{F}_{\sigma u})_x u_{\delta\sigma x} + \varepsilon^2 \iint\limits_{\Omega_T} (u_{\delta\sigma xx})^2 = \iint\limits_{\Omega_T} H_{\delta\sigma}^2$$

$$= \iint\limits_{\Omega_T} (H_{\delta\sigma} - \overline{H_{\delta\sigma}})^2 + \iint\limits_{\Omega_T} \overline{H_{\delta\sigma}}^2 \leqslant c_P \iint\limits_{\Omega_T} (H_{\delta\sigma x})^2 + \iint\limits_{\Omega_T} (\mathcal{F}_{\sigma u})^2,$$

where c_P denotes the Poincaré coefficient. Summing this inequality with (3.9) we obtain

$$\iint\limits_{\Omega_T} ((\mathcal{F}_{\sigma v})^2 + 2\varepsilon \iint\limits_{\Omega_T} [(\mathcal{F}_{\sigma u})_x u_{\delta\sigma x} + (\mathcal{F}_{\sigma v})_x v_{\delta\sigma x}] + \varepsilon^2 \iint\limits_{\Omega_T} \left[(u_{\delta\sigma xx})^2 + (v_{\delta\sigma xx})^2 \right] \leqslant c_{16}.$$

From this estimate, the Lemma follows recalling that $\varphi_\sigma > 0$ and observing that

$$\begin{aligned}
(\mathcal{F}_{\sigma u})_x u_{\delta\sigma x} + (\mathcal{F}_{\sigma v})_x v_{\delta\sigma x} \; &= \varphi_\sigma(u_{\delta\sigma} + v_{\delta\sigma})\, [(u_{\delta\sigma} + v_{\delta\sigma})_x]^2 \\
&+ \varphi_\sigma(u_{\delta\sigma} - v_{\delta\sigma})\, [(u_{\delta\sigma} - v_{\delta\sigma})_x]^2 \\
&+ U''(u_{\delta\sigma})\,(u_{\delta\sigma x})^2 + V''(v_{\delta\sigma})\,(v_{\delta\sigma x})^2
\end{aligned} \tag{3.10}$$

and using the definition of U, V and Lemma 3.1 (i), (ii) to bound the last two terms.   □

We can now state the following result.

PROPOSITION 3.5   For each $0 < \delta < \delta_0$ there exists a triplet $(u_\delta, v_\delta, w_\delta)$ such that:

   (i) $u_\delta, v_\delta \in L^\infty(0, T; \ H^1(\Omega))$ uniformly in δ;
  (ii) $u_\delta \in C^{0, \frac{1}{2}, \frac{1}{8}}(\overline{\Omega}_T)$ and $v_\delta \in C^{0, \frac{1}{2}, \frac{1}{4}}(\overline{\Omega}_T)$ uniformly in δ;
 (iii) $u_{\delta t} \in L^2(0, T; (H^1(\Omega))')$ and $Q_\delta^{1/2} v_{\delta t} \in L^2(\Omega_T)$ uniformly in δ;
  (iv) $u_{\delta xx}, v_{\delta xx} \in L^2(\Omega_T)$;
   (v) $|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| = 0$;
  (vi) $\mathcal{F}_v(u_\delta, v_\delta) \in L^2(\Omega_T)$ and $\mathcal{F}_u(u_\delta, v_\delta) \in L^2_{\mathrm{loc}}(\mathcal{D}(u_\delta))$;
 (vii) $w_\delta \in L^2(\Omega_T)$ uniformly in δ;
(viii) $u_\delta(x, 0) = u_0(x), v_\delta(x, 0) = v_0(x)$;
  (ix) $u_{\delta x}|_{S_T} = v_{\delta x}|_{S_T} = 0$ in $L^2(S_T)$;
   (x) $(u_\delta, v_\delta, w_\delta)$ solves $(\mathbf{P}_\delta)$ in the following sense:

$$\int_0^T \langle u_{\delta t}, \phi \rangle \, \mathrm{d}t = -\iint_{\Omega_T} Q_\delta^{1/2}(u_\delta, v_\delta) w_\delta \phi_x \quad \forall \phi \in L^2(0, T; H^1(\Omega)), \qquad (3.11)$$

$$\iint_{\Omega_T} v_{\delta t} \psi = -\iint_{\Omega_T} Q_\delta(u_\delta, v_\delta)(\mathcal{F}_v(u_\delta, v_\delta) - \varepsilon v_{\delta xx})\psi \quad \forall \psi \in L^2(\Omega_T), \qquad (3.12)$$

where $w_\delta = Q_\delta^{1/2}(\mathcal{F}_u - \varepsilon u_{\delta xx})_x$ in the sense that

$$\iint_{\Omega_T} w_\delta \phi = -\iint_{\Omega_T} (\mathcal{F}_u(u_\delta, v_\delta) - \varepsilon u_{\delta xx})(Q_\delta^{1/2}(u_\delta, v_\delta)\phi)_x \qquad (3.13)$$

for all $\phi \in \text{\L}^2(0, T; H_0^1(\Omega))$ with compact support in $\mathcal{D}(u_\delta)$.

