Interfaces and Free Boundaries **2**, (2000) 181–200

Distribution of vortices in a Type-II superconductor as a free boundary problem: existence and regularity via Nash–Moser theory

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[Received 22 January 1999 and in revised form 15 October 1999]

This paper is concerned with a model describing the distribution of vortices in a Type-II superconductor. These vortices are distributed continuously and occupy an unknown region D with ∂D representing the free boundary. The problem is set as follows: two constants $H_0 > H_1 > 0$ are given, to find an open subset D of the smooth bounded open set $\Omega \subset \mathbb{R}^2$ and a function H defined on $\overline{\Omega \backslash D}$ such that:

$$\begin{cases} \operatorname{div}(F(|\nabla H|^2)\nabla H) - H = 0 \text{ in } \Omega\backslash\overline{D} \\ \text{where the function } F \text{ is analytic positive increasing} \\ H = H_0 \text{ on } \partial\Omega \\ H = H_1 \text{ on } \partial D \\ \frac{\partial H}{\partial n} = 0 \text{ on } \partial D. \end{cases}$$

Here we prove the existence of a solution H with a domain D having an analytic boundary. We use the Nash–Moser inverse function theorem applied to a degenerate case.

1. Introduction

1.1 *Physical motivation*

We are interested in a model of a Type-II superconductor submerged in a uniform magnetic field H_0. We examine different states of the material as a function of values of the applied magnetic field H_0. In [2], Berestycki, Bonnet & Chapman have studied the following model:

$$\begin{cases} \operatorname{div}(\frac{\nabla H}{1-u}) - H = 0 \text{ on } \Omega \subset \mathbb{R}^2 \\ \text{where } u(1-u)^2 = |\nabla H|^2 \\ H = H_0 \text{ on } \partial\Omega \end{cases}$$

which describes a cylindrical Type-II superconductor $\Omega \times \mathbb{R} \subset \mathbb{R}^3$. They were interested in solutions that are invariant by translation along the axis of the cylinder. It then reduces to a two-dimensional problem posed on the section Ω of the cylindrical superconductor. The equations are

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formally derived in [2] by passing to the limit when $\kappa \to +\infty$ in Ginzburg–Landau equations. These authors have proved that for fields $H_0 \in [0, H_0^*)$ there is a unique solution corresponding to the superconducting state. The value H_0^* corresponds to the superheating field beyond which the superconducting solution ceases to be stable (Chapman [3]). This instability appears on the boundary $\partial\Omega$ of the open set when the variable u reaches the value $\frac{1}{3}$ (meaning that $|\nabla H|^2$ reaches its maximal possible value which is $\frac{4}{27}$).

In the present work we consider the superconductor as submerged in a magnetic field superior to the latter superheating field H_0^*. In this case we see that a vorticity zone appears, in which some thin filaments of normal material are surrounded by superconducting currents. This state is referred to in the literature as the mixed state. In the limit $\kappa \to +\infty$ it is explained in [7] how one can formally derive a model where there exists a density of vortices in a certain zone D of the superconductor. This model is the following:

$$\begin{cases} \operatorname{div}(\frac{\nabla H}{1-u}) - H = 0 \text{ on } \Omega\backslash\overline{D} \\ \text{where } u(1-u)^2 = |\nabla H|^2 \\ H = H_0 = \text{const. on } \partial\Omega \\ H = H_1 = \text{const. on } \partial D \\ \partial_n H = 0 \text{ on } \partial D, \end{cases} \tag{1.1}$$

where n is the normal vector to ∂D, and the new parameter H_1 ($\leqslant H_0$) plays the role of a Lagrange multiplier for a constraint on the quantity of vortices in the superconductor.

Although problem (1.1) can be posed in higher dimensions as well, it has a physical meaning for our present paper only in dimension 2. Thus we only study the problem in this physical case. From a mathematical point of view, our method could be adapted to problems in higher dimensions.

The open set D could have a finite number of smooth connected components $\{D_i\}$ and our approach could also apply to the free boundary problem (1.1) with the following more general boundary conditions,

$$\left\{ \begin{array}{l} H = H_i \\ \partial_n H = 0 \end{array} \right| \text{ on each } D_i.$$

Nevertheless, in our article we only consider the special case where $H_i = H_1$ for all index i.

1.2 *Main results and ideas of proofs*

In this article we state existence and analytic regularity of the solution (D, H) to problem (1.1). Assuming $\partial\Omega \in C^\infty$, we prove the existence of a branch of solutions, applying the Nash–Moser inverse function theorem to the degenerate case: $H_1 = H_0$, $\Omega\backslash D = \emptyset$. We also prove some estimates for an associated degenerate elliptic problem.

As in [2], we are interested in solutions of (1.1) such that $u < \frac{1}{3}$. The analysis of Chapman [3] shows linear instability for solutions with $u \geqslant \frac{1}{3}$. We restrict ourselves to the study of solutions (D, H) to the following nonlinear equation:

$$\begin{cases} \operatorname{div}(F(|\nabla H|^2)\nabla H) - H = 0 \text{ on } \Omega\backslash\overline{D} \\ H = H_0 = \text{const. on } \partial\Omega \\ H = H_1 = \text{const. on } \partial D \\ \partial_n H = 0 \text{ on } \partial D, \end{cases} \tag{1.2}$$

where $F(s)$ is defined for $s \in [0, \frac{4}{27})$ by $F(s) = \frac{1}{1-u}$ and u is the unique solution to $u(1-u)^2 = s$ such that $0 \leqslant u < \frac{1}{3}$. The existence of a weak solution of (1.2) is proved in [7] such that $|\nabla H|^2 < \frac{4}{27} - \delta$, with some $\delta > 0$. This is a solution of a nonlinear obstacle problem. The uniqueness of this solution is proved also.

We are interested in the regularity of the free boundary ∂D. Our main result is the following.

THEOREM 1.1 $\forall H_0 > 0,\ \exists \underline{H_1} \in (0, H_0),\ \forall H_1 \in (\underline{H_1}, H_0)$: the free boundary ∂D of the solution to (1.2) is C^∞ and homeomorphic to $\partial \Omega$.

REMARK 1.2 If we know that ∂D is C^∞, the analyticity of ∂D is then a consequence of Kinderlehrer & Nirenberg's results [6], which deal with the case of analytic elliptic equations, with analytic boundary conditions on the free boundary.

The interest of our article lies on the one hand in the result of C^∞ regularity of the free boundary ∂D, and on the other hand in the use of the Nash–Moser inverse function theorem applied to a degenerate case. For references on this theorem, see the book by Alinhac & Gérard [1], and Hamilton [5]. Another proof of Theorem 1.1 can be derived by arguments coming from free boundary theory for the obstacle problem (see [7]).

In Section 3 we prove in particular the following result.

THEOREM 1.3 Let (D^*, H^*) be a particular solution of (1.2) for $H_1 = H_1^* \in (0, H_0)$ such that $D^* \subset\subset \Omega$ and $\partial D^* \in C^\infty$. Then problem (1.2) has a solution (D, H) with $\partial D \in C^\infty$, for every H_1 in a neighbourhood of H_1^*.

Some results similar to Theorem 1.3 are proved in Hamilton [5] and Schaeffer [10], and correspond to the use of the Nash–Moser theorem for a nondegenerate elliptic problem. Some degenerate cases are studied in Schaeffer [11] (the capacitor problem) and in Plotnikov [9] with methods different from those used in this article.

Problem (1.2) that we study is an extension of both Berestycki, Bonnet & Chapman [2] and Chapman, Rubinstein & Schatzman [4] combining the nonlinear field with the existence of a free boundary. In [4], Chapman, Rubistein & Schatzman consider a linearized version of model (1.2) for small H_0. They formally prove a result similar to Theorem 1.1 using asymptotic analysis.

Here, to prove Theorem 1.1, we apply the inverse function theorem beginning with the solution for $H_1^* = H_0$

$$\begin{cases} D^* = \Omega \\ H^* \equiv H_0 \text{ on } \partial\Omega = \partial D^*. \end{cases}$$

This solution is unusual because $\Omega \backslash D^* = \emptyset$. We obtain a branch of solutions for H_1 in an interval $(\underline{H_1}, H_0]$, where $\underline{H_1}$ is chosen close enough to $H_1^* = H_0$.

More precisely, we proceed in the following way.

To simplify, we set $\Gamma = \partial D$. We split up problem (1.2) into two sub-problems

Step 1. Given Γ, find H such that

$$\begin{cases} \operatorname{div}(F(|\nabla H|^2)\nabla H) - H = 0 \text{ on } \Omega \backslash \overline{D} \\ H = H_0 = \text{const. on } \partial\Omega \\ \partial_n H = 0 \text{ on } \Gamma. \end{cases} \tag{1.3}$$

We obtain the solution $H = H(\Gamma)$, by local inversion of problem (1.3), for Γ in a neighbourhood of $\Gamma^* = \partial D^*$.

