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Evolution of compressible and incompressible fluids separated by a closed interface

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This work solves the problem governing the simultaneous motion of two viscous liquids of different kinds: compressible and incompressible. The boundary between the fluids is considered as an unknown (free) interface where the surface tension is taken into account. Although the fluids occupy the whole space $\mathbb{R}^3$, one of them should have a finite volume.

Local (in time) unique solvability of this problem is obtained in the Sobolev–Slobodetskiĭ spaces of functions. Estimates of the solution of a model problem for the Stokes equations are considered in detail, the interface between the fluids being a plane. The Schauder method is used to study a linear problem with a compact boundary. The passage to the nonlinear problem is made by successive approximations.

Keywords: free boundary problem, Navier–Stokes equations, two immiscible fluids

1. Introduction

This paper considers the unsteady motion of a bubble in an incompressible fluid, or that of a drop in a compressible one. On one hand, the study of a simultaneous motion of two viscous capillary liquids of different kinds (compressible and incompressible) is interesting from the pure mathematical point of view. The present work can be considered as a continuation of [2–5] where the evolution of two viscous capillary fluids of single type, two compressible [4], or two incompressible [2, 3, 5] have been considered. On the other hand, this problem arises in many important physical phenomena. The evolution of a bubble in an incompressible fluid appears, for example, in the case of an ejection of gas into water, for example, after an explosion in the ocean or after a volcanic eruption at the ocean bed. Another example of the physical interpretation of our problem may be the presence of many small bubbles in a big volume of liquid with large distances between them. Then we can also consider the motion of a single bubble in an infinite liquid medium. A similar situation arises for drops in a volume of gas.

Here, we study the problem governing the motion of two different liquids in full generality. The local unique solvability of this free boundary problem is the main result of the present work. An essential difference in the proof of the existence of a unique solution to the problem is contained in the analysis of a linear model problem with a plane interface between the fluids, so we examine this model problem in detail. Section 2 is devoted to the homogeneous problem and Section 3 deals with the non-homogeneous one. We describe briefly the scheme of the nonlinear problem in Section 4. Its key features are the same as in the case of a single type fluid. The detailed proofs can be found in articles [7–9] where the motion of a finite volume of a single viscous capillary fluid was analysed.

We obtain our results under some restrictions (the relations (1.9)) for the coefficients of the liquid viscosities. These inequalities are imposed for mathematical reasons but are physically reasonable in certain cases. They are satisfied, for instance, for the pairs air–water, air–alcohol, air–mercury, etc. but not for air– glycerin, air–lubricating oil. More generally, they are valid for fluids with low viscosity. Incompressible liquids with large viscosity at ordinary temperatures may also satisfy (1.9) when the temperature increases above certain values, lowering the viscosity. This is because for capillary liquids, both viscosities μ and ν decrease quickly with temperature raising whereas for gases, conversely, μ and ν increase with the rise of temperature (see for example [6]). For instance, inequalities (1.9) become true for the pair lubricating oil and air at temperature $\approx 80°$C.

We consider, for definiteness, the case when at the initial moment $t = 0$ a compressible fluid is situated in an interior, bounded, domain $\Omega_0^+ \subset \mathbb{R}^3$. Let $\mu^+ > 0$, $\lambda^+ > 0$ be its kinematic viscosities. Let the 'exterior' domain $\Omega_0^- \equiv \mathbb{R}^3 \setminus \overline{\Omega_0^+}$ be occupied by an incompressible fluid with the kinematic viscosity $\nu^- > 0$ and density $\rho^- > 0$. We assume that the compressible fluid is barotropic. We note that we could also assume the compressible fluid to be exterior to the incompressible one.

The problem consists in determining, for each $t > 0$, the free interface Γ_t between the liquids evolving in the domains Ω_t^- and Ω_t^+. In addition, it is necessary to find the density function $\rho^+(x,t) > 0$ of the compressible fluid, the pressure function $p^-(x,t)$ of the incompressible fluid, as well as the velocity vector field of both liquids $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$ satisfying the initial-boundary value problem for the Navier–Stokes system:

$$
\begin{gathered}
\rho^+(\mathcal{D}_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) - \nabla\mathbb{T} = \rho^+\boldsymbol{f},\\
\mathcal{D}_t\rho^+ + \nabla\cdot(\rho^+\boldsymbol{v}) = 0 \quad \text{in} \quad \Omega_t^+, \quad t > 0,\\
\rho^+|_{t=0} = \rho_0^+, \quad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \quad \text{in} \quad \Omega_0^+,
\end{gathered} \tag{1.1}
$$

$$
\begin{gathered}
\mathcal{D}_t\boldsymbol{v} - \nu^-\nabla^2\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \frac{1}{\rho^-}\nabla p^- = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = 0 \quad \text{in} \quad \Omega_t^-, \quad t > 0,\\
\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \quad \text{in} \quad \Omega_0^-; \quad \boldsymbol{v} \xrightarrow[|x|\to\infty]{} 0, \quad p^- \xrightarrow[|x|\to\infty]{} 0;
\end{gathered} \tag{1.2}
$$

$$
\begin{gathered}
[\boldsymbol{v}]\Big|_{\Gamma_t} \equiv \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^+}} \boldsymbol{v}(x) - \lim_{\substack{x\to x_0\in\Gamma_t,\\ x\in\Omega_t^-}} \boldsymbol{v}(x) = 0,\\
[\mathbb{T}\boldsymbol{n}]\Big|_{\Gamma_t} = \sigma H\boldsymbol{n} \quad \text{on} \quad \Gamma_t, \quad t > 0.
\end{gathered} \tag{1.3}
$$

Here $\mathcal{D}_t = \partial/\partial t$, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, $\nabla^2 = \nabla\cdot\nabla$, and the stress tensor is given by

$$
\mathbb{T} = \begin{cases} \big(-p^+(\rho^+) + \lambda\nabla\cdot\boldsymbol{v}\big)\,\mathbb{I} + \mu^+\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^+,\\ -p^-\mathbb{I} + \mu^-\mathbb{S}(\boldsymbol{v}) & \text{in } \Omega_t^-, \end{cases}
$$

$(\mathbb{S}(\boldsymbol{v}))_{ik} = \partial v_i/\partial x_k + \partial v_k/\partial x_i$, $i,k = 1,2,3$; $\mathbb{I}$ is the unit matrix; $\mu^- = \nu^-\rho^-$; $p^+(\rho^+)$ is the pressure of the compressible fluid given by a smooth function of its density; $\boldsymbol{f}$ is the given vector field of mass forces; $\boldsymbol{v}_0$ is the initial value of the velocity vector field; ρ_0^+ is the initial density distribution of the compressible fluid; $\sigma \geqslant 0$ is the surface tension coefficient, $\boldsymbol{n}$ is the outward normal vector to Ω_t^+, $H(x,t)$ is twice the mean curvature of Γ_t ($H < 0$ at the points where Γ_t is convex towards Ω_t^-); $\nabla\mathbb{T}$ means the vector with the components

$$
(\nabla\mathbb{T})_j = \frac{\partial T_{ij}}{\partial x_i}, \quad T_{ij} = (\mathbb{T})_{ij}, \quad j = 1,2,3.
$$

We use the standard convention on the summation from 1 to 3 with respect to repeated indices. A Cartesian coordinate system $\{x\}$ is introduced in $\mathbb{R}^3$.

Since we suppose the liquids to be immiscible, it is natural to impose on Γ_t a condition excluding the mass transportation through this surface. Mathematically, this condition means that Γ_t consists of the points $x(\xi,t)$ whose radius vector $\boldsymbol{x}(\xi,t)$ is a solution of the Cauchy problem

$$\mathcal{D}_t\boldsymbol{x} = \boldsymbol{v}(x(\xi,t),t), \quad \boldsymbol{x}(\xi,0) = \boldsymbol{\xi}, \quad \xi \in \Gamma, \quad t > 0, \tag{1.4}$$

where $\Gamma \equiv \Gamma_0 = \partial\Omega_0^+$ is a surface given at the initial moment. Hence, $\Omega_t^{\pm} = \{x = x(\xi,t) | \xi \in \Omega_0^{\pm}\}$.

The condition (1.4) completes the system (1.1)–(1.3).

As usual, we transform the Eulerian coordinates $\{x\}$ into the Lagrangian ones $\{\xi\}$ by the formula

$$\boldsymbol{x}(\xi,t) = \boldsymbol{\xi} + \int_0^t \boldsymbol{u}(\xi,\tau)\,\mathrm{d}\tau \equiv \boldsymbol{X}_{\boldsymbol{u}}(\xi,t). \tag{1.5}$$

Here $\boldsymbol{u}(\xi,t)$ is the velocity vector field in the Lagrangian coordinates.

The Jacobian of the transformation (1.5)

$$\mathcal{J}_{\boldsymbol{u}}(\xi,t) = \det\{a_{ij}\}_{i,j=1}^3,$$

$$a_{ij}(\xi,t) = \delta_j^i + \int_0^t \frac{\partial u_i}{\partial \xi_j}\,\mathrm{d}\tau,$$

being a solution of the Cauchy problem

$$\mathcal{D}_t\mathcal{J}_{\boldsymbol{u}}(\xi,t) = A_{ij}\frac{\partial u_i}{\partial \xi_j} \equiv \mathcal{J}_{\boldsymbol{u}}(\xi,t)(\nabla\cdot\boldsymbol{v}|_{x=X_{\boldsymbol{v}}}),$$
$$\mathcal{J}_{\boldsymbol{u}}(\xi,0) = 1,$$

may be expressed by the formula

$$\mathcal{J}_{\boldsymbol{u}}(\xi,t) = \exp\left(\int_0^t \nabla\cdot\boldsymbol{v}|_{x=X_{\boldsymbol{u}}}\,\mathrm{d}\tau\right) \equiv \exp\left(\int_0^t \nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}\,\mathrm{d}\tau\right). \tag{1.6}$$

Here $\{\delta_j^i\}_{i,j=1}^3$ denotes the Kronecker symbol,

$$\nabla_{\boldsymbol{u}} \equiv \left\{\frac{\partial \xi_i}{\partial x_k}\frac{\partial}{\partial \xi_i}\right\}_{k=1}^3 = \mathcal{J}_{\boldsymbol{u}}^{-1}\mathbb{A}\nabla,$$

and $\mathbb{A} \equiv \{A_{ij}\}_{i,j=1}^3$ is the cofactor matrix for the Jacobi matrix $\{a_{ij}\}$ of (1.5). We remark that $\mathcal{J}_{\boldsymbol{u}}(\xi,t) \equiv 1$ in the domain Ω_t^-.

After the tranformation (1.5) of the system (1.1)–(1.3), the second equation in (1.1) in Lagrangean coordinates takes the form

$$\mathcal{D}_t\widehat{\rho^+} + \widehat{\rho^+}\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u} = 0$$

from where, in virtue of (1.6), we obtain the following expression for the density $\widehat{\rho^+}$

$$\widehat{\rho^+}(\xi,t) = \rho_0^+(\xi)\exp\left(-\int_0^t \nabla_{\boldsymbol{u}}\cdot\boldsymbol{u}\,\mathrm{d}\tau\right) = \rho_0^+(\xi)\mathcal{J}_{\boldsymbol{u}}^{-1}(\xi,t). \tag{1.7}$$

We substitute (1.7) in the first equation of (1.1) and apply the well-known formula

$$Hn = \Delta(t)x = \Delta(t)X_u$$

where $\Delta(t)$ is the Beltrami–Laplace operator on Γ_t. Moreover, we separate the last vector boundary condition in (1.3) in the tangential and in the normal components. To this end, we project it first onto the tangent plane of Γ_t and then onto that of Γ by means of projectors Π and Π_0, respectively.

Next, let $\boldsymbol{n}_0$ be the outward normal to Γ. It is connected with $\boldsymbol{n}$ by the relation

$$\boldsymbol{n} = \frac{\mathcal{J}_{\boldsymbol{u}}^{-1}\mathbb{A}\boldsymbol{n}_0}{|\mathcal{J}_{\boldsymbol{u}}^{-1}\mathbb{A}\boldsymbol{n}_0|} = \frac{\mathbb{A}\boldsymbol{n}_0}{|\mathbb{A}\boldsymbol{n}_0|}.$$

For $\boldsymbol{n}\cdot\boldsymbol{n}_0 > 0$, problem (1.1)–(1.4), as a result of the above transformation, is changed into an equivalent system:

$$\mathcal{D}_t\boldsymbol{u} - \frac{1}{\rho_0^+(\xi)}\mathbb{A}\nabla\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{u}) = \boldsymbol{f}(X_{\boldsymbol{u}},t) - \frac{1}{\rho_0^+(\xi)}\mathbb{A}\nabla p^+(\rho_0^+\mathcal{J}_{\boldsymbol{u}}^{-1})$$
$$\text{in}\quad Q_T^+ \equiv \Omega_0^+\times(0,T),$$

$$\begin{gathered}
\mathcal{D}_t\boldsymbol{u} - \nu^-\nabla_{\boldsymbol{u}}^2\boldsymbol{u} + \frac{1}{\rho^-}\nabla_{\boldsymbol{u}} q = \boldsymbol{f}(X_{\boldsymbol{u}},t),\\
\nabla_{\boldsymbol{u}}\cdot\boldsymbol{u} = 0 \quad\text{in}\quad Q_T^-\equiv\Omega_0^-\times(0,T),\\
\boldsymbol{u}\big|_{t=0} = \boldsymbol{v}_0 \quad\text{in}\quad \Omega_0^-\cup\Omega_0^+,\quad \boldsymbol{u}\xrightarrow[|\xi|\to\infty]{}0,\quad q\xrightarrow[|\xi|\to\infty]{}0,\\
[\boldsymbol{u}]\big|_{G_T} = 0,\quad [\mu^\pm\Pi_0\Pi\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{u})\boldsymbol{n}]\big|_{G_T} = 0\quad (G_T\equiv\Gamma\times(0,T)),\\
[\boldsymbol{n}_0\cdot\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{u},q)\boldsymbol{n}]\big|_{G_T} - \sigma\boldsymbol{n}_0\cdot\Delta(t)X_{\boldsymbol{u}}\big|_{G_T} = (\boldsymbol{n}_0\cdot\boldsymbol{n})p^+(\rho_0^+\mathcal{J}_{\boldsymbol{u}}^{-1})\big|_{G_T}.
\end{gathered}\tag{1.8}$$

In (1.8) $q(\xi,t)$ was the pressure function in the Lagrangean coordinates, and we used the following notation:

$$\begin{aligned}
&(\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{w},q))_{i,j} = \begin{cases}(\lambda^+\nabla_{\boldsymbol{u}}\cdot\boldsymbol{w})\delta_j^i + \mu^+(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))_{ij} & \text{in } Q_T^+,\\ -\delta_j^i q + \mu^-(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))_{ij} & \text{in } Q_T^-;\end{cases}\\
&(\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w}))_{ij} = \mathcal{J}_{\boldsymbol{u}}^{-1}\left(A_{ik}\frac{\partial w_j}{\partial\xi_k} + A_{jk}\frac{\partial w_i}{\partial\xi_k}\right);\\
&\Pi_0\boldsymbol{\omega} = \boldsymbol{\omega} - (\boldsymbol{n}_0\cdot\boldsymbol{\omega})\boldsymbol{n}_0,\quad \Pi\boldsymbol{\omega} = \boldsymbol{\omega} - (\boldsymbol{n}\cdot\boldsymbol{\omega})\boldsymbol{n}.
\end{aligned}$$

The main result of this paper is a theorem on the unique solvability of problem (1.8) in the Sobolev–Slobodetskiĭ spaces. We now recall the definition of these spaces.

Let Ω be a domain in $\mathbb{R}^n$, $n\in\mathbb{N}$, and let $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n)$ be the multi-index of order $|\boldsymbol{\alpha}| = \alpha_1+\cdots+\alpha_n$ with integer non-negative components α_i, $i=1,\dots,n$. We denote the generalized derivative of a function u by $\mathcal{D}_x^{\boldsymbol{\alpha}}u = \frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$.

We determine the Sobolev–Slobodetskiĭ space $W_2^m(\Omega)$ for $m>0$ as the space of functions u with finite norm

$$\|u\|_{W_2^m(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}|<m}\|\mathcal{D}_x^{\boldsymbol{\alpha}}u\|_\Omega^2 + \|u\|^2_{\dot W_2^m(\Omega)}\right)^{1/2}$$

where $\|\cdot\|_\Omega$ is the norm of $L_2(\Omega)$ and

$$\|u\|^2_{\dot W_2^m(\Omega)} = \begin{cases} \sum\limits_{|\alpha|=m} \|\mathcal{D}_x^\alpha u\|^2_\Omega & \text{for } m \in \mathbb{N}, \\ \sum\limits_{|\alpha|=[m]} \int_\Omega \int_\Omega \dfrac{|\mathcal{D}_x^\alpha u(x) - \mathcal{D}_y^\alpha u(y)|^2}{|x-y|^{n+2(m-[m])}}\,\mathrm{d}x\,\mathrm{d}y & \text{for } m \notin \mathbb{N}, \end{cases}$$

$[m]$ being the integer part of m.