Proof. Let $\{(u_{\delta\sigma}, v_{\delta\sigma})\}$ be the sequence given in (3.5). By Lemmas 3.1 and 3.4, it can be seen that

$$
\begin{array}{llll}
u_{\delta\sigma}, v_{\delta\sigma} & \xrightarrow{\;*\;} & u_\delta, v_\delta & \text{in} \quad L^\infty(0, T; \ H^1(\Omega)), \\
u_{\delta\sigma}, v_{\delta\sigma} & \longrightarrow & u_\delta, v_\delta & \text{in} \quad L^2(0, T; H^2(\Omega)), \\
u_{\delta\sigma t} & \longrightarrow & u_{\delta t} & \text{in} \quad L^2(0, T; (H^1(\Omega))'), \\
Q_\delta^{-1/2}(u_{\delta\sigma}, v_{\delta\sigma}) v_{\delta\sigma t} & \longrightarrow & Q_\delta^{-1/2}(u_\delta, v_\delta) v_{\delta t} & \text{in} \quad L^2(\Omega_T), \\
Q_\delta^{1/2}(u_{\delta\sigma}, v_{\delta\sigma})(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x & \longrightarrow & w_\delta & \text{in} \quad L^2(\Omega_T).
\end{array}
$$

Recalling also Corollary 3.2 and Lemma 3.3, (i)–(v) and (vii) now follow. Lemma 3.4 (i) implies that

$$\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma}) \longrightarrow g \quad \text{in} \quad L^2(\Omega_T);$$

since $\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma})$ converges pointwise in $\mathcal{B}(u_\delta, v_\delta)$ and $|\Omega_T \setminus \mathcal{B}(u_\delta, v_\delta)| = 0$, it follows that $g = \mathcal{F}_v(u_\delta, v_\delta)$, which proves the first part of (vi). In order to prove the second part, we recall that

$$
\begin{aligned}
\mathcal{F}_{\sigma u}(u, v) &= F_\sigma'(u + v) + F_\sigma'(u - v) - F_\sigma'(1 - (u + v)) - F_\sigma'(1 - (u - v)) + U'(u) \\
\mathcal{F}_{\sigma v}(u, v) &= F_\sigma'(u + v) - F_\sigma'(u - v) - F_\sigma'(1 - (u + v)) + F_\sigma'(1 - (u - v)) + V'(v).
\end{aligned}
$$

Lemma 3.4 (i) and the structure of $\mathcal{F}_{\sigma v}$ yield a uniform bound in $L^2(\mathcal{D}(u_\delta))$ for each of its terms. Then this implies in turn a uniform bound in $L^2(\mathcal{D}(u_\delta))$ for each of the terms in $\mathcal{F}_{\sigma u}$: we prove this in detail for the first term. Let $K \subset \mathcal{D}(u_\delta)$ be compact; then

$$K \subset \{\eta < u_\delta < 1 - \eta\}, \quad \frac{\eta}{2} < u_{\delta\sigma} < 1 - \frac{\eta}{2} \quad \text{in } K$$

for a suitable $\eta > 0$ and for σ sufficiently small. Let us consider the sets

$$K'_\sigma = K \cap \{u_{\delta\sigma} + v_{\delta\sigma} < \eta/2\}, \quad K''_\sigma = K \setminus K'_\sigma;$$

in K''_σ we have

$$0 \geqslant F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) - 1 \geqslant F'_\sigma(\eta/2) - 1 \geqslant c_{17}. \tag{3.14}$$

In order to estimate $F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma})$ in K'_σ, observe that the following relation holds:

$$\begin{aligned} 0 \geqslant F'_\sigma(u + v) - 1 &\geqslant F'_\sigma(u + v) + F'_\sigma(1 - (u - v)) - 2 \\ &= \mathcal{F}_{\sigma v} + F'_\sigma(u - v) + F'_\sigma(1 - (u + v)) - V'(v) - 2. \end{aligned}$$

From its definition, in K'_σ we have $u_{\delta\sigma} - v_{\delta\sigma} > \eta/2$ and $1 - (u_{\delta\sigma} + v_{\delta\sigma}) > 1 - \eta/2$; hence

$$\begin{aligned} 0 \geqslant F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) - 1 &\geqslant \mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma}) + F'_\sigma(\eta/2) + F'_\sigma(1 - \eta/2) - V'(v_{\delta\sigma}) - 2 \\ &\geqslant \mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma}) + c_{18} \end{aligned}$$

which, together with (3.14), implies

$$\iint\limits_K \left[F'_\sigma(u_{\delta\sigma} + v_{\delta\sigma}) - 1 \right]^2 \leqslant c_{19} + 2 \iint\limits_{\Omega_T} (\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma}))^2.$$

Repeating the same argument for the other terms, we conclude that

$$\iint\limits_K \mathcal{F}^2_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}) \leqslant c_{20} + 8 \iint\limits_{\Omega_T} (\mathcal{F}_{\sigma v}(u_{\delta\sigma}, v_{\delta\sigma}))^2$$

and therefore

$$\mathcal{F}_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}) \longrightarrow \mathcal{F}_u(u_\delta, v_\delta) \quad \text{in } L^2_{\text{loc}}(\mathcal{D}(u_\delta)), \tag{3.15}$$

which completes the proof of (vi).
The property (viii) is straightforward; by compactness we have that

$$u_{\delta\sigma} \longrightarrow u_\delta \quad \text{in } L^2(0, T; H^{2-\lambda}(\Omega)), \quad \lambda > 0, \tag{3.16}$$
$$v_{\delta\sigma} \longrightarrow v_\delta \quad \text{in } L^2(0, T; H^{2-\lambda}(\Omega)), \quad \lambda > 0, \tag{3.17}$$

which imply (ix).
To prove (x), we pass to the limit as $\sigma \to 0$ in the first equation of $(\mathbf{P}_{\delta\sigma})$, obtaining

$$\int\limits_0^T \langle u_{\delta t}, \phi \rangle \, dt = - \iint\limits_{\Omega_T} Q_\delta^{1/2}(u_\delta, v_\delta) w_\delta \phi_x \quad \forall \phi \in L^2(0, T; H^1(\Omega));$$

passing to the limit in the second equation of $(\mathbf{P}_{\delta\sigma})$ we have

$$\iint\limits_{\Omega_T} v_{\delta t}\psi = -\iint\limits_{\Omega_T} Q_\delta(u_\delta, v_\delta)(\mathcal{F}_v(u_\delta, v_\delta) - \varepsilon v_{\delta xx})\psi \quad \forall \psi \in L^2(\Omega_T).$$