Step 2. We impose on $H(\Gamma)$ the constraint

$$H(\Gamma)_{|\Gamma} = H_1 \tag{1.4}$$

which allows us to obtain $\Gamma = \Gamma(H_1)$, once more by local inversion, for H_1 in a neighbourhood of H_1^*.

This method presents two main difficulties, which we will examine briefly:

(i) Why does the usual inverse function theorem not apply?

(ii) How can we deal with the degeneracy: $\Omega\backslash\overline{D} \longrightarrow \emptyset$ as $H_1 \longrightarrow H_0$?

(i) Application of the inverse function theorem

The inverse function theorem in Banach spaces $C^{q,\alpha}$ can be applied to step 1 and gives $H = H(\Gamma)$. In contrast, if we note that $\Phi : C^{q,\alpha} \longrightarrow C^{q,\alpha}$, $\Gamma \longmapsto H(\Gamma)_{|\Gamma}$, a computation proves that the differential $D\Phi$ is a 1–1 map from $C^{q-1,\alpha}$ into $C^{q,\alpha}$ (and thus it is not invertible from $C^{q,\alpha}$ to $C^{q,\alpha}$), which in step 2 prevents us from using the inverse function theorem in Banach spaces $C^{q,\alpha}$.

Actually, the loss of derivative of one unity on the inverse of the differential of Φ comes from the fact that we impose two limit conditions on the free boundary Γ: a Dirichlet condition $H = H_1$ and a Neumann condition $\partial_n H = 0$ (see Subsection 3.2).

However, if we consider Φ as a map $\Phi : C^\infty \longrightarrow C^\infty$, then $D\Phi$ is a 1–1 map from C^∞ into C^∞, and the Nash–Moser inverse function theorem can be applied.

(ii) Degeneracy : $\Omega\backslash\overline{D} \longrightarrow \emptyset$ when $H_1 \longrightarrow H_0$

To apply the Nash–Moser theorem, we must prove that $(D\Phi)^{-1}$ is regular (more precisely C^∞-*tame*, see Subsection 2.1) until $H_1 = H_0$.

If ϵ is the order of the distance between Γ and $\partial\Omega$, then when $H_1 \longrightarrow H_0$, we have $\epsilon \longrightarrow 0$, and the difficulty is proving that $(D\Phi)^{-1}$ is C^∞-tame until $\epsilon = 0$.

The main idea is to scale the coordinate normal to the boundary so that the problem of the domain shrinking as $\epsilon \to 0$ is transformed to that of the elliptic constant tending to zero.

Hence, to inverse the differential $D\Phi$ is reduced to solving an elliptic equation on the fixed domain $\mathbb{S}^1 \times (0, 1)$ whose coordinates are $x = (s, \rho)$. Given the coefficients $A = (a_{ij}, b_j, c)$ and on the right-hand side $k = (k_0, k_1, k_2)$, we search for the solution w to the following equation

$$\begin{cases} a_{ij}(x)\partial_{ij} w + b_j(x)\partial_j w + c(x)w = k_0 \\ \partial_\rho w(\rho = 1, s) = k_1 \\ w(\rho = 0, s) = k_2. \end{cases} \tag{1.5}$$

We use the classical repeated index summation convention with $i, j \in \{\sigma, \rho\}$ and $\partial_\sigma = \epsilon\partial_s$. To be clear we have, for example,

$$a_{ij}(x)\partial_{ij} w = \epsilon^2 a_{\sigma\sigma}(x)\partial_{ss} w + 2\epsilon a_{\sigma\rho}\partial_{s\rho} w + a_{\rho\rho}(x)\partial_{\rho\rho} w.$$

In particular, for $\epsilon = 0$, problem (1.5) is degenerate. We assume that:

$$\exists c_0 > 0,\ \forall x \in \mathbb{S}^1 \times [0, 1], \forall \xi \in \mathbb{R}^2, a_{ij}(x)\xi_i\xi_j \geqslant c_0|\xi|^2. \tag{1.6}$$

We prove the following result of independent interest.

THEOREM 1.4 (Tame ellipticity in a degenerate case). Let us consider a solution w of problem (1.5) which we can write as

$$\mathcal{L}(A, \epsilon)w = k.$$

If there exist more particular coefficients $A^* \in C^\infty$ which satisfy (1.6) and such that $\mathcal{L}(A^*, 0)$ satisfies the maximum principle (see Subsection 5.1 for details), then $w = w(A, k, \epsilon)$ exists and is C^∞-tame on the right-hand side $k \in C^\infty$ of the equation and on the coefficients (A, ϵ) in a neighbourhood of $(A^*, 0)$.

1.3 *Organization of the article*

In Section 2 we recall some elements on tame maps and the Nash–Moser theorem, for the reader who is not familiar with this theory. The proof of Theorem 1.3 is given in Section 3. In particular, we explain in detail why the inverse function theorem in spaces $C^{q,\alpha}$ does not apply. The proof of Theorem 1.1 is given in Section 4. The last section, Section 5, is devoted to the proof of Theorem 1.4.

2. Preliminaries: tame maps and the Nash–Moser theorem

We recall below elements of the Nash–Moser theory that are used to prove the results presented in this article. For this presentation we have taken our inspiration from Hamilton [5]. In the following sections we will often refer to these preliminaries, but those readers who know the Nash–Moser theory can skip Section 2 and start with Section 3.

2.1 *Definitions*

Let us choose $\alpha \in (0, 1)$ which will be fixed hereafter. We recall some standard notations. For an integer q and a function u defined on a set $\Omega \subset \mathbb{R}^n$, $D^q u$ denotes all the partial derivatives of total order equal to q; $[u]_{q,\alpha;\Omega} = \sup_{x,y\in\Omega, x\neq y} \frac{|D^q u(x) - D^q u(y)|}{|x-y|^\alpha}$, $|u|_{0;\Omega} = \sup_{x\in\Omega} |u(x)|$, and $|u|_{q,\alpha;\Omega} = \sum_{j=0}^{q} |D^j u|_{0;\Omega} + [u]_{q,\alpha;\Omega}$.

Continuity. Let $\Phi : C^\infty(M) \to C^\infty(N)$, where M and N are C^∞ compact manifolds, possibly with boundaries. The sequences of seminorms $(|\cdot|_{q,\alpha})_{q\in\mathbb{N}}$ on M and N, allow us to define the continuity of the map Φ in $u_0 \in C^\infty(M)$, as

$$\begin{aligned} &\forall q \in \mathbb{N},\ \eta > 0,\ \exists q' \in \mathbb{N},\ \eta' > 0,\ \forall v \in C^\infty(M), \\ &(|v - u_0|_{q',\alpha;M} < \eta') \Longrightarrow (|\Phi(v) - \Phi(u_0)|_{q,\alpha;N} < \eta). \end{aligned} \tag{2.1}$$

This is the definition of the continuity of maps between Frechet spaces.

Classically, if we consider $\Phi : U \to C^\infty(N)$ where U is an open set of $C^\infty(M)$, we will say that Φ is C^0 (i.e. Φ is continuous) if and only if it is continuous in every point $u \in U$.

Differentiation. The map Φ is C^1 if and only if Φ is continuous and if there exists a continuous map denoted $D\Phi : U \times C^\infty(M) \to C^\infty(N)$ linear in the second variable and such that: $\forall u \in U,\ \forall v \in C^\infty(M),\ \lim_{t\to 0} \dfrac{\Phi(u+tv)-\Phi(u)}{t} = D\Phi(u)\cdot v$. We define in the same way the successive differentiations and we say that Φ is C^q if $D^{q-1}\Phi$ exists and is C^1. We say that Φ is C^∞ if it is C^q for all q.

In particular, to verify the continuity of $D^q\Phi$ on $U \times (C^\infty(M))^q \ni (u, v_1, \ldots, v_q)$ we use the natural grading (in the sense of Hamilton [5]) $|(u, v_1, \ldots, v_q)|_{p,\alpha;M} = |u|_{p,\alpha;M} + \sum_{i=1}^q |v_i|_{p,\alpha;M}$.

