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On a constrained variational problem with an arbitrary number of free boundaries

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We study the problem of minimizing the Dirichlet integral among all functions $u \in \mathrm{H}^1(\Omega)$ whose level sets $\{u = l_i\}$ have prescribed Lebesgue measure α_i. This problem was introduced in connection with a model for the interface between immiscible fluids. The existence of minimizers is proved with an arbitrary number of level-set constraints, and their regularity is investigated. Our technique consists in enlarging the class of admissible functions to the whole space $\mathrm{H}^1(\Omega)$, penalizing those functions whose level sets have measures far from those required; in fact, we study the minimizers of a family of penalized functionals F_λ, $\lambda > 0$, showing that they are Hölder continuous, and then we prove that such functions minimize the original functional also, provided the penalization parameter λ is large enough. In the case where only two levels are involved, we prove Lipschitz continuity of the minimizers.

1. Introduction

Given a connected bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, a natural number $m \geqslant 1$ and real numbers α_i, l_i such that

$$l_0 < l_1 < \cdots < l_m, \qquad \sum_{i=0}^{m} \alpha_i < |\Omega|, \qquad \alpha_i > 0, \qquad i = 0, \dots, m, \tag{1}$$

consider the following minimum problem:

$$\begin{cases} \min\limits_{u \in K} \displaystyle\int_\Omega |\nabla u|^2 \\ K = \{u \in \mathrm{H}^1(\Omega) : \ |\{u = l_i\}| = \alpha_i, \ i = 0, \dots, m\}, \end{cases} \tag{2}$$

that is, minimize the Dirichlet integral among all functions whose level sets $\{u = l_i\}$ have Lebesgue measure equal to α_i, $i = 0, \dots, m$ (observe that no boundary condition is given).

The above problem was first introduced by Ambrosio *et al*. in [4], motivated by a question posed by Gurtin in connection with a problem on the interface between immiscible fluids.

In [4], the authors established existence of minimizers under certain assumptions on the vectors l_i, namely that they should be extreme points of a convex set of $\mathbb{R}^n$; in the scalar case this condition reduces to considering only two phases, i.e. $m = 1$.

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In addition, several properties of minimizers where obtained in [4]; in particular, the asymptotic behavior of solutions was identified, when the total volume of the different phases exhaust the total measure of Ω.

In this note we focus our attention on the scalar case (i.e. u is real valued), and our goal is twofold: on one hand, we prove the existence of minimizers of (2) for an arbitrary number of levels; on the other hand, we study their regularity, which was not investigated in [4].

Our main result is the following theorem.

THEOREM 1.1 Suppose $\Omega \subset \mathbb{R}^n$ is a connected bounded domain with Lipschitz boundary, $m \geqslant 1$ is a natural number and l_i, α_i satisfy (1). Then problem (2) has a solution. Each solution is locally Hölder continuous in Ω, and is harmonic in the open set $\Omega \setminus \bigcup_i \{u = l_i\}$. Moreover, if $m = 1$ (i.e. only two levels are involved), then each solution is locally Lipchitz continuous.

Our technique consists in adding to the Dirichlet integral a quantity which penalizes those functions u whose level sets have a measure which is less than the prescribed quantities α_i, and minimizing the penalized functional over the whole space $\mathrm{H}^1(\Omega)$. More precisely, for $\lambda > 0$ we define the functional

$$F_\lambda(u) = \int_\Omega |\nabla u|^2 + \lambda \sum_{i=0}^{m} f_i\big(|\{u = l_i\}|\big), \tag{3}$$

where

$$f_i(x) = (\alpha_i - x)_+, \qquad i = 0, \ldots, m \tag{4}$$

are the penalization functions, and we consider the problem

$$\min_{u \in \mathrm{H}^1(\Omega)} F_\lambda(u) \tag{5}$$

in place of (2). We show that the minimizers of F_λ exist for all $\lambda > 0$ and are Hölder continuous (Lipschitz continuous in the case of two levels). Furthermore, we show that when λ is sufficiently large (depending only on Ω, α_i and l_i according to the explicit condition (21)), then u minimizes F_λ if and only if u is a solution to (2). As a consequence, we obtain that the minimizers of (2) exist for all $m \geqslant 1$, and they are Hölder continuous (Lipschitz continuous when $m = 1$).