To identify w_δ, we observe that for all $\sigma > 0$ and all $\phi \in L^2(0, T; H_0^1(\Omega))$ the following identity holds:

$$\iint\limits_{\Omega_T} Q_\delta^{1/2}(u_{\delta\sigma}, v_{\delta\sigma})(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x\phi = -\iint\limits_{\Omega_T} (\mathcal{F}_{\sigma u}(u_{\delta\sigma}, v_{\delta\sigma}) - \varepsilon u_{\delta\sigma xx})(Q_\delta^{1/2}(u_{\delta\sigma}, v_{\delta\sigma})\phi)_x.$$

$$(3.18)$$

By (3.15), (3.16) and (3.17) we can pass to the limit on the right hand side of (3.18), provided ϕ is compactly supported in $\mathcal{D}(u_\delta)$. $\qquad\square$

3.3   *The degenerate system*

Proposition 3.5 allows us to select a subsequence, which we still denote by $(u_\delta, v_\delta, w_\delta)$, such that

$$u_\delta, v_\delta \to u, v \text{ uniformly in } \overline{\Omega}_T,$$

$$u \in C^{0,\frac{1}{2},\frac{1}{8}}(\overline{\Omega}_T), \quad v \in C^{0,\frac{1}{2},\frac{1}{4}}(\overline{\Omega}_T), \tag{3.19}$$

$$w_\delta \longrightarrow w \text{ in } L^2(\Omega_T).$$

We already know by Lemma 3.3 that $(u, v) \in \overline{B}$; in the following Lemma we prove an estimate which shall enable us to improve these bounds. Let χ_A denote the characteristic function of the set A, and let

$$G(u, v) = v\left[\ln(u - v) + \ln(1 - (u + v))\right]\chi_{v\geqslant 0} - v\left[\ln(u + v) + \ln(1 - (u - v))\right]\chi_{v\leqslant 0}.$$

LEMMA 3.6   For all $0 < \delta < \delta_0$, $G(u_\delta, v_\delta) \in L^2(\Omega_T)$, and for $\beta \in (1, 4]$ there exist two positive constants β_0, C_4 independent of δ such that for any $t \in (0, T]$

$$\frac{1}{\beta-1}\int\limits_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - 2\iint\limits_{\Omega_t} Q_{1\delta}(u_\delta)Q_{2\delta}^{1-\beta}(v_\delta)G(u_\delta, v_\delta) +$$

$$+\varepsilon\beta_0\iint\limits_{\Omega_t} Q_{1\delta}(u_\delta)Q_{2\delta}^{-\beta}(v_\delta)(v_{\delta x})^2 \leqslant \frac{1}{\beta-1}\int\limits_\Omega Q_{2\delta}^{1-\beta}(v_0) + C_4 + C_4\iint\limits_{\Omega_t} Q_{2\delta}^{1-\beta}(v_\delta).$$

$$(3.20)$$

Proof. We choose $\psi = -Q_{2\delta}(v_\delta)^{-\beta}Q_{2\delta}'(v_\delta)\chi_{[0,t]}$ as a test function in (3.12). Observe that, since $(u_\delta, v_\delta) \in \overline{B}$, we have

$$Q_{1\delta}'(u_\delta) = 1 - 2u_\delta, \quad Q_{2\delta}'(v_\delta) = -2v_\delta.$$

Since $v_{\delta t} \in L^2(\Omega_T)$, for all $t \in (0, T]$ we obtain

$$\frac{1}{\beta-1}\int\limits_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - \frac{1}{\beta-1}\int\limits_\Omega Q_{2\delta}^{1-\beta}(v_0) = -\iint\limits_{\Omega_t} Q_{2\delta}^{-\beta}Q_{2\delta}'v_{\delta t}$$

$$= -2\iint\limits_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}v_\delta\mathcal{F}_v + 2\varepsilon\iint\limits_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}v_\delta v_{\delta xx}. \tag{3.21}$$

From the definition of G and $\mathcal{F}_v$ we can write the identity

$$G(u_\delta, v_\delta) = -v_\delta \mathcal{F}_v(u_\delta, v_\delta) + v_\delta \ln(u_\delta + v_\delta)\chi_{v_\delta > 0} - v_\delta \ln(u_\delta - v_\delta)\chi_{v_\delta < 0}$$
$$-v_\delta \ln(1 - (u_\delta + v_\delta))\chi_{v_\delta < 0} + v_\delta \ln(1 - (u_\delta - v_\delta))\chi_{v_\delta > 0} + v_\delta V'(v_\delta).$$

Since $u_\delta \geqslant 0$, we get

$$0 \geqslant v_\delta \ln(u_\delta + v_\delta)\chi_{v_\delta > 0} \geqslant v_\delta \ln(v_\delta)\chi_{v_\delta > 0} \geqslant -1/e,$$

and similar inequalities hold for the other logarithmic terms on the right hand side. Hence

$$0 \geqslant G(u_\delta, v_\delta) \geqslant -\frac{4}{e} - v_\delta \mathcal{F}_v(u_\delta, v_\delta) + v_\delta V'(v_\delta), \tag{3.22}$$

which implies that $G(u_\delta, v_\delta) \in L^2(\Omega_T)$. Substituting (3.22) into (3.21), we obtain

$$\frac{1}{\beta - 1}\int_\Omega Q_{2\delta}^{1-\beta}(v_\delta(t)) - \frac{1}{\beta - 1}\int_\Omega Q_{2\delta}^{1-\beta}(v_0) \leqslant$$
$$\leqslant 2\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}G + c_{21}\iint_{\Omega_t} Q_{2\delta}^{1-\beta} + 2\varepsilon\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}v_\delta v_{\delta xx}. \tag{3.23}$$

It remains to estimate the last integral at the right hand side of (3.23). Integrating by parts, we write

$$\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{1-\beta}v_\delta v_{\delta xx} = -\iint_{\Omega_t} Q_{1\delta}'Q_{2\delta}^{1-\beta}v_\delta u_{\delta x}v_{\delta x}$$
$$-\iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta}\left[Q_{2\delta} + 2(\beta - 1)v_\delta^2\right](v_{\delta x})^2 =: I_1 + I_2.$$

