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Metastability in a flame front evolution equation

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A weakly nonlinear parabolic equation pertinent to the flame front dynamics subject to the buoyancy effect, and to a statistical description of biological evolution is considered. In the context of combustion it is shown that the parabolic interface occurring in upward propagating flames in vertical channels may actually be merely quasi-equilibrium transient states which eventually collapse to a stable configuration in which the flame tip slides along the channel wall.

Keywords: Upward propagating flames; interfaces; Burgers type equation; parabolic equations; stationary solutions; hyperbolic equation; metastability

1. Introduction

The premixed flame can be considered to be a self-sustained wave of an exothermic chemical reaction propagating through a reactive gas mixture. It is a classical case of a free interface system. Indeed, in the flame, the bulk of the heat release normally occurs in a narrow layer, the *reaction zone*. This zone separates the cold combustible mixture from hot combustion products. The width of the reaction zone is often much smaller than the typical length scale of the underlying flow field. This leads one to consider classically the flame as a geometric interface. The dynamics and geometry of this surface are strongly coupled with those of the background gas flow.

The main motivation of the present study is a certain dynamic phenomenon occurring in premixed gas flames in vertical tubes subject to the buoyancy effect.

Thermal expansion of a gas accompanying flame propagation makes the latter sensitive to external acceleration. In upward propagating flames, the cold (denser) mixture is superimposed over the hot (less dense) combustion products. Hence, the plane flame front separating the cold and hot gases is subjected to the classical effect of Rayleigh–Taylor instability. (In combustion, in contradistinction to the Rayleigh–Taylor problem, the interface is permeable, since here the gas has a nonzero normal velocity relative to the flame front.) As a result, the flame front becomes convex toward the cold gas [17, 22] (Fig. 1).

As is known from many experimental observations, upward propagating flames often assume a characteristic shape with the tip of the paraboloid located somewhere near the channel's centerline

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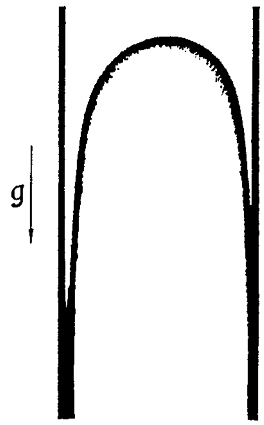

FIG. 1. An upward propagating methane–air flame in a 5.1 cm diameter tube. Adapted from a Schlieren photograph presented in [22].

(compare Fig. 1). Flames where the tip slides along the channel's wall have also been observed [21], however, this type of flame configuration has received less attention. Upward flame propagation, thus, may occur through different but seemingly stable geometrical realizations. The present study is intended to give a better understanding of the pertinent nonlinear phenomenology, which transpires to be rather interesting.

As a mathematical model we shall employ the weakly nonlinear flame interface evolution equation similar to that proposed by Rakib and Sivashinsky [18]. In Appendix B, we specify the framework of approximation and carry out the formal derivation of the equation—which appears here for the first time. Within the framework of the one-dimensional slab geometry, which will be discussed here, this equation reads

$$F_t - \tfrac{1}{2} U_b F_x^2 = D_M F_{xx} + \frac{\gamma g}{2U_b}(F - \langle F \rangle). \tag{1.1}$$

Here $y = F(x, t)$ is the perturbation of the planar flame front $y = U_b t$;

$$\langle F \rangle = \frac{1}{L} \int_0^L F(x, t)\,\mathrm{d}x$$

is the space average over the gap between vertical walls $x = 0$ and $x = L$. The walls are assumed to be thermally insulating. Hence, (1.1) should be solved subject to the adiabatic boundary conditions

$$F_x(t, 0) = F_x(t, L) = 0. \tag{1.2}$$

Here U_b is the flame speed relative to the burned gas; g is acceleration due to gravity; $\gamma = (\rho_u - \rho_b)/\rho_u$ is the thermal expansion parameter; ρ_u, ρ_b are densities of the unburned (cold) and burned (hot) gas, respectively; $D_M = D_{\mathrm{th}}[\frac{1}{2}\beta(Le - 1) - 1]$ is the Markstein diffusivity; D_{th}, thermal diffusivity of the mixture; β is the Zeldovich number and Le is the Lewis number assumed to be high enough to ensure the positive sign of D_M.