Tame maps. The map $\Phi : U \subset C^\infty(M) \to C^\infty(N)$ is C^0-tame if and only if it is C^0 and if it satisfies the tame inequality

$$\exists r \in \mathbb{N},\ \forall q \in \mathbb{N},\ \exists C_{q,\alpha} > 0,\ \forall u \in U,\ |\Phi(u)|_{q,\alpha} \leqslant C_{q,\alpha}(1 + |u|_{q+r,\alpha}). \tag{2.2}$$

It requires that we have a linear estimate on a nonlinear map Φ, and this is for all seminorms $|\cdot|_{q,\alpha;N}$. We may have a loss of derivatives $r \geqslant 0$ possibly large but independent on q. In addition, we recall that the Nash–Moser theorem stays true for the following more general definition of tame maps where (2.2) in the previous definition is replaced by

$$\begin{aligned} &\forall u_0 \in U,\ \exists \text{ neighbourhood } \mathcal{V}(u_0) \subset U,\ \exists r \in \mathbb{N},\ \exists b \in \mathbb{N},\ \forall q \geqslant b,\ \exists C_{q,\alpha} > 0,\\ &\forall u \in \mathcal{V}(u_0),\ |\Phi(u)|_{q,\alpha} \leqslant C_{q,\alpha}(1 + |u|_{q+r,\alpha}). \end{aligned} \tag{2.3}$$

In the same way Φ is C^q-tame if and only if the maps $(D^j\Phi)_{0\leqslant j\leqslant q}$ are C^0-tame, and Φ is C^∞-tame if and only if it is C^q-tame for all q.

REMARK 2.1 It is easy to see that: if Φ_1 and Φ_2 are tame, then the sum $\Phi_1 + \Phi_2$, the product $\Phi_1 \cdot \Phi_2$, the quotient $\frac{\Phi_1}{\Phi_2}$, the composition $f \circ \Phi_1$ by a function $f \in C^\infty$, the composition of tame maps $\Phi_1 \circ \Phi_2$, are tame maps.

2.2 *The Nash–Moser theorem*

THEOREM 2.2 (Nash–Moser). If M and N are C^∞ compact manifolds, possibly with boundaries, if $\Phi : U \subset C^\infty(M) \to C^\infty(N)$ is C^∞-tame, and if $\forall u \in U,\ \forall k \in C^\infty(N),\ D\Phi(u)\cdot w = k$ is invertible, with a unique solution $w = (D\Phi(u))^{-1}\cdot k$, and $(D\Phi(\cdot))^{-1}$ is a C^∞-tame map in (u, k), then Φ is locally invertible and Φ^{-1} is locally C^∞-tame.

REMARK 2.3 Contrary to the classical inverse function theorem in Banach spaces, here we must (for the spaces C^∞) check that the linear map $D\Phi(u)$ is invertible in a neighbourhood of one point u_0, and not only in the point u_0 (see Hamilton [5] for a counter-example).

2.3 *The tame ellipticity theorem*

The following theorem will be used several times in the following sections. It gives the solution to a nondegenerate elliptic equation as a C^∞-tame map of the coefficients and the right-hand side of the equation.

THEOREM 2.4 (Tame ellipticity, Hamilton [5]). Let the linear elliptic operator L_A of order 2, defined on a C^∞ compact manifold M, possibly with boundaries, such that denoting $A = (a_{ij}, b_j, c)$ which satisfies (1.6), we have $L_A w = a_{ij}(x)\partial_{ij} w + b_j(x)\partial_j w + c(x)w$, with boundary conditions

$$Bw = \begin{cases} w & \text{Dirichlet} \\ \text{or} & \\ \partial_n w & \text{Neumann} \end{cases}$$

where n is the exterior unit normal.

If the coefficients are C^∞, i.e. $A \in U \subset C^\infty(M)$, and if the open set U is chosen such that the system

$$\begin{cases} L_A w = k_0 \text{ on } M \\ Bw = k_B \text{ on } \partial M \end{cases}$$

has a unique solution w, then this solution, $w = w(A, k_0, k_B) \in C^\infty(M)$, is C^∞-tame in A and in $k = (k_0, k_B)$. That is, w is C^∞-tame relative to the coefficients and the right-hand side of the equation.

In particular, if we set $t = 0$ for Dirichlet conditions, $t = 1$ for Neumann conditions, and $|k|_{q,\alpha} = |k_0|_{q,\alpha} + |k_B|_{q+2-t,\alpha}$, then we have the tame elliptic inequality

$$\begin{aligned} &\forall A^* \in U,\ \exists \eta > 0,\ \forall A \in U, \\ &\big(|A - A^*|_{0,\alpha} < \eta\big) \Longrightarrow \big(\forall q \in \mathbb{N},\ |w|_{q+2,\alpha} \leqslant C_{q+2,\alpha}\left(|k|_{q,\alpha} + |A|_{q,\alpha}|k|_{0,\alpha}\right)\big). \end{aligned} \tag{2.4}$$

REMARK 2.5 Theorem 2.4 is true if the Neumann condition is changed into an oblique derivative condition. Moreover, in Theorem 2.4 the boundary conditions can be different on different connected components of the boundary of the manifold M. In this case, inequality (2.4) should naturally be adapted.

3. Existence result by perturbation of a smooth free boundary

In this section we will prove Theorem 1.3. This kind of result is referred to in the literature as the stability of the free boundary (see, for example, Schaeffer [10]). This approach consists of finding a solution to problem (1.2) by perturbation of a particular solution.

3.1 *Setting of the problem*

Let us assume that a particular solution (D^*, H^*) is given with $D^* \subset\subset \Omega$, $\Gamma^* = \partial D^* \in C^\infty$ and $H^*_{|\Gamma^*} = H_1^* < H_0$. For each value of the parameter H_1 in a neighbourhood of H_1^*, we will build a smooth solution (D, H) to problem (1.2).

To simplify (without loss of generality), we assume that Ω and D^* are diffeomorphic to a disk. Then we can construct a local curvilinear parametrization (r, s) in a neighbourhood of $\Gamma^* \subset\subset \Omega$: $s \in \mathbb{S}^1$ parametrizes Γ^* proportionally to its length, and r denotes the transversal coordinate positively oriented inward to the interior of D^*, with $r \in (-r_0, r_0), r_0 > 0$. Thus the boundary $\Gamma = \partial D$ of every smooth open set D close to D^* can be described in local coordinates by $r = G(s)$ where $G \in C^\infty(\mathbb{S}^1)$, and in particular, Γ^* is characterized by the equation $r = G^*(s) := 0$.

We follow the method of the proof given in the introduction.

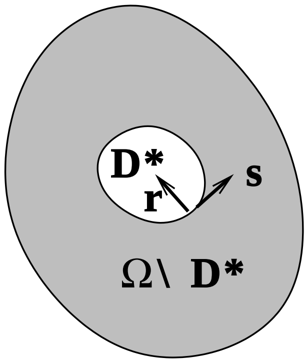

FIG. 1. Local coordinates near $\Gamma^* = \partial D^*$.

Step 1. Let us introduce the map

$$\begin{array}{llll} \Phi_1 : & C^\infty(\mathbb{S}^1) \times C^\infty(\overline{\Omega \setminus D}) & \longrightarrow & C^\infty(\Omega \setminus \overline{D}) \times C^\infty(\mathbb{S}^1) \times C^\infty(\mathbb{S}^1) \\ & (G, H) & \longmapsto & \Phi_1(G, H) = \left\{ \begin{array}{l} (\mathcal{Q}H)_{|\Omega\setminus\overline{D}} \\ H_{|\partial\Omega} - H_0 \\ \partial_r H_{|\Gamma} \end{array} \right. \end{array} \tag{3.1}$$

where $\mathcal{Q}$ is a quasilinear elliptic operator given by

$$\mathcal{Q}H = a^0_{ij}(\nabla H)\partial_{ij} H - H$$

with for every vector $V \in \mathbb{R}^2$

$$a^0_{ij}(V) := F(|V|^2)\delta_{ij} + 2F'(|V|^2)V_i V_j \tag{3.2}$$

and $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. With these notations we rewrite (1.3) (here $\partial_r H_{|\Gamma}$ stands in place of $\partial_n H_{|\Gamma}$) as

$$\Phi_1(G, H) = 0 \tag{3.3}$$

and we will be able to obtain the solution $H(G)$ by a proper inverse function theorem.

Step 2. We will set $\Phi_2(G) := H(G)_{|\Gamma} \in C^\infty(\mathbb{S}^1)$ and rewrite (1.4) as

$$\Phi_2(G) = H_1 \tag{3.4}$$

where H_1 is a constant close to H_1^*. Solving (3.4) will give the parametrization $G(H_1)$ of the free boundary Γ.

This method will give *a posteriori* $\partial_n H_{|\Gamma} = 0$, since $\partial_r H_{|\Gamma} = 0$ and $H_{|\Gamma} = H_1 =$ const. imply that $\nabla H = 0$ on Γ. Let us remark that we have chosen to write equations (3.1) and (3.3) with the condition $\partial_r H = 0$ on Γ, and not $\partial_n H = 0$ on Γ, because the vector field ∂_r is independent on Γ contrary to the normal vector n.