A penalization approach was also adopted in [1], where the Dirichlet integral is minimized among all functions assuming a given boundary value and having a prescribed measure of the set $\{u > 0\}$. Here we deal with an arbitrary number of free boundaries and no boundary condition: the main difference with respect to [1], however, consists in the way we penalize the functionals and, consequently, in the way we prove that for large λ the measures of the level sets adjust to the prescribed values. The penalized functional in [1] was

$$\mathcal{F}_\epsilon(u) = \int_\Omega |\nabla u|^2 + \epsilon^{\mathrm{sgn}(\omega - |\{u>0\}|)}(\omega - |\{u > 0\}|) \tag{6}$$

where ω is the prescribed value of $|\{u > 0\}|$; in other words, the difference $\omega - |\{u > 0\}|$ was given a weight ϵ if positive and $1/\epsilon$ if negative. In [1], after proving Lipschitz continuity of the minimizers and their linear growth away from the free boundary $\partial\{u > 0\}$, the standalone theory from [2] was invoked in order to obtain the smoothness of $\partial\{u > 0\}$ and of the normal derivative of u, which was shown to be equal to some constant λ_ϵ along $\partial\{u > 0\}$. With such tools available and an estimate $0 < c \leqslant \lambda_\epsilon \leqslant C$ with c, C independent of ϵ, it was possible to make smooth inward and outward perturbations of the set $\{u > 0\}$ via diffeomorphisms and prove that, for ϵ small enough, the measure of $\{u > 0\}$ adjusts exactly to ω.

The only disavantage of the approach [1], perhaps, was the fact that the smoothness of the free boundary was needed in order to study the behavior for small ϵ, and this made the technique in [1] highly dependent on the measure-theoretic machinery from [2].

In this paper we present an entirely different approach, which allows one to tackle the problem without relying on the smoothness of the free boundary and the normal derivative. Observe that, according to (3) and (4), our penalization does not affect those functions whose level sets $\{u = l_i\}$ exceed the prescribed measures, since we *a priori* show that, for any minimizer of F_λ, the measures of the level sets do not exceed the prescribed values (Theorem 2.1). This fact plays a central role in our argument, since it allows us to avoid smooth inward perturbations of the sets $\{u = l_i\}$ (which would require strong smoothness properties of free boundaries and normal derivatives). The main idea is that outward perturbations of $\{u = l_i\}$ can be done in a natural way, without diffeomorphisms or smoothness, just replacing u, in the set $l_i < u < l_{i+1}$, by $w_\delta = l_i + c(u - l_i - \delta)^+$, where $\delta \in (l_i, l_{i+1})$ and c is chosen in such a way that the other levels $\{u = l_j\}$, $j \neq i$ are preserved (whereas $\{u = l_i\}$ is replaced by the larger set $\{l_i \leqslant u \leqslant l_i + \delta\}$).

When λ is large enough, we rule out the case $|\{u = l_i\}| < \alpha_i$ showing that, in this case, it would hold $F_\lambda(w_\delta) < F_\lambda(u)$ for small δ, contrary to the minimality of u. More precisely, replacing u by w_δ, the Dirichlet integral increases by a quantity

$$\frac{l_{i+1} - l_i}{l_{i+1} - l_i - \delta} \int_{\{l_i < u < l_i + \delta\}} |\nabla u|^2 ,$$

whereas penalization changes by $-\lambda\, |\{l_i < u < l_i + \delta\}|$. The crucial point here is the fact that the average of $|\nabla u|^2$ over $\{l_i < u < l_i + \delta\}$ is bounded by a quantity independent of λ (Lemma 3.1).

Our approach seems to be quite general and we feel it might be of use in other settings also, dealing with one or several free boundaries (in particular, it might be applied to problems of the kind in [1] as well).

2. Existence of minimizers and basic properties

We begin with the definition of some quantities which depend only on Ω, α_i and l_i, which will be used throughout the paper without further reference.

REMARK 2.1 Given Ω, $m \geqslant 1$, α_i and l_i as in (1), we let

$$\mu = \inf_{u \in \mathcal{K}} \int_\Omega |\nabla u|^2 \tag{7}$$

where $\mathcal{K}$ is as defined in (2). Moreover, we define the numbers

$$\alpha = \min_{0 \leqslant i < m} \alpha_i, \qquad h = \min_{0 \leqslant i < m} (l_{i+1} - l_i), \qquad M = \max(|l_0|\,, |l_m|). \tag{8}$$

We denote by C_S the Sobolev–Poincaré constant relative to Ω, that is, the smallest number such that

$$\int_\Omega |u - \bar{u}| \leqslant C_S\, |\Omega|^{1/n}\, |Du|\,(\Omega), \qquad \bar{u} = \fint_\Omega u, \qquad \forall u \in \mathrm{BV}(\Omega), \tag{9}$$

where $|Du|\,(\Omega)$ is the total variation of the distributional derivative of u, and equals $\int_\Omega |\nabla u|$ if $u \in \mathrm{H}^1(\Omega)$. We remark that C_S depends only on the shape of Ω and not on its volume.