Equation (1.1) was derived within the framework of the Boussinesq-type model for flame–buoyancy interaction which neglects density variation everywhere but in the external forcing term.

The weakly nonlinear dynamics described by (1.1) corresponds to the limit

$$U_b^2/\gamma g L \gg 1, \tag{1.3}$$

which is easily attainable in many realistic situations.

For example, at $L = 5$ cm, $U_b = 500$ cm s^{-1}, $g = 1000$ cm s^{-2}, $\gamma = 0.8$, the left-hand side of (1.3) comes to 62.5.

In non-dimensional formulation the problem (1.1), (1.2) may be written as

$$\begin{gathered} \Phi_\tau - \tfrac{1}{2}\Phi_\xi^2 = \varepsilon\Phi_{\xi\xi} + \Phi - \langle\Phi\rangle\,, \\ \Phi_\xi(0,\tau) = \Phi_\xi(1,\tau) = 0, \end{gathered} \tag{1.4}$$

where

$$\xi = x/L, \qquad \tau = \gamma g t/2U_b, \qquad \varepsilon = 2D_M U_b/\gamma g L^2.$$

Problem (1.4) admits a basic planar solution, $\Phi = \text{const}$, which, however, becomes unstable at $\varepsilon < \varepsilon_0 = \pi^{-2} \simeq 0.10$. For many experimentally typical situations ε is significantly smaller than ε_0. For example, at $L = 5$ cm, $U_b = 500$ cm s^{-1}, $D_M = 0.1$ cm^2 s^{-1}, $\gamma = 0.8$, (1.4) yields $\varepsilon = 0.05$.

The numerical simulations conducted with the above system show the following basic trends in the flame dynamics [13].

At $\varepsilon \lesssim \varepsilon_0$ any initial perturbation rapidly leads to an equilibrium solution where the flame slope appears as a monotonic function of ξ.

At $\varepsilon \ll \varepsilon_0$ the final result is qualitatively the same as for $\varepsilon \lesssim \varepsilon_0$. However, depending on the initial conditions, the character of the transient behavior here may be markedly different for a rather wide class of initial data. At the early stage of its development the solution is rapidly attracted to some intermediate state where the flame assumes a somewhat asymmetric parabolic shape (Fig. 2a). The subsequent evolution occurs at a low rate which may become even extremely slow, provided ε is small enough (Fig. 2c). In the process of this quasi-steady development the tip of the parabola gradually moves towards one of the walls. As it comes close enough to the wall the rate of flame evolution again increases. The final equilibrium state is formed when the tip touches the wall (Fig. 2b).

As is shown in the present communication, the above numerical observations are not accidental but indeed reflect the genuine nature of the pertinent dynamics.

It is worth mentioning that apart from the context of premixed gas combustion, equation (1.4) also arises in the statistical description of biological evolution, where $(-\Phi)$ plays the role of the system's scaled entropy ($\Phi \sim \ln f$), f being the distribution function [9: equation (1.17.4)].

2. Description of the main results

In (1.4), we set

$$u = -\Phi_\xi.$$

We thus obtain the equation (and setting x instead of ξ)

$$u_t - \varepsilon u_{xx} + u u_x - u = 0 \quad \text{for} \quad x \in (0,1), \qquad t > 0 \tag{2.1}$$