Although the inverse function theorem in the Banach spaces $C^{q,\alpha}$ applies in step 1, we will see why it does not apply in step 2.

3.2 *The inverse function theorem in spaces $C^{q,\alpha}$ does not apply*

Let us assume that step 1 is solved in the spaces $C^{q,\alpha}$, i.e. that the function $H(G)$ has been obtained. Then it is not possible to solve step 2 in spaces $C^{q,\alpha}$ because of the following lemma.

LEMMA 3.1 If $H : G \in C^{q,\alpha}(\mathbb{S}^1) \longmapsto H(G) \in C^{q,\alpha}(\overline{\Omega\backslash D})$ satisfies $\Phi_1(G, H(G)) = 0$ and if $\Phi_2 : G \in C^{q,\alpha}(\mathbb{S}^1) \longmapsto \Phi_2(G) = H(G)_{|\Gamma} \in C^{q,\alpha}(\mathbb{S}^1)$, then the differential $D_G\Phi_2(G^*)$ is a 1–1 map from $C^{q,\alpha}(\mathbb{S}^1)$ into $C^{q+1,\alpha}(\mathbb{S}^1)$.

In particular, $D_G\Phi_2(G^*)$ is not invertible from $C^{q,\alpha}(\mathbb{S}^1)$ into $C^{q,\alpha}(\mathbb{S}^1)$.

For the proof of Lemma 3.1, we use the following result.

LEMMA 3.2 Under the assumptions of Lemma 3.1, we obtain

$$D_G\Phi_2 \cdot g = m$$

if and only if

$$g = -\left(\frac{\partial_r h}{\partial_{rr} H}\right)_{|\Gamma}$$

where h satisfies

$$\begin{cases} D_H(\mathcal{Q}H) \cdot h = 0_{|\Omega\backslash\overline{D}} \\ h_{|\partial\Omega} = 0 \\ h_{|\Gamma} = m. \end{cases} \tag{3.5}$$

Proof of Lemma 3.2. Let us introduce $h := D_G H \cdot g$ where $g \in C^\infty(\mathbb{S}^1)$. Thus

$$D_G\Phi_2 \cdot g = (D_G H \cdot g + g\, \partial_r H)_{|\Gamma}$$

because $\frac{d}{dt}(u_t \circ v_t) = (\frac{du_t}{dt}) \circ v_t + (Du_t \circ v_t) \cdot \frac{dv_t}{dt}$. Here we know that H satisfies $\Phi_1(G, H(G)) = 0$, and then, in particular, $\partial_r H_{|\Gamma} = 0$. Consequently, by the definition of h we have

$$D_G\Phi_2 \cdot g = h_{|\Gamma}.$$

Differentiating $\Phi_1(G, H(G)) = 0$ with respect to G, we obtain

$$D_H(\mathcal{Q}H) \cdot h = 0_{|\Omega\backslash\overline{D}} \tag{3.6}$$

$$h_{|\partial\Omega} = 0 \tag{3.7}$$

$$(\partial_r h + g\partial_{rr} H)_{|\Gamma} = 0. \tag{3.8}$$

Therefore, equation $D_G\Phi_2 \cdot g = m$, written as

$$h_{|\Gamma} = m, \tag{3.9}$$

can be inverted into:

$$g = -\left(\frac{\partial_r h}{\partial_{rr} H}\right)_{|\Gamma}$$

where h is a solution of system (3.6)–(3.8). This ends the proof of Lemma 3.2. □

Proof of Lemma 3.1. For $H = H^*$, $H_1 = H_1^*$, $D = D^*$, we have $0 \neq \partial_{rr} H^*_{|\Gamma^*} \in C^\infty(\mathbb{S}^1)$. Therefore if $m \in C^{q+1,\alpha}$, we have from Lemma 3.2 that $(D_G \Phi_2(G^*))^{-1}(m) = g \in C^{q,\alpha}$, and therefore $D_G\Phi_2(G^*)$ is a 1–1 map from $C^{q,\alpha}(\mathbb{S}^1)$ to $C^{q+1,\alpha}(\mathbb{S}^1)$ which proves Lemma 3.1. □

There is a loss of derivatives on the inverse of the differential of Φ_2 which corresponds to the fact that we impose on Γ two boundary conditions of different orders: $H_{|\Gamma} = H_1$ and $\partial_r H_{|\Gamma} = 0$. We will therefore use the Nash–Moser theorem which applies in the spaces C^∞ to problems with loss of derivatives.

3.3 *Proof of Theorem 1.3*

Step 1: the function $H(G)$ obtained by the Nash–Moser theorem. Let us recall that applying the implicit function theorem to the equation $\Phi_1(G, H) = 0$ is the same as applying the inverse function theorem to the map $\Psi : (G, H) \longmapsto (G, \Phi_1(G, H))$. In particular, $D\Psi$ is invertible if and only if $D_H\Phi_1$ is.

There is a small technical difficulty: on the one hand, a variation of G changes the domain of definition of H, and on the another hand, the notion of tame map that we want to use to apply the Nash–Moser theorem has only been defined on fixed compacts. To be rigorous we should use a diffeomorphism $f_G : \overline{\Omega\backslash D} \to \overline{\Omega\backslash D^*}$ which depends smoothly on G. This would allows to bring back Φ_1 in a C^∞-tame map $\tilde{\Phi}_1$ defined on the fixed compact set $\overline{\Omega\backslash D^*}$.

Nevertheless, to avoid tedious computations, we will work with Φ_1 as if it was $\tilde{\Phi}_1$.

Now the tame ellipticity Theorem 2.4 applies to the following elliptic problem

$$D_H\Phi_1(G, H) \cdot h = k$$

and therefore gives $h = h(G, H, k)$ as a C^∞-tame map of its arguments.

Consequently, $D\Psi$ is invertible with a C^∞-tame inverse, and the Nash–Moser inverse function Theorem 2.2 applies. This proves that $\Psi(G, H) = (G, 0)$ is solved in $H = H(G)$ which is a C^∞-tame map.

Step 2: the solution $G = G(H_1)$. Let us consider the map $\Phi_2 : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ defined by $\Phi_2(G) := H(G)_{|\Gamma}$. Thus from Lemma 3.2, equation $D_G\Phi_2 \cdot g = m$ is solved setting $g = -(\frac{\partial_r h}{\partial_{rr} H})_{|\Gamma}$ where h is a solution of (3.5).

As in step 1 we prove that $(D_G\Phi_2)^{-1}$ is C^∞-tame which allows us to apply the Nash–Moser Theorem 2.2. This proves that Φ_2 is invertible in a neighbourhood of G^* which satisfies $\Phi_2(G^*) = H_1^*$. This ends the proof of Theorem 1.3. □

REMARK 3.3 In the proof of Theorem 1.3 we have assumed that D and Ω are diffeomorphic to a disk. It is straightforward to adapt the proof to cases where D and Ω have other smooth shapes and topologies (and not necessary the same).

4. Existence of a smooth free boundary close to $\partial\Omega$.

In this section we prove the main result of this article: Theorem 1.1.

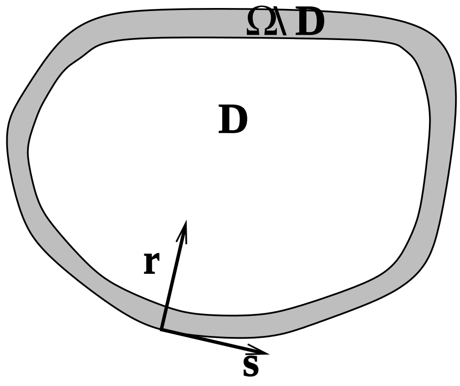

FIG. 2. Local coordinates near $\partial\Omega$.

4.1 *Setting of the problem*

Here we start from the particular solution that we know when $\Gamma = \partial\Omega$

$$\begin{cases} D^* = \Omega \\ H^* = H_0 = H_1^*. \end{cases}$$

We search for free boundaries $\Gamma = \partial D$ close to $\partial\Omega$. As in Section 3 we construct a local curvilinear parametrization (r, s) in a neighbourhood of $\partial D^* = \partial\Omega$: $s \in \mathbb{S}^1$ parametrizes $\partial\Omega$ proportionally to its length, and r denotes the transversal coordinate positively oriented inward to the interior of Ω, with $r \in (-r_0, r_0)$, $r_0 > 0$. We will note that

$$f(y_1, y_2) = (s, r),$$

the diffeomorphism associated to these coordinates.