The anisotropic space $W_2^{m,m/2}(Q_T)$ consists of functions defined in the cylinder $Q_T = \Omega \times (0,T)$, $0 < T \leqslant \infty$, and having finite norm

$$\|u\|_{W_2^{m,m/2}(Q_T)} = \left(\int_0^T \|u\|^2_{W_2^m(\Omega)} \mathrm{d}t + \int_\Omega \|u\|^2_{W_2^{m/2}(0,T)} \mathrm{d}x \right)^{1/2}.$$

To formulate intermediate results, we need the Sobolev–Slobodetskiĭ spaces with exponential weight introduced by M. Agranovich and M. Vishik in [1] and also considered by V. Solonnikov in [7].

Let $\gamma \geqslant 0$. The weighted space $H_\gamma^{m,m/2}(Q_T)$ is the space of functions admitting zero continuation to the domain $t < 0$ without loss of regularity. This space is equipped with the norm

$$\|u\|_{H_\gamma^{m,m/2}(Q_T)} = \left(\|u\|^2_{H_\gamma^{m,0}(Q_T)} + \|u\|^2_{H_\gamma^{0,m/2}(Q_T)} \right)^{1/2}.$$

Here

$$\|u\|^2_{H_\gamma^{m,0}(Q_T)} = \int_0^T \mathrm{e}^{-2\gamma t} \left(\|u\|^2_{\dot W_2^m(\Omega)} + \gamma^m \|u\|^2_\Omega \right) \mathrm{d}t,$$

$$\|u\|^2_{H_\gamma^{0,m/2}(Q_T)} = \int_0^T \mathrm{e}^{-2\gamma t} \left\| \frac{\partial^{m/2} u}{\partial t^{m/2}} \right\|^2_\Omega \mathrm{d}t \qquad \text{for } m/2 \in \mathbb{N}$$

and

$$\|u\|^2_{H_\gamma^{0,m/2}(Q_T)} = \int_0^T \mathrm{e}^{-2\gamma t} \int_0^\infty \left\| \frac{\partial^k u_0(\cdot,t)}{\partial t^k} - \frac{\partial^k u_0(\cdot,t-\tau)}{\partial t^k} \right\|^2_\Omega \frac{\mathrm{d}\tau}{\tau^{1+m-2k}}\,\mathrm{d}t$$

for $k \equiv [m/2] < m/2$ where u_0 is the extension of u by zero in the domain $t < 0$. In addition, in the case $m > 1$,

$$\left. \frac{\partial^i u}{\partial t^i} \right|_{t=0} = 0, \quad i = 0, \dots, \left[\frac{m-1}{2} \right].$$

On the cylinder boundary $G_T = \partial\Omega \times (0,T)$ of the domain Q_T, we define the space $H_\gamma^{m+1/2,1/2,m/2}(G_T)$ with the norm whose square is determined by the formula

$$\|u\|^2_{H_\gamma^{m+1/2,1/2,m/2}(G_T)} = \int_0^T \mathrm{e}^{-2\gamma t} \Bigg(\|u\|^2_{W_2^{m+1/2}(\partial\Omega)} + \gamma^m \|u\|^2_{W_2^{1/2}(\partial\Omega)} + \int_0^\infty \left\| \frac{\partial^k u_0(\cdot,t)}{\partial t^k} - \frac{\partial^k u_0(\cdot,t-\tau)}{\partial t^k} \right\|^2_{W_2^{1/2}(\partial\Omega)} \frac{\mathrm{d}\tau}{\tau^{1+m-2k}} \Bigg) \mathrm{d}t,$$

$k \equiv [m/2] < m/2$; if $k = m/2$ the last term under the first integral sign should be changed into $\left\| \frac{\partial^{m/2} u}{\partial t^{m/2}} \right\|^2_{W_2^{1/2}(\partial\Omega)}$. (Here u_0 is again the extension of u by zero and $\left.\frac{\partial^i u}{\partial t^i}\right|_{t=0} = 0, \quad i = 0, \dots, \left[\frac{m-1}{2}\right]$.)

All these norms can be introduced on any smooth manifold by means of local maps and partitions of unity.

As usual, for a function defined in two domains Q_T^- and Q_T^+, we set

$$\|u\|_{\bigcup_{i=-,+} H_\gamma^{m,m/2}(Q_T^i)} = \|u\|_{H_\gamma^{m,m/2}(Q_T^-)} + \|u\|_{H_\gamma^{m,m/2}(Q_T^+)}.$$

We say that a vector field belongs to a certain space if each of its components belongs to this space and we define its norm as the sum of the norms of its components. The numeration of constants is individual for each section.

Now we define three norms necessary for formulating the main result of this paper. The first of them is

$$\|u\|^{(m,m/2)}_{Q_T^-\cup Q_T^+} = \left(\|u\|^2_{\bigcup_{i=-,+} W_2^{m,m/2}(Q_T^i)} + T^{-m}\|u\|^2_{\mathbb{R}^3_T} \right)^{1/2}.$$

It is equivalent to $\|u\|^2_{\bigcup_i H_\gamma^{m,m/2}(Q_T^i)}$ if $m < 1$ and to $\|u\|^2_{\bigcup_i W_2^{m,m/2}(Q_T^i)}$ for $\forall T < \infty$. The square of the second norm is determined by the formula

$$\left(\|u\|^{(2+l,1+l/2)}_{Q_T^-\cup Q_T^+} \right)^2 = \|u\|^2_{\bigcup_{i=-,+} W_2^{2+l,1+l/2}(Q_T^i)} + T^{-l} \Bigg\{ \|\mathcal{D}_t u\|^2_{Q_T^-\cup Q_T^+}$$
$$+ \sum_{|\alpha|=2} \|\mathcal{D}_x^\alpha u\|^2_{Q_T^-\cup Q_T^+} \Bigg\} + \sup_{t\leqslant T} \|u(\cdot,t)\|^2_{\bigcup_i W_2^{1+l}(\Omega_0^i)}.$$

For $\beta \in (0,1)$ we will consider the following Hölder norm of $u \in \mathbb{R}^3_T \equiv \mathbb{R}^3 \times (0,T)$:

$$\||u|\|_{\mathbb{R}^3_T} = \sup_{\mathbb{R}^3_T} |u| + \max_k \sup_{(x,t)\in\mathbb{R}^3_T} |\mathcal{D}_{x_k} u(x,t)| + \sup_{(x,t),\tau\leqslant T} \frac{|u(x,t)-u(x,\tau)|}{|\tau - t|^\beta}.$$

Let B_d be the ball $\{x : |x| < d\}$. We choose a coordinate system $\{x\}$ so that Ω_0^+ is contained in the ball $B_d,\ d < \infty$, and we set $B_{dT}^- \equiv (B_d \setminus \overline{\Omega_0^+}) \times (0,T)$.

THEOREM 1.1 Assume that for some $l \in (1/2,1)$ we have $\Gamma \in W_2^{5/2+l}$, $\rho_0^+ \in W_2^{1+l}(\Omega_0^+)$, $0 < R_0 \leqslant \rho_0^+(\xi) \leqslant R_\infty < \infty$, $\xi \in \Omega_0^+$, $p^+ \in C^3(\mathbb{R}_+)$, $\boldsymbol{f} \in \boldsymbol{W}_2^{l,l/2}(\mathbb{R}^3_T)$, $0 < T < \infty$, $\boldsymbol{f}(\cdot,t) \in \boldsymbol{C}^2(\mathbb{R}^3)$ for $\forall t \in [0,T]$, $\boldsymbol{f}(\xi,\cdot)$, $\nabla \boldsymbol{f}(\xi,\cdot) \in \boldsymbol{C}^\beta(0,T)$ for $\forall \xi \in \mathbb{R}^3$ with some $\beta \in (1/2,1)$. In addition, let the initial velocity vector $\boldsymbol{v}_0 \in \bigcup_{i=-,+} \boldsymbol{W}_2^{1+l}(\Omega_0^i)$ satisfy the compatibility conditions

$$\nabla \cdot \boldsymbol{v}_0 = 0 \quad \text{in} \quad \Omega_0^-,$$

$$[\boldsymbol{v}_0]\Big|_\Gamma = 0, \quad [\Pi_0 \mathbb{S}(\boldsymbol{v}_0)\boldsymbol{n}_0]\Big|_\Gamma = 0,$$

and let the viscosities of the liquids satisfy the inequalities

$$\mu^- > \mu^+, \quad \nu^- < \mu^+/R_\infty\,. \tag{1.9}$$

Under these hypotheses, there exists a constant $T_0 \in (0, T]$ such that problem (1.8) is uniquely solvable on the interval $(0, T_0)$, and its solution $(\boldsymbol{u}, q)$ has the properties: $\boldsymbol{u} \in \bigcup_{i=-,+} \boldsymbol{W}_2^{2+l,1+l/2}(Q_{T_0}^i)$, $q \in W_{2,loc}^{l,l/2}(Q_{T_0}^-)$, $\nabla q \in \boldsymbol{W}_2^{l,l/2}(Q_{T_0}^-)$, $q|_{G_{T_0}} \in W_2^{l+1/2,l/2+1/4}(G_{T_0})$ and

$$\begin{aligned}
&\|\boldsymbol{u}\|_{Q_{T_0}^-\cup Q_{T_0}^+}^{(2+l,1+l/2)} + \|\nabla q\|_{Q_{T_0}^-}^{(l,l/2)} + \|q\|_{B_{dT_0}^-}^{(l,l/2)} + \|q\|_{W_2^{l+1/2,l/2+1/4}(G_{T_0})} \\
&\leqslant c_1\Big(c_2 + c_3 T_0^{\frac{1-l}{2}} \|\boldsymbol{v}_0\|_{\bigcup_i \boldsymbol{W}_2^{1+l}(\Omega_0^i)}\Big)\Big\{|||\boldsymbol{f}|||_{\mathbb{R}_{T_0}^3} + \|\boldsymbol{v}_0\|_{\bigcup_i \boldsymbol{W}_2^{1+l}(\Omega_0^i)} \\
&\qquad + \sigma\|H_0\|_{W_2^{l+1/2}(\Gamma)} + \left\|\frac{1}{\rho_0^+}\nabla p^+(\rho_0^+)\right\|_{\boldsymbol{W}_2^l(\Omega_0^+)} + \|p^+(\rho_0^+)\|_{W_2^{1+l}(\Omega_0^+)}\Big\}.
\end{aligned}$$

The value of T_0 depends on the norms of $\boldsymbol{f}$, $\boldsymbol{v}_0$, ρ_0, p^+ and on the curvature magnitude of Γ.

This theorem is proved by successive approximations in the same way as the analogous theorems for the case of a single incompressible fluid [8] or for the case of a single compressible one [9]. We will consider the main steps of the proof in Section 4. The role of the successive approximations will be played by the solutions of the following linearized problems:

$$\begin{gathered}
\mathcal{D}_t \boldsymbol{w} - \frac{1}{\rho_0^+(\xi)}\mathbb{A}\nabla\mathbb{T}_{\boldsymbol{u}}'(\boldsymbol{w}) = \boldsymbol{f} \quad \text{in} \quad Q_T^+, \\
\mathcal{D}_t \boldsymbol{w} - \nu^-\nabla_{\boldsymbol{u}}^2\boldsymbol{w} + \frac{1}{\rho_0^-}\nabla_{\boldsymbol{u}} s = \boldsymbol{f}, \quad \nabla_{\boldsymbol{u}}\cdot\boldsymbol{w} = r \quad \text{in} \quad Q_T^-, \\
\boldsymbol{w}\Big|_{t=0} = \boldsymbol{w}_0 \quad \text{in} \quad \Omega_0^-\cup\Omega_0^+, \quad \boldsymbol{w} \xrightarrow[|\xi|\to\infty]{} 0, \quad s \xrightarrow[|\xi|\to\infty]{} 0, \qquad (1.10) \\
[\boldsymbol{w}]\Big|_{G_T} = 0, \quad [\mu^\pm \Pi_0 \Pi \mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w})\boldsymbol{n}]\Big|_{G_T} = \Pi_0 a, \\
[\boldsymbol{n}_0\cdot\mathbb{T}_{\boldsymbol{u}}'(\boldsymbol{w}, s)\boldsymbol{n}]\Big|_\Gamma - \sigma\boldsymbol{n}_0\cdot\Delta(t)\int_0^t \boldsymbol{w}\Big|_\Gamma \,\mathrm{d}\tau = b + \sigma\int_0^t B\,\mathrm{d}\tau, \quad t\in(0,T).
\end{gathered}$$

THEOREM 1.2 Let $\Gamma \in W_2^{3/2+l}$, $\rho_0^+ \in W_2^{1+l}(\Omega_0^+)$, for some $l \in (1/2, 1)$ and let $0 < R_0 \leqslant \rho_0^+(\xi) \leqslant R_\infty < \infty$, $\xi \in \Omega_0^+$. In addition, assume that the vector field $\boldsymbol{u}$ is continuous across the boundary Γ and that for some $T < \infty$ it satisfies the inequality

$$T^{1/2}\|\boldsymbol{u}\|_{Q_T^-\cup Q_T^+}^{(2+l,1+l/2)} \leqslant \delta \qquad (1.11)$$

with a small number δ. We suppose also that inequalities (1.9) hold for the viscosities $\mu^\pm$, ν^-.

Then for any $\boldsymbol{f} \in \bigcup_{i=-,+} \boldsymbol{W}_2^{l,l/2}(Q_T^i)$, $r \in W_2^{1+l,1/2+l/2}(Q_T^-)$, $r = \nabla\cdot\boldsymbol{R}$, $\boldsymbol{R} \in \boldsymbol{W}_2^{0,1+l/2}(Q_T^-)$, $\boldsymbol{w}_0 \in \bigcup_{i=-,+} \boldsymbol{W}_2^{1+l}(\Omega_0^i)$, $\boldsymbol{a} \in \boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)$, $b \in W_2^{l+1/2,l/2+1/4}(G_T)$, and $B \in W_2^{l-1/2,l/2-1/4}(G_T)$ for which the compatibility conditions

$$[\boldsymbol{w}_0]\Big|_\Gamma = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\boldsymbol{w}_0)\boldsymbol{n}_0]\Big|_\Gamma = \Pi_0 \boldsymbol{a}\Big|_{t=0},$$

$$\nabla\cdot\boldsymbol{w}_0=r\Big|_{t=0}\quad\text{on}\quad\Omega_0^-$$

hold, there exists a unique solution $(\boldsymbol{w},s)$ of problem (1.10) such that $\boldsymbol{w}\in\bigcup_{i=-,+}\boldsymbol{W}_2^{2+l,1+l/2}(Q_T^i)$, $s\in W_{2,loc}^{l,l/2}(Q_T^-)$, $\nabla s\in\boldsymbol{W}_2^{l,l/2}(Q_T^-)$, $s\big|_{G_T}\in W_2^{l+1/2,l/2+1/4}(G_T)$, and

$$\begin{aligned}
&\|\boldsymbol{w}\|^{(2+l,1+l/2)}_{Q_T^-\cup Q_T^+}+\|\nabla s\|^{(l,l/2)}_{Q_T^-}+\|s\|^{(l,l/2)}_{B_{dT}^-}+\|s\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\\
&\leqslant c_1(T)\bigg\{\|\boldsymbol{f}\|^{(l,l/2)}_{Q_T^-\cup Q_T^+}+\|\boldsymbol{w}_0\|_{\bigcup_i W_2^{1+l}(\Omega_0^i)}+\|r\|_{W_2^{1+l,0}(Q_T^-)}\\
&\qquad+\|\boldsymbol{R}\|_{\boldsymbol{W}_2^{0,1+l/2}(Q_T^-)}+T^{-l/2}\|\mathcal{D}_t\boldsymbol{R}\|_{Q_T^-}+\|\boldsymbol{a}\|_{\boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)}\\
&\qquad+\|b\|_{W_2^{l+1/2,l/2+1/4}(G_T)}+T^{-l/2}\|b\|_{W_2^{1/2,0}(G_T)}\\
&\qquad+\sigma\|B\|_{W_2^{l-1/2,l/2-1/4}(G_T)}\bigg\},
\end{aligned}\tag{1.12}$$

$c_1(T)$ being a non-decreasing function of T.

The existence of a unique smooth solution of problem (1.10) is based on the analysis of the model problem when $\boldsymbol{u}\equiv 0$ and when the interface Γ is a plane.