together with

$$u(t,0) = u(t,1) = 0 \tag{2.2}$$

$$u(0,x) = u_0(x). \tag{2.3}$$

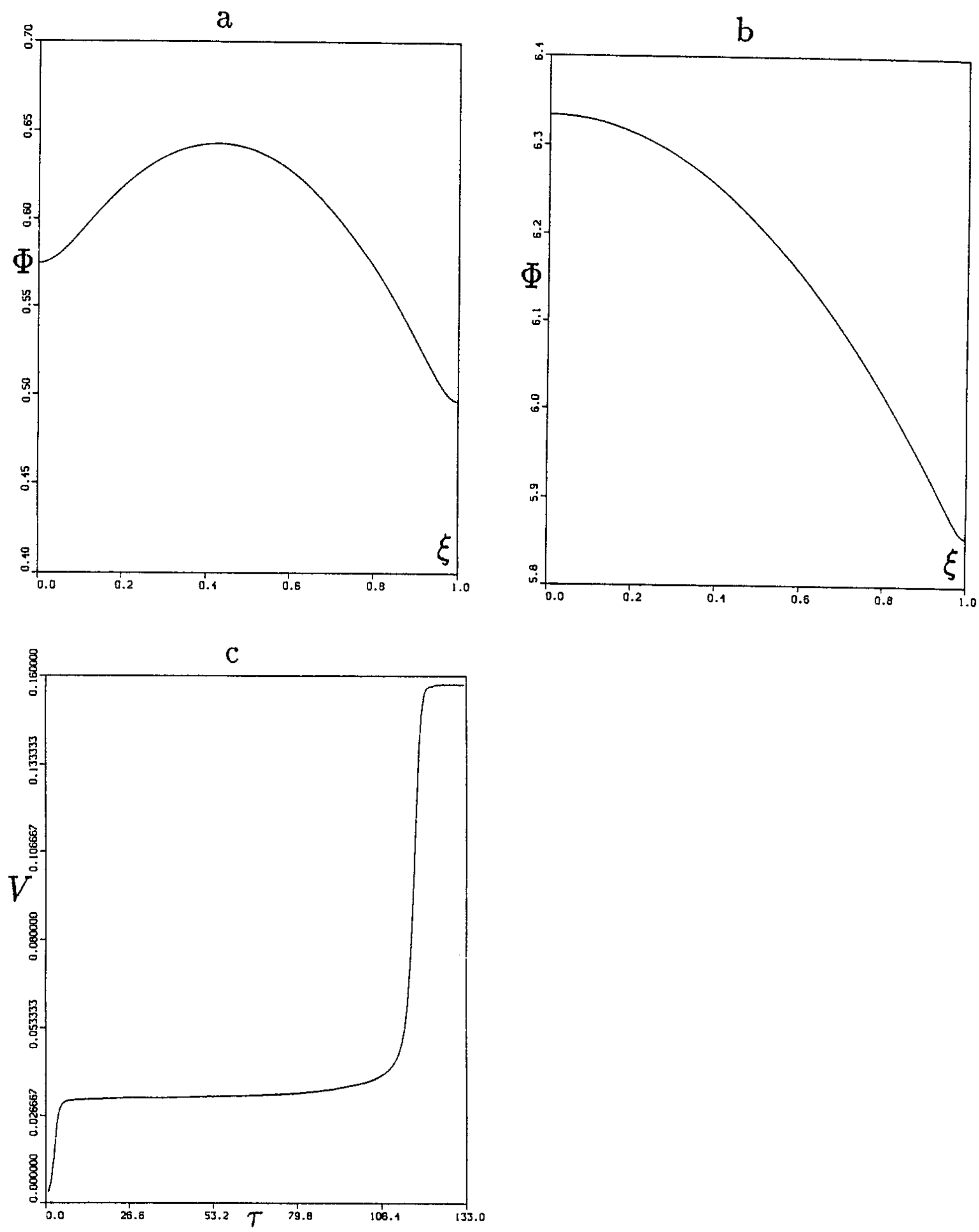

FIG. 2. Numerical simulation of the initial-boundary value problem (1.4) at $\Phi(\xi, 0) = 0.01(\xi - 0.3)\exp[100(\xi - 0.3)^2]$ and $\epsilon = 0.0115$. (a) Quasi-equilibrium solution $\Phi(\xi, \tau)$ at $\tau = 22$. (b) Equilibrium solution $\Phi(\xi, \tau)$ at $\tau = 132$. (c) Temporal evolution of the flame speed $V(\tau) = \frac{1}{2}\langle \Phi_\xi^2 \rangle$.

Nonlinear equations of Burgers' type have been well studied (see e.g. [10, 16], the book [20] and references therein). Notice, however, that differing from the classical Burgers' case, here there is a term $-u$ in the equation. It turns out that for asymptotic behavior—which is the object of the study here—this term plays an essential role.

In the following, we will essentially discuss our results in the framework of (2.1). These can be immediately translated into results for (1.4). It is worthwhile to keep in mind that when $u(\cdot, t)$ is close to linear, then Φ is close to a parabola. The top of this parabola corresponds with the point where u vanishes. In particular, when u does not change sign, then Φ is monotonic and the top of the parabola is at one of the endpoints.