For $u \in \mathcal{K}$, we observe that

$$\int_{\Omega} |u - \bar{u}| \geqslant \int_{\{u=l_0\}} |l_0 - \bar{u}| + \int_{\{u=l_m\}} |l_m - \bar{u}| \geqslant \alpha(l_m - l_0) \geqslant \alpha h.$$

Since Ω is connected, from the Sobolev inequality it follows that $\mu > 0$ (and a positive lower bound for μ can be explicitly determined in terms of the data). Moreover, since $F_\lambda(v) = \int_\Omega |\nabla v|^2$ when $v \in \mathcal{K}$, it holds that

$$\inf_{v \in \mathrm{H}^1(\Omega)} F_\lambda(v) \leqslant \mu, \qquad \forall \lambda > 0. \tag{10}$$

PROPOSITION 2.1 If $\lambda > 0$ and $u \in \mathrm{H}^1(\Omega)$ is a minimizer of F_λ, then u is locally Hölder continuous in Ω for every exponent $\theta \in (0, 1)$. More precisely, we have

$$|u(x) - u(y)| \leqslant C_K |x - y| \log \frac{C_K}{|x - y|}, \tag{11}$$

for every compact set $K \subset \Omega$ and every $x, y \in K$, where C_K is some positive constant depending on K.

Proof. Let B_r be an open ball such that $\bar{B}_r \subset \Omega$, and let v_r be the harmonic function on B_r which coincides with u on ∂B_r. Replacing u with v_r inside B_r, the Dirichlet integral on Ω decreases by

$$\int_{B_r} |\nabla u|^2 \,\mathrm{d}x - \int_{B_r} |\nabla v_r|^2 \,\mathrm{d}x = \int_{B_r} |\nabla(u - v_r)|^2 \,\mathrm{d}x.$$

On the other hand, since this variation affects the values of u only inside B_r, the penalization cannot increase more than $\lambda\,|B_r|$, hence we obtain from the minimality of u

$$\int_{B_r} |\nabla(u - v_r)|^2 \,\mathrm{d}x \leqslant \lambda \omega_n r^n,$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Then (11) follows from the arbitrariness of B_r, reasoning as in Theorem 2.1 of [3]. □

THEOREM 2.1 For all $\lambda > 0$, the functional F_λ in (3) achieves a minimum u over $\mathrm{H}^1(\Omega)$, and each minimum u satisfies

$$l_0 \leqslant u(x) \leqslant l_m, \qquad \text{for a.e. } x \in \Omega. \tag{12}$$

If, moreover,

$$\lambda > \frac{\mu}{\alpha} \tag{13}$$

and $u \in \mathrm{H}^1(\Omega)$ is a minimum for F_λ, then it holds that

$$\alpha_i - \frac{\mu}{\lambda} \leqslant |\{u = i\}| \leqslant \alpha_i, \qquad i = 0, \ldots, m. \tag{14}$$

Proof. Let u_k be a minimizing sequence for F_λ. Replacing, if necessary, u_k with $\min(l_m, \max(l_0, u_k))$, it is not restrictive to assume that $|u_k| \leqslant M$. Then, since

$$\int_\Omega u_k^2 \leqslant |\Omega|\, M^2, \qquad \sup_k \int_\Omega |\nabla u_k|^2 \leqslant \sup_k F_\lambda(u_k) < +\infty,$$

passing to a subsequence (not relabeled), there exists $u \in \mathrm{H}^1(\Omega)$ such that

$$\nabla u_k \to \nabla u \qquad \text{weakly in } \mathrm{L}^2(\Omega), \qquad u_k(x) \to u(x) \qquad \text{for a.e. } x \in \Omega.$$

Then u is a minimum for F_λ by semicontinuity. In order to prove (12), it suffices to observe that replacing u by $w = \max(l_0, \min(u, l_m))$ would yield $F_\lambda(w) < F_\lambda(u)$ in case (12) should not hold, contrary to the minimality of u.

Moreover, from (4), (3) and (10) it follows that

$$\lambda(\alpha_j - |\{u = j\}|) \leqslant \lambda \sum_{i=0}^{m} f_i\big(|\{u = i\}|\big) \leqslant F_\lambda(u) \leqslant \mu, \qquad j = 0, \ldots, m,$$

which proves the first inequality in (14).

It remains to prove the second inequality in (14). In order to do so, we suppose that $\big|\{u = l_j\}\big| > \alpha_j$ holds for some j and we seek a contradiction. Letting $E = \{u = l_j\}$, we observe that E is closed in Ω, according to Proposition 2.1. Then we can choose a point $x_0 \in \Omega \cap \partial E$ such that $|B(x_0, r) \cap E| > 0$ for all sufficiently small $r > 0$ (the existence of such x_0 follows from elementary measure-theoretic arguments). In particular, since $u(x_0) = l_j$ and u is continuous, there is r such that, letting $B_r = B(x_0, r)$, it holds that $B_r \subset \Omega$ and

$$0 < |B_r \cap E| < |B_r| \leqslant |E| - \alpha_j, \qquad |B_r \cap \{u = l_i\}| = 0, \qquad i \neq j. \tag{15}$$

Then, if $v \in \mathrm{H}^1(\Omega)$ is harmonic in B_r and coincides with u outside B_r, we have

$$\big|\{v = l_j\}\big| \geqslant |E| - |B_r| \geqslant \alpha_j, \qquad |\{v = l_i\}| = |\{u = l_i\}|, \qquad i \neq j.$$

Then the penalization of v is the same as that of u, and its Dirichlet integral is less than that of u (observe that u is not harmonic inside B_r, by virtue of (15)), thus violating the minimality of u. Since this is a contradiction, it must hold that $\big|\{u = l_j\}\big| \leqslant \alpha_j$ for all j, and (14) is completely established. □

The following elementary observation will allow us to simplify all our arguments, since the level sets $\{u = l_i\}$ and $\{u = l_{m-i}\}$ play complementary roles.