The constant H_0 is fixed and H_1 is a parameter. For H_1 close to H_0, Γ is close to $\partial\Omega$, at a distance of order ϵ. Problem (1.2) can be approximated by the one-dimensional equation $H'' = H_1$, which should give

$$H_0 - H_1 = \frac{H_1}{2}\epsilon^2. \tag{4.1}$$

We take (4.1) as the definition of ϵ. Then it is natural to search for free boundaries $\Gamma = \partial D$ parametrized by $r = \epsilon G(s)$, such that, in the limit

$$G^* := 1.$$

A difficulty is that the open set $\Omega\backslash\overline{D}$ degenerates as $\epsilon \to 0$. Our goal is to obtain a branch of solutions starting from the obvious solution $G^* = 1$ in $\epsilon = 0$, and using the Nash–Moser inverse function theorem.

4.2 *The problem is reduced on the fixed compact set $\overline{\omega} = \mathbb{S}^1 \times [0, 1]$*

We reduce the problem on the compact set $\overline{\omega}$ where $\omega = \mathbb{S}^1 \times (0, 1)$. We will note (s, ρ) as a point of this set. For this we use the following diffeomorphism from $\overline{\Omega\backslash D}$ into $\overline{\omega} = \mathbb{S}^1 \times [0, 1]$ with

$$\rho = \frac{r}{\epsilon G(s)}.$$

We denote $\tilde{H}(s,\rho) = \frac{H(y_1,y_2)-H_1}{\epsilon^2 H_1}$. As in Section 3, we aim to solve equation $\Phi_1(G,H) = 0$, or equivalently

$$\tilde{\Phi}_1(G,\tilde{H},\epsilon) = 0$$

where

$$\begin{array}{llll} \tilde{\Phi}_1: & C^\infty(\mathbb{S}^1)\times C^\infty(\overline{\omega})\times(-\epsilon_0,\epsilon_0) & \longrightarrow & C^\infty(\omega)\times C^\infty(\mathbb{S}^1)\times C^\infty(\mathbb{S}^1) \\ & (G,\tilde{H},\epsilon) & \longmapsto & \tilde{\Phi}_1(G,\tilde{H},\epsilon) = \left\{ \begin{array}{l} (\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H})_{|\omega} - 1 \\ \tilde{H}_{|\partial\Omega} - \frac{1}{2} \\ \partial_\rho \tilde{H}_{|\Gamma^*} \end{array}\right. \end{array}$$

and we denote $\partial\Omega := \{\rho = 0\}$, $\Gamma^* := \{\rho = 1\}$. A straightforward computation gives:

$$\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H} = \tilde{a}_{ij}\partial_{ij}\tilde{H} + \epsilon\tilde{b}_j\partial_j\tilde{H} + \epsilon^2\tilde{c}\tilde{H}$$

where $i,j \in \{\sigma,\rho\}$, $\partial_\sigma = \epsilon\partial_s$ and the coefficients $\tilde{A} = (\tilde{a}_{ij},\tilde{b}_j,\tilde{c})$ are C^∞ functions of several arguments $\tilde{A} = \tilde{A}(\partial_\rho\tilde{H},\ \epsilon\partial_s\tilde{H},\ G, G', G'', Df, D^2 f, \epsilon, \rho, H_0)$, and

$$\tilde{a} = P\cdot a^0(V)\cdot {}^tP \text{ for some vector field } V:\omega\to\mathbb{R}^2$$

where $P = M\cdot(Df)$, $M = \begin{pmatrix} 1 & 0 \\ -\epsilon\rho\frac{G'(s)}{G(s)} & \frac{1}{G(s)} \end{pmatrix}$, a^0 is defined in (3.2) and $a^0_{ij}(V)\xi_i\xi_j \geqslant |\xi|^2$ because F is increasing and $F(0) = 1$. Therefore the quasilinear operator $\tilde{\mathcal{Q}}_{G,\epsilon}$ stays elliptic while $\epsilon > 0$ and G is close to $G^* = 1$.

Let us remark that the particular solution

$$\tilde{H}^*(s,\rho) = \frac{(\rho-1)^2}{2}$$

satisfies $\tilde{\Phi}_1(G^*,\tilde{H}^*,0) = 0$.

The domain of definition is now fixed, and we would like to apply the same method as in Section 3 (an ad hoc tame ellipticity theorem and the Nash–Moser theorem). For every $\epsilon > 0$, the tame ellipticity Theorem 2.4 applies. However, on the one hand our problem degenerates as $\epsilon \to 0$, because the ellipticity constant of the operator $\tilde{H} \longmapsto \tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}$ tends to 0. And on the other hand we want to find solutions starting from the case $\epsilon = 0$. We remove this difficulty by applying Theorem 1.4 which is another version of the tame ellipticity Theorem 2.4 which is valid in this degenerate case.

4.3 *Proof of Theorem 1.1*

Step 1: the map $(G,\epsilon) \longmapsto \tilde{H}(G,\epsilon)$ is C^∞-tame. We want to solve

$$\tilde{\Phi}_1(G,\tilde{H},\epsilon) = 0. \tag{4.2}$$

We obtain the straightforward result on the differential of $\tilde{\Phi}_1$:

LEMMA 4.1 We have

$$D_{\tilde{H}}\tilde{\Phi}_1 \cdot \tilde{h} = \begin{cases} D_{\tilde{H}}(\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}) \cdot \tilde{h}_{|\omega} \\ \tilde{h}_{|\partial\Omega} \\ \partial_\rho \tilde{h}_{|\Gamma^*} \end{cases}$$

where $\tilde{h} \in C^\infty(\overline{\varpi})$. For $A = (a_{ij}, b_j, c)$, let

$$\mathcal{L}(A, \epsilon)\tilde{h} := a_{ij}\partial_{ij}\tilde{h} + b_j\partial_j\tilde{h} + c\tilde{h}.$$

Then

$$D_{\tilde{H}}(\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}) \cdot \tilde{h} = \mathcal{L}(A, \epsilon)\tilde{h}$$

where coefficients $A = (a_{ij}, b_j, c)$ are C^∞ functions

$$A = A(\nabla\tilde{H},\ D^2\tilde{H},\ G, G', G'', Df, D^2 f, \epsilon,\ \rho, H_0).$$

Moreover, for $A^* = A(G^*, \tilde{H}^*, 0)$, we have

$$a^* = \begin{pmatrix} \left(\frac{2\pi}{|\partial\Omega|}\right)^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b^* = c^* = 0$$

where $|\partial\Omega|$ denotes the length of $\partial\Omega$.

Because $(G, \tilde{H}, \epsilon) \longmapsto A(G, \tilde{H}, \epsilon)$ is a C^∞-tame map, it is clear that for $(G, \tilde{H}, \epsilon)$ in a neighbourhood of $(G^*, \tilde{H}^*, 0)$, the coefficients A are close to A^*, and therefore we can apply Theorem 1.4. By composition we deduce that the solution $\tilde{h} = \tilde{h}(G, \tilde{H}, \epsilon, \tilde{k})$ of

$$\mathcal{L}(A(G, \tilde{H}, \epsilon), \epsilon)\tilde{h} = \tilde{k}$$

is C^∞-tame. Then the Nash–Moser Theorem 2.2 applies to the map $(G, \tilde{H}, \epsilon) \longmapsto (G, \tilde{\Phi}_1(G, \tilde{H}, \epsilon), \epsilon)$, defined in a neighbourhood of $(G^*, \tilde{H}^*, 0)$. We deduce the existence of a C^∞-tame solution $\tilde{H} = \tilde{H}(G, \epsilon)$ to (4.2).

Step 2: the solution $G = G(\epsilon)$. To obtain the solution $G(\epsilon)$ of our problem, we have to impose the condition $H_{|\Gamma} = H_1$, i.e. $\tilde{H}_{|\Gamma^*} = 0$. We introduce the map $\Phi_2(G, \epsilon) := \tilde{H}(G, \epsilon)_{|\Gamma^*}$ which is C^∞-tame. To inverse $D_G\Phi_2 \cdot g = m$, we go back to the initial equations written in $\Omega\backslash\overline{D}$ which do not depend explicitly on (G, ϵ).

We proceed as in the proof of Lemma 3.2 and a straightforward computation gives the following lemma.

LEMMA 4.2 With the notations of this section we have

$$D_G\Phi_2 \cdot g = m$$

if and only if

$$g = -G \cdot \left(\frac{\partial_\rho \tilde{h}}{\partial_{\rho\rho}\tilde{H}}\right)_{|\Gamma^*} \tag{4.3}$$

where $\tilde{h} \in C^{\infty}(\overline{\omega})$ satisfies

$$\begin{cases} D_{\tilde{H}}(\tilde{\mathcal{Q}}_{G,\epsilon}\tilde{H}) \cdot \tilde{h} = 0 \\ \tilde{h}_{|\partial\Omega} = 0 \\ \tilde{h}_{|\Gamma^*} = m. \end{cases}$$

We conclude as previously with the help of Theorem 1.4 (written here with two Dirichlet conditions), that $\tilde{h} = \tilde{h}(G, \epsilon, m)$ is C^{∞}-tame and therefore from (4.3): $g = g(G, \epsilon, m)$ is C^{∞}-tame.