We start this section with a description of our results. For the sake of simplicity, we will consider initial data which change sign at most once (other types of initial data could be studied as well, using the methods of this paper).

The goal of this work is to analyse the long-time dynamics of this equation when $\varepsilon > 0$ is very small, but fixed. Of particular interest is to describe the motion of the point $a(t)$ where u vanishes in $(0, 1)$: $u(t, a(t)) = 0$. This point will be called 'the interface' and corresponds to the top of the parabola.

We assume in the following that the initial datum u_0 is a continuous function which vanishes at $x = 0$ and $x = 1$. We then say that u_0 is in C_0^0. Hence, there exists a unique classical solution $u(t, x)$ of (2.1)–(2.3).

The first part of our study concerns the stationary solutions, i.e. the solutions $f = f(x)$ of the ODE boundary value problem,

$$\begin{cases} \varepsilon f'' - ff' + f = 0 & \text{in } (0, 1), \\ f(0) = f(1) = 0. \end{cases} \tag{2.4}$$

The first theorem gives a complete description of solutions of (2.4).

THEOREM 1 There exists no nontrivial solution (i.e. $f \not\equiv 0$) of (2.4) when $\varepsilon \geqslant \pi^{-2}$. For every ε, $0 < \varepsilon < \pi^{-2}$, there exists a unique positive solution f_ε^+. Likewise, there exists a unique negative solution f_ε^-. For any $\varepsilon > 0$ such that $\varepsilon < (2\pi)^{-2}$, there also exist two solutions $f_{1,\varepsilon}^+$ and $f_{1,\varepsilon}^-$ which have one zero in $(0, 1)$. These solutions are determined uniquely by this property and $(f_{1,\varepsilon}^+)'(0) > 0$ and $(f_{1,\varepsilon}^-)'(0) < 0$.

REMARK. Here and below we call a solution f positive (negative) if $f(x) > 0$ ($f(x) < 0$) for all $x \in (0, 1)$.

The next result is about the behavior of these solutions when $\varepsilon \to 0$.

THEOREM 2

(i) The solutions $f_\varepsilon^+ \equiv f_\varepsilon$ converge uniformly on compact sets of $[0, 1)$ to the function $\varphi^+(x) \equiv x$ as $\varepsilon \searrow 0$. Moreover, there exists some positive $\alpha > 0$ such that

$$f_\varepsilon'(0) = 1 - \mathrm{O}(\mathrm{e}^{-\alpha/\varepsilon}), \qquad |f_\varepsilon'(1)| = \mathrm{O}\left(\frac{1}{\varepsilon}\right).$$

(ii) As $\varepsilon \to 0$, the solution $f_{1,\varepsilon}^+$ converges uniformly on compact sets of $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ to the function φ_1^+ defined by $\varphi_1^+(x) = x$ for $x < \frac{1}{2}$ and $\varphi_1^+(x) = x - 1$ for $\frac{1}{2} < x \leqslant 1$. Lastly, the solution $f_{1,\varepsilon}^-$ converges uniformly on compact sets of $(0, 1)$ to the function $\varphi_1^-(x) = x - \frac{1}{2}$.

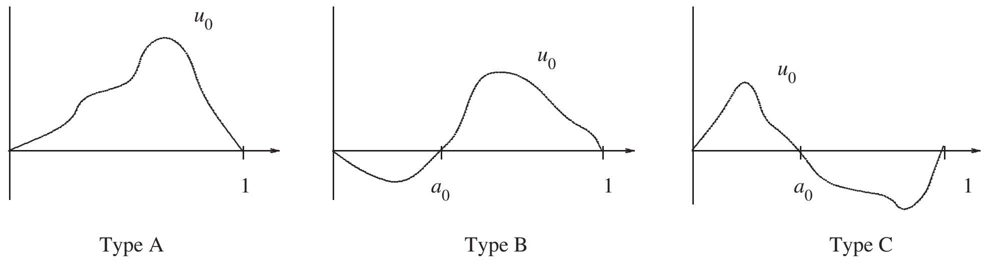

FIG. 3. Different classes of initial data.