REMARK 2.2 If u minimizes F_λ for some $\lambda > 0$, then the function $l_m - u$ minimizes the functional obtained from (3) by replacing l_i with $l_m - l_i$ and α_i with α_{m-i}, $i = 0, \ldots, m$.

Now we establish some properties of the minimizers of F_λ, which generalize those obtained in [4] for the case of two levels.

THEOREM 2.2 If $u \in \mathrm{H}^1(\Omega)$ is a minimum for F_λ and (13) holds, then

$$\int_\Omega |\nabla u|^2 \phi(x) f'(u) + \int_\Omega f(u) \nabla u \nabla \phi = 0 \tag{16}$$

for all $\phi \in \mathrm{C}^1(\bar{\Omega})$ and all Lipschitz f such that $f(l_i) = 0$, $i = 0, \ldots, m$, and

$$\int_\Omega |\nabla u|^2 g(u) = \sum_{i=0}^{m-1} \left(\fint_{l_i}^{l_{i+1}} g(s)\,\mathrm{d}s \right) \int_{\{l_i < u < l_{i+1}\}} |\nabla u|^2 \tag{17}$$

for all $g \in \mathrm{L}^\infty(\mathbb{R})$.

Proof. For $\epsilon > 0$, let $u_\epsilon = u + \epsilon\phi f(u)$, and observe that $u_\epsilon(x) = l_i$ whenever $u(x) = l_i$. Therefore, the penalization of u_ϵ does not exceed that of u, and $F_\lambda(u) \leqslant F_\lambda(u_\epsilon)$ implies

$$\int_\Omega |\nabla u|^2 \leqslant \int_\Omega |\nabla u_\epsilon|^2 .$$

Then (16) follows from the arbitrarity of ϵ. Finally, given $g \in \mathrm{L}^\infty(\mathbb{R})$, let

$$f(t) = \sum_{i=0}^{m-1} \left(\int_{l_i}^{t} g(s)\,\mathrm{d}s - (t - l_i) ⨍_{l_i}^{l_{i+1}} g(s)\,\mathrm{d}s \right) \chi_{[l_i, l_{i+1})}(t)$$

and observe that f is Lipschitz continuous and $f(l_i) = 0$. Then (17) follows from (16) with this choice of f and $\phi \equiv 1$. □

3. Behaviour for large λ

The following lemma is crucial in order to prove that, for large λ, the measures of the level sets adjust to the prescribed values.

LEMMA 3.1 If $\lambda \geqslant 2\mu/\alpha$ and u minimizes F_λ, then for any $\delta \in (0, l_{i+1} - l_i)$ it holds that

$$⨍_{\{l_i < u < l_i + \delta\}} |\nabla u|^2 \leqslant C_1^2 \tag{18}$$

where

$$C_1 = 2\mu \frac{|\Omega|^{1+\frac{1}{n}}}{h\alpha^2} C_S \tag{19}$$

depends only on Ω, l_i and α_i.

Proof. Letting $A_t = \{u < t\}$ for $t \in (l_i, l_{i+1})$, we have $\{u = l_i\} \subset A_t \subset \Omega \setminus \{u = l_{i+1}\}$ and

$$\int_\Omega \left| \chi_{A_t} - ⨍_\Omega \chi_{A_t} \right| = 2\frac{|A_t|}{|\Omega|}(|\Omega| - |A_t|) \geqslant 2\frac{|\{u = l_{i+1}\}|\ |\{u = l_i\}|}{|\Omega|} \geqslant \frac{\alpha^2}{2\,|\Omega|}$$

(the last inequality follows from (14)). For almost every $t \in (l_i, l_{i+1})$ it holds that $\chi_{A_t} \in \mathrm{BV}(\Omega)$, $\partial A_t = \{u = t\}$, and the isoperimetric inequality (9), combined with the last inequality, yields

$$\frac{\alpha^2}{2\,|\Omega|} \leqslant C_S\,|\Omega|^{\frac{1}{n}}\,\mathcal{H}^{n-1}(\{u = t\}), \qquad \text{i.e. } \mu \leqslant hC_1\mathcal{H}^{n-1}(\{u = t\}). \tag{20}$$

Therefore, for $\delta \in (0, l_{i+1} - l_i)$ and for a.e. $t \in (l_i, l_{i+1})$, from (17) with $g = \chi_{(l_i, l_i+\delta)}$, (10) and (20) we obtain

$$\int_{\{l_i < u < l_i + \delta\}} |\nabla u|^2 = \frac{\delta}{l_{i+1} - l_i} \int_{\{l_i < u < l_{i+1}\}} |\nabla u|^2 \leqslant \frac{\delta\mu}{h} \leqslant \delta C_1\,\mathcal{H}^{n-1}(\{u = t\}).$$