A straightforward calculation shows that

$$Q_2(v) + 2(\beta - 1)v^2 \geqslant \beta_0 := \min\left\{\frac{\beta - 1}{2}, \frac{1}{4}\right\} > 0, \quad -\frac{1}{2} \leqslant v \leqslant \frac{1}{2}.$$

Since, by its definition, $Q_{2\delta}(v) \geqslant Q_2(v)$ for $v \in [-1/2, 1/2]$, it follows therefore that

$$I_2 \leqslant -\beta_0 \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta}(v_{\delta x})^2.$$

Applying Cauchy–Schwarz and Young's inequalities to I_1, we have

$$|I_1| \leqslant \frac{1}{2}\beta_0 \iint_{\Omega_t} Q_{1\delta}Q_{2\delta}^{-\beta}(v_{\delta x})^2 + c_{22}\iint_{\Omega_t} Q_{1\delta}^{-1}(Q_{1\delta}')^2Q_{2\delta}^{2-\beta}v_\delta^2(u_{\delta x})^2.$$

Since $(u_\delta, v_\delta) \in \overline{B}$, we can make use of the following simple properties of B:

$$(u, v) \in \overline{B} \Longrightarrow \begin{cases} |v| \leqslant 2u(1 - u) \\ |1 - 2u| \leqslant 4(\frac{1}{4} - v^2), \end{cases}$$

which imply that

$$|v_\delta| \leqslant 2u_\delta(1-u_\delta) \leqslant 2Q_{1\delta},$$
$$(Q_{1\delta}')^2 = (1-2u_\delta)^2 \leqslant 16(\tfrac{1}{4}-v_\delta^2)^2 \leqslant 16Q_{2\delta}^2.$$

Therefore

$$|I_1| \leqslant \frac{1}{2}\beta_0 \iint\limits_{\Omega_t} Q_{1\delta} Q_{2\delta}^{-\beta} (v_{\delta x})^2 + c_{23} \iint\limits_{\Omega_t} |v_\delta| Q_{2\delta}^{4-\beta} (u_{\delta x})^2 \leqslant \frac{1}{2}\beta_0 \iint\limits_{\Omega_t} Q_{1\delta} Q_{2\delta}^{-\beta} (v_{\delta x})^2 + c_{24}$$

since $\beta \leqslant 4$. Collecting the estimates of I_1 and I_2, we finally obtain from (3.23)

$$\frac{1}{\beta-1}\int\limits_{\Omega} Q_{2\delta}^{1-\beta}(v_\delta(t)) - \frac{1}{\beta-1}\int\limits_{\Omega} Q_{2\delta}^{1-\beta}(v_0)$$
$$\leqslant c_{25} + 2\iint\limits_{\Omega_t} Q_{1\delta} Q_{2\delta}^{1-\beta} G + c_{21}\iint\limits_{\Omega_t} Q_{2\delta}^{1-\beta} - \varepsilon\beta_0 \iint\limits_{\Omega_t} Q_{1\delta} Q_{2\delta}^{-\beta} (v_{\delta x})^2, \tag{3.24}$$

which completes the proof of the Lemma. $\qquad\square$

COROLLARY 3.7   There exists a constant $d > 0$ such that

$$v(x,t) \in \left[-\frac{1}{2}+d, \frac{1}{2}-d\right] \quad \forall (x,t) \in \overline{\Omega}_T; \tag{3.25}$$

in addition,

$$|\mathcal{D}(u) \setminus \mathcal{B}(u,v)| = 0.$$

Proof. Applying a Gronwall argument to (3.20) with $\beta = 3$ and noting that $G(u_\delta, v_\delta) \leqslant 0$, we obtain

$$\int\limits_{\Omega} Q_{2\delta}^{-2}(v_\delta(t)) - \iint\limits_{\Omega_t} Q_{1\delta}(u_\delta) Q_{2\delta}^{-2}(u_\delta) G(u_\delta, v_\delta) + \iint\limits_{\Omega_t} Q_{1\delta}(u_\delta) Q_{2\delta}^{-3}(v_\delta) (v_{\delta x})^2 \leqslant c_{26}. \tag{3.26}$$

Suppose by contradiction that $v(x_0, t_0) = -\frac{1}{2}$. The uniform convergence of v_δ implies that for any $\gamma > 0$ there exists $\delta_\gamma > 0$ such that

$$|v_\delta(x,t_0) + \frac{1}{2}| \leqslant |v_\delta(x,t_0) - v(x,t_0)| + |v(x,t_0) + \frac{1}{2}| \leqslant \gamma + C_1|x-x_0|^{1/2}$$

for all $\delta < \delta_\gamma, x \in \overline{\Omega}$. Therefore, by (3.26) and (3.22)

$$c_{26} \geqslant \int\limits_{\Omega} \left(\left(\frac{1}{2}+\delta\right)^2 - v_\delta^2\right)^{-2} \geqslant \int\limits_{\Omega} (1+\delta)^{-2}\left(\gamma + C_1|x-x_0|^{1/2} + \delta\right)^{-2} \quad \forall \delta < \delta_\gamma,$$

which implies

$$c_{26} \geqslant \int\limits_{\Omega} (\gamma + C_1|x-x_0|^{1/2})^{-2}$$

a contradiction for γ sufficiently small. Thus $v > -\frac{1}{2}$ in $\overline{\Omega}_T$. Similarly we may show that $v < \frac{1}{2}$ in $\overline{\Omega}_T$, and the first assertion follows.