2. The homogeneous model problem

In this section, we consider the problem

$$\begin{aligned}
&\mathcal{D}_t\boldsymbol{v}-\nu^+\nabla^2\boldsymbol{v}+\frac{1}{\rho_0^+}\nabla p=0,\quad \nabla\cdot\boldsymbol{v}=0,\quad\text{in}\quad D_\infty^+=\mathbb{R}_+^3\times(0,\infty),\\
&\mathcal{D}_t\boldsymbol{v}-\nu^-\nabla^2\boldsymbol{v}-(\nu^-+\kappa^-)\nabla(\nabla\cdot\boldsymbol{v})=0\quad\text{in}\quad D_\infty^-=\mathbb{R}_-^3\times(0,\infty),\\
&\boldsymbol{v}\Big|_{t=0}=0\quad\text{on}\quad\mathbb{R}_-^3\cup\mathbb{R}_+^3,\quad \boldsymbol{v}\xrightarrow[|x|\to\infty]{}0,\quad p\xrightarrow[|x|\to\infty]{}0,\\
&[\boldsymbol{v}]\Big|_{x_3=0}=0,\quad -\left[\mu^\pm\left(\frac{\partial v_\alpha}{\partial x_3}+\frac{\partial v_3}{\partial x_\alpha}\right)\right]\Big|_{x_3=0}=b_\alpha(x',t),\quad\alpha=1,2;\\
&-(p+\lambda^-\nabla\cdot\boldsymbol{v})\Big|_{x_3=0}+\left[2\mu^\pm\frac{\partial v_3}{\partial x_3}\right]\Big|_{x_3=0}+\sigma\Delta'\int_0^t v_3\Big|_{x_3=0}\,\mathrm{d}\tau\\
&\qquad\qquad=b_3+\sigma\int_0^t B\,\mathrm{d}\tau\equiv b_3'\quad\text{on}\ \mathbb{R}_\infty^2.
\end{aligned}\tag{2.1}$$

Here we have used the notation $\mathbb{R}_\pm^3=\{\pm x_3>0\}$, $\mathbb{R}_\infty^2=\mathbb{R}^2\times(0,\infty)$, $\kappa^-=\lambda^-/\rho_0^-$, $\rho_0^-=constant>0$, $\nu^-=\mu^-/\rho_0^-$, $\Delta'=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$, $x'=(x_1,x_2)$.

We take the Fourier transform on the tangent space variables $(x_1,x_2)=x'$ and the Laplace transform with respect to t given by the formula

$$\widetilde{f}(\xi,x_3,s)=\int_0^\infty e^{-st}\int_{\mathbb{R}^2}f(x,t)\mathrm{e}^{-\mathrm{i}x'\cdot\xi}\,\mathrm{d}x'\,\mathrm{d}t,\ \mathrm{Re}\,s\geqslant 0,\quad \xi=(\xi_1,\xi_2).\tag{2.2}$$

For a function $f \in H_\gamma^{l,l/2}(D_\infty^\pm)$, transfomation (2.2) defines an analytic function $\widetilde{f}$ in the half-plane $\{\operatorname{Re} s > \gamma\}$.

The problem (2.1) is then transformed into the system of ordinary differential equations

$$\begin{aligned}
&\frac{\mathrm{d}^2\widetilde{v}_\alpha}{\mathrm{d}x_3^2} - \left(\frac{s}{\nu^+} + \xi^2\right)\widetilde{v}_\alpha - \frac{\mathrm{i}\xi_\alpha}{\mu^+}\widetilde{p} = 0, \qquad \alpha = 1, 2,\\
&\frac{\mathrm{d}^2\widetilde{v}_3}{\mathrm{d}x_3^2} - \left(\frac{s}{\nu^+} + \xi^2\right)\widetilde{v}_3 - \frac{1}{\mu^+}\frac{\widetilde{p}}{\mathrm{d}x_3} = 0,\\
&\frac{\mathrm{d}\widetilde{v}_3}{\mathrm{d}x_3} + \mathrm{i}\xi_\alpha\widetilde{v}_\alpha = 0, \qquad\qquad x_3 > 0,\\
&\frac{\mathrm{d}^2\widetilde{v}_\alpha}{\mathrm{d}x_3^2} - \left(\frac{s}{\nu^-} + \xi^2\right)\widetilde{v}_\alpha + (1+\beta^-)\mathrm{i}\xi_\alpha\left(\mathrm{i}\xi_1\widetilde{v}_1 + \mathrm{i}\xi_2\widetilde{v}_2 + \frac{\mathrm{d}\widetilde{v}_3}{\mathrm{d}x_3}\right) = 0,\ \alpha = 1, 2,\\
&(2+\beta^-)\frac{\mathrm{d}^2\widetilde{v}_3}{\mathrm{d}x_3^2} - \left(\frac{s}{\nu^-} + \xi^2\right)\widetilde{v}_3 + (1+\beta^-)\frac{\mathrm{d}}{\mathrm{d}x_3}(\mathrm{i}\xi_1\widetilde{v}_1 + \mathrm{i}\xi_2\widetilde{v}_2) = 0, \quad x_3 < 0,
\end{aligned} \tag{2.3}$$

with the boundary conditions

$$\begin{aligned}
&\widetilde{\boldsymbol{v}} \xrightarrow[|x_3|\to\infty]{} 0, \quad \widetilde{p} \xrightarrow[|x_3|\to\infty]{} 0, \quad [\widetilde{\boldsymbol{v}}]\Big|_{x_3=0} = 0,\\
&-\left[\mu^\pm\left(\frac{\mathrm{d}\widetilde{v}_\alpha}{\mathrm{d}x_3} + \mathrm{i}\xi_\alpha\widetilde{v}_3\right)\right]\Big|_{x_3=0} = \widetilde{b}_\alpha, \quad \alpha = 1, 2,\\
&s\left(-\widetilde{p}^+ + \left[2\mu^\pm\frac{\mathrm{d}\widetilde{v}_3}{\mathrm{d}x_3}\right]\Big|_{x_3=0} - \lambda^-\left(\sum_{\beta=1}^{2}\mathrm{i}\xi_\beta\widetilde{v}_\beta^- + \frac{\mathrm{d}\widetilde{v}_3^-}{\mathrm{d}x_3}\right)\right) - \sigma\xi^2\widetilde{v}_3^+ = s\widetilde{b}_3'
\end{aligned} \tag{2.4}$$

where $\xi^2 = \xi_1^2 + \xi_2^2$, $\beta^- \equiv \kappa^-/\nu^- \equiv \lambda^-/\mu^-$, $\widetilde{\boldsymbol{v}}^\pm = \lim_{x_3\to 0\pm}\widetilde{\boldsymbol{v}}$, $\widetilde{p}^+ = \lim_{x_3\to 0+}\widetilde{p}$.

As in the case of two one-type liquids (see [2, 4]), we write the general solution of (2.3) in the two half-spaces:

$$\begin{aligned}
&\widetilde{\boldsymbol{v}} = C_1^+\begin{pmatrix} r^+ \\ 0 \\ \mathrm{i}\xi_1\end{pmatrix}\mathrm{e}^{-r^+x_3} + C_2^+\begin{pmatrix} 0 \\ r^+ \\ \mathrm{i}\xi_2\end{pmatrix}\mathrm{e}^{-r^+x_3} + C_3^+\begin{pmatrix}\mathrm{i}\xi_1 \\ \mathrm{i}\xi_2 \\ -|\xi|\end{pmatrix}\mathrm{e}^{-|\xi|x_3},\\
&\widetilde{p} = -C_3^+\rho_0^+ s\mathrm{e}^{-|\xi|x_3} \qquad\qquad \text{for } x_3 > 0,\\
&\widetilde{\boldsymbol{v}} = C_1^-\begin{pmatrix} -r^- \\ 0 \\ \mathrm{i}\xi_1\end{pmatrix}\mathrm{e}^{r^-x_3} + C_2^-\begin{pmatrix} 0 \\ -r^- \\ \mathrm{i}\xi_2\end{pmatrix}\mathrm{e}^{r^-x_3} + C_3^-\begin{pmatrix}\mathrm{i}\xi_1 \\ \mathrm{i}\xi_2 \\ r_1^-\end{pmatrix}\mathrm{e}^{r_1^-x_3}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x_3 < 0.
\end{aligned} \tag{2.5}$$

Here $C_i^\pm$, $i = 1, 2, 3$, are arbitrary constants, $r^\pm = \sqrt{\frac{s}{\nu^\pm} + \xi^2}$, $r_1^- = \sqrt{\frac{s}{(2+\beta^-)\nu^-} + \xi^2}$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$ and we assume $|\arg\sqrt{z}| < \pi/2$ for $\forall z \in \mathbb{C}$.

We substitute (2.5) in the boundary conditions (2.4) and solve the system obtained. Then we find a solution of problem (2.3), (2.4) which we write in the form convenient for the following estimates:

$$\widetilde{\boldsymbol{v}} = \boldsymbol{W}e_0^{\pm} + \boldsymbol{V}^{\pm}e_1^{\pm}, \quad \pm x_3 > 0, \tag{2.6}$$
$$\widetilde{p} = -C_3^+\rho_0^+ s\mathrm{e}^{-|\xi|x_3} = -\mu^+C_3^+(r^+ - |\xi|)(r^+ + |\xi|)\mathrm{e}^{-|\xi|x_3}, \quad x_3 > 0,$$

where

$$e_0^{\pm} = \mathrm{e}^{\mp r^{\pm}x_3}, \quad e_1^+ = \frac{\mathrm{e}^{-r^+x_3} - \mathrm{e}^{-|\xi|x_3}}{r^+ - |\xi|}, \quad e_1^- = \frac{\mathrm{e}^{r^-x_3} - \mathrm{e}^{r_1^-x_3}}{r^- - r_1^-},$$

$$\boldsymbol{W} = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}, \quad \boldsymbol{V}^+ = -C_3^+(r^+ - |\xi|)\begin{pmatrix}\mathrm{i}\xi_1\\ \mathrm{i}\xi_2\\ -|\xi|\end{pmatrix}, \quad \boldsymbol{V}^- = -C_3^-(r^- - r_1^-)\begin{pmatrix}\mathrm{i}\xi_1\\ \mathrm{i}\xi_2\\ r_1^-\end{pmatrix},$$

$$\begin{aligned}
C_3^+ = -\frac{1}{(r^+ - |\xi|)P}\Bigg\{&A\bigg\{\Big[2\mu^+r^+ + \frac{\sigma}{s}\xi^2\Big](r^-r_1^- - \xi^2)\\
&\quad + \mu^-r^+(r^{-2} - 2r^-r_1^- + \xi^2) + \rho_0^- sr^-\bigg\}\\
&+ \widetilde{b}_3'\{\mu^+(r^-r_1^- - \xi^2)(r^{+2} + \xi^2)\\
&\quad + \mu^-\xi^2(r^{-2} + \xi^2 - 2r^-r_1^-) + \rho_0^- sr^+r_1^-\}\Bigg\},
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
C_3^- = &-\frac{A}{P}\Big\{\mu^+[r^+(r^+ + |\xi|) + r^-(r^+ - |\xi|)] + 2\mu^-r^-|\xi| + \frac{\sigma}{s}|\xi|^3\Big\}\\
&+ \frac{\widetilde{b}_3'}{P}\{\mu^+[r^+|\xi|(r^- + |\xi|) + (r^- - |\xi|)\xi^2] + \mu^-(r^{-2} + \xi^2)|\xi|\},
\end{aligned}$$

$$\begin{aligned}
\omega_\alpha = \frac{1}{\mu^+r^+ + \mu^-r^-}\{&\widetilde{b}_\alpha + \mathrm{i}\xi_\alpha[\mu^+(r^+ - |\xi|)C_3^+ + \mu^-(r^- - r_1^-)C_3^-\\
&+ (\mu^+ - \mu^-)\omega_3]\}, \quad \alpha = 1, 2,
\end{aligned}$$

$$\omega_3 = \frac{-(r^+ - |\xi|)|\xi|C_3^+ + (r^-r_1^- - \xi^2)C_3^-}{r^+ + r^-}.$$

In formulas (2.7) we have used the notation:

$$A = \mathrm{i}\xi_1\widetilde{b}_1 + \mathrm{i}\xi_2\widetilde{b}_2, \quad \widetilde{b}_3' = \widetilde{b}_3 + \frac{\sigma}{s}\widetilde{B},$$

$$\begin{aligned}
P = \;&\rho_0^+\mu^+s(r^+ + |\xi|)(r^-r_1^- - \xi^2) + \rho_0^{-2}s^2|\xi|\\
&+ \rho_0^-\mu^+s(r^+ + |\xi|)(r^-|\xi| + r^+r_1^- + 2\xi^2)\\
&+ 4(\mu^+ - \mu^-)\xi^2\{\mu^+r^+(r^-r_1^- - \xi^2) - \mu^-r^-(r^- - r_1^-)|\xi|\}\\
&+ \frac{\sigma|\xi|^3}{s}\{\mu^+(r^+ + |\xi|)(r^-r_1^- - \xi^2) + \rho_0^-r_1^-s\}.
\end{aligned}$$

We observe that solution (2.6), (2.7) may be obtained by the passage to the limit $r_1^+ \to |\xi|$ from a solution of the model problem with a plane interface between two compressible fluids (see [4]).

On the other hand, it goes over as $r_1^- \to |\xi|$ into a solution of the corresponding problem for two incompressible liquids [2]. This passage may also be demonstrated in the system of equations. Let us turn r_1^- to $|\xi|$ that corresponds to $\beta^- \to \infty$, in system (2.3). In this case, the Navier–Stokes equations for a compressible fluid in the domain $\{x_3 < 0\}$ become the equations for the incompressible one.

We now move on to the estimates of solution (2.6), (2.7). As C_3^+ is contained in all the expressions with multiplier $(r^+ - |\xi|)$, which is cancelled with the denominator of C_3^+, the estimate of solution (2.6), (2.7) depends only on the lower bound of P.

We rewrite P in the form:

$$P = \sum_{j=1}^{6} I_j \tag{2.8}$$

where

$$\begin{aligned}
I_1 &= \left(\rho_0^+ + \frac{\rho_0^-|\xi|}{r_1^-}\right) sq, \\
I_2 &= (\mu^+ - \mu^-)\mu^+\xi^2 r^+(r^- r_1^- - \xi^2), \\
I_3 &= 4{\mu^-}^2 r^-(r^- - r_1^-)|\xi|^3, \\
I_4 &= \mu^+\rho_0^- s(r^+ + |\xi|)(r_1^-|\xi| + |\xi|^3/r^-), \\
I_5 &= 2\mu^+\rho_0^- s\xi^2\left\{r^+ + |\xi| - 2\nu^-|\xi|\frac{r^-(r^- - r_1^-)}{s}\right\} \\
&= 2\mu^+\rho_0^- s\xi^2\left\{r^+ + |\xi| - 2\frac{1+\beta^-}{2+\beta^-}\frac{r^-|\xi|}{r^- + r_1^-}\right\}, \\
I_6 &= \frac{\sigma|\xi|^3}{s} q, \\
q &= \mu^+(r^+ + |\xi|)(r^- r_1^- - \xi^2) + \rho_0^- r_1^- s.
\end{aligned}$$

LEMMA 2.1 For $\forall\xi \in \mathbb{R}^2$, $\forall s \in \mathbb{C}$, $\operatorname{Re} s > 0$, the absolute value of the expression q is estimated by

$$|q| \geqslant \frac{\rho_0^-}{\sqrt{2}}|r_1^-||s|, \tag{2.9}$$

$$|q| \leqslant c_1(|s|^{3/2} + |s||\xi|). \tag{2.10}$$

Moreover, if the viscosities of the fluids satisfy the inequality

$$\nu^+ < \nu^-$$

then $\arg(q/s) \in (0, (\arg s)/2]$.

Proof. For definiteness, we assume from now on that $\arg s \geqslant 0$. (For $\arg s \leqslant 0$ the picture will be symmetric with respect to the axis $\operatorname{Im} s = 0$.) Under this hypothesis, $\arg r^\pm$, $\arg r_1^- \in [0, \pi/4)$ and

$$\arg\frac{r^- r_1^- - \xi^2}{s} = \arg\frac{1}{(2+\beta^-)\nu^-}\left(1 + \frac{(1+\beta^-)r_1^-}{r^- + r_1^-}\right) \in \left(-\frac{\pi}{4}, 0\right]. \tag{2.11}$$

Hence, $\arg q/s \in (-\pi/4, \pi/4)$ and

$$|q| \geqslant |s|\mathrm{Re}\left\{\mu^+(r^+ + |\xi|)\frac{r^- r_1^- - \xi^2}{s} + \rho_0^- r_1^-\right\} \geqslant \frac{\rho_0^-|s||r_1^-|}{\sqrt{2}}.$$

Inequality (2.10) follows from the estimate

$$\begin{aligned}|q| &\leqslant |s|\left\{c_2(|r^+| + |\xi|)\left(1 + \sup_{s,\xi}\frac{|r_1^-|}{|r^- + r_1^-|}\right) + \rho_0^-|r_1^-|\right\}\\ &\leqslant c_1|s|(|s|^{1/2} + |\xi|).\end{aligned}$$

Next, it follows from the inequality $\nu^+ < \nu^-$ that $\arg(r^+ + |\xi|) > \arg(r^- + |\xi|) > \arg\frac{r^- + r_1^-}{r_1^-} = \arg(r^- + |r_1^-|)$ because $|\xi| < |r_1^-|$. So, $\arg q/s > 0$.