Integrating the last inequality with respect to t over $(l_i, l_i + \delta)$, dividing by δ and using the coarea formula yields

$$\int_{\{l_i < u < l_i + \delta\}} |\nabla u|^2 \leqslant C_1 \int_{l_i}^{l_i+\delta} \mathcal{H}^{n-1}(\{u = t\})\,\mathrm{d}t = C_1 \int_{\{l_i < u < l_i + \delta\}} |\nabla u|\,,$$

and (18) follows from the Hölder inequality. □

THEOREM 3.1 If λ satisfies

$$\lambda > \max(2\mu/\alpha, C_1^2) \tag{21}$$

where C_1 is given by (19), and u minimizes F_λ, then

$$|\{u = l_i\}| = \alpha_i, \qquad i = 0, 1, \ldots, m. \tag{22}$$

As a consequence, u is a solution to (5) if and only if it is a solution to (2).

Proof. From $|\{u = l_i\}| < \alpha_i$ for a given $i < m$ we derive a contradiction (the case $i = m$ easily follows from Remark 2.2). Indeed, if $|\{u = l_i\}| < \alpha_i$, then for all sufficiently small $\delta \in (0, l_{i+1} - l_i)$ it holds that

$$|\{u = l_i\}| + |\{l_i < u \leqslant l_i + \delta\}| \leqslant \alpha_i. \tag{23}$$

For such δ we let

$$w_\delta(x) = \begin{cases} u(x) & \text{if } \ u(x) \leqslant l_i \\ l_i + \dfrac{l_{i+1} - l_i}{l_{i+1} - l_i - \delta}(u - l_i - \delta)_+ & \text{if } \ l_i < u(x) < l_{i+1} \\ u(x) & \text{if } \ u(x) \geqslant l_{i+1} \end{cases}$$

and we observe that

$$\begin{aligned} &\{w_\delta = l_j\} = \{u = l_j\}, \qquad j = 0, \ldots, m, \qquad j \neq i, \\ &\{w_\delta = l_i\} = \{u = l_i\} \cup \{l_i < u \leqslant l_i + \delta\}. \end{aligned} \tag{24}$$

Therefore, recalling (23), $F_\lambda(u) \leqslant F_\lambda(w_\delta)$ simplifies to

$$\lambda\, |\{l_i < u \leqslant l_i + \delta\}| \leqslant \int_\Omega |\nabla w_\delta|^2 - \int_\Omega |\nabla u|^2 . \tag{25}$$

On the other hand, from (17) with $g = \chi_{(l_i+\delta, l_{i+1})}$ we obtain

$$\int_\Omega |\nabla w_\delta|^2 = \int_{\{u<l_i\}} |\nabla u|^2 + \int_{\{u>l_{i+1}\}} |\nabla u|^2 + \frac{l_{i+1} - l_i}{l_{i+1} - l_i - \delta} \int_{l_i<u<l_{i+1}} |\nabla u|^2$$

and (25) reduces to

$$\lambda\, |\{l_i < u \leqslant l_i + \delta\}| \leqslant \frac{\delta}{l_{i+1} - l_i - \delta} \int_{l_i<u<l_{i+1}} |\nabla u|^2 = \frac{l_{i+1} - l_i}{l_{i+1} - l_i - \delta} \int_{l_i<u\leqslant l_i+\delta} |\nabla u|^2$$

where we have used (17) again with $g = \chi_{(l_i, l_i+\delta]}$. Dividing both sides by $|\{l_i < u < l_i + \delta\}|$ and using (18) we have

$$\lambda \leqslant \frac{l_{i+1} - l_i}{l_{i+1} - l_i - \delta} \fint_{\{l_i<u<l_i+\delta\}} |\nabla u|^2 \leqslant \frac{l_{i+1} - l_i}{l_{i+1} - l_i - \delta} C_1^2$$

which is a contradiction, since it violates (21) after taking the limit for $\delta \to 0$. Therefore, $|\{u = l_i\}| \geqslant \alpha_i$ and equality must hold by virtue of (14). □

4. The case of two levels

In this section we suppose that $m = 1$, that is, we consider only two levels l_0 and l_1. In order to simplify the notation, we can assume $l_0 = 0$ and $l_1 = 1$ without losing generality. Our goal here is proving that, in the case of two levels, minimizers are Lipschitz continuous. In the general case, however, we already know Hölder continuity of the minimizers (Proposition 2.1).

REMARK 4.1 Observe that in Section 2 the only use we made of Proposition 2.1 was in the proof of Theorem 2.1, while proving the second inequality in (14). However, in the case of two levels the proof of (14) can be completed without relying on continuity, as follows.