We shall consider the stability properties of these stationary solutions with respect to the evolution problem (2.1)–(2.3). They have very different behaviors. For small $\varepsilon > 0$, the solutions f_ε^+ and f_ε^- will be shown to be stable and $f_{1,\varepsilon}^+$ and $f_{1,\varepsilon}^-$ are unstable. The most interesting case is that of $f_{1,\varepsilon}^-$ which turns out to be unstable but exhibits a metastable behavior.

This brings us to the evolution problem. We consider three different classes of initial data (Fig. 3):

- u_0 is of type A means that $u_0 > 0$ in $(0, 1)$ (or that $u_0 < 0$ in $(0, 1)$).
- u_0 is of type B means that $u < 0$ in $(0, a_0)$, $u > 0$ in $(a_0, 1)$.
- Lastly, u_0 is of type C refers to the case $u > 0$ in $(0, a_0)$ and $u < 0$ in $(a_0, 1)$.

In order to understand the behavior when $\varepsilon > 0$ is small, we start with the discussion of the case when $\varepsilon = 0$: that is, of the first-order hyperbolic equation

$$\begin{cases} U_t + UU_x - U = 0 & \text{in } (0,1) \\ U(t,0) = U(t,1) = 0 \\ U(0,x) = u_0(x). \end{cases} \tag{2.5}$$

Since we intend to use (2.5) as a limit of (2.1)–(2.3) when $\varepsilon \to 0$, we consider viscosity solutions of (2.5) as it is classically defined. Furthermore, because of the special boundary condition appearing in (2.5), it is easy to derive (2.5) from a Cauchy problem set on the whole line by extending the solution anti-symmetrically and periodically.

For (2.5) as well, we consider all three cases of initial data A, B and C. We focus mainly on case B which is the most delicate.

From the work of Lyberopoulos [11], we can infer that $U(t, x)$ has a limit $\varphi(x)$ as $t \nearrow +\infty$. Using the equation it is easily seen that φ is piecewise linear with $\varphi'(x) = 1$ at almost every point, but φ may have jumps inside $(0, 1)$. By the maximum principle it follows that if u_0 does not change sign more than once, then neither does $U(t, x)$ for all $t > 0$. Hence, it follows that for the initial data that we consider here, $U(t, x)$ converges to one of the following four limits when $t \nearrow \infty$:

$$\varphi^+(x) = x \tag{2.6}$$

$$\varphi^-(x) = x - 1 \tag{2.7}$$

$$\varphi_1^-(x; a) = x - a \quad \text{for some } 0 < a < 1 \tag{2.8}$$

$$\varphi_1^+(x) = \begin{cases} x & \text{for } 0 < x < \frac{1}{2} \\ x - 1 & \text{for } \frac{1}{2} < x < 1. \end{cases} \tag{2.9}$$

In fact, one can be more precise in cases A and B. If $u_0 > 0$ or $u_0 < 0$ (case A), then $U(t,x)$ converges to φ^+ or to φ^- respectively. In case B, one can further show that $U(t,x)$ converges to a function $\varphi_1^-(x; a_0)$. In the last case, case C, the solution may converge to either φ^+, φ^- or to φ_1^+ (but not to φ_1^-).

The preceding description allows us to understand the limiting behavior (as $t \to \infty$) of the solution of problem (2.1)–(2.3), with viscosity $\varepsilon > 0$, for small $\varepsilon > 0$. We first state the results for case B which is the one of interest here since it exhibits the metastable behavior.

Our first result shows that $u(t,x)$ eventually becomes close to the linear function.

THEOREM 3 Suppose u_0 satisfies condition B. Then, for any (arbitrarily small) $\delta > 0$ and $0 < \gamma < \frac{1}{2}$, there is a time T and $\varepsilon_0 > 0$ depending on γ and δ such that for $\varepsilon < \varepsilon_0$

$$\left|u_\varepsilon(T,x) - (x - a_0)\right| < \delta$$

for all $x \in [\gamma, 1-\gamma]$.

This theorem rests on the fact that, for small $\varepsilon > 0$, u_ε is close to a viscosity solution of (2.5).