It is not difficult to see that the arguments of vectors $r^\pm, r_1^-, r^\pm + |\xi|$ do not exceed $\frac{1}{2}\arg s$. Taking (2.11) into account again, we conclude that $\arg(q/s) \leqslant (\arg s)/2 < \pi/4$. □

LEMMA 2.2 Assume that for the viscosities of the fluids the inequalities

$$\nu^+ < \nu^-, \quad \mu^+ > \mu^- \tag{2.12}$$

hold and that $\sigma \geqslant 0$. Then for $\forall\xi \in \mathbb{R}^2$, $\forall s \in \mathbb{C}$, $\mathrm{Re}\, s = \gamma > 0$, we have

$$\begin{aligned}&|P| \geqslant c_3(\rho_0^\pm, \nu^-, \beta^-)\gamma^{5/2}, \qquad (2.13)\\ &|P| \geqslant c_4\left(|s|^2 + |s|^{3/2}|\xi| + |s|\xi^2 + \sigma|\xi|^3\right)\left(|s|^{1/2} + |\xi|\right).\end{aligned}$$

Proof. We will show that each $\mathrm{Re}\,(I_j/q)$ is non-negative. Then for the modulus of the expression P, we can use the inequality

$$|P| \geqslant |q|\left|\mathrm{Re}\sum_{j=1}^{6}\frac{I_j}{q}\right| = |q|\sum_{j=1}^{6}\mathrm{Re}\,\frac{I_j}{q}. \tag{2.14}$$

We have

$$\begin{aligned}\mathrm{Re}\,\frac{I_1}{q} &= \mathrm{Re}\left(\rho_0^+ s + \rho_0^-|\xi|\frac{s}{r_1^-}\right) = \rho_0^+\mathrm{Re}\, s\\ &\quad + \rho_0^-|\xi|\frac{|s|}{|r_1^-|}\{\cos(\arg s)\cos(\arg r_1^-) + \sin(\arg s)\sin(\arg r_1^-)\}\\ &\geqslant \gamma\left(\rho_0^+ + \rho_0^-\frac{|\xi|}{\sqrt{2}|r_1^-|}\right) > 0 \qquad (2.15)\end{aligned}$$

because the arguments of s and r_1^- have the same sign and consequently the product of their sines is non-negative; in addition $\cos(\arg r_1^-) > 1/\sqrt{2}$.

Next,

$$\begin{aligned}\mathrm{Re}\,\frac{I_2}{q} &= \mathrm{Re}\left\{4(\mu^+ - \mu^-)\mu^+\xi^2\frac{r^+(r^- r_1^- - \xi^2)}{s}\frac{s}{q}\right\} \geqslant 0,\\ \mathrm{Re}\,\frac{I_3}{q} &= \mathrm{Re}\left\{4\mu^-\frac{1+\beta^-}{\nu^-(2+\beta^-)}\frac{r^-|\xi|^3}{r^- + r_1^-}\frac{s}{q}\right\} \geqslant 0,\end{aligned}$$

as $\arg \frac{r^+(r^-r_1^- - \xi^2)}{s}$, $\arg \frac{s}{q} \in (-\frac{\pi}{4}, \frac{\pi}{4})$, and $\arg \frac{r^-}{r^-+r_1^-} \in [0, \frac{\pi}{4})$.

For $\frac{I_4}{q}$, we have

$$\begin{aligned}\operatorname{Re}\frac{I_4}{q} &= \mu^+\rho_0^-|\xi|\left\{\operatorname{Re}\frac{(r^+ + |\xi|)r_1^- s}{q} + \operatorname{Re}\left(\frac{(r^+ + |\xi|)\xi^2}{r_1^-}\frac{s}{q}\right)\right\} \\ &\geqslant \mu^+\rho_0^-|\xi|\operatorname{Re}\frac{1}{\mu^+\frac{r^-r_1^- - \xi^2}{sr_1^-} + \frac{\rho_0^-}{r^+ + |\xi|}} \geqslant 0\end{aligned}$$

since the argument of the last fraction's denominator belongs to the interval $(-\pi/2, 0]$.

We transform I_5 in the following way:

$$\begin{aligned}I_5 &= 2\mu^+\rho_0^- s\xi^2\left\{r^+ + \frac{(2+\beta^-)r_1^-|\xi| - \beta^- r^-|\xi|}{(2+\beta^-)(r^- + r_1^-)}\right\} \\ &= 2\mu^+\rho_0^- s\xi^2\left\{\frac{2}{2+\beta^-}r^+ + \frac{2(1+\beta^-)r_1^-|\xi|}{(2+\beta^-)(r^- + r_1^-)} + \frac{\beta^-(r^+ - |\xi|)}{2+\beta^-}\right\}.\end{aligned}$$

Then taking the preceding arguments into account, we verify that

$$\begin{aligned}\operatorname{Re}\frac{I_5}{q} &= 2\mu^+\rho_0^-\xi^2\operatorname{Re}\left\{\frac{2}{2+\beta^-}\frac{r^+}{q/s} + \frac{2(1+\beta^-)r_1^-|\xi|}{(2+\beta^-)(r^- + r_1^-)q/s}\right. \\ &\quad \left. + \frac{\beta^-}{(2+\beta^-)\nu^+}\frac{s}{(r^+ + |\xi|)q/s}\right\} > 0. \end{aligned} \tag{2.16}$$

Finally,

$$\operatorname{Re}\frac{I_6}{q} = \operatorname{Re}\frac{\sigma|\xi|^3}{s} = \frac{\sigma|\xi|^3\gamma}{|s|^2} > 0.$$

Now let us return to estimate (2.14). Dropping all the terms except the first one, by virtue of (2.9) and (2.15) we obtain:

$$|P| \geqslant \gamma\rho_0^+\frac{\rho_0^-|r_1^-||s|}{\sqrt{2}} \geqslant c_3(\rho_0^\pm, \nu^-, \beta^-)\gamma^{5/2}$$

which proves the strict positiveness of $|P|$ for $\gamma > 0$. In addition, from (2.16), it follows an inequality useful for the further estimates of the absolute value of P. As

$$|P| \geqslant \frac{4\mu^+\rho_0^- s\xi^2}{2+\beta^-}|q|\operatorname{Re}\frac{r^+}{q/s}$$

and $\arg \frac{r^+}{q/s} \in (-\frac{\pi}{4}, \frac{\pi}{4})$, we have

$$\begin{aligned}|P| &\geqslant \frac{4\mu^+\rho_0^-\xi^2}{2+\beta^-}\frac{|r^+||s|}{\sqrt{2}} \\ &= c_5|s||r^+|\xi^2 \geqslant c_6(|s|^{3/2}\xi^2 + |s||\xi|^3).\end{aligned} \tag{2.17}$$

In order to prove the second inequality in (2.13), we consider two cases.

Case 1. Let $|s| < \xi^2$. Then

$$|s|^{5/2} + |s|^2|\xi| \leqslant |s|^{3/2}\xi^2 + |s||\xi|^3 \leqslant \frac{|P|}{c_6}.$$

In view of formula (2.8) for P, we have

$$\sigma|\xi|^3\left|\frac{q}{s}\right| \leqslant |P| + \sum_{j=1}^{5}|I_j|.$$

Let us estimate $|I_j|$, $j = 1, \dots, 5$. Using (2.10), we obtain:

$$\begin{aligned}
|I_1| &\leqslant \left(\rho_0^+ + \frac{\rho_0^-|\xi|}{|r_1^-|}\right)|s||q| \leqslant c_7\left(|s|^{5/2} + |s|^2|\xi|\right) \leqslant \frac{c_7}{c_6}|P|,\\
|I_2| &\leqslant \frac{4(\mu^+ - \mu^-)\mu^+}{(2+\beta^-)\nu^-}|s||r^+|\xi^2 \max_{s,\xi}\left|1 + \frac{(1+\beta^-)r_1^-}{r^- + r_1^-}\right| \leqslant c_8|s||r^+|\xi^2\\
&\leqslant \frac{c_8}{c_5}|P|,\\
|I_3| &\leqslant c_9 \max_{s,\xi}\left|\frac{r^-}{r^- + r_1^-}\right||s||\xi|^3 \leqslant \frac{c_9}{c_6}|P|,\\
|I_4| &\leqslant c_{10}|s|\left(|r^+| + |\xi|\right)\left(|r_1^-||\xi| + \xi^2\right) \leqslant c_{11}\left(|s|^2|\xi| + |s||\xi|^3\right) \leqslant \frac{c_{11}}{c_6}|P|,\\
|I_5| &\leqslant c_{12}|s|\xi^2\left(|r^+| + |\xi|\right) \leqslant c_{12}\left(\frac{1}{c_5} + \frac{1}{c_6}\right)|P|.
\end{aligned}$$

Hence,

$$\sigma|\xi|^4 \leqslant \sigma|\xi|^3|r_1^-| \leqslant \sigma\frac{\sqrt{2}}{\rho_0^-}|\xi|^3\left|\frac{q}{s}\right| \leqslant c_{13}|P|.$$

Thus, (2.13) is proved for the first case.

Case 2. Now consider $|s| \geqslant \xi^2$. From (2.17) it follows that

$$\sigma|\xi|^3(|s|^{1/2} + |\xi|) \leqslant 2\sigma\frac{|s||\xi|^3}{\sqrt{\gamma}} \leqslant \frac{2\sigma|P|}{c_6\sqrt{\gamma}}. \tag{2.18}$$

We gather all terms in P of orders $|s|^{5/2}$ and $|s|^2|\xi|$ and introduce the function

$$I_0 = I_1 + \mu^+\rho_0^- s\left(r^+ - |\xi|\right)r_1^-|\xi|.$$

Evaluating its absolute value by its real part, we deduce the estimate

$$\begin{aligned}
|I_0| &\geqslant |s|^2 \mathrm{Re}\left\{\rho_0^+\frac{q}{s} + \frac{\rho_0^-|\xi|}{r_1^-}\frac{q}{s} + \rho_0^+\rho_0^-\frac{r_1^-|\xi|}{r^+ + |\xi|}\right\}\\
&\geqslant \frac{1}{\sqrt{2}}|s|^2\left\{\frac{\rho_0^+\rho_0^-}{\sqrt{2}}|r_1^-| + \frac{\rho_0^{-2}}{\sqrt{2}}|\xi| + \rho_0^+\rho_0^- \min_{s,\xi}\left|\frac{r_1^-}{r^+ + |\xi|}\right||\xi|\right\}.
\end{aligned}$$

We note that in the last inequality we have used Lemma 2.1 and the fact that the argument of each term lies in the interval $(-\pi/4, \pi/4)$.

At last, with regard to (2.17), (2.18) we have:

$$c_{14}\left(|s|^{5/2}+|s|^2|\xi|\right) \leqslant |I_0| \leqslant |P| + \sum_{j=2,\, j\neq 4}^{6} |I_j| + 2\mu^+\rho_0^-|s||r_1^-|\xi^2 + \mu^+\rho_0^-|s|\frac{|r^+ + |\xi||}{|r_1^-|}|\xi|^3 \leqslant c_{15}|P|.$$

So, (2.13) is completely proved.

□

Now we may estimate the solution (2.6) easily.

LEMMA 2.3 For the coefficients $\boldsymbol{W}$, $\boldsymbol{V}^{\pm}$ defined by formulas (2.7), the inequalities

$$|\boldsymbol{V}^{\pm}| \leqslant c_{16}(\gamma)\left(\sum_{\alpha=1}^{2}|\widetilde{b_\alpha}| + \frac{|\widetilde{b_3}||\xi| + \sigma|\widetilde{B}|}{\sqrt{|s|+\xi^2}}\right), \tag{2.19}$$

$$|\boldsymbol{W}| \leqslant c_{17}\left(\sum_{\alpha=1}^{2}\frac{|\widetilde{b_\alpha}|}{\sqrt{|s|+\xi^2}} + \frac{|\widetilde{b_3}||\xi| + \sigma|\widetilde{B}|}{|s|+\xi^2}\right) \tag{2.20}$$

hold.

Proof. Taking into account the evident inequality

$$\frac{|r^-r_1^- - \xi^2|}{|s|} \leqslant \frac{1}{(2+\beta)\nu^-}\left(1 + (1+\beta)\sup_{s,\xi}\frac{|r_1^-|}{|r^- + r_1^-|}\right) \leqslant c_{18}(\nu^-, \beta^-),$$

we obtain

$$\begin{aligned}
|\boldsymbol{V}^+| &\leqslant c_{19}\sum_{\alpha=1}^{2}\frac{|\widetilde{b_\alpha}|\xi^2}{|P|}|s|\left(|r^+| + |r^-|\right) \\
&\quad + c_{20}\frac{\left(|\widetilde{b_3}||s| + \sigma|\widetilde{B}|\right)}{|P|}|\xi|\left\{|s| + \xi^2 + |r^+||r_1^-|\right\}, \\
|\boldsymbol{V}^-| &\leqslant \frac{(1+\beta^-)|s||r_1^-|}{(2+\beta^-)\nu^-|r^- + r_1^-|}|C_3^-| \\
&\leqslant \frac{c_{21}|s|}{|P|}\left\{\sum_{\alpha=1}^{2}|\widetilde{b_\alpha}||\xi|\left[|s| + \xi^2 + \left(|r^+| + |r_1^-|\right)|\xi| + \frac{\sigma|\xi|^3}{|s|}\right]\right. \\
&\quad \left. + \left(|\widetilde{b_3}| + \frac{\sigma|\widetilde{B}|}{|s|}\right)|\xi|\left[|s| + \xi^2 + |r^+||\xi|\right]\right\}.
\end{aligned} \tag{2.21}$$

From here, in view of (2.13), we obtain the estimate

$$|\boldsymbol{V}^{\pm}| \leqslant c_{22} \left\{ \sum_{\alpha=1}^{2} |\widetilde{b_\alpha}| + \frac{|\widetilde{b_3}||\xi| + \sigma|\widetilde{B}|(\frac{1}{\sigma} + \frac{1}{\sqrt{\gamma}})}{|s|^{1/2} + |\xi|} \right\},$$

(the term with $|\widetilde{B}|$ drops out for $\sigma = 0$).

The inequality (2.20) follows from the relations

$$|\omega_3| \leqslant \frac{c_{23}}{|r^+ + r^-|} \left(|\boldsymbol{V}^+| + c_{18}|s||C_3^-| \right),$$

$$|\omega_\alpha| \leqslant \frac{1}{|\mu^+ r^+ + \mu^- r^-|} \left(|\widetilde{b_\alpha}| + \mu^+|\boldsymbol{V}^+| + \mu^-|\boldsymbol{V}^-| + (\mu^+ - \mu^-)|\omega_3||\xi| \right),$$

$$\alpha = 1, 2,$$

and from estimates (2.19), (2.21). □

In the dual Fourier–Laplace space, we introduce normalizations corresponding to the norms defined in Section 1. It is possible to do so by means of a Parceval equality for the transform (2.2)

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^2} |\widetilde{f}(\xi, x_3, \gamma + \mathrm{i}\xi_0)|^2 \,\mathrm{d}\xi \,\mathrm{d}\xi_0 = (2\pi)^3 \int_0^{\infty} \mathrm{e}^{-2\gamma t} \int_{\mathbb{R}^2} |f(x,t)|^2 \,\mathrm{d}x' \,\mathrm{d}t.$$

Thus in Section 2 of [7] it was shown that for $\forall\gamma \geqslant 0$ the norm $\|u\|_{\bigcup_i H_\gamma^{m,m/2}(D_\infty^i)}$ is equivalent to the norm whose square is given by

$$|||u|||^2_{m,\gamma,D^3_\infty} \equiv \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left\{ \sum_{j<m} \left\| \frac{\partial^j}{\partial x_3^j} \widetilde{u}(\xi, x_3, s) \right\|^2_{\mathbb{R}_- \cup \mathbb{R}_+} |r|^{2m-2j} + \|\widetilde{u}(\xi, \cdot, s)\|^2_{\dot{W}_2^m(\mathbb{R}_-) \cup \dot{W}_2^m(\mathbb{R}_+)} \right\} \mathrm{d}\xi_0 \,\mathrm{d}\xi$$

where $s = \gamma + \mathrm{i}\xi_0$, $r = \sqrt{s + \xi^2}$, $\mathbb{R}_\pm = \{y \in \mathbb{R} | \pm y > 0\}$, $D^3_\infty = D^-_\infty \cup D^+_\infty$.