Let $\beta_i = \min(\alpha_i, |\{u = i\}|)$, $i = 0, 1$, where u minimizes F_λ. From (13) and the first inequality in (14), it follows that $\beta_0, \beta_1 > 0$, and u is also a solution of the problem

$$\begin{cases} \displaystyle\min_{v \in \mathcal{K}'} \int_\Omega |\nabla v|^2 \\ \mathcal{K}' = \{v \in \mathrm{H}^1(\Omega) : |\{v = i\}| \geqslant \beta_i,\ i = 0, 1\} \end{cases} \tag{26}$$

(indeed, for $v \in \mathcal{K}'$ it holds $F_\lambda(u) \leqslant F_\lambda(v)$, but due to the choice of β_i the penalization of v does not exceed that of u). From the results in [4] concerning the relaxed problem (26), it follows that $|\{u = i\}| = \beta_i$, $i = 0, 1$, and the second inequality in (14) follows, without assuming continuity.

The techniques of this section essentially follow the guideline of [2], Section 3. However, since we are dealing with two independent free boundaries, comparison with harmonic functions (as in [2], where minimizers are subharmonic) is no longer fruitful. Here, minimizers turn out to be subharmonic near $\{u = 0\}$ and superharmonic near $\{u = 1\}$, and the natural comparison functions are the following.

REMARK 4.2 Given $u \in \mathrm{H}^1(\Omega)$ and a ball $B_r \subset \Omega$, we denote by $\mathcal{H}u$ the unique harmonic function such that $u - \mathcal{H}u \in \mathrm{H}^1_0(B_r)$, and we let $\mathcal{H}u \equiv u$ in $\Omega \setminus B_r$, so that $\mathcal{H}u \in \mathrm{H}^1(\Omega)$. Moreover, we let

$$\mathcal{H}^- u = \min(u, \mathcal{H}u), \qquad \mathcal{H}_+ u = \max(u, \mathcal{H}u).$$

The dependence of $\mathcal{H}u$, $\mathcal{H}^- u$ and $\mathcal{H}_+ u$ on B_r will always be clear from the context and will not be pointed out explicitly.

We now state a very useful lemma on harmonic functions; the proof is omitted since it is just the last part of the proof of Lemma 3.2 in [2] (see also [5], pp. 276–277).

LEMMA 4.1 If $g \in \mathrm{H}^1(\Omega)$, $B_r \subset \Omega$ is a ball of radius r and $g \geqslant 0$ in B_r, it holds that

$$|\{g = 0\} \cap B_r| \left(\fint_{\partial B_r} g \right)^2 \leqslant C r^2 \int_{B_r} |\nabla (g - \mathcal{H}g)|^2 , \tag{27}$$

where C depends only on n.

REMARK 4.3 Given $u \in \mathrm{H}^1(\Omega)$, we denote

$$\Omega_i = \{x \in \Omega : |\{u = i\} \cap B_r(x)| > 0, \forall r > 0\}, \qquad i = 0, 1,$$
$$\Omega^* = \Omega \setminus (\Omega_0 \cup \Omega_1).$$

It is easy to see that Ω_0 and Ω_1 are closed in Ω, whereas Ω^* is open.

Throughout this section, we assume that $\lambda > 0$ satisfies (13), so that Theorem 2.1 is available.

LEMMA 4.2 If u minimizes F_λ, then

$$\int_{B_r} |\nabla(u - \mathcal{H}_+u)|^2 \leqslant \lambda\, |\{u = 0\} \cap B_r|\,, \tag{28}$$

$$\left(\fint_{\partial B_r} u\right)^2 |\{u = 0\} \cap B_r| \leqslant \lambda C r^2\, |\{u = 0\} \cap B_r|\,, \tag{29}$$

for any ball $B_r \subset \Omega$. Here C depends only on n.

Proof. For any ball $B_r \subset \Omega$, an elementary computation yields

$$\int_{B_r} |\nabla(u - \mathcal{H}_+u)|^2 = \int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla \mathcal{H}_+u|^2$$

and, since $\mathcal{H}_+u \equiv u$ outside B_r, $F_\lambda(u) \leqslant F_\lambda(\mathcal{H}_+u)$ reads

$$\int_{B_r} |\nabla(u - \mathcal{H}_+u)|^2 \leqslant \sum_{i=0}^{1} f_i\big(|\{\mathcal{H}_+u = i\}|\big) - f_i\big(|\{u = i\}|\big).$$

Since $u \leqslant \mathcal{H}_+u \leqslant 1$ and f_1 is nonincreasing, the above inequality implies

$$\int_{B_r} |\nabla(u - \mathcal{H}_+u)|^2 \leqslant f_0\big(|\{\mathcal{H}_+u = 0\}|\big) - f_0\big(|\{u = 0\}|\big). \tag{30}$$

Now, from (14) we have $|\{\mathcal{H}_+u = 0\}| \leqslant |\{u = 0\}| \leqslant \alpha_0$ and, since $f_0(x) - f_0(y) = \lambda(y - x)$ when $x \leqslant y \leqslant \alpha_0$, (30) reduces to

$$\int_{B_r} |\nabla(u - \mathcal{H}_+u)|^2 \leqslant \lambda\, |\{u = 0\} \cap \{\mathcal{H}_+u > 0\}|\,. \tag{31}$$

Since u and $\mathcal{H}_+u$ coincide outside B_r, (28) follows from (31).