We apply the Nash–Moser theorem to the map $(G, \epsilon) \longmapsto (\Phi_2(G, \epsilon), \epsilon)$ which satisfies $\Phi_2(G^*, 0) = \tilde{H}^*_{|\Gamma^*} = 0$. This proves the existence of the solution $G(\epsilon)$ to the equation

$$\Phi_2(G, \epsilon) = 0.$$

Theorem 1.1 is proved. □

5. The tame ellipticity theorem in a degenerate case

In this section we prove Theorem 1.4.

5.1 *Setting of the problem*

For $A = (a_{ij}, b_j, c)$ let us introduce the elliptic operator defined for $(s, \rho) \in \omega = \mathbb{S}^1 \times (0, 1)$

$$L_{A,\epsilon} w = a_{ij} \partial_{ij} w + b_j \partial_j w + cw$$

where $i, j \in \{\sigma, \rho\}$, $\partial_\sigma = \epsilon \partial_s$. We assume that $\epsilon \in \mathbb{R}$, that $A \in C^{\infty}(\overline{\omega})$, and that the coefficients A satisfy

$$\exists c_0 > 0, \ \forall x \in \mathbb{S}^1 \times [0, 1], \forall \xi \in \mathbb{R}^2, a_{ij}(x)\xi_i\xi_j \geqslant c_0|\xi|^2. \tag{5.1}$$

For $k_0 \in C^{\infty}(\overline{\omega})$ and $(k_1, k_2) \in C^{\infty}(\partial\omega)$ we consider the following system

$$\begin{cases} L_{A,\epsilon} w = k_0 \text{ on } \omega \\ \partial_\rho w(s, \rho = 1) = k_1(s) \\ w(s, \rho = 0) = k_2(s). \end{cases} \tag{5.2}$$

Let $k = (k_0, k_1, k_2)$. System (5.2) will be denoted as

$$\mathcal{L}(A, \epsilon) w = k.$$

For $\epsilon \neq 0$, let us recall that from the Fredholm alternative theorem, $\mathcal{L}(A, \epsilon)$ is invertible if and only if $\mathcal{L}(A, \epsilon) w = 0$ implies $w = 0$. Let us denote $k \leqslant 0$ if and only if $k_0 \leqslant 0$, $k_1 \geqslant 0$, $k_2 \geqslant 0$. Then let us recall that $\mathcal{L}(A, \epsilon)$ satisfies the maximum principle on ω if and only if for every $w \in C^{2,\alpha}(\overline{\omega})$, $\mathcal{L}(A, \epsilon) w \leqslant 0$ implies $w \geqslant 0$. In particular, if $\mathcal{L}(A, \epsilon)$ satisfies the maximum principle, then $\mathcal{L}(A, \epsilon)$ is invertible.

For $\epsilon = 0$, the operator is degenerate. For $s_0 \in \mathbb{S}^1$ we define $A_{s_0} = A(s_0, \cdot)$, $w_{s_0} = w(s_0, \cdot)$, $k_{s_0} = k(s_0, \cdot)$. We introduce the following one-dimensional elliptic operator defined for $v \in C^{\infty}([0, 1])$ by

$$\mathcal{L}_0(A_{s_0}) v = \begin{cases} a_{\rho\rho}(s_0, \rho) \partial_{\rho\rho} v(\rho) + b_\rho(s_0, \rho) \partial_\rho v(\rho) + c(s_0, \rho) v(\rho) \\ \partial_\rho v(\rho = 1) \\ v(\rho = 0). \end{cases}$$

With these notations we see that for $\epsilon = 0$, we have a continuous family of one-dimensional elliptic problems well posed on $(0, 1)$

$$\mathcal{L}(A, 0)w = k \Longleftrightarrow \forall s \in \mathbb{S}^1, \qquad \mathcal{L}_0(A_s)w_s = k_s.$$

In particular, we will say that the operator $\mathcal{L}(A, 0)$ satisfies the maximum principle on ω if and only if for every $s \in \mathbb{S}^1$, $\mathcal{L}_0(A_s)$ satisfies the maximum principle on $(0, 1)$.

REMARK 5.1 The main reason for which the limit at $\epsilon = 0$ is well posed is that the problem remains nondegenerate in one direction transversal to the boundary of the annulus $\mathbb{S}^1 \times (0, 1)$. In contrast, there is no general existence nor uniqueness of solutions to the limit problem on the same annulus for the operator $\partial_s^2 + \epsilon^2\partial_\rho^2$.

Let us denote $P = (A, k)$ and for $q \in \mathbb{N}$, $|P|_{q,\alpha} = |A|_{q,\alpha} + |k|_{q,\alpha}$ with $|k|_{q,\alpha} = |k_0|_{q,\alpha} + |k_1|_{q+1,\alpha} + |k_2|_{q+2,\alpha}$. We denote by $w(P, \epsilon)$ the solution to $\mathcal{L}(A, \epsilon)w = k$.

For $P^* = (A^*, 0)$ with A^* as in Theorem 1.4, we introduce the following open set

$$\mathcal{B} = \{P \in C^\infty, |P - P^*|_{0,\alpha} < R\}$$

with $R > 0$. We chose R small enough such that the coefficients A satisfy (5.1) for $(A, k) \in \overline{\mathcal{B}}$, and $\mathcal{L}(A, 0)$ satisfy the maximum principle on the same set of coefficients. We will prove

THEOREM 5.2 (i) $w(P, 0)$ is C^0-tame on $\mathcal{B}$
(ii) There exists $\epsilon_0 \in (0, 1)$ such that

$$\begin{aligned}&\forall q \in \mathbb{N},\ \exists C_{q,\alpha} > 0,\ \forall (P, \epsilon) \in \mathcal{B} \times (-\epsilon_0, \epsilon_0) :\\ &|w(P, \epsilon) - w(P, 0)|_{q,\alpha} \leqslant C_{q,\alpha}|\epsilon|^{1-\alpha}|P|_{q+3,\alpha}.\end{aligned}$$

REMARK 5.3 There appears to be a loss of derivatives in the degenerate case, in contrast to the tame ellipticity Theorem 2.4 for the nondegenerate case.

5.2 *Theorem 5.2 implies Theorem 1.4*

In this subsection, we prove in two steps that Theorem 5.2 implies Theorem 1.4.

Step A: if w is C^0-tame, then w is C^∞-tame. We denote $\mathcal{P} = (A, k, \epsilon)$ and $W = (w, \partial_s w, \partial_\rho w, \partial_{ss} w, \partial_{s\rho} w, \partial_{\rho\rho} w)$. These notations will only be used in this paragraph. In particular, W is C^0-tame if and only if w is. We have

$$\mathcal{L}(A, \epsilon)w = k \Longleftrightarrow \mathcal{L}(\mathcal{P})W(\mathcal{P}) = l_0(\mathcal{P})$$

where $\mathcal{L}(\mathcal{P})$ is a vector whose coordinates are polynomials in $\mathcal{P}$, $\mathcal{L}(\mathcal{P})W(\mathcal{P})$ is a bilinear vector function in $\mathcal{L}(\mathcal{P})$ and $W(\mathcal{P})$, and $l_0(\mathcal{P})$ is the linear polynomial in $\mathcal{P}$ defined by $l_0(\mathcal{P}) = k$.
If $W(\mathcal{P})$ is C^0-tame and satisfies $\mathcal{L}(\mathcal{P})W(\mathcal{P}) = l(\mathcal{P})$ where $l(\mathcal{P})$ is C^1-tame, then we obtain by differentiation

$$\mathcal{L}(\mathcal{P})\,(D_{\mathcal{P}}W(\mathcal{P}) \cdot p) = D_{\mathcal{P}}l(\mathcal{P}) \cdot p - (D_{\mathcal{P}}\mathcal{L}(\mathcal{P}) \cdot p)W(P).$$

Because the polynomial $\mathcal{L}(\mathcal{P})$ is C^∞-tame in $\mathcal{P}$ we deduce that $D_{\mathcal{P}}W(\mathcal{P})$ is C^0-tame and then $W(\mathcal{P})$ is C^1-tame.