To prove the second statement, suppose by contradiction that the set $\mathcal{D}(u) \setminus \mathcal{B}(u, v)$ has positive measure. Without loss of generality we can assume that the set

$$A = \{(x, t) \in \Omega_T : u + v = 0, \ u > 0\}$$

has positive measure. Then, defining $A' = A \cap \mathcal{D}_\eta(u)$, we have $|A'| \geqslant |A|/2$ for η sufficiently small. By (3.19) and Lemma 3.3 it follows that

$$\forall \, \gamma > 0 \, \exists \, \delta_\gamma > 0 \ \text{s.t.} \ 0 < u_\delta + v_\delta < \gamma \ \text{and} \ |u_\delta - u| < \gamma \ \text{a.e. in} \ A', \ \forall \, \delta < \delta_\gamma; \quad (3.27)$$

it is now easy to check that almost everywhere in A'

$$\frac{2\eta}{3} < u_\delta < \frac{1}{2} + \frac{\eta}{3} \quad \text{and} \quad v_\delta < -\frac{\eta}{3} \quad \forall \, \gamma \in \left(0, \frac{\eta}{3}\right), \, \delta < \delta_\gamma. \quad (3.28)$$

From (3.26) and the definition of G, we obtain

$$\iint\limits_{\Omega_T} Q_{1\delta} Q_{2\delta}^{-2} |v_\delta \ln(u_\delta + v_\delta)| \chi_{v_\delta \leqslant 0} \leqslant -\iint\limits_{\Omega_T} Q_{1\delta} Q_{2\delta}^{-2} G(u_\delta, v_\delta) \leqslant c_{26}. \quad (3.29)$$

Since

$$|v_\delta \ln(u_\delta + v_\delta) \chi_{v_\delta > 0}| \leqslant |v_\delta \ln(v_\delta) \chi_{v_\delta > 0}| \leqslant \frac{1}{e},$$

using (3.29) and (3.25) we conclude that

$$\iint\limits_{\Omega_T} Q_{1\delta} |v_\delta \ln(u_\delta + v_\delta)| \leqslant c_{27},$$

and a contradiction now follows for γ sufficiently small, taking (3.28) into account and arguing as in the proof of Lemma 3.3. $\qquad \square$

In order to pass to the limit as $\delta \to 0$ we need an estimate independent of δ on the second derivatives of u_δ, v_δ and for the singular terms $\mathcal{F}_u(u_\delta, v_\delta)$, $\mathcal{F}_v(u_\delta, v_\delta)$. The next Lemma provides such an estimate on any compact subset of $\mathcal{D}(u)$.

LEMMA 3.8 For any $K \subset \mathcal{D}(u)$ compact there exist positive constants δ_K and C_K such that the following estimates hold for all $\delta < \delta_K$:

$$\|u_{\delta xx}\|_{L^2(K)} \leqslant C_K, \quad \|\mathcal{F}_u(u_\delta, v_\delta)\|_{L^2(K)} \leqslant C_K,$$
$$\|v_{\delta xx}\|_{L^2(K)} \leqslant C_K, \quad \|\mathcal{F}_v(u_\delta, v_\delta)\|_{L^2(K)} \leqslant C_K.$$

Proof. Let $K \subset \mathcal{D}(u)$ compact; then $K \subseteq \mathcal{D}_\eta(u)$ for some $\eta > 0$. Since there exists a positive distance between $\mathcal{D}_\eta(u)$ and $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$, it is possible to select $\zeta \in C^\infty(\overline{\Omega}_T)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $\mathcal{D}_\eta(u)$, $\zeta = 0$ in $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$ and $\zeta_x = 0$ over S_T.
Choosing $\psi = [Q_\delta(u_{\delta\sigma}, v_{\delta\sigma})]^{-1} \zeta^2 v_{\delta\sigma xx}$ as a test function in the second equation in $(\mathbf{P}_{\delta\sigma})$, we have

$$\iint\limits_{\Omega_T} \zeta^2 Q_\delta^{-1} v_{\delta\sigma xx} v_{\delta\sigma t} = -\iint\limits_{\Omega_T} \zeta^2 (\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx}) v_{\delta\sigma xx}. \quad (3.30)$$

By the choice of ζ, Corollary 3.7 and the equicontinuity of $\{u_{\delta\sigma}\}$ and $\{v_{\delta\sigma}\}$, the following properties hold for $0 < \delta < \delta_K$ and $0 < \sigma < \sigma(\delta)$:

$$u_{\delta\sigma} \in \left[\tfrac{\eta}{4}, 1 - \tfrac{\eta}{4}\right] \text{ in } \operatorname{supp} \zeta,$$
$$v_{\delta\sigma} \in \left[-\tfrac{1}{2} + \tfrac{d}{4}, \tfrac{1}{2} - \tfrac{d}{4}\right].$$

Hence

$$\left|\iint\limits_{\Omega_T} \zeta^2 Q_\delta^{-1} v_{\delta\sigma xx} v_{\delta\sigma t}\right| \leqslant \frac{1}{2\varepsilon} Q_{1\delta}^{-1}\left(\frac{\eta}{4}\right) Q_{2\delta}^{-1}\left(\frac{1}{2} - \frac{d}{4}\right) \iint\limits_{\Omega_T} \zeta^2 Q_\delta^{-1} (v_{\delta\sigma t})^2 + \frac{\varepsilon}{2}\iint\limits_{\Omega_T} \zeta^2 (v_{\delta\sigma xx})^2,$$

where we have used the Cauchy–Schwarz and Young's inequalities. Therefore, it follows from (3.30) and Lemma 3.1 (vi) that

$$\frac{\varepsilon}{2}\iint\limits_{\Omega_T} \zeta^2 (v_{\delta\sigma xx})^2 - \iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma v} v_{\delta\sigma xx} \leqslant c_{28} \quad \forall\ \delta < \delta_K,\ \sigma < \sigma(\delta). \tag{3.31}$$