For the trace of a function u on the plane $\mathbb{R}^2$, the norm $\|u\|_{H_\gamma^{m,m/2}(\mathbb{R}^2_\infty)}$ is equivalent to the norm

$$|||u|||_{m,\gamma,\mathbb{R}^2_\infty} \equiv \left(\int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |\widetilde{u}(\xi, 0, s)|^2 |r|^{2m} \,\mathrm{d}\xi_0 \,\mathrm{d}\xi \right)^{1/2}$$

while the norm $\|u\|_{H_\gamma^{m+1/2,1/2,m/2}(\mathbb{R}^2_\infty)}$ is equivalent to the norm

$$|u|_{m,\gamma,\mathbb{R}^2_\infty} \equiv \left(\int_{\mathbb{R}^2} |\xi| \int_{-\infty}^{\infty} |\widetilde{u}(\xi, 0, s)|^2 |r|^{2m} \,\mathrm{d}\xi_0 \,\mathrm{d}\xi \right)^{1/2}.$$

Consider now the velocity vector $\widetilde{\boldsymbol{v}}$ from (2.6). We can write:

$$\left| \frac{\mathrm{d}^j \widetilde{\boldsymbol{v}}(x_3)}{\mathrm{d}x_3^j} \right| \leqslant c_{24} \left(|\boldsymbol{W}| \left| \frac{\mathrm{d}^j e_0^{\pm}(x_3)}{\mathrm{d}x_3^j} \right| + |\boldsymbol{V}^{\pm}| \left| \frac{\mathrm{d}^j e_1^{\pm}(x_3)}{\mathrm{d}x_3^j} \right| \right), \quad \pm x_3 > 0,$$

where $j = 0, 1, \ldots$. We estimate the derivatives of functions $e_0^{\pm}, e_1^{\pm}$ according to the inequalities (see Lemma 3.1 from [7]):

$$\pm \int_0^{\pm\infty} \left| \frac{\mathrm{d}^j e_0^{\pm}(x_3)}{\mathrm{d}x_3^j} \right|^2 \mathrm{d}x_3 \leqslant \frac{|r^{\pm}|^{2j-1}}{\sqrt{2}}, \tag{2.22}$$

$$\pm \int_0^{\pm\infty} \int_0^{\infty} \left| \frac{\mathrm{d}^j e_0^{\pm}(x_3 \pm z)}{\mathrm{d}x_3^j} - \frac{\mathrm{d}^j e_0^{\pm}(x_3)}{\mathrm{d}x_3^j} \right|^2 \frac{\mathrm{d}z\,\mathrm{d}x_3}{z^{1+2\alpha}} \leqslant c_{26}^{\pm} |r|^{2(j+\alpha)-1},$$

$$\pm \int_0^{\pm\infty} \left| \frac{\mathrm{d}^j e_1^{\pm}(x_3)}{\mathrm{d}x_3^j} \right|^2 \mathrm{d}x_3 \leqslant c_{27}^{\pm} |r|^{2j-3}, \tag{2.23}$$

$$\pm \int_0^{\pm\infty} \int_0^{\infty} \left| \frac{\mathrm{d}^j e_1^{\pm}(x_3 \pm z)}{\mathrm{d}x_3^j} - \frac{\mathrm{d}^j e_1^{\pm}(x_3)}{\mathrm{d}x_3^j} \right|^2 \frac{\mathrm{d}z\,\mathrm{d}x_3}{z^{1+2\alpha}} \leqslant c_{28}^{\pm} |r|^{2(j+\alpha)-3}$$

where $j = 0, 1, \ldots,\ \alpha \in (0, 1),\ r = \sqrt{s + \xi^2},\ s = \gamma + \mathrm{i}\xi_0,\ \gamma > 0,\ \xi \in \mathbb{R}^2$, and the constants $c_{26}^{\pm}, c_{27}^{\pm}, c_{28}^{\pm}$ do not depend on $|r|$. Then for the norm of vector $\boldsymbol{v}$ the relation

$$|||\boldsymbol{v}|||^2_{2+l,\gamma,D^3_\infty} \leqslant c_{29} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} (|\boldsymbol{W}|^2 |r|^{2l+3} + |\boldsymbol{V}^{\pm}|^2 |r|^{2l+1})\, \mathrm{d}\xi_0\, \mathrm{d}\xi \tag{2.24}$$

holds.

As the function $\mathrm{e}^{-|\xi|x_3}$ satisfies inequalities (2.22) with $|r|$ replaced by $|\xi|$, for the pressure gradient

$$\nabla \widetilde{p} = \mu^+ \boldsymbol{V}^+ (r^+ + |\xi|) \mathrm{e}^{-|\xi|x_3}, \quad x_3 > 0,$$

the inequality

$$|||\nabla p|||^2_{l,\gamma,D^+_\infty} \leqslant c_{30} \int_{\mathbb{R}^2} \int_{-\infty}^{\infty} |\boldsymbol{V}^+|^2 |r^+|^2 |\xi|^{2l-1}\, \mathrm{d}\xi_0\, \mathrm{d}\xi$$

holds.

And finally, as a consequence of Lemma 2.3 we conclude that

$$\begin{aligned}
&|||\boldsymbol{v}|||_{2+l,\gamma,D^3_\infty} + |||\nabla p|||_{l,\gamma,D^+_\infty} \\
&\leqslant c_{31} \left(\int_{\mathbb{R}^2} \int_{-\infty}^{\infty} \left(\sum_{\alpha=1}^{2} |\widetilde{b_\alpha}|^2 + \frac{|\widetilde{b_3}|^2 |\xi|^2 + \sigma^2 |\widetilde{B}|^2}{|r|^2} \right) |r|^{2l+1}\, \mathrm{d}\xi_0\, \mathrm{d}\xi \right)^{1/2} \\
&\leqslant c_{32} \left\{ \sum_{\alpha=1}^{2} |||b_\alpha|||_{l+1/2,\gamma,\mathbb{R}^2_\infty} + |b_3|_{l,\gamma,\mathbb{R}^2_\infty} + \sigma |||B|||_{l-1/2,\gamma,\mathbb{R}^2_\infty} \right\}.
\end{aligned} \tag{2.25}$$

Thus, we have proved the unique solvability of problem (2.1).

THEOREM 2.4 Assume that $l > 1/2,\ \gamma \geqslant \gamma_0 > 0,\ \sigma \geqslant 0$ and that inequalities (2.12) hold for the viscosities of the liquids. In addition, we suppose that $b_\alpha \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}^2_\infty)$, $\alpha = 1, 2$, $b_3 \in H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}^2_\infty)$, and $B \in H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}^2_\infty)$.

Then there exists a unique solution $(\boldsymbol{v}, p)$ of problem (2.1) such that $\boldsymbol{v} \in \bigcup_{i=-,+} \boldsymbol{H}_\gamma^{2+l,1+l/2}(D^i_\infty)$, $\nabla p \in \boldsymbol{H}_\gamma^{l,l/2}(D^+_\infty)$, and inequality (2.25) holds.

REMARK 2.5 It should be noted that the uniqueness of solution (2.6), (2.7) for $\gamma \geqslant \gamma_0 > 0$ is guaranteed by the first inequality in (2.13), i.e. by the fact that in this case P is separated from zero.

3. The non-homogeneous model problem

In this section we will prove the unique solvability of the non-homogeneous problem with plane interface between the fluids by reducing it to problem (2.1).

Let $T \leqslant \infty$ and consider the system

$$\begin{gathered}
\mathcal{D}_t \boldsymbol{w} - \nu^+ \nabla^2 \boldsymbol{w} + \frac{1}{\rho_0^+} \nabla q = \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{w} = g \quad \text{in} \quad D_T^+ = \mathbb{R}_+^3 \times (0, T), \\
\mathcal{D}_t \boldsymbol{w} - \nu^- \nabla^2 \boldsymbol{w} - (\nu^- + \kappa^-) \nabla(\nabla \cdot \boldsymbol{w}) = \boldsymbol{f} \quad \text{in} \quad D_T^- = \mathbb{R}_-^3 \times (0, T), \\
\boldsymbol{w}\Big|_{t=0} = 0, \quad \boldsymbol{w} \xrightarrow[|x| \to \infty]{} 0, \quad q \xrightarrow[|x| \to \infty]{} 0, \quad [\boldsymbol{w}]\Big|_{x_3=0} = 0, \qquad (3.1) \\
-\left[\mu^\pm \left(\frac{\partial w_\alpha}{\partial x_3} + \frac{\partial w_3}{\partial x_\alpha} \right)\right]\Big|_{x_3=0} = a_\alpha(x', t), \ x' = (x_1, x_2), \ t \in (0, T), \ \alpha = 1, 2; \\
-(q + \lambda^- \nabla \cdot \boldsymbol{w})\Big|_{x_3=0} + \left[2\mu^\pm \frac{\partial w_3}{\partial x_3}\right]\Big|_{x_3=0} + \sigma \Delta' \int_0^t w_3\Big|_{x_3=0} \mathrm{d}\tau \\
= a_3 + \sigma \int_0^t A \, \mathrm{d}\tau \qquad \text{on} \ \mathbb{R}_T^2 \equiv \mathbb{R}^2 \times (0, T).
\end{gathered}$$

THEOREM 3.1 Let $T > 0$, $\gamma \geqslant \gamma_0 > 0$, $\sigma \geqslant 0$, and let the viscosities of the liquids $\mu^\pm$, $\nu^\pm$ satisfy inequalities (2.12). In addition, we assume that, for some $l > 1/2$, $\boldsymbol{f} \in \bigcup_{i=-,+} \boldsymbol{H}_\gamma^{l,l/2}(D_T^i)$, $g \in H_\gamma^{1+l,1/2+l/2}(D_T^+)$, $g = \nabla \cdot \boldsymbol{R}$, $\boldsymbol{R} \in H_\gamma^{0,l+1/2}(D_T^+)$, $a_\alpha \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)$, $\alpha = 1, 2$, $a_3 \in H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}_T^2)$, and $A \in H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)$. Then problem (3.1) is uniquely solvable on the interval $(0, T]$ and for its solution $(\boldsymbol{w}, q)$, $\boldsymbol{w} \in \bigcup_{i=-,+} \boldsymbol{H}_\gamma^{2+l,1+l/2}(D_T^i)$, $\nabla q \in \boldsymbol{H}_\gamma^{l,l/2}(D_T^+)$, the inequality

$$\begin{aligned}
&\|\boldsymbol{w}\|_{\bigcup_i \boldsymbol{H}_\gamma^{2+l,1+l/2}(D_T^i)} + \|\nabla q\|_{\boldsymbol{H}_\gamma^{l,l/2}(D_T^+)} \leqslant c_1 \Bigg\{ \|\boldsymbol{f}\|_{\bigcup_i \boldsymbol{H}_\gamma^{l,l/2}(D_T^i)} \\
&+ \|g\|_{H_\gamma^{1+l,1/2+l/2}(D_T^+)} + \|\boldsymbol{R}\|_{\boldsymbol{H}_\gamma^{0,1+l/2}(D_T^+)} + \sum_{\alpha=1}^2 \|a_\alpha\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)} \\
&+ \|a_3\|_{H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}_T^2)} + \sigma \|A\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)} \Bigg\} \qquad (3.2)
\end{aligned}$$

holds.

Proof. We shall look for the velocity vector $\boldsymbol{w}$ in the form: $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{w}' + \boldsymbol{v}'$ where

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}_1^+, & x_3 > 0, \\ \boldsymbol{u}_1^- + \boldsymbol{u}_2^-, & x_3 < 0, \end{cases}$$

$$\boldsymbol{w}' = \begin{cases} \nabla \Phi, & x_3 > 0, \\ 0, & x_3 < 0, \end{cases}$$

$\Phi(x, t)$ being a solution of the Dirichlet problem for the Poisson equation

$$\begin{aligned} &\Delta\Phi(x,t) = g - \nabla\cdot \boldsymbol{u}_1^+ \equiv g'(x,t), \quad x\in\mathbb{R}^3_+, \quad t>0,\\ &\Phi|_{x_3=0} = 0. \end{aligned} \tag{3.3}$$

Next, let $(\boldsymbol{v}', p')$ be a solution of the homogeneous problem (2.1) with the boundary functions

$$\begin{aligned} b_\alpha &= a_\alpha + \left[\mu^\pm\left(\frac{\partial u_\alpha}{\partial x_3} + \frac{\partial u_3}{\partial x_\alpha}\right)\right]\Big|_{x_3=0} + \mu^+\left(\frac{\partial w'_\alpha}{\partial x_3} + \frac{\partial w'_3}{\partial x_\alpha}\right)\Big|_{x_3=0}, \quad \alpha = 1,2,\\ b_3 &= a_3 - \left[2\mu^\pm \frac{\partial(u_3 + w'_3)}{\partial x_3}\right]\Big| + \lambda^-\nabla\cdot(\boldsymbol{u}_1^- + \boldsymbol{u}_2^-)\Big|_{x_3=0} + \mu^+ g'\Big|_{x_3=0},\\ B &= A - \Delta'(u_3 + w'_3)\Big|_{x_3=0} \end{aligned} \tag{3.4}$$

then as a pressure function associated with $\boldsymbol{w}$ we can take $q = p' + \rho_0^+(\nu^+ g' - \partial\Phi/\partial t)$ ($\Phi = 0$ in $\mathbb{R}^3_-$).

We define the vector-function $\boldsymbol{u}$ as follows. $\boldsymbol{u}_1^+$ is a solution of the Cauchy problem for the heat equation

$$\begin{aligned} &\mathcal{D}_t\boldsymbol{u}_1^+ - \nu^+\Delta\boldsymbol{u}_1^+ = \boldsymbol{f}^+ \quad \text{in } \mathbb{R}^3_T \equiv \mathbb{R}^3\times(0,T),\\ &\boldsymbol{u}_1^+|_{t=0} = 0; \end{aligned} \tag{3.5}$$

$\boldsymbol{u}_1^-$ is a solution of the Cauchy problem too:

$$\begin{aligned} &\mathcal{D}_t\boldsymbol{u}_1^- - \nu^-\Delta\boldsymbol{u}_1^- - (\nu^- + \kappa^-)\nabla(\nabla\cdot\boldsymbol{u}_1^-) = \boldsymbol{f}^- \qquad \text{in } \mathbb{R}^3_T,\\ &\boldsymbol{u}_1^-|_{t=0} = 0. \end{aligned} \tag{3.6}$$

(Here $\boldsymbol{f}^+$ and $\boldsymbol{f}^-$ are continuous extensions of restrictions $\boldsymbol{f}|_{x_3>0}$ and $\boldsymbol{f}|_{x_3<0}$, respectively, on the whole $\mathbb{R}^3_\infty$.) We determine $\boldsymbol{u}_2^-$ as a solution of the initial-boundary value problem

$$\begin{aligned} &\mathcal{D}_t\boldsymbol{u}_2^- - \nu^-\Delta\boldsymbol{u}_2^- - (\nu^- + \kappa^-)\nabla(\nabla\cdot\boldsymbol{u}_2^-) = 0 \qquad \text{in } D_T^-,\\ &\boldsymbol{u}_2^-|_{t=0} = 0, \quad \boldsymbol{u}_2^-|_{x_3=0} = (\boldsymbol{u}_1^+ - \boldsymbol{u}_1^- - \boldsymbol{w}')|_{x_3=0} \equiv \boldsymbol{\varphi}. \end{aligned} \tag{3.7}$$

Now, we show that every function introduced above is well defined and that its norm is bounded by the norms of the data of problem (3.1).

We consider $T = \infty$ because every given function admits an extension to $t > T$ in the same class.