Finally, because $l_0(\mathcal{P})$ is C^∞-tame, it is then straightforward to prove by recurrency that $D^m_{\mathcal{P}}W(\mathcal{P})$ is C^0-tame, for every $m \in \mathbb{N}$.

This proves that w is C^∞-tame.

Step B: Theorem 5.2 implies that w is C^0-tame. Let us denote $\theta(P,\epsilon) = w(P,\epsilon) - w(P,0)$ where we recall that $P = (A,k)$.

From Theorem 5.2 (i) and the tame ellipticity Theorem 2.4, it is easy to prove that θ is continuous for $\epsilon \neq 0$. Theorem 5.2 (ii) states that θ is continuous until $\epsilon = 0$ and then is C^0-tame on $\mathcal{B} \times (-\epsilon_0, \epsilon_0)$.

The solution $w = \theta + w(\cdot, 0)$ is the sum of two C^0-tame maps and therefore is C^0-tame on $\mathcal{B} \times (-\epsilon_0, \epsilon_0)$. Finally, from the following straightforward Lemma 5.4, $w(P,\epsilon) = w(A,k,\epsilon)$ is C^0-tame on a larger open set without bounds on $k \in C^\infty$.

This proves Theorem 1.4. □

We recall the following straightforward lemma, which is used in the proof (Step B) of Theorem 1.4.

LEMMA 5.4 Suppose that $\Phi(u,k)$ is C^0, linear in k, and satisfies a tame estimate when k is bounded, i.e.

$$\begin{aligned} &\exists R_0 > 0,\ \forall u \in U \subset C^\infty,\ \forall k \in C^\infty :\\ &\big(|k|_{b,\alpha} \leqslant R_0\big) \Longrightarrow \big(\forall q \in \mathbb{N},\ |\Phi(u,k)|_{q,\alpha} \leqslant C_{q,\alpha}\left(1 + |(u,k)|_{q+r,\alpha}\right)\big) \end{aligned} \tag{5.3}$$

then for all $k \in C^\infty$, we have a tame inequality without bound on k

$$\begin{aligned} &\forall u \in U \subset C^\infty,\ \forall k \in C^\infty :\ \forall q \in \mathbb{N},\\ &|\Phi(u,k)|_{q,\alpha} \leqslant \tfrac{C_{q,\alpha}}{R_0} |k|_{b,\alpha}(1 + |u|_{q+r,\alpha}) + C_{q,\alpha}|k|_{q+r,\alpha}. \end{aligned}$$

In other words, because Φ is linear in k, it is enough for Φ to satisfy (5.3) to be C^0-tame in $U \times C^\infty$ in the sense of Definition (2.3).

5.3 *Proof of Theorem 5.2 (i)*

We recall some useful tame inequalities. In particular, we will use them to prove that the map $P \longmapsto w(P,0)$ is C^0-tame.

5.3.1 *Fundamental tame inequalities.* Let u and v be some C^∞ functions on a smooth bounded open set Ω in $\mathbb{R}^n$; then we have the following tame inequalities:

product (see [1])

$$\forall q \in \mathbb{N}, \qquad |u \cdot v|_{q,\alpha} \leqslant C_{q,\alpha}(|u|_0 |v|_{q,\alpha} + |u|_{q,\alpha}|v|_0). \tag{5.4}$$

interpolation (see [5]). If (i,j) belongs to the segment of extremities (k,l) and (m,n), then for every positive or null integers i, j, k, l, m, n we have

$$|u|_{i,\alpha}|v|_{j,\alpha} \leqslant C_{(i,j,k,l,m,n),\alpha}(|u|_{k,\alpha}|v|_{l,\alpha} + |u|_{m,\alpha}|v|_{n,\alpha}).$$

In particular, we will use this inequality under the following form

$$\forall i,\ j \geqslant 0,\ \ |P|_{i,\alpha}|P|_{j,\alpha} \leqslant C|P|_{i+j,\alpha} \quad \text{if } |P|_{0,\alpha} \text{ is bounded.} \tag{5.5}$$

5.3.2 *The map $P \longmapsto w(P,0)$ is C^0-tame.* In all that follows C will denote a generic constant. To simplify the notations let us introduce $w^0(P) = w(P,0)$. For every $s \in \mathbb{S}^1$, we apply the tame ellipticity Theorem 2.4 to each one-dimensional elliptic problem

$$\mathcal{L}_0(A_s)w_s^0 = k_s \tag{5.6}$$

where we recall that $A_s = A(s,\cdot)$, $w_s^0 = w^0(s,\cdot)$, $k_s = k(s,\cdot)$. For every $s \in \mathbb{S}^1$ we find that with $P_s = P(s,\cdot)$,

$$|w_s^0|_{q+2,\alpha} \leqslant C|P_s|_{q,\alpha}. \tag{5.7}$$

Let us emphasize that here the coordinate s is fixed and then the Hölder norms $|\cdot|_{q,\alpha}$ apply to functions that only depend on the coordinate ρ.

The solution w_s^0 is C^∞ relative to P_s. In particular, it is derivable relative to s because $\partial_s w^0 = D_P w^0 \cdot \partial_s P$. Derivating equation (5.6), we obtain

$$\mathcal{L}_0(A_s)(\partial_s w_s^0) = K_s := \partial_s k_s - \mathcal{L}_0(\partial_s A_s)(w_s^0).$$

From the tame ellipticity estimate (2.4) we obtain

$$|\partial_s w_s^0|_{q+2,\alpha} \leqslant C(|K_s|_{q,\alpha} + |K_s|_{0,\alpha}|A_s|_{q,\alpha})$$

and

$$\begin{aligned}|K_s|_{q,\alpha} &\leqslant C(|\partial_s k_s|_{q,\alpha} + |\mathcal{L}_0(\partial_s A_s)\cdot w_s^0|_{q,\alpha})\\ &\leqslant C(|k_s|_{q+1,\alpha} + |\partial_s A_s|_{0,\alpha}|w_s^0|_{q+2,\alpha} + |\partial_s A_s|_{q,\alpha}|w_s^0|_{2,\alpha})\\ &\leqslant C(|P_s|_{q+1,\alpha} + |P_s|_{1,\alpha}|P_s|_{q,\alpha} + |P_s|_{q+1,\alpha}|P_s|_{0,\alpha})\\ &\leqslant C|P_s|_{q+1,\alpha}\end{aligned}$$

where we have used (5.4) for the second line, (5.7) for the third line, and for the last line: (5.5) and the fact that $|P_s|_{0,\alpha}$ is bounded because $P \in \mathcal{B}$. We find that for every $s \in \mathbb{S}^1$

$$|\partial_s w_s^0|_{q+2,\alpha} \leqslant C|P_s|_{q+1,\alpha}.$$

By a straightforward recurrency we obtain

$$|\partial_s^m w_s^0|_{q+2,\alpha} \leqslant C|P_s|_{q+m,\alpha}.$$

In particular, this implies with the Hölder norms on $\omega = \mathbb{S}^1 \times (0,1)$ that

$$|w^0|_{q,\alpha} \leqslant C_{q,\alpha}|P|_{q+1,\alpha}$$

which proves that w^0 is C^0-tame. □

5.4 *Proof of Theorem 5.2 (ii)*

We want to find estimates on $\theta(P,\epsilon) = w(P,\epsilon) - w^0(P)$ with $w^0(P) = w(P,0)$.

Let us remark that

$$\mathcal{L}(A,\epsilon)\theta = -(\mathcal{L}(A,\epsilon) - \mathcal{L}(A,0))w^0.$$

The linear differential operator $\mathcal{L}(A,\epsilon)$ is linear in A and polynomial in ϵ, therefore we can write

$$\mathcal{L}(A,\epsilon)\theta = K := -\epsilon\mathcal{M}(A,\epsilon)w^0$$

where $\mathcal{M}(A,\epsilon)$ is a linear differential operator, whose coefficients are linear in A and polynomials in ϵ. To obtain elliptic estimates with constants independent on ϵ, it is useful to introduce the change of coordinates: $\sigma = \frac{s}{\epsilon}, \omega_\epsilon = \frac{\mathbb{S}^1}{\epsilon}\times(0,1), \overline{A}(\sigma,\rho) = A(s,\rho), \overline{\theta}(\sigma,\rho) = \theta(s,\rho), \overline{K}(\sigma,\rho) = K(s,\rho)$. We deduce that

$$\mathcal{L}(\overline{A},1)\overline{\theta} = \overline{K} \qquad \text{on } \omega_\epsilon. \tag{5.8}$$