We now choose $\phi = \zeta^2 \tilde{Q}_{1\delta}(u_{\delta\sigma})[Q_{2\delta}(v_{\delta\sigma})]^{-1}$ as a test function in the first equation of $(\mathbf{P}_{\delta\sigma})$, where $\tilde{Q}_{1\delta}(u)$ is defined by

$$\tilde{Q}_{1\delta}(u) = \int\limits_{1/2}^{u} \frac{1}{Q_{1\delta}(s)}\,\mathrm{d}s,$$

obtaining

$$\int\limits_{0}^{T} \langle u_{\delta\sigma t}, \zeta^2 \tilde{Q}_{1\delta} Q_{2\delta}^{-1}\rangle\,\mathrm{d}t = -\iint\limits_{\Omega_T} \zeta^2 u_{\delta\sigma x}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x$$
$$+ \iint\limits_{\Omega_T} \zeta^2 Q_{1\delta} \tilde{Q}_{1\delta} Q_{2\delta}^{-1} Q_{2\delta}' v_{\delta\sigma x}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x \tag{3.32}$$
$$- \iint\limits_{\Omega_T} (\zeta^2)_x Q_{1\delta} \tilde{Q}_{1\delta}(\mathcal{F}_{\sigma u} - \varepsilon u_{\delta\sigma xx})_x =: I_1 + I_2 + I_3.$$

It easily follows from the energy estimates in Lemma 3.1 and the definition of ζ that

$$\left|\int\limits_{0}^{T} \langle u_{\delta\sigma t}, \zeta^2 \tilde{Q}_{1\delta} Q_{2\delta}^{-1}\rangle\,\mathrm{d}t\right| + |I_2| + |I_3| \leqslant c_{29} \quad \forall\ \delta < \delta_K,\ \sigma < \sigma(\delta).$$

Let $\gamma > 0$ be a constant to be chosen later: integrating I_1 by parts and using the Cauchy–Schwarz and Young's inequalities we find that

$$I_1 \leqslant \iint\limits_{\Omega_T} \zeta^2 \mathcal{F}_{\sigma u} u_{\delta\sigma xx} - \frac{\varepsilon}{2}\iint\limits_{\Omega_T} \zeta^2 (u_{\delta\sigma xx})^2 + \frac{\gamma}{2}\iint\limits_{\Omega_T} \zeta^2 (\mathcal{F}_{\sigma u})^2 + 2\left(\varepsilon + \frac{1}{\gamma}\right)\iint\limits_{\Omega_T} (\zeta_x)^2 (u_{\delta\sigma x})^2.$$

Note that the last integral on the right hand side is uniformly bounded. Because of the definition of ζ and arguing as in the proof of Proposition 3.5, we find that

$$\frac{\gamma}{2} \iint\limits_{\Omega_T} \zeta^2 \left(\mathcal{F}_{\sigma u}\right)^2 \leqslant c_{30} + 4\gamma \iint\limits_{\Omega_T} \zeta^2 \left(\mathcal{F}_{\sigma v}\right)^2. \tag{3.33}$$

We now proceed to bound the right hand side of (3.33):

$$4\gamma \iint\limits_{\Omega_T} \zeta^2 \left(\mathcal{F}_{\sigma v}\right)^2 \leqslant 8\gamma \iint\limits_{\Omega_T} \zeta^2 \left(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx}\right)^2 + 8\gamma\varepsilon^2 \iint\limits_{\Omega_T} \zeta^2 \left(v_{\delta\sigma xx}\right)^2$$

$$\leqslant 8\gamma c_{31} \iint\limits_{\Omega_T} \zeta^2 Q_\delta \left(\mathcal{F}_{\sigma v} - \varepsilon v_{\delta\sigma xx}\right)^2 + 8\gamma\varepsilon^2 \iint\limits_{\Omega_T} \zeta^2 \left(v_{\delta\sigma xx}\right)^2,$$

and therefore, choosing $\gamma < 1/(32\varepsilon)$ and recalling Lemma 3.1 (iv) it turns out that

$$4\gamma \iint\limits_{\Omega_T} \zeta^2 \left(\mathcal{F}_{\sigma v}\right)^2 \leqslant c_{32} + \frac{\varepsilon}{4} \iint\limits_{\Omega_T} \zeta^2 \left(v_{\delta\sigma xx}\right)^2. \tag{3.34}$$

Summing (3.31) and (3.32) and using the estimates above we have

$$\frac{\varepsilon}{2} \iint\limits_{\Omega_T} \zeta^2 \left(u_{\delta\sigma xx}\right)^2 + \frac{\varepsilon}{4} \iint\limits_{\Omega_T} \zeta^2 \left(v_{\delta\sigma xx}\right)^2 - \iint\limits_{\Omega_T} \zeta^2 \left[\mathcal{F}_{\sigma u} u_{\delta\sigma xx} + \mathcal{F}_{\sigma v} v_{\delta\sigma xx}\right] \leqslant c_{33}. \tag{3.35}$$

Integrating by parts the last integral at the left hand side and recalling the identity (3.10), we obtain

$$-\iint\limits_{\Omega_T} \zeta^2 \left[\mathcal{F}_{\sigma u} u_{\delta\sigma xx} + \mathcal{F}_{\sigma v} v_{\delta\sigma xx}\right] =$$

$$= -\iint\limits_{\Omega_T} (\zeta^2)_{xx} \mathcal{F}_\sigma + \iint\limits_{\Omega_T} \zeta^2 \left[U''(u_{\delta\sigma}) \left(u_{\delta\sigma x}\right)^2 + V''(v_{\delta\sigma}) \left(v_{\delta\sigma x}\right)^2\right]$$

$$+ \iint\limits_{\Omega_T} \zeta^2 \left[\varphi_\sigma (u_{\delta\sigma} + v_{\delta\sigma}) \left[(u_{\delta\sigma} + v_{\delta\sigma})_x\right]^2 + \varphi_\sigma (u_{\delta\sigma} - v_{\delta\sigma}) \left[(u_{\delta\sigma} - v_{\delta\sigma})_x\right]^2\right].$$