Now we apply Lemma 4.1 with $g = \mathcal{H}^-u$. Observe that $g \geqslant 0$, $\mathcal{H}g = \mathcal{H}u$ and $\fint_{\partial B_r} g = \fint_{\partial B_r} u$; moreover, $g - \mathcal{H}g = u - \mathcal{H}_+u$ and $\{u = 0\} \cap B_r \subset \{g = 0\} \cap B_r$, and (27) yields

$$|\{u = 0\} \cap B_r| \left(\fint_{\partial B_r} u\right)^2 \leqslant C r^2 \int_{B_r} |\nabla(u - \mathcal{H}_+u)|^2\,. \tag{32}$$

Combining (32) and (28) we prove (29). □

THEOREM 4.1 Suppose λ satisfies (13) and u is a minimum for F_λ. Then $\Omega_0 \cap \Omega_1 = \emptyset$, and there exists a constant $C = C(n)$ such that for any ball $B_r = B(x, r) \subset \Omega$ with $r^2 \lambda C < 1$ it holds that

$$\text{either} \qquad B_r \cap \Omega_0 = \emptyset \qquad \text{or} \qquad B_r \cap \Omega_1 = \emptyset. \tag{33}$$

Moreover, u is subharmonic in the open set $\Omega \setminus \Omega_1$, superharmonic in the open set $\Omega \setminus \Omega_0$ and harmonic in the open set Ω^*. In particular, u can be defined pointwise by

$$u(x) = \lim_{r \to 0} \fint_{B_r(x)} u \qquad \forall x \in \Omega \tag{34}$$

and, according to this pointwise definition,

$$\Omega_0 = \{u = 0\}, \qquad \Omega_1 = \{u = 1\}, \qquad \text{and} \qquad \Omega^* = \{0 < u < 1\}. \tag{35}$$

Proof. Consider a ball $B_r = B(x,r) \subset \Omega$, and suppose $|\{u=0\}\cap B_r| > 0$ and $|\{u=1\}\cap B_r| > 0$ hold simultaneously. Then (29) implies

$$\fint_{\partial B_r} u \leqslant r\sqrt{\lambda C}. \tag{36}$$

According to Remark 2.2, (29) also holds with $1-u$ in place of u, which implies that

$$\left(1 - \fint_{\partial B_r} u\right) \leqslant r\sqrt{\lambda C} \tag{37}$$

Adding (36) and (37) we obtain $1 \leqslant r^2\lambda C$, where C depends only on n, and the first part of the theorem is proved.

Now we prove that u is superharmonic in $\Omega \setminus \Omega_0$. For any ball $B_r \subseteq \Omega \setminus \Omega_0$, it clearly holds that $|\{u=0\}\cap B_r| = 0$ and from (28) it follows that $u \equiv \mathcal{H}_+ u$ inside B_r. Therefore, it holds that $u \geqslant \mathcal{H}u$ in B_r, hence u is superharmonic in $\Omega \setminus \Omega_0$ since B_r was arbitrary.

The subharmonicity in $\Omega \setminus \Omega_1$ now follows from Remark 2.2, whereas in $\Omega^* = \Omega \setminus (\Omega_0 \cup \Omega_1)$ u is harmonic, being both sub and superharmonic. The limit in (34) exists for all $x \in \Omega$, since u is superharmonic or subharmonic near x.

Now let $x_0 \in \Omega_0$, and let $B_r = B(x_0, r)$. Then (29) implies (36) for all $r > 0$ such that $B_r \subset \Omega$. Multiplying both sides of (36) by r^{n-1} and integrating over $(0,\rho)$ yields

$$\fint_{B_\rho} u \leqslant \sqrt{\lambda C}\rho,$$

and from (34) we have $u(x_0) = 0$. Furthermore, from Remark 2.2 we obtain $u(x_1) = 1$ whenever $x_1 \in \Omega_1$. Finally, if $x \in \Omega^*$, then u is harmonic near x, hence the strong maximum principle implies that $0 < u(x) < 1$. □

THEOREM 4.2 If u is a minimum for F_λ and $m = 1$, then u is locally Lipschitz continuous in Ω.