Let us take the Laplace transform on t and the Fourier transform on the all spatial variables in problems (3.5), (3.6). Then it is easy to see that for their solutions the estimate

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{u}_1^\pm}(\xi,s)|^2(|s| + \xi^2)^{l+2}\,\mathrm{d}\xi\,\mathrm{d}\xi_0 \leqslant c_2\int_{-\infty}^{\infty}\int_{\mathbb{R}^3} |\widetilde{\boldsymbol{f}^\pm}(\xi,s)|^2(|s|+\xi^2)^l\,\mathrm{d}\xi\,\mathrm{d}\xi_0 \tag{3.8}$$

holds. Here $\xi = (\xi_1, \xi_2, \xi_3)$, $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$, $s = \gamma + \mathrm{i}\xi_0$. Estimate (3.8) is equivalent to the inequalities

$$\|\boldsymbol{u}_1^\pm\|_{\boldsymbol{H}_\gamma^{2+l,1+l/2}(\mathbb{R}^3_\infty)} \leqslant c_3\|\boldsymbol{f}^\pm\|_{\boldsymbol{H}_\gamma^{l,l/2}(\mathbb{R}^3_\infty)} \leqslant c_4^\pm\|\boldsymbol{f}\|_{\bigcup_i \boldsymbol{H}_\gamma^{l,l/2}(D^i_\infty)}. \tag{3.9}$$

After the partial Fourier–Laplace transform (2.2), a solution of system (3.7) may be expressed in form (2.6):

$$\widetilde{\boldsymbol{u}_2^-} = \boldsymbol{W} e_0^- + \boldsymbol{V} e_1^-.$$

It is not difficult to show that

$$\boldsymbol{W} = \widetilde{\boldsymbol{\varphi}}, \quad \boldsymbol{V} = -C_3(r^- - r_1^-)\begin{pmatrix} \mathrm{i}\xi_1 \\ \mathrm{i}\xi_2 \\ r_1^- \end{pmatrix}$$

where $C_3 = \frac{\mathrm{i}\xi_1\widetilde{\varphi}_1 + \mathrm{i}\xi_2\widetilde{\varphi}_2 + r^-\widetilde{\varphi}_3}{r^- r_1^- - \xi^2}$. Taking inequalities (2.22)–(2.24) into consideration, we conclude that

$$\begin{aligned} |||\boldsymbol{u}_2^-|||_{2+l,\gamma,D_\infty^-} &\leqslant c_5 |||\boldsymbol{\varphi}|||_{3/2+l,\gamma,\mathbb{R}_\infty^2} \\ &\leqslant c_5(|||\boldsymbol{u}_1^+|||_{2+l,\gamma,D_\infty^+} + |||\boldsymbol{u}_1^-|||_{2+l,\gamma,D_\infty^-} + |||\boldsymbol{w}'|||_{2+l,\gamma,D_\infty^+}). \end{aligned} \tag{3.10}$$

Thus, it remains to estimate only the norm of $\boldsymbol{w}' = \nabla\Phi$. To this end, we turn our attention to problem (3.3). By means of the Green formula for the Poisson equation in the half-space

$$\Phi(x,t) = \int_{\mathbb{R}_+^3} G(x,y) g'(y,t)\,\mathrm{d}y = -\int_{\mathbb{R}_+^3} \nabla_y G(x,y)\cdot(\boldsymbol{R}(y,t) - \boldsymbol{u}_1^+(y,t))\,\mathrm{d}y,$$

where $G(x,y) = -\frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|x-y*|}\right)$, $y* = (y_1, y_2, -y_3)$, it was proved in [7] that

$$\begin{aligned} \|\boldsymbol{w}'\|_{\boldsymbol{H}_\gamma^{2+l,1+l/2}(D_T^+)} \leqslant c_6 \Big\{ &\|g\|_{H_\gamma^{1+l,0}(D_T^+)} + \|\boldsymbol{R}\|_{\boldsymbol{H}_\gamma^{0,1+l/2}(D_T^+)} \\ &+ \|\boldsymbol{u}_1^+\|_{\boldsymbol{H}_\gamma^{2+l,1+l/2}(D_T^+)} \Big\}. \end{aligned} \tag{3.11}$$

Finally, the evalution of the solution of the initial-boundary value problem (2.1), (3.4) is given by Theorem 2.4. According to inequality (2.25), we have:

$$\begin{aligned} |||\boldsymbol{v}'|||_{2+l,\gamma,D_\infty^3} + |||\nabla p'|||_{l,\gamma,D_\infty^+} &\leqslant c_7 \Big\{ \sum_{\alpha=1}^{2} |||a_\alpha|||_{l+1/2,\gamma,\mathbb{R}_\infty^2} + |a_3|_{l,\gamma,\mathbb{R}_\infty^2} \\ &\quad + \sigma |||A|||_{l-1/2,\gamma,\mathbb{R}_\infty^2} + |||\boldsymbol{u} + \boldsymbol{w}'|||_{l+3/2,\gamma,\mathbb{R}_\infty^2} + |g'|_{l,\gamma,\mathbb{R}_\infty^2} \Big\} \\ &\leqslant c_7 \Big\{ \sum_{\alpha=1}^{2} |||a_\alpha|||_{l+1/2,\gamma,\mathbb{R}_\infty^2} + |a_3|_{l,\gamma,\mathbb{R}_\infty^2} + \sigma |||A|||_{l-1/2,\gamma,\mathbb{R}_\infty^2} \\ &\quad + |||\boldsymbol{u}_1^+|||_{2+l,\gamma,D_\infty^+} + |||\boldsymbol{u}_1^-|||_{2+l,\gamma,D_\infty^-} + |||\boldsymbol{u}_2^-|||_{2+l,\gamma,D_\infty^-} \\ &\quad + |||\boldsymbol{w}'|||_{2+l,\gamma,D_\infty^3} + |||g|||_{1+l,\gamma,D_\infty^+} \Big\}. \end{aligned} \tag{3.12}$$

Taking the equivalence of norms $|||f|||_{m,\gamma}$ and $\|f\|_{H_\gamma^{m,m/2}}$ and also inequalities (3.9)–(3.12) into account, we arrive at the estimate (3.2). The existence of a solution of (3.1) is proved. Its uniqueness in the spaces $H_\gamma^{m,m/2}$, $\gamma \geqslant \gamma_0 > 0$, follows from the triviality of the solution of problem (2.1) with homogeneous boundary conditions. $\square$

REMARK 3.2 We note that the symmetry transformation with respect to the plane $\{x_3 = 0\}$ makes the liquids change places. It is clear that this change preserves Theorems 2.4 and 3.1. Thus, our results on the linear problem are independent of the location of the liquids.

4. Problems (1.8) and (1.10)

Now we suppose that $\Omega^+ \equiv \Omega_0^+$ is contained in the ball B_{d-2}, $d < \infty$. We introduce the notation: $B_d^\pm \equiv B_d \cap \Omega^\pm$, $B_{dT}^\pm = B_d^\pm \times (0,T)$; $\Omega^- \equiv \Omega_0^-$.

First, we consider system (1.10) with $\boldsymbol{u} = 0$. We rewrite this problem with respect to the functions $\boldsymbol{v}$ and p :

$$\begin{gathered}
\mathcal{D}_t \boldsymbol{v} - (\rho_0^+(x))^{-1}\nabla\mathbb{T}'(\boldsymbol{v}) = \boldsymbol{f} \quad \text{in} \quad Q_T^+,\\
\mathcal{D}_t \boldsymbol{v} - \nu^-\nabla^2\boldsymbol{v} + \frac{1}{\rho_0^-}\nabla p = \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = r \quad \text{in} \quad Q_T^-,\\
\boldsymbol{v}\Big|_{t=0} = \boldsymbol{w}_0 \quad \text{in} \quad \Omega^-\cup\Omega^+, \quad \boldsymbol{v}\xrightarrow[|x|\to\infty]{} 0, \quad p\xrightarrow[|x|\to\infty]{} 0,\\
[\boldsymbol{v}]\Big|_{G_T} = 0, \quad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{v})\boldsymbol{n}_0]\Big|_{G_T} = \Pi_0 a,\\
-(p+\lambda^+\nabla\cdot\boldsymbol{v})\Big|_\Gamma + [\mu^\pm\boldsymbol{n}_0\cdot\mathbb{S}(\boldsymbol{v})\boldsymbol{n}_0]\Big|_\Gamma - \sigma\boldsymbol{n}_0\cdot\Delta_\Gamma\int_0^t \boldsymbol{v}\Big|_\Gamma \,\mathrm{d}\tau\\
= b + \sigma\int_0^t B\,\mathrm{d}\tau, \quad t\in(0,T).
\end{gathered} \tag{4.1}$$

Here Δ_Γ is the Beltrami–Laplace operator on the given interface Γ and $\boldsymbol{n}_0$ is its outward normal with respect to Ω_0^+.

We formulate the existence theorem for (4.1) with homogeneous initial data.

THEOREM 4.1 Assume that $\boldsymbol{w}_0 = 0$ in (4.1). Moreover, let $\Gamma \in W_2^{3/2+l}$, $\rho_0^+ \in W_2^{1+l}(\Omega^+)$ for some $l > 1/2$, $0 < R_0 \leqslant \rho_0^+(x) \leqslant R_\infty < \infty$, $x \in \Omega^+$, and let inequalities (1.9) hold. Then for arbitrary $\boldsymbol{f} \in \bigcup_{i=-,+}\boldsymbol{H}_\gamma^{l,l/2}(Q_T^i)$, $r \in H_\gamma^{1+l,1/2+l/2}(Q_T^-)$, $r = \nabla\cdot\boldsymbol{R}$, $\boldsymbol{R} \in \boldsymbol{H}_\gamma^{0,l+1/2}(Q_T^-)$, $\boldsymbol{a} \in \boldsymbol{H}_\gamma^{l+1/2,l/2+1/4}(G_T)$, $b \in H_\gamma^{l+1/2,1/2,l/2}(G_T)$, and $A \in H_\gamma^{l-1/2,l/2-1/4}(G_T)$ with $T \leqslant \infty$, problem (4.1) is uniquely solvable on the interval $(0,T]$ provided that γ is large enough. Its solution $(\boldsymbol{v}, p)$ has the properties: $\boldsymbol{v} \in \bigcup_{i=-,+}\boldsymbol{H}_\gamma^{2+l,1+l/2}(Q_T^i)$, $p \in H_\gamma^{l,l/2}(B_T^-)$, $\nabla p \in \boldsymbol{H}_\gamma^{l,l/2}(Q_T^-)$, and satisfies the inequality

$$\begin{aligned}
&\|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}_\gamma^{2+l,1+l/2}(Q_T^i)} + \|\nabla p\|_{\boldsymbol{H}_\gamma^{l,l/2}(Q_T^-)} + \|p\|_{H_\gamma^{l,l/2}(B_T^-)} \leqslant c_1\Big\{\|\boldsymbol{f}\|_{\bigcup_i \boldsymbol{H}_\gamma^{l,l/2}(Q_T^i)}\\
&\quad + \|r\|_{H_\gamma^{1+l,1/2+l/2}(Q_T^-)} + \|\boldsymbol{R}\|_{\boldsymbol{H}_\gamma^{0,1+l/2}(Q_T^-)} + \|\boldsymbol{a}\|_{\boldsymbol{H}_\gamma^{l+1/2,l/2+1/4}(G_T)}\\
&\quad + \|b\|_{H_\gamma^{l+1/2,1/2,l/2}(G_T)} + \sigma\|B\|_{H_\gamma^{l-1/2,l/2-1/4}(G_T)}\Big\} \equiv c_1 M
\end{aligned} \tag{4.2}$$

where the constant c_1 is independent of T.

Proof. We prove this theorem in two steps. At first, we obtain the estimate (4.2) as an *a priori* one. Next, we show that system (4.1) with $\boldsymbol{w}_0 = 0$ has a unique solution belonging to the spaces $H_\gamma^{l,l/2}$.

Step 1. We use the Schauder method to prove the validity of (4.2).

We first estimate the solution outside some ball containing Ω^+. We multiply equations (4.1) by a smooth function $\eta : \eta = 0$ in the ball B_{d-1} and $\eta = 1$ outside the ball B_d. Let us introduce the new unknown functions $\boldsymbol{u} = \boldsymbol{v}\eta$ and $q = p\eta$. It is necessary to evaluate a solution of the following Cauchy problem:

$$\begin{aligned}
&\mathcal{D}_t\boldsymbol{u} - \nu^-\nabla^2\boldsymbol{u} + \frac{1}{\rho_0^-}\nabla q = \boldsymbol{f}\eta - \nu^-(\boldsymbol{v}\nabla^2\eta + 2(\nabla\eta\cdot\nabla)\boldsymbol{v}) + \frac{1}{\rho_0^-}p\nabla\eta,\\
&\nabla\cdot\boldsymbol{v} = r\eta + \boldsymbol{v}\cdot\nabla\eta,\\
&\boldsymbol{u}\Big|_{t=0} = 0, \quad \boldsymbol{u}\xrightarrow[|x|\to\infty]{} 0, \quad q\xrightarrow[|x|\to\infty]{} 0.
\end{aligned}$$

This problem was analysed in [2] where was obtained the estimate

$$\begin{aligned}
\|\boldsymbol{u}\|_{\boldsymbol{H}_\gamma^{2+l,1+l/2}(\mathbb{R}_T^3)} + \|\nabla q\|_{\boldsymbol{H}_\gamma^{l,l/2}(\mathbb{R}_T^3)} \leqslant c_2\Big\{&\|\boldsymbol{f}\|_{\boldsymbol{H}_\gamma^{l,l/2}(Q_T^-)} + \|r\|_{H_\gamma^{1+l,\frac{1+l}{2}}(Q_T^-)} + \|\boldsymbol{R}\|_{\boldsymbol{H}_\gamma^{0,1+l/2}(Q_T^-)}\\
&+ \|\boldsymbol{v}\|_{\boldsymbol{H}_\gamma^{2+l,1+l/2}(K_{dT})} + \|p\|_{H_\gamma^{l,l/2}(K_{dT})}\Big\}. \qquad (4.3)
\end{aligned}$$

Here $K_{dT} = (B_d \setminus B_{d-1}) \times (0, T)$.

Next, by a partition of unity, on the base of local estimates for compressible [9] and incompressible [7] fluids, as well as of the boundary estimate (3.2) and inequality (4.3), we deduce that

$$\begin{aligned}
&\|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}_\gamma^{2+l,1+l/2}(Q_T^i)} + \|\nabla p\|_{\boldsymbol{H}_\gamma^{l,l/2}(Q_T^-)} + \|p\|_{H_\gamma^{l,l/2}(B_{dT}^-)}\\
&\leqslant c_3\Big\{M + \|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}_\gamma^{1+l,1/2+l/2}(B_{dT}^i)} + \|p\|_{H_\gamma^{l,l/2}(B_{dT}^-)} + \|p\|_{H_\gamma^{0,l/2}(G_T)}\Big\}\\
&\leqslant c_4\Big\{M + \|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}_\gamma^{1+l,1/2+l/2}(B_{dT}^i)} + \varepsilon\|\nabla p\|_{\boldsymbol{H}_\gamma^{l,l/2}(B_{dT}^-)}\\
&\qquad\qquad + c_5(\varepsilon)\|p\|_{H_\gamma^{0,l/2}(B_{dT}^-)}\Big\}. \qquad (4.4)
\end{aligned}$$

In the last relation, we have used interpolation inequalities from [2: p. 21].

Thus, it remains to estimate only $\|p\|_{H_\gamma^{0,l/2}(B_{dT}^-)}$. To this end, we consider the Dirichlet problem

$$\begin{aligned}
\frac{\nabla^2 p}{\rho_0^-} &= \nabla\cdot(\boldsymbol{f} - \mathcal{D}_t\boldsymbol{v} + \nu^-\nabla^2\boldsymbol{v}) = \nabla\cdot(\boldsymbol{f} - \mathcal{D}_t\boldsymbol{R} + \nu^-\nabla r)\\
&\equiv \nabla\cdot\boldsymbol{F} \quad \text{in} \quad \Omega^-,\\
p\Big|_\Gamma &= \left[2\mu^\pm\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}_0}\right)_{\boldsymbol{n}_0}\right]\Big|_\Gamma - \lambda^+\nabla\cdot\boldsymbol{v}\Big|_\Gamma - \sigma\boldsymbol{n}_0\cdot\Delta_\Gamma\int_0^t\boldsymbol{v}\Big|_\Gamma\,\mathrm{d}\tau \qquad (4.5)\\
&\quad - b - \sigma\int_0^t B\,\mathrm{d}\tau \equiv p_0, \quad p\xrightarrow[|x|\to\infty]{} 0
\end{aligned}$$

where $\boldsymbol{b}_{\boldsymbol{n}_0} = \boldsymbol{b} \cdot \boldsymbol{n}_0$.

We extend the functions $\boldsymbol{f}, r, \boldsymbol{R}, \boldsymbol{v}, b, B$ with preservation of class into the region $t > T$ in the case when $T < \infty$. After the Laplace transform $\widehat{f}(x,s) = \int_0^\infty \mathrm{e}^{-st} f(x,t)\,\mathrm{d}t$ in (4.5)we arrive at the problem

$$\begin{aligned} &\frac{\nabla^2 \widehat{p}}{\rho_0^-} = \nabla \cdot (\widehat{\boldsymbol{f}} - s\widehat{\boldsymbol{R}} + \nu^- \nabla \widehat{r}) \equiv \nabla \cdot \widehat{\boldsymbol{F}} \quad \text{in} \quad \Omega^-, \\ &\widehat{p}\Big|_\Gamma = \left[2\mu^\pm \left(\frac{\partial \widehat{\boldsymbol{v}}}{\partial \boldsymbol{n}_0} \right)_{\boldsymbol{n}_0} \right]\Big|_\Gamma - \lambda^+ \nabla \cdot \widehat{\boldsymbol{v}}\Big|_\Gamma - \frac{\sigma}{s} (\Delta_\Gamma \widehat{\boldsymbol{v}})_{\boldsymbol{n}_0}\Big|_\Gamma \\ &\qquad - \widehat{b} - \frac{\sigma}{s} \widehat{B} \equiv \widehat{p}_0, \qquad \widehat{p} \xrightarrow[|x|\to\infty]{} 0. \end{aligned} \tag{4.6}$$

We shall need also the solution Φ of the Dirichlet problem with homogeneous boundary condition

$$\frac{\nabla^2 \Phi}{\rho_0^-} = \overline{\widehat{p}} \zeta^2 \quad \text{in} \quad \Omega^-, \qquad \Phi\Big|_\Gamma = 0, \qquad \Phi \xrightarrow[|x|\to\infty]{} 0. \tag{4.7}$$

Here ζ is a smooth function with compact support equal to 1 in B_d and to 0 outside B_{d+1}. The bar means complex conjugation.