The quantity ϵ_0 introduced in Theorem 5.2 is given by

LEMMA 5.5 (Schauder estimate). There exists $\epsilon_0 \in (0,1)$ and a constant $C_0 > 0$ such that for each $\epsilon \in (-\epsilon_0,\epsilon_0)\backslash\{0\}$, for each coefficient $\overline{A}(\sigma,\rho) = A(\epsilon\sigma,\rho)$ such that $(A,0) \in \mathcal{B}$, and for every $\overline{\theta} \in C^{2,\alpha}$ solution to (5.8), we have

$$|\overline{\theta}|_{2,\alpha} \leqslant C_0|\overline{K}|_{0,\alpha}.$$

Proof deferred

Estimate on $|\overline{\theta}|_{2,\alpha}$. For every $(P,\epsilon) \in \mathcal{B}\times((-\epsilon_0,\epsilon_0)\backslash\{0\})$, we deduce the straightforward estimates

$$\begin{aligned} |\overline{\theta}|_{2,\alpha} &\leqslant C|\overline{K}|_{0,\alpha} \\ &\leqslant C|K|_{0,\alpha} \\ &\leqslant C|\epsilon||\mathcal{M}(A,\epsilon)w^0|_{0,\alpha} \\ &\leqslant C|\epsilon||A|_{0,\alpha}|w^0|_{2,\alpha}. \end{aligned}$$

It gives

$$|\overline{\theta}|_{2,\alpha} \leqslant C|\epsilon||P|_{3,\alpha}.$$

Estimate on $|\partial_\sigma\overline{\theta}|_{2,\alpha}$. We derive equation $\mathcal{L}(\overline{A},1)\overline{\theta} = \overline{K}$ relative to σ, and we obtain

$$\mathcal{L}(\overline{A},1)(\partial_\sigma\overline{\theta}) = \partial_\sigma\overline{K} - \mathcal{L}(\partial_\sigma\overline{A},1)\overline{\theta}.$$

We proceed similarly as in Subsection 5.3.2, and this gives

$$|\partial_\sigma\overline{\theta}|_{2,\alpha} \leqslant C\epsilon^2|P|_{4,\alpha}.$$

Estimate on $|\partial_\rho\overline{\theta}|_{2,\alpha}$. We have:

$$|\partial_\rho\overline{\theta}|_{2,\alpha} \leqslant C(|\overline{\theta}|_{2,\alpha} + |\partial_\sigma\overline{\theta}|_{2,\alpha} + |\partial^3_\rho\overline{\theta}|_{0,\alpha}).$$

Then we only need to estimate $|\partial^3_\rho\overline{\theta}|_{0,\alpha}$. Let us note that $\overline{K} = (\overline{K}_0,\overline{K}_1,\overline{K}_2)$. With these notations we have $L_{\overline{A},1}\overline{\theta} = \overline{K}_0$, i.e.

$$\overline{a}_{ij}\partial_{ij}\overline{\theta} + \overline{b}_j\partial_j\overline{\theta} + \overline{c}\,\overline{\theta} = \overline{K}_0$$

where $i,j \in \{\sigma,\rho\}$. We deduce that

$$\partial^2_\rho\overline{\theta} = \overline{K}'_0$$

where $\overline{K}'_0$ is a function on $(\overline{K}_0,\overline{A},\overline{\theta},\nabla\overline{\theta},\partial_\rho\nabla\overline{\theta})$. Thus by derivation relative to ρ we can estimate $\partial^3_\rho\overline{\theta}$. A straightforward computation gives

$$|\partial^3_\rho\overline{\theta}|_{0,\alpha} \leqslant C|\epsilon||P|_{4,\alpha}$$

and then

$$|\partial_\rho \overline{\theta}|_{2,\alpha} \leqslant C|\epsilon||P|_{4,\alpha}.$$

Estimates on higher derivatives of $\overline{\theta}$

A straightforward recurrency allows us to prove that for every $j \geqslant i \geqslant 0$ we have

$$|\partial_\sigma^i \partial_\rho^{j-i} \overline{\theta}|_{2,\alpha} \leqslant C_{j,\alpha}|\epsilon|^{i+1}|P|_{j+3,\alpha}. \tag{5.9}$$

We deduce that

$$\forall q \geqslant 0,\ \exists C_{q,\alpha} > 0,\ \forall (P,\epsilon) \in \mathcal{B} \times (-\epsilon_0, \epsilon_0),$$
$$|\theta|_{q,\alpha} \leqslant C_{q,\alpha}|\epsilon|^{1-\alpha}|P|_{q+3,\alpha}.$$

This ends the proof of Theorem 5.2 (ii). □

REMARK 5.6 In fact we can also obtain the following estimate: $|\theta|_q \leqslant C_{q,\alpha}|\epsilon||P|_{q+4,\alpha}$.

Proof of Lemma 5.5. For this proof we take our inspiration from some ideas in [7], which were based on Morrey [8].

If the lemma is false, then we can find sequences $(\epsilon_n)_n \in (-1,1)\backslash\{0\}$, $(A_n, k_n)_n \in \mathcal{B}$, $(\overline{K}_n)_n$ and $(\overline{\theta}_n)_n \in C^{2,\alpha}$ such that for $\overline{A}_n(\sigma,\rho) = A_n(\epsilon_n\sigma,\rho)$

$$\mathcal{L}(\overline{A}_n, 1)\overline{\theta}_n = \overline{K}_n \qquad \text{on } \omega_{\epsilon_n} = \frac{\mathbb{S}^1}{\epsilon_n} \times (0,1)$$

and $\epsilon_n \to 0$, $|\overline{\theta}_n|_{2,\alpha} = 1$, $|\overline{K}_n|_{0,\alpha} \to 0$. For $\sigma_0 \in \mathbb{R}$ and $\mu > 0$ let us define the box $C_\mu(\sigma_0) = \{(\sigma,\rho), \rho \in [0,1], |\sigma - \sigma_0| \leqslant \mu\}$. Then, on the one hand, there exists a constant $C > 1$ and $\sigma_n \in \frac{\mathbb{S}^1}{\epsilon_n}$ such that

$$|\overline{\theta}_n|_{2,\alpha;C_1(\sigma_n)} \leqslant |\overline{\theta}_n|_{2,\alpha} \leqslant C|\overline{\theta}_n|_{2,\alpha;C_1(\sigma_n)}.$$

On the other hand, from elliptic estimates (see Morrey [8]) there exists a constant $C_0 > 0$ such that

$$|\overline{\theta}_n|_{2,\alpha;C_1(\sigma_n)} \leqslant C_0(|\overline{\theta}_n|_{L^1(C_2(\sigma_n))} + |\overline{K}_n|_{0,\alpha;C_2(\sigma_n)}).$$

Up to extraction of some subsequence, we have $\epsilon_n\sigma_n \to s_\infty \in \mathbb{S}^1$, $A_n \to A_\infty$ and $\overline{A}_n \to \overline{A}_\infty$ where $\overline{A}_\infty(\sigma,\rho) = \overline{A}_{\infty,s_\infty}(\rho) = \lim_{n\to+\infty} A_n(\epsilon_n\sigma + \epsilon_n\sigma_n, \rho) = A_\infty(s_\infty,\rho)$. Moreover, $\overline{\theta}_n(\sigma+\sigma_n,\rho) \to \overline{\theta}_\infty(\sigma,\rho)$ which satisfies $|\overline{\theta}_\infty|_{2,\alpha} \leqslant 1$, and then

$$\frac{1}{CC_0} \leqslant |\overline{\theta}_\infty|_{L^1(C_2(0))}. \tag{5.10}$$

We also have

$$\mathcal{L}(\overline{A}_\infty, 1)\overline{\theta}_\infty = 0 \qquad \text{on } \omega_0 = \mathbb{R} \times (0,1).$$

Let the subsolution $\underline{v}(\rho) = \sup_{\sigma\in\mathbb{R}} \overline{\theta}_\infty(\sigma,\rho)$ (resp. the supersolution $\overline{v}(\rho) = \inf_{\sigma\in\mathbb{R}} \overline{\theta}_\infty(\sigma,\rho)$): it satisfies $\mathcal{L}(\overline{A}_\infty, 1)\underline{v} \geqslant 0$ (resp. $\mathcal{L}(\overline{A}_\infty, 1)\overline{v} \leqslant 0$). From the maximum principle applied to $\mathcal{L}_0(\overline{A}_{\infty,s_\infty})$ we obtain

$$0 \leqslant \overline{v} \leqslant \overline{\theta}_\infty \leqslant \underline{v} \leqslant 0.$$

This gives a contradiction with (5.10) and proves the lemma. □

Acknowledgements

The authors are very grateful to Patrick Gérard for stimulating discussions on Nash–Moser theory. The authors are also very grateful to Henri Berestycki and Claude Le Bris for discussions and suggestions in the preparation of this work. We also thank the referees for their valuable suggestions that helped improving the presentation of this article.

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