Since $\mathcal{F}_\sigma$ is uniformly bounded in $L^1(\Omega_T)$ by Lemma 3.1 (vii), the second integral is uniformly bounded and the third is positive, we obtain

$$\iint\limits_{\Omega_T} \zeta^2 \left[\mathcal{F}_{\sigma u} u_{\delta\sigma xx} + \mathcal{F}_{\sigma v} v_{\delta\sigma xx}\right] \leqslant c_{34}.$$

Therefore, from (3.35), we can conclude that

$$\frac{\varepsilon}{2} \iint\limits_{\Omega_T} \zeta^2 \left(u_{\delta\sigma xx}\right)^2 + \frac{\varepsilon}{4} \iint\limits_{\Omega_T} \zeta^2 \left(v_{\delta\sigma xx}\right)^2 \leqslant c_{35} \quad \forall \; \delta < \delta_K, \; \sigma < \sigma(\delta).$$

Combining this estimate with (3.34) and (3.33) we obtain, moreover, that

$$\iint\limits_{\Omega_T} \zeta^2 \, (\mathcal{F}_\sigma u)^2 \leqslant c_{36}, \quad \text{and} \quad \iint\limits_{\Omega_T} \zeta^2 \, (\mathcal{F}_\sigma v)^2 \leqslant c_{37}.$$

The assertion now follows by means of the definition of ζ, the lower semicontinuity of L^2-norms and the arbitrariness of K. $\qquad\square$

Proof of Theorem 1.   Let (u_δ, v_δ) be the sequence given in (3.19). Using Proposition 3.5 and Lemma 3.8 we have

$$
\begin{aligned}
u_\delta, v_\delta &\xrightarrow{\;*\;} u, v &&\text{in} && L^\infty(0, T; H^1(\Omega)), \\
u_{\delta xx}, v_{\delta xx} &\longrightarrow u_{xx}, v_{xx} &&\text{in} && L^2_{\text{loc}}(\mathcal{D}(u)), \\
u_{\delta t} &\longrightarrow u_t &&\text{in} && L^2(0, T; (H^1(\Omega))'), \\
v_{\delta t} &\longrightarrow v_t &&\text{in} && L^2(\Omega_T),
\end{aligned}
$$

and (i)–(iii) follow. Property (iv) is a consequence of Corollary 3.7. From Lemma 3.8 we obtain

$$
\begin{aligned}
\mathcal{F}_u(u_\delta, v_\delta) &\longrightarrow g_1 \quad \text{in } L^2_{\text{loc}}(\mathcal{D}(u)), \\
\mathcal{F}_v(u_\delta, v_\delta) &\longrightarrow g_2 \quad \text{in } L^2_{\text{loc}}(\mathcal{D}(u)).
\end{aligned}
$$

Since $|\mathcal{D}(u) \setminus \mathcal{B}(u, v)| = 0$ and $u_\delta, v_\delta \to u, v$ uniformly in $\overline{\Omega}_T$, we have

$$\mathcal{F}_u(u_\delta, v_\delta) \longrightarrow \mathcal{F}_u(u, v) \quad \text{a.e. in } \mathcal{D}(u),$$

and therefore $g_1 = \mathcal{F}_u(u, v)$. By the same argument $g_2 = \mathcal{F}_v(u, v)$ and (v) is proved.
Property (vi) is straightforward. To prove (vii), let $K \subset \mathcal{D}(u)$ compact; then $K \subseteq \mathcal{D}_\eta(u)$ for some $\eta > 0$. Since $u \in C(\overline{\Omega}_T)$, we can select $\zeta \in C^\infty(\overline{\Omega}_T)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta = 1$ in $\mathcal{D}_\eta(u)$ and $\zeta = 0$ in $\overline{\Omega}_T \setminus \mathcal{D}_{\eta/2}(u)$. Hence we have

$$\|\zeta u_\delta\|_{L^2(0,T;H^2(\Omega))} \leqslant c_{38},$$

$$\|(\zeta u_\delta)_t\|_{L^2(0,T;(H^1(\Omega))')} \leqslant c_{39},$$

which implies that

$$(\zeta u_\delta)_x \longrightarrow (\zeta u)_x \quad \text{in } L^2(0, T; H^{1-\lambda}(\Omega)), \quad \lambda > 0. \tag{3.36}$$

The following continuous embedding and linear mapping

$$L^2(0, T; H^{1-\lambda}(\Omega)) \to L^2(0, T; H^{\frac{1}{2}-\lambda}(\partial\Omega)) \subset L^2(0, T; L^2(\partial\Omega)), \quad \lambda < \frac{1}{2},$$

yield

$$(\zeta u_\delta)_x|_{S_T} \to (\zeta u)_x|_{S_T} \quad \text{in } L^2(S_T)$$

and therefore $u_x|_{S_T \cap K} = 0$. Now (vii) follows applying the same argument to v_x.
Finally we pass to the limit as $\delta \to 0$ in equations (3.11)–(3.13) to prove (viii). The limit is straightforward in equations (3.11) and (3.12). With respect to (3.13) we observe that

$$(Q_\delta^{1/2}\phi)_x = \frac{1}{2}Q_{1\delta}^{-1/2}Q_{1\delta}'Q_{2\delta}^{1/2}u_{\delta x}\phi + \frac{1}{2}Q_{2\delta}^{-1/2}Q_{2\delta}'Q_{1\delta}^{1/2}v_{\delta x}\phi + Q_\delta^{1/2}\phi_x$$

and, for all $\phi \in L^2(0, T; H_0^1(\Omega))$ with compact support in $\mathcal{D}(u)$, each term converges strongly in $L^2(\Omega_T)$ because of (3.36) (with the analogous convergence for v) and the fact that $v \in (-\frac{1}{2}, \frac{1}{2})$. This completes the proof of the Theorem. □

4. Conclusions

Further questions regarding existence might focus around extending our results to higher space dimensions, as well as, attaining control on the measure and perhaps boundary of the set on which $u = 0$ or 1.

We remark that our analysis does not depend heavily on the choice of the polynomial terms in the free energy; in particular, an appropriate third or fourth order polynomial could be taken to model the behaviour of the free energy in the vicinity of the triple point as in [15].