We multiply the equation in (4.7) by Φ and integrate by parts twice. Then we have

$$\int_{\Omega^-} |\widehat{p}|^2 \zeta^2 \,\mathrm{d}x = -\int_{\Omega^-} \widehat{\boldsymbol{F}} \cdot \nabla \Phi \,\mathrm{d}x + \int_\Gamma \widehat{p}_0 \frac{\partial \Phi}{\partial \boldsymbol{n}_0} \,\mathrm{d}\Gamma$$

whence it follows that

$$\|\widehat{p}\zeta\|^2_{\Omega^-} \leqslant c_6 \big(\|\widehat{\boldsymbol{F}}\|^2_{\Omega^-} + \varepsilon \|\nabla \Phi\|^2_{\Omega^-} + \|\widehat{p}_0\|^2_\Gamma + \varepsilon \|\frac{\partial \Phi}{\partial \boldsymbol{n}_0}\|^2_\Gamma \big). \tag{4.8}$$

Applying the known estimates for $\nabla \Phi\Big|_\Gamma$ and $\nabla\nabla\Phi$

$$\|\nabla \Phi\|^2_\Gamma \leqslant c_7 \big(\|\nabla\nabla\Phi\|^2_{B_d^-} + \|\nabla\Phi\|^2_{\Omega^-} \big), \quad \|\nabla\nabla\Phi\|^2_{B_d^-} \leqslant c_8 \big(\|\widehat{p}\zeta\|^2_{\Omega^-} + \|\nabla\Phi\|^2_{\Omega^-} \big),$$

we see that it is only necessary to estimate $\|\nabla\Phi\|^2_{\Omega^-}$. To this end, we multiply the equation of (4.7) by $\overline{\Phi}$ and integrate by parts over Ω^-. We then obtain the equality $\int_{\Omega^-} \frac{|\nabla\Phi|^2}{\rho_0^-}\,\mathrm{d}x = -\int_{\Omega^-} \overline{\widehat{p}}\zeta^2 \overline{\Phi}\,\mathrm{d}x$ from which it follows that

$$\|\nabla\Phi\|^2_{\Omega^-} \leqslant \varepsilon_1 \|\Phi\zeta\|^2_{\Omega^-} + c_{\varepsilon_1} \|\widehat{p}\zeta\|^2_{\Omega^-}. \tag{4.9}$$

Next, by the embedding theorems we obtain: $\|\Phi\zeta\|^2_{\Omega^-} \leqslant c_9 \|\Phi\zeta\|^2_{L^6(B^-_{d+1})} \leqslant c_9 \|\nabla\Phi\|^2_{\Omega^-}$. Finally, choosing sufficiently small ε_1 and ε, we deduce from (4.9)

$$\|\nabla\Phi\|^2_{\Omega^-} \leqslant c_{10} \|\widehat{p}\zeta\|^2_{\Omega^-},$$

and then from (4.8)

$$\|\widehat{p}\zeta\|^2_{\Omega^-} \leqslant c_{11}\big(\|\widehat{\boldsymbol{F}}\|^2_{\Omega^-} + \|\widehat{p}_0\|^2_{\Gamma}\big) \leqslant c_{12}\bigg\{\|\widehat{\boldsymbol{f}}\|^2_{\Omega^-} + \|\widehat{r}\|^2_{\Omega^-} + |s|^2\|\widehat{\boldsymbol{R}}\|^2_{\Omega^-} + \|\nabla\widehat{\boldsymbol{v}}\|^2_{\Gamma} + \frac{\sigma^2}{|s|^2}\|(\Delta_\Gamma\widehat{\boldsymbol{v}})_{\boldsymbol{n}_0} + \widehat{B}\|^2_{\Gamma} + \|\widehat{b}\|^2_{\Gamma}\bigg\}. \tag{4.10}$$

Let us multiply inequality (4.10) by $|s|^l$ and integrate over the line $\operatorname{Im} s = \xi_0$. By the Parseval equality, we obtain the estimate

$$\|p\|^2_{H^{0,l/2}_\gamma(B^-_{d\infty})} \leqslant c_{13}\bigg\{\|\boldsymbol{f}\|^2_{\boldsymbol{H}^{0,l/2}_\gamma(Q^-_\infty)} + \|r\|^2_{H^{l/2}_\gamma(Q^-_\infty)} + \|\boldsymbol{R}\|^2_{\boldsymbol{H}^{0,1+l/2}_\gamma(Q^-_\infty)} + \|b\|_{H^{0,l/2}_\gamma(G_\infty)} + \sigma^2\|(\Delta_\Gamma\boldsymbol{v})_{\boldsymbol{n}_0} + B\|_{H^{0,l/2-1}_\gamma(G_\infty)} + \|\nabla\boldsymbol{v}\|_{H^{0,l/2}_\gamma(G_\infty)}\bigg\}. \tag{4.11}$$

For $l < 2$ the term with σ^2 must be replaced by $\sigma^2\gamma^{l-2}\|e^{-\gamma t}\big[(\Delta_\Gamma\boldsymbol{v})_{\boldsymbol{n}_0} + B\big]\|_{G_\infty}$.

By means of the embedding theorems for $H^{m,m/2}_\gamma$ [7], in the same way as in [2], we obtain from (4.4), (4.11) that

$$\|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}^{2+l,1+l/2}_\gamma(Q^i_T)} + \|\nabla p\|_{\boldsymbol{H}^{l,l/2}_\gamma(Q^-_T)} + \|p\|^2_{H^{0,l/2}_\gamma(B^-_{dT})} \leqslant c_{14}\bigg\{M + \|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}^{3/2+j,3/4+j/2}_\gamma(B^i_{dT})}\bigg\}.$$

Here $j = \max(1, l)$ which is always less than $l + 1/2$, since $l > 1/2$, and hence we have the weak norm of $\boldsymbol{v}$ on the right. It can be estimated by $\theta\|\boldsymbol{v}\|_{\bigcup_i \boldsymbol{H}^{2+l,1+l/2}_\gamma(Q^i_T)}$ with a small coefficient θ for a sufficiently large value of the parameter γ (see [7: Section 4]). Thus, we arrive at estimate (4.2).

Step 2. The proof of existence of a solution with the mentioned properties is proved by constructing a regularizer in the same manner as in the case of single liquid [7, 9]. The uniqueness of the constructed solution follows from the triviality of the homogeneous problem solution which is due to inequality (4.2). □

The passage from the weighted spaces $H^{m,m/2}_\gamma(Q^\pm_T)$, $\gamma \geqslant 0$, to the ordinary Sobolev spaces $W^{m,m/2}_2(Q^\pm_T)$ is easily made because they are equivalente for $\forall T < \infty$ [7: Section 6].

Problem (4.1) with $\boldsymbol{w}_0 \neq 0$ can be reduced to the problem with homogeneous initial conditions considered above by constructing a vector $\boldsymbol{V} \in \bigcup_i \boldsymbol{W}^{2+l,1+l/2}_2(Q^i_T)$ such that $\boldsymbol{V}(x, 0) = \boldsymbol{w}_0$, $[\boldsymbol{V}]\big|_{G_T} = 0$ and

$$\|\boldsymbol{V}\|_{\bigcup_i \boldsymbol{W}^{2+l,1+l/2}_2(Q^i_T)} \leqslant c_{15}\|\boldsymbol{w}_0\|_{\bigcup_i \boldsymbol{W}^{1+l}_2(\Omega^i)}.$$

In this way, we arrive at Theorem 1.2 with $\boldsymbol{u} = 0$.

THEOREM 4.2 Suppose that $\Gamma \in W^{3/2+l}_2$, $\rho^+_0 \in W^{1+l}_2(\Omega^+_0)$ for some $l \in (1/2, 1)$, $0 < R_0 \leqslant \rho^+_0(x) \leqslant R_\infty < \infty$, $x \in \Omega^+_0$, and that inequalities (1.9) hold. In addition, assume that $\boldsymbol{f} \in \bigcup_{i=-,+}\boldsymbol{W}^{l,l/2}_2(Q^i_T)$, $r \in W^{1+l,1/2+l/2}_2(Q^-_T)$, $r = \nabla\cdot\boldsymbol{R}$, $\boldsymbol{R} \in \boldsymbol{W}^{0,1+l/2}_2(Q^-_T)$,

$\boldsymbol{w}_0 \in \bigcup_{i=-,+} \boldsymbol{W}_2^{1+l}(\Omega_0^i)$, $\boldsymbol{a} \in \boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)$, $b \in W_2^{l+1/2,l/2+1/4}(G_T)$, and $B \in W_2^{l-1/2,l/2-1/4}(G_T)$ with $T < \infty$ and that the compatibility conditions

$$[\boldsymbol{w}_0]\Big|_\Gamma = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\boldsymbol{w}_0)\boldsymbol{n}_0]\Big|_\Gamma = \Pi_0 \boldsymbol{a}\Big|_{t=0},$$
$$\nabla \cdot \boldsymbol{w}_0 = r\Big|_{t=0} \quad \text{on} \quad \Omega_0^-$$

are satisfied.

Then problem (4.1) is uniquely solvable and its solution $(\boldsymbol{v}, p)$ has the properties: $\boldsymbol{v} \in \bigcup_{i=-,+} \boldsymbol{W}_2^{2+l,1+l/2}(Q_T^i)$, $p \in W_{2,loc}^{l,l/2}(Q_T^-)$, $\nabla p \in \boldsymbol{W}_2^{l,l/2}(Q_T^-)$, $p\big|_{G_T} \in W_2^{l+1/2,l/2+1/4}(G_T)$, and

$$\begin{aligned}
N[v, p] &\equiv \|\boldsymbol{v}\|^{(2+l,1+l/2)}_{Q_T^-\cup Q_T^+} + \|\nabla p\|^{(l,l/2)}_{Q_T^-} + \|p\|^{(l,l/2)}_{B_{dT}^-} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\
&\leqslant c_{16}(T)\Big\{ \|\boldsymbol{f}\|^{(l,l/2)}_{Q_T^-\cup Q_T^+} + \|\boldsymbol{w}_0\|_{\bigcup_i \boldsymbol{W}_2^{1+l}(\Omega_0^i)} + \|r\|_{W_2^{1+l,0}(Q_T^-)} \\
&\quad + \|\boldsymbol{R}\|_{\boldsymbol{W}_2^{0,1+l/2}(Q_T^-)} + T^{-l/2}\|\mathcal{D}_t \boldsymbol{R}\|_{Q_T^-} + \|\boldsymbol{a}\|_{\boldsymbol{W}_2^{l+1/2,l/2+1/4}(G_T)} \\
&\quad + \|b\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + T^{-l/2}\|b\|_{W_2^{1/2,0}(G_T)} \\
&\quad + \sigma\|B\|_{W_2^{l-1/2,l/2-1/4}(G_T)}\Big\} \equiv c_{16}(T)F,
\end{aligned} \tag{4.12}$$

$c_{16}(T)$ being a non-decreasing function of T.

We will solve problem (1.10) by successive approximations taking $\boldsymbol{w}^{(0)} = 0$, $s^{(0)} = 0$ and determining $(\boldsymbol{w}^{(m+1)}, s^{(m+1)})$, $m \geqslant 0$, as solutions of the problems

$$\mathcal{D}_t \boldsymbol{w}^{(m+1)} - (\rho_0^+(x))^{-1}\nabla \mathbb{T}'(\boldsymbol{w}^{(m+1)}) = \boldsymbol{f} + \boldsymbol{l}_1(\boldsymbol{w}^{(m)}) \quad \text{in} \quad Q_T^+,$$

$$\begin{aligned}
&\mathcal{D}_t \boldsymbol{v} - \nu^- \nabla^2 \boldsymbol{w}^{(m+1)} + \frac{1}{\rho_0^-}\nabla s^{(m+1)} = \boldsymbol{f} + \boldsymbol{l}_1(\boldsymbol{w}^{(m)}, s^{(m)}), \\
&\nabla \cdot \boldsymbol{w}^{(m+1)} = r + l_2(\boldsymbol{w}^{(m)}) = \nabla \cdot (\boldsymbol{\mathcal{L}}(\boldsymbol{w}^{(m)}) + \boldsymbol{R}) \quad \text{in} \quad Q_T^-,
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{w}^{(m+1)}\Big|_{t=0} = \boldsymbol{w}_0, \qquad \boldsymbol{w}^{(m+1)} \xrightarrow[|\xi|\to\infty]{} 0, \quad s^{(m+1)} \xrightarrow[|\xi|\to\infty]{} 0, \\
&[\boldsymbol{w}^{(m+1)}]\Big|_{G_T} = 0, \quad [\mu^\pm \Pi_0 \mathbb{S}(\boldsymbol{w}^{(m+1)})\boldsymbol{n}_0]\Big|_{G_T} = \boldsymbol{l}_3(\boldsymbol{w}^{(m)}) + \Pi_0 a,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&[\boldsymbol{n}_0 \cdot \mathbb{T}'(\boldsymbol{w}^{(m+1)}, s^{(m+1)})\boldsymbol{n}_0]\Big|_\Gamma - \sigma \boldsymbol{n}_0 \cdot \Delta_\Gamma \int_0^t \boldsymbol{w}^{(m+1)}\Big|_\Gamma \,\mathrm{d}\tau \\
&= l_4(\boldsymbol{w}^{(m)}, s^{(m)}) + b + \sigma \int_0^t (l_5(\boldsymbol{w}^{(m)}) + B)\,\mathrm{d}\tau, \quad t \in (0, T).
\end{aligned}$$

Here we use the notation

$$
\begin{aligned}
\boldsymbol{l}_1(\boldsymbol{w}, s) &= \begin{cases} (\rho_0^+(\xi))^{-1}[\mathbb{A}\nabla\mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{w}) - \nabla\mathbb{T}'(\boldsymbol{w})] & \text{in } \Omega_t^+, \\ \nu^-(\nabla_{\boldsymbol{u}}^2 - \nabla^2)\boldsymbol{w} + (\nabla - \nabla_{\boldsymbol{u}})s & \text{in } \Omega_t^-, \end{cases} \\
l_2(\boldsymbol{w}) &= (\nabla - \nabla_{\boldsymbol{u}})\boldsymbol{w} = \nabla \cdot \boldsymbol{\mathcal{L}}(\boldsymbol{w}), \qquad \boldsymbol{\mathcal{L}}(\boldsymbol{w}) = (\mathbb{I} - \mathbb{A}^T)\boldsymbol{w}, \\
\boldsymbol{l}_3(\boldsymbol{w}) &= [\mu^{\pm}\Pi_0(\mathbb{S}(\boldsymbol{w})\boldsymbol{n}_0 - \Pi_{\boldsymbol{u}}\mathbb{S}_{\boldsymbol{u}}(\boldsymbol{w})\boldsymbol{n})]\Big|_{G_T}, \\
l_4(\boldsymbol{w}, s) &= [\boldsymbol{n}_0 \cdot (\mathbb{T}'(\boldsymbol{w}, s)\boldsymbol{n}_0 - \mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{w}, s)\boldsymbol{n})]\Big|_{G_T}, \\
l_5(\boldsymbol{w}) &= \boldsymbol{n}_0 \cdot \mathcal{D}_t(\Delta(t) - \Delta_\Gamma)\int_0^t \boldsymbol{w}\Big|_\Gamma \mathrm{d}\tau \\
&= \boldsymbol{n}_0 \cdot \left\{(\Delta(t) - \Delta_\Gamma)\boldsymbol{w} + \dot{\Delta}(t)\int_0^t \boldsymbol{w}\Big|_\Gamma \mathrm{d}\tau\right\},
\end{aligned} \tag{4.14}
$$

where $\dot{\Delta}(t)$ is the derivative of $\Delta(t)$ with respect to time.

Operators $\boldsymbol{l}_1, \ldots, l_5$ and $\boldsymbol{\mathcal{L}}$ were considered in [8] and [9] where the following estimates were obtained.