Once the solution u_ε is close to this line $(x - a_0)$ for $t = T$, u_ε will stay close to it for an exponentially long interval of time. Hence, even though $x - a_0$ is close to an unstable solution of (2.1), it exhibits a metastable character on exponential long intervals of time. Here is the precise result.

We denote by $a_\varepsilon(t)$, $0 < t$, the curve of zeros of $u_\varepsilon(t,\cdot)$ in the interval (0, 1)

$$u_\varepsilon\big(t, a_\varepsilon(t)\big) = 0, \qquad 0 < a_\varepsilon(t) < 1. \tag{2.10}$$

THEOREM 4 Suppose u_0 satisfies condition B. Fix some $\eta \in (0, \min[a_0, 1-a_0])$ and let δ be any small positive number less than $\min\{(a_0 - \eta), (1 - a_0 - \eta)\}$. Then there are constants $\alpha > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$\left|a_\varepsilon(t) - a_0\right| < \delta/2 \quad \text{for all} \quad 0 \leqslant t \leqslant T_\varepsilon := \mathrm{e}^{\alpha/\varepsilon} \tag{2.11}$$

and

$$\left|u_\varepsilon(t,x) - (x - a_0)\right| \leqslant \delta \quad \text{for all} \quad x \in [\eta, 1-\eta] \quad \text{and all } t,\ T \leqslant t \leqslant T_\varepsilon \tag{2.12}$$

where T is defined in Theorem 3.

THEOREM 5 Let u_0 satisfy condition B with $a_0 \in (0, \frac{1}{2})$. Then for all ε small enough, $u_\varepsilon(t,x) \to f_\varepsilon^+(x)$ as $t \to \infty$. If $a_0 \in (\frac{1}{2}, 1)$ then $u_\varepsilon(t,x) \to f_\varepsilon^-(x)$.

THEOREM 6 Let u_0 satisfy condition A. Then if $u_0 > 0$ in (0, 1), $u_\varepsilon(t,x) \to f_\varepsilon^+(x)$ as $t \to \infty$ and $u_\varepsilon(t,x) \to f_\varepsilon^-(x)$ if $u_0 < 0$ in (0, 1).

COROLLARY The trivial solution of problem (3.1), (3.2) $f \equiv 0$ is unstable.

Theorem 3 will be proved in Section 7 and Theorems 4, 5 and 6 in Section 8.

The results presented in this paper were announced in [4]. The linearized stability properties of stationary solutions will be the subject of a separate study.

There are several works dealing with metastable behavior in various evolution equations starting with the paper of Carr and Pego [7]. We now briefly indicate some of these previous works.

In [7] the Allen–Cahn equation is considered and it is proved that the solution moves exponentially slow to its equilibrium state. The approach they used is based on spectral methods. In [1] the slow motion for the Cahn–Hilliard equation is studied by a similar method to that used in [7].

A different approach was suggested in [6] for the Allen–Cahn equation. These authors use the energy method and construct the appropriate Lyapunov functional. The same method was used in [5] for the Cahn–Hilliard equation. In the last two papers mentioned above it is proved that the solution changes slowly in time of order ε^{-m}.

In the paper [8], the system of Cahn–Hilliard equations is considered. By the improved version of [6] it is proved that if the initial data are close to equilibrium the solution changes exponentially slow. There have also been several works dealing with this type of question relying on formal asymptotic methods in connection with our and mathematically related problems. We refer in particular to the papers of Laforgue and O'Malley [12], Ward and Reyna [25] and Sun and Ward [23].

3. Stationary solutions

We consider, in this section, the stationary problem

$$-\varepsilon f'' + ff' - f = 0 \quad \text{on} \quad (0, 1) \tag{3.1}$$

$$f(0) = f(1) = 0 \tag{3.2}$$

where $\varepsilon > 0$ is a fixed parameter. We will start with the analysis of positive solutions. The set of all solutions, including the ones which change sign will be described in Section 5.

For positive solutions, our main result is the following.

THEOREM 3.1 For $\varepsilon \geqslant \pi^{-2}$ there is no nontrivial solution (i.e. with $f \not\equiv 0$) of (3.1), (3.2). For any ε, $0 < \varepsilon < \pi^{-2}$, there exists a positive solution of (3.1), (3.2).