With regard to limiting motion for the Allen–Cahn/Cahn–Hilliard system (**P**), it would be nice to be able to justify the predictions of the formal asymptotic analysis. But, as the asymptotic analysis has yet to be justified even for the case of a single Cahn–Hilliard equation with degenerate mobility and logarithmic free energy, such a task is at present somewhat remote. However, our anaylsis provides a step in this direction, and later on, hopefully, numerical schemes can be developed.

Finally, it would be nice to be able to treat other systems which couple degenerate fourth and second order equations which have a less distinct structure.

Acknowledgements

REFERENCES

1. ALLEN, S. E. & CAHN, J. W. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Met.* **27**, (1979) 1085–1095.
2. BARRETT, J. W. & BLOWEY, J. F. Finite element approximation of the Cahn–Hilliard equation with concentration dependent mobility. *Mathematics of Computation* **68**, (1999) 487–517.
3. BARRETT, J. W., BLOWEY, J. F., & GARCKE, H. Finite element approximation of the Cahn–Hilliard equation with degenerate mobility. *SIAM J. Num. Anal.* (to appear).
4. BERETTA, E., BERTSCH, M., & DAL PASSO, R. Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Rational Mech. Anal.* **129**, (1995) 175–200.
5. BERNIS, F. & FRIEDMAN, A. Higher order nonlinear degenerate parabolic equations. *J. Differential Equations* **83**, (1990) 179–206.
6. BERTSCH, M., DAL PASSO, R., & UGHI, M. Discontinuous "viscosity" solutions of a degenerate parabolic equation. *Trans. Amer. Math. Soc.* **320**, (1990) 779–798.
7. BLOWEY, J. F. & ELLIOTT, C. M. The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy - Part I: Mathematical analysis. *Euro. J. Appl. Math.* **2**, (1991) 233–279.
8. BLOWEY, J. F. & ELLIOTT, C. M. The Cahn–Hilliard gradient theory for phase separation with non-smooth free energy - Part II: Numerical analysis. *Euro. J. Appl. Math.* **3**, (1992) 147–179.
9. BROCHET, D., HILHORST, D., & NOVICK-COHEN, A. Inertial sets for Cahn–Hilliard/Cahn–Allen systems. *Appl. Math. Lett.* **7**, (1994) 83–87.

10. BRONSARD, L. GARCKE, H., & STOTH, B. A multi-phase Mullins–Sekerka system: matched asymptotic expansions and an implicit time discretization for the geometric evolution problem. *Proc. Roy. Soc. Edinburgh* **128 A**, (1998) 481–506.

11. BRONSARD, L. & REITICH, F. On three-phase boundary motion and the singular limit of a vector-valued Ginzburg–Landau equation. *Arch. Rat. Mech. Anal.* **124**, (1993) 355–379.

12. CAHN, J. W., ELLIOTT, C. M., & NOVICK-COHEN, A. The Cahn–Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature. *Euro. J. Appl. Math.* **7**, (1996) 287–301.

13. CAHN, J. W. & NOVICK-COHEN, A. Evolution equations for phase separation and ordering in binary alloys. *J. Stat. Phys.* **76**, (1994) 877–909.

14. CAHN, J. W. & NOVICK-COHEN, A. Limiting motion for an Allen–Cahn/Cahn–Hilliard system. In NIEZGODKA, N. & STRZELECKI, P. (eds), *Free Boundary Problems and Applications*. Addison Wesley Longman, London (1996).

15. CAHN, J. W. & NOVICK-COHEN, A. Motion by curvature and impurity drag: resolution of a paradox, Preprint.

16. CHEN, X. Global asymptotic limit of the Cahn–Hilliard equation, Preprint.

17. DAL PASSO, R., GARCKE H., & GRÜN, G. On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behaviour of solutions. *SIAM J. Math. Anal.* **29**, (1998) 321–342.

18. DAL PASSO, R. & LUCKHAUS, S. A degenerate diffusion problem not in divergence form. *J. Differential Equations* **69**, (1987) 1–14.

19. ELLIOTT, C. M. & GARCKE, H. On the Cahn–Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.* **27**, (1996) 404–423.

20. ELLIOTT, C. M. & GARCKE, H. Diffusional phase transitions in multicomponent systems with a concentration dependent mobility matrix. *Physica* D **109**, (1997) 242–256.

21. ELLIOTT, C. M. & LUCKHAUS, S. A generalized diffusion equation for phase separation of a multi-component mixture with interfacial free energy. IMA, University of Minnesota, Preprint 887 (1991).

22. EVANS, L. C., SONER, H. M., & SOUGANIDIS, P. Phase transitions and generalized mean curvature flow equations. *Comm. Pure Appl. Math.* **45**, (1992) 1097–1123.

23. GARCKE, H. & NOVICK-COHEN, A. A singular limit for a system of degenerate Cahn–Hilliard equations. *Advances Diff. Eqns* (to appear).

24. NOVICK-COHEN, A. The Cahn–Hilliard equation: mathematical and modeling perspectives. *Adv. Math. Sci. Appl.* **8**, (1998) 965–985.

25. NOVICK-COHEN, A. Triple junction motion for Allen–Cahn/Cahn–Hilliard systems. *Physica* D, (to appear).

26. NOVICK-COHEN, A. Energy methods for the Cahn–Hilliard equation. *Quart. Appl. Math.* **46**, (1988) 681–690.

27. PEGO, R. L. Front migration in the nonlinear Cahn–Hilliard equation. *Proc. Roy. Soc. London* A **422**, (1989) 116–133.

28. RUBINSTEIN, J., STERNBERG, P., & KELLER, J. Fast reaction, slow diffusion, and curve shortening. *SIAM J. Appl. Math.* **49**, (1989) 116–133.

29. SIMON, J. Compact sets in the space $L^p(0, T; B)$. *Annali Mat. Pura Appl.* **146**, (1987) 65–97.