Proof. Fix a compact set $K \subset \Omega$, a point $x \in \Omega^* \cap K$, and let $d = \mathrm{dist}(x, \partial\Omega)$, $\rho = \mathrm{dist}(x, \Omega_0 \cup \Omega_1)$ and $B_\rho = B(x,\rho)$; if $\rho \geqslant d$, then u is harmonic in $B(x,d)$, hence

$$|\nabla u(x)| \leqslant \frac{C(n)}{d} \fint_{\partial B_d} u \leqslant \frac{C(n)}{d} = C(n,K).$$

If $\rho < d$, then at least one of either $|\{u=0\}\cap B_r| > 0$ or $|\{u=1\}\cap B_r| > 0$ is satisfied for all $\rho < r < d$ (depending on whether x is closer to Ω_0 or Ω_1); in the former case, from (29) we obtain that (36) holds for all $\rho < r < d$, whereas in the latter case from Remark 2.2 and (29) applied with $1-u$ in place of u we obtain that (37) is satisfied for all $\rho < r < d$. Taking the limit for $r \to \rho$ in (36) or in (37), we obtain that

$$\min\left(\frac{1}{\rho}\fint_{\partial B_\rho} u, \frac{1}{\rho}\fint_{\partial B_\rho} 1-u\right) \leqslant \sqrt{\lambda C}.$$

Since u and $1-u$ are harmonic inside B_ρ, and $|\nabla u| = |\nabla(1-u)|$, we have

$$|\nabla u(x)| \leqslant C(n,\lambda).$$

Finally, if $x \in K \cap (\Omega \setminus \Omega^*)$, then either $u(x) = 0$ or $u(x) = 1$, and we can assume that $|\nabla u(x)| = 0$. In all cases, $|\nabla u(x)|$ is bounded by a constant $C = C(n,\lambda,K)$ when $x \in K$. □

5. Concluding remarks and open questions

By the techniques introduced in this paper, one can probably handle problems of the kind (2) with functionals more general than the Dirichlet integral, with only minor changes. However, even in the simplest case, several questions remain unsolved.

We point out that we were not able to prove Lipschitz continuity of the minimizers when more than two levels are involved. Indeed, reasoning as in Section 4, one can only prove that for arbitrary m any minimizer u grows linearly away from the extreme free boundaries $\partial\{u = l_0\}$ and $\partial\{u = l_m\}$, thus proving Lipschitz continuity in a neighborhood of the extreme levels $\{u = l_0\}$ and $\{u = l_m\}$. We remark that Lemma 4.1 cannot be applied near a middle level l_i (i.e. $0 < i < m$), since if $x_0 \in \partial\{u = l_i\}$, then the function $u - l_i$ might take, as far as we know, both positive and negative values near x_0 (although we believe that this cannot happen if u is a minimizer, we were not able to rule out such a behavior). On the other hand, (27) is false if one drops the assumption $g \geqslant 0$, even if one puts $|g|$ in place of g in the left-hand side of (27), as one can see from the following example.

EXAMPLE 5.1 Let $B \subset \mathbb{R}^2$ be the unitary ball, let k be an odd natural number and define, in polar coordinates,

$$g(r, \theta) = \chi_{[\mathrm{e}^{-\pi/2},1]}(r) \left(r^k + \frac{\mathrm{e}^{-k\pi/2}}{\sin(k\pi/2)} \sin\log r^k \right) \sin(k\theta).$$

Then, clearly, $\mathcal{H}g = r^k \sin(k\theta)$. Moreover, a simple computation yields

$$|\{g = 0\} \cap B| \left(\fint_{\partial B} |g| \right)^2 = \frac{4}{\pi}\mathrm{e}^{-\pi}, \qquad \int_B |\nabla(g - \mathcal{H}g)|^2 = \pi k \left(1 + k\frac{\pi}{2}\right) \mathrm{e}^{-k\pi},$$

hence (27) does not hold for large k, even with $|g|$ in the left-hand side.

Proving that minimizers are Lipschitz continuous would be an important step in order to prove the linear growth away from the free boundaries, which would lead to regularity theorems for the free boundaries and the normal derivative, following the ideas in [1]. Indeed, since any minimizer u is harmonic in the open set $\Omega \setminus \bigcup_i \{u = l_i\}$, the theory of Alt and Caffarelli [2] on harmonic functions with linear growth could be succesfully applied in a neighborhood of each level set $\{u = l_i\}$. Moreover, the Neumann condition in the formal Euler equation of (2), i.e.

$$\triangle u = 0 \quad \text{in} \quad \Omega \setminus \bigcup_i \{u = l_i\}, \qquad -\frac{\partial u}{\partial \nu} = \lambda_i \quad \text{on} \quad \partial\{u = l_i\} \cap \Omega \tag{38}$$

could be given a precise meaning, reasoning as in [1]. At present, we can do this only in the case of two levels, where one can prove linear growth proceeding as in Lemma 3.4 from [2], thus obtaining that the Neumann condition in (38) is satisfied $\mathcal{H}^{n-1}$-almost everywhere along the reduced boundary of each level set (see [1], Theorem 3).

We point out that Lipschitz continuity (and even linear growth) could be proved for an arbitrary number of levels, if only one could show that any point on any free boundary $\partial\{u = l_i\}$ is either a local maximum or a local minimum.

Finally, we remark that we do not know any explicit example of a minimizer except when $n = 1$ and Ω is an interval, where each minimizer is a monotone and piecewise linear function (see [4] for more details).

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