LEMMA 4.3 If $\boldsymbol{u}$ and $\boldsymbol{u}'$ satisfy inequality (1.11) then

$$
\begin{aligned}
&\|\boldsymbol{l}_1(\boldsymbol{w}, s) - \boldsymbol{l}'_1(\boldsymbol{w}, s)\|^{(l,l/2)}_{Q_T^- \cup Q_T^+} + \|l_2(\boldsymbol{w}) - l'_2(\boldsymbol{w})\|^{(l,l/2)}_{W_2^{1+l,\frac{1+l}{2}}(Q_T^-)} \\
&+ \|\boldsymbol{l}_3(\boldsymbol{w}) - \boldsymbol{l}'_3(\boldsymbol{w})\|_{W_2^{1/2+l,1/4+l/2}(G_T)} + \|l_5(\boldsymbol{w}) - l'_5(\boldsymbol{w})\|^{(l-1/2,l/2-1/4)}_{G_T} \\
&\qquad \leqslant c_{17}\sqrt{T}\|\boldsymbol{u} - \boldsymbol{u}'\|^{(2+l,1+l/2)}_{Q_T^- \cup Q_T^+}\left\{\|\boldsymbol{w}\|^{(2+l,1+l/2)}_{Q_T^- \cup Q_T^+} + \|\nabla s\|^{(l,l/2)}_{Q_T^-}\right\}, \\
&\|\mathcal{D}_t(\boldsymbol{\mathcal{L}}(\boldsymbol{w}) - \boldsymbol{\mathcal{L}}'(\boldsymbol{w}))\|^{(0,l/2)}_{Q_T^-} \leqslant c_{18}\left\{\sqrt{T}\|\boldsymbol{u} - \boldsymbol{u}'\|^{(2+l,1+l/2)}_{Q_T^-}\right. \\
&\qquad\qquad \left. + T^{\frac{1-l}{2}}\|\boldsymbol{u}(\cdot, 0) - \boldsymbol{u}'(\cdot, 0)\|_{W_2^1(\Omega^-)}\right\}\|\boldsymbol{w}\|^{(2+l,1+l/2)}_{Q_T^-},
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
&\|l_4(\boldsymbol{w}, s) - l'_4(\boldsymbol{w}, s)\|_{W^{1/2+l,1/4+l/2}(G_T)} \leqslant c_{19}\sqrt{T}\|\boldsymbol{u} - \boldsymbol{u}'\|^{(2+l,1+l/2)}_{Q_T^- \cup Q_T^+} \\
&\times\left\{\|\boldsymbol{w}\|^{(2+l,1+l/2)}_{Q_T^- \cup Q_T^+} + \|\nabla s\|^{(l,l/2)}_{Q_T^-} + \|s\|^{(l,l/2)}_{B_T^-} + \|s\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\right\}.
\end{aligned} \tag{4.16}
$$

Here operators $\boldsymbol{l}'_1, \ldots, l'_5$ and $\boldsymbol{\mathcal{L}}'$ are calculated according to formulas (4.14) where vector $\boldsymbol{u}$ is replaced by $\boldsymbol{u}'$. If $\boldsymbol{w}\Big|_{t=0} = 0$, then inequality (4.15) is valid without $T^{\frac{1-l}{2}}\|\boldsymbol{u}(\cdot, 0) - \boldsymbol{u}'(\cdot, 0)\|_{W_2^1(\Omega^-)}$ in the right-hand side.

REMARK 4.4 We note also that in (4.16) it is sufficient to take the norm $\|s\|^{(l,l/2)}$ over the bounded domain B_T^-.

Lemma 4.5 follows on from Lemma 4.3.

LEMMA 4.5 If $\boldsymbol{u}$ satisfies inequality (1.11), then

$$\|\boldsymbol{l}_1(\boldsymbol{w}, s)\|^{(l,l/2)}_{Q_T^-\cup Q_T^+} + \|l_2(\boldsymbol{w})\|^{(l,l/2)}_{W_2^{1+l,\frac{1+l}{2}}(Q_T^-)} + \|\boldsymbol{l}_3(\boldsymbol{w})\|_{W_2^{1/2+l,1/4+l/2}(G_T)}$$
$$+ \|l_4(\boldsymbol{w}, s)\|_{W^{1/2+l,1/4+l/2}(G_T)} + \|l_5(\boldsymbol{w})\|^{(l-1/2,l/2-1/4)}_{G_T}$$
$$\leqslant c_{20}\delta\left\{\|\boldsymbol{w}\|^{(2+l,1+l/2)}_{Q_T^-\cup Q_T^+} + \|\nabla s\|^{(l,l/2)}_{Q_T^-} + \|s\|^{(l,l/2)}_{B_T^-} + \|s\|_{W_2^{l+1/2,l/2+1/4}(G_T)}\right\},$$
$$\|\mathcal{D}_t\boldsymbol{\mathcal{L}}(\boldsymbol{w})\|^{(0,l/2)}_{Q_T^-} \leqslant c_{21}\left\{\delta + T^{\frac{1-l}{2}}\|\boldsymbol{u}(\cdot,0)\|_{W_2^1(\Omega^-)}\right\}\|\boldsymbol{w}\|^{(2+l,1+l/2)}_{Q_T^-}. \tag{4.17}$$

If $\boldsymbol{w}(\cdot,0) = 0$ in Ω^-, then the term with $\|\boldsymbol{u}(\cdot,0)\|_{W_2^1(\Omega^-)}$ may be dropped in the last inequality.

Proof of Theorem 1.2. We return to problem (4.13). It follows from Theorem 4.2 and Lemma 4.5 that $(\boldsymbol{w}^{(m+1)}, s^{(m+1)})$, $m > 0$, are uniquely determined, $(\boldsymbol{w}^{(1)}, s^{(1)})$ being a solution of (1.10) with $\boldsymbol{u} = 0$ and satisfying inequality (4.12).

Let us consider the differences $\boldsymbol{z}^{(m+1)} = \boldsymbol{w}^{(m+1)} - \boldsymbol{w}^{(m)}$, $g^{(m+1)} = s^{(m+1)} - s^{(m)}$, $m \geqslant 1$. We have for them the problem

$$\mathcal{D}_t\boldsymbol{z}^{(m+1)} - (\rho_0^+(x))^{-1}\nabla\mathbb{T}'(\boldsymbol{z}^{(m+1)}) = \boldsymbol{l}_1(\boldsymbol{z}^{(m)}) \quad \text{in} \quad Q_T^+,$$

$$\mathcal{D}_t\boldsymbol{v} - \nu^-\nabla^2\boldsymbol{z}^{(m+1)} + \frac{1}{\rho_0^-}\nabla g^{(m+1)} = \boldsymbol{l}_1(\boldsymbol{z}^{(m)}, g^{(m)}),$$
$$\nabla\cdot\boldsymbol{z}^{(m+1)} = l_2(\boldsymbol{z}^{(m)}) = \nabla\cdot\boldsymbol{\mathcal{L}}(\boldsymbol{z}^{(m)}) \quad \text{in} \quad Q_T^-,$$

$$\boldsymbol{z}^{(m+1)}\Big|_{t=0} = 0, \qquad \boldsymbol{z}^{(m+1)} \xrightarrow[|\xi|\to\infty]{} 0, \quad g^{(m+1)} \xrightarrow[|\xi|\to\infty]{} 0, \tag{4.18}$$
$$[\boldsymbol{z}^{(m+1)}]\Big|_{G_T} = 0, \quad [\mu^\pm\Pi_0\mathbb{S}(\boldsymbol{z}^{(m+1)})\boldsymbol{n}_0]\Big|_{G_T} = \boldsymbol{l}_3(\boldsymbol{z}^{(m)}),$$

$$[\boldsymbol{n}_0\cdot\mathbb{T}'(\boldsymbol{z}^{(m+1)}, g^{(m+1)})\boldsymbol{n}_0]\Big|_\Gamma - \sigma\boldsymbol{n}_0\cdot\Delta_\Gamma\int_0^t\boldsymbol{z}^{(m+1)}\Big|_\Gamma\,\mathrm{d}\tau$$
$$= l_4(\boldsymbol{z}^{(m)}, g^{(m)}) + \sigma\int_0^t l_5(\boldsymbol{z}^{(m)})\,\mathrm{d}\tau, \quad t\in(0,T).$$

If $m \geqslant 1$, then $\boldsymbol{z}^{(m+1)}\big|_{t=0} = 0$, and we deduce from (4.12) and Lemma 4.5 that

$$N[\boldsymbol{z}^{(m+1)}, g^{(m+1)}] \leqslant c_{22}\delta N[\boldsymbol{z}^{(m)}, g^{(m)}].$$

If $m = 1$ then in virtue of (4.17) we obtain

$$N[\boldsymbol{z}^{(2)}, g^{(2)}] \leqslant (c_{22}\delta + c_{21}\delta_1)N[\boldsymbol{w}^{(1)}, s^{(1)}]$$

with $\delta_1 = T^{\frac{1-l}{2}}\|\boldsymbol{u}(\cdot,0)\|_{W_2^1(\Omega^-)}$. Next, for $\Sigma_m = \sum_{j=2}^m N[\boldsymbol{z}^{(j)}, g^{(j)}]$ the following inequality

$$\Sigma_{m+1} \leqslant c_{22}\delta\Sigma_m + N[\boldsymbol{z}^{(2)}, g^{(2)}]$$

holds. Let us take $c_{22}\delta < 1$. It is obvious that

$$\Sigma_{m+1} \leqslant (1 - c_{22}\delta)^{-1} N[\boldsymbol{z}^{(2)}, g^{(2)}].$$

Because of Theorem 4.2 we have:

$$\begin{aligned} N[\boldsymbol{w}^{(m+1)}, s^{(m+1)}] &\leqslant \Sigma_{m+1} + N[\boldsymbol{w}^{(1)}, s^{(1)}] \\ &\leqslant \left(\frac{1}{1 - c_{22}\delta} + \frac{c_{21}}{1 - c_{22}\delta} T^{\frac{1-l}{2}} \|\boldsymbol{u}(\cdot, 0)\|_{W_2^1(\Omega^-)} \right) c_{16} F \end{aligned}$$

where F is the sum of the right-hand-side norms in (4.12). Hence, the sequence $\{\boldsymbol{w}^{(m+1)}, s^{(m+1)}\}$ is convergent and its limit $(\boldsymbol{w}, s)$ is a solution of (1.10) satisfying inequality (1.12) with

$$c_1(T) = \frac{c_{16}F}{1 - c_{22}\delta} \left(1 + c_{21} T^{\frac{1-l}{2}} \|\boldsymbol{u}(\cdot, 0)\|_{W_2^1(\Omega^-)} \right).$$

In a similar way, we can conclude that the difference $(\boldsymbol{z} = \boldsymbol{w} - \boldsymbol{w}', g = s - s')$ of two solutions of (1.10) satisfies the estimate

$$N[\boldsymbol{z}, g] \leqslant c_{22}\delta N[\boldsymbol{z}, g]$$

whence it follows $\boldsymbol{z} = 0$, $g = 0$. Thus, the uniqueness of the solution is also proved. □

Now, we give the main ideas of the Proof of Theorem 1.1.

Using the formula for twice the mean curvature,

$$\Delta(t)\boldsymbol{X_u} = \Delta(0)\boldsymbol{\xi} + \int_0^t \dot{\Delta}(\tau)\boldsymbol{\xi}\, \mathrm{d}\tau + \Delta(t) \int_0^t \boldsymbol{u}\, \mathrm{d}\tau,$$

we rewrite the last boundary condition in (1.8) as follows:

$$\begin{aligned} [\boldsymbol{n}_0 \cdot \mathbb{T}'_{\boldsymbol{u}}(\boldsymbol{u}, q)\boldsymbol{n}]\Big|_{G_T} - \sigma \boldsymbol{n}_0 \cdot \Delta(t) \int_0^t \boldsymbol{u}\Big|_{G_T} &= \sigma H_0(\xi) + \sigma \int_0^t \boldsymbol{n}_0 \cdot \dot{\Delta}(\tau)\boldsymbol{\xi}\, \mathrm{d}\tau \\ &\quad + (\boldsymbol{n}_0 \cdot \boldsymbol{n}) p^+(\rho_0^+ \mathcal{J}_{\boldsymbol{u}}^{-1})\Big|_{G_T}. \end{aligned}$$

Au.Q. open face 'ℕ'? Here $H_0(\xi) = \boldsymbol{n}_0 \cdot \Delta(0)\boldsymbol{\xi}$ is twice the mean curvature of Γ.

Next, we use again the successive approximation method, now for solving system (1.8). We put

$\boldsymbol{u}^{(0)} = 0,\ q^{(0)} = 0$ and define $\boldsymbol{u}^{(m+1)},\ q^{(m+1)},\ m \in \mathbb{N}$, as a solution of the problem

$$\begin{aligned}
&\mathcal{D}_t \boldsymbol{u}^{(m+1)} - \frac{1}{\rho_0^+(\xi)} \mathbb{A}_m \nabla \mathbb{T}'_m(\boldsymbol{u}^{(m+1)}) = \boldsymbol{f}(X_m, t) - \frac{1}{\rho_0^+(\xi)} \mathbb{A}_m \nabla p^+(\rho_0^+ \mathcal{J}_m^{-1}) \text{ in } \quad Q_T^+,\\
&\mathcal{D}_t \boldsymbol{u}^{(m+1)} - \nu^- \nabla_m^2 \boldsymbol{u}^{(m+1)} + \frac{1}{\rho^-} \nabla_m q^{(m+1)} = \boldsymbol{f}(X_m, t), \nabla_m \cdot \boldsymbol{u}^{(m+1)} = 0 \quad \text{in} \quad Q_T^-,\\
&\boldsymbol{u}^{(m+1)}\Big|_{t=0} = \boldsymbol{v}_0 \quad \text{in} \quad \Omega_0^- \cup \Omega_0^+, \quad \boldsymbol{u}^{(m+1)} \xrightarrow[|\xi| \to \infty]{} 0, \quad q^{(m+1)} \xrightarrow[|\xi| \to \infty]{} 0,\\
&[\boldsymbol{u}^{(m+1)}]\Big|_{G_T} = 0, \quad [\mu^\pm \Pi_0 \Pi_m \mathbb{S}_m(\boldsymbol{u}^{(m+1)}) \boldsymbol{n}_m]\Big|_{G_T} = 0,\\
&[\boldsymbol{n}_0 \cdot \mathbb{T}'_m(\boldsymbol{u}^{(m+1)}, q^{(m+1)}) \boldsymbol{n}_m]\Big|_{G_T} - \sigma \boldsymbol{n}_0 \cdot \Delta_m(t) \int_0^t \boldsymbol{u}^{(m+1)}\Big|_{G_T}\\
&\quad = \sigma H_0(\xi) + \sigma \int_0^t \boldsymbol{n}_0 \cdot \dot{\Delta}_m(\tau) \boldsymbol{\xi}\, \mathrm{d}\tau + (\boldsymbol{n}_0 \cdot \boldsymbol{n}_m) p^+(\rho_0^+ \mathcal{J}_m^{-1})\Big|_{G_T}.
\end{aligned} \tag{4.19}$$

Here we have used the notation $\nabla_m = \nabla_{\boldsymbol{u}^{(m)}}$ etc.; $\Delta_m(t)$ is the Beltrami–Laplace operator on $\Gamma_m(t)$; $\boldsymbol{n}_m$ is an outward normal to the surface $\Gamma_m(t) = \{x = X_m(\xi, t) | \xi \in \Gamma\}$; Π_m is the projector on its tangential plane; $\mathbb{A}_m$ is the cofactor matrix for the Jacobi matrix $\{a_{ij}^{(m)}\}$ of (1.5) corresponding to the vector field $\boldsymbol{u}^{(m)}$.

Since $\boldsymbol{n}_0 \cdot \dot{\Delta}_m(\tau) \boldsymbol{\xi} \in W_2^{l-1/2, l/2-1/4}(G_T)$ if $\boldsymbol{u}^{(m)} \in \cup_i \boldsymbol{W}_2^{2+l, 1+l}(Q_T^i)$ [8], and since $H_0 \in W_2^{l+1/2}(\Gamma)$ then, by Theorem 1.2, problem (4.19) is solvable in an interval $(0, T_m)$ in which $\boldsymbol{u}^{(m)}, q^{(m)}$ are determined and $\boldsymbol{u}^{(m)}$ satisfies condition (1.11) with a sufficiently small $\delta > 0$. It is necessary to show that for $\forall m\ T_m \geqslant T' > 0,\ N[\boldsymbol{u}^{(m)}, q^{(m)}]$ are uniformly bounded in $(0, T')$ and that the sequence $\{\boldsymbol{u}^{(m)},\ q^{(m)}\}$ converges to a solution of problem (1.8). The proof of these facts is the same as in the case of a single fluid. It was presented in detail in [8].

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