A necessary condition for the existence of solutions f which are nontrivial (i.e. $f \not\equiv 0$) is easily obtained multiplying the (3.1) by f and integrating by parts which yields

$$\varepsilon \int_0^1 (f')^2 = \int_0^1 f^2 \leqslant \frac{1}{\pi^2} \int_0^1 (f')^2. \tag{3.3}$$

The right-hand side inequality in (3.3) is just the Poincaré–Sobolev inequality in $H_0^1(0, 1)$. Hence, $\varepsilon \leqslant \frac{1}{\pi^2}$ is a necessary condition. Suppose now that $\varepsilon = \frac{1}{\pi^2}$. This implies that in (3.3) the inequality is an equality and hence that $f(x) = C \sin(\pi x)$ for some constant C. A direct computation in the equation then shows that $C = 0$. Therefore, we find that $\varepsilon < \pi^{-2}$ is a necessary condition.

Denote by N the operator $Nf = -\varepsilon f'' + ff' - f$. The function $v = x$ satisfies $Nv \geqslant 0$, i.e. v is a supersolution. Moreover, for any $\alpha \leqslant \frac{1}{\pi}(1 - \varepsilon\pi^2)$ the function $w = \alpha \sin \pi x$ satisfies $Nw = \alpha \sin \pi x(\varepsilon\pi^2 + \alpha\pi \cos \pi x - 1) \leqslant \alpha \sin \pi x(\varepsilon\pi^2 + \alpha\pi - 1) \leqslant 0$, i.e. w is a subsolution. It is obvious that for α as above, $\alpha \sin \pi x \leqslant x$, hence there exists a positive solution for $\varepsilon < \frac{1}{\pi^2}$.

4. Uniqueness of positive solutions

This section is devoted to the proof of the following result.

THEOREM 4.1 For any $\varepsilon > 0$, ($\varepsilon < \pi^{-2}$) the positive solution f of

$$\begin{cases} -\varepsilon f'' + ff' - f = 0, \quad f > 0 \quad \text{in} \quad (0,1) \\ f(0) = f(1) = 0 \end{cases} \tag{2.4}$$

is unique.

With the change of variables $f(x) = \frac{1}{\lambda} v(\lambda x)$ and $\lambda = \frac{1}{\sqrt{\varepsilon}}$, the equation is transformed into

$$\begin{cases} -v'' + vv' - v = 0, \quad \text{on} \quad (0,\lambda), \quad v > 0, \\ v(0) = v(\lambda) = 0. \end{cases} \tag{4.1}$$

Hence, it suffices to show that the solution of (4.1) is unique. To this end, we consider the initial value problem

$$\begin{cases} -v'' + vv' - v = 0, \quad x > 0, \\ v(0) = 0, \quad v'(0) = \alpha. \end{cases} \tag{4.2}$$

It is easily seen that for any α, $0 < \alpha < 1$, the solution $v = v_\alpha(x)$ of (4.2) has a first zero which we denote by $x = L(\alpha) > 0$, i.e. $v > 0$ on $(0, L(\alpha))$ and $v(0) = v(L(\alpha)) = 0$. To prove our claim, i.e. the uniqueness for (4.1)—implying uniqueness for (2.3)—it suffices to prove the following proposition.

PROPOSITION 4.2 The function $L(\alpha)$ is strictly increasing with respect to $\alpha \in (0, 1)$.

We now proceed to prove this proposition. Consider a solution v of (4.2). It is easily seen (compare the arguments in the next section) that v is concave and that there exists a unique $a = a(\alpha) \in (0, L(\alpha))$ such that $v' > 0$ on $(0, a)$, $v'(a) = 0$ and $v' < 0$ on (a, L). We denote $L = L(\alpha)$, $a = a(\alpha)$. We let $m = m_\alpha := \max v_\alpha = v_\alpha(a_\alpha)$. We make the change of variables $s = v(x)$, $x = x(s)$, which is defined separately on the two monotone branches $(0, m) \to (0, a)$ and $(0, m) \to (a, L)$, and $p = p(s) = v'(x(s))$. Writing the equation for p and integrating it leads to the explicit relation

$$p + \ln(1 - p) = \frac{s^2}{2} - \frac{m^2}{2} \tag{4.3}$$

(indeed $p = p(m) = 0$). Let us denote by $p_+(s)$ (resp. $p_-(s)$) the expression of p corresponding to the branch $x(s) \in (0, a)$ (resp. (a, L)) so that $p_+(s) \in (0, 1)$, $p_-(s) < 0$.

From (4.3), we have the expression $m = m(\alpha)$. A straightforward computation shows that m as a function of α is strictly increasing. We can now use m as a parameter instead of α and it will be enough to show that, as a function of m, L is strictly increasing. Let us now compute this function.

Since $x'(s) = \frac{1}{p(s)}$, we see that

$$a = x(m) = x(m) - x(0) = \int_0^m \frac{1}{p_+(s)} \, \mathrm{d}s.$$

Let us denote by $g_+(t)$ the function defined by

$$g_+(t) = p \Leftrightarrow p + \ln(1-p) = -t. \tag{4.4}$$

This function g_+ is defined on $\mathbb{R}^+$ with values in [0,1). Thus, from (4.3) we see that

$$a(m) = \int_0^m \frac{1}{g_+(\frac{m^2}{2} - \frac{s^2}{2})}\,\mathrm{d}s. \tag{4.5}$$

Likewise, we can compute, using the second branch, i.e. $p_-(s)$, the quantity $L(m) - a(m)$. We get

$$L(m) - a(m) = \int_0^m \frac{-1}{p_-(s)}\,\mathrm{d}s = \int_0^m \frac{1}{g_-(\frac{m^2}{2} - \frac{s^2}{2})}\,\mathrm{d}s \tag{4.6}$$

where the function $g_-(t)$ is defined by

$$g_-(t) = p \Leftrightarrow p - \ln(1+p) = t \quad \text{and} \quad g : \mathbb{R}^+ \to \mathbb{R}^+. \tag{4.7}$$

Therefore, by (4.5) and (4.6) we get

$$L(m) = \int_0^m \left[\frac{1}{g_+\big(\frac{m^2}{2} - \frac{s^2}{2}\big)} + \frac{1}{g_-\big(\frac{m^2}{2} - \frac{s^2}{2}\big)}\right]\mathrm{d}s. \tag{4.8}$$

Let us make a change of variable $t = 1 - \frac{s^2}{m^2}$ in (4.8). This yields

$$L(m) = \frac{1}{2}\int_0^1 \left[\frac{m}{g_+(m^2 t/2)} + \frac{m}{g_-(m^2 t/2)}\right]\frac{\mathrm{d}t}{\sqrt{1-t}}. \tag{4.9}$$

Therefore, to complete our proof, it suffices to show that for each fixed $t > 0$, the function

$$m \mapsto m\left(\frac{1}{g_+(m^2 t/2)} + \frac{1}{g_-(m^2 t/2)}\right) \tag{4.10}$$

is monotonic. We fix $t > 0$. Taking as a new variable $\tau = m^2 t/2$, we see that the monotonicity of (4.10) is the same as the monotonicity of the function

$$\tau \mapsto \frac{\sqrt{\tau}}{g_+(\tau)} + \frac{\sqrt{\tau}}{g_-(\tau)} := H(\tau). \tag{4.11}$$

For the sake of convenience let us denote $x(\tau) = g_+(\tau)$ and $y(\tau) = g_-(\tau)$. Recall that

$$\begin{cases} -x - \ln(1-x) = \tau \\ y - \ln(1+y) = \tau \\ H(\tau) = \sqrt{\tau}\Big(\frac{1}{x(\tau)} + \frac{1}{y(\tau)}\Big). \end{cases} \tag{4.12}$$

The proof is completed by showing that the function $H(\tau)$ is monotonic. This is carried out in Appendix A.

5. Behavior of stationary solutions for small ε

In this section, we describe the limiting behavior, as $\varepsilon \searrow 0$, for the stationary solutions of problem (3.1), (3.2).

We denote by f_ε the positive solution, that is

$$\begin{cases} -\varepsilon f_\varepsilon'' + f_\varepsilon f_\varepsilon' - f_\varepsilon = 0, & f_\varepsilon > 0 \text{ in } (0,1) \\ f_\varepsilon(0) = 0, \quad f_\varepsilon(1) = 0. \end{cases} \tag{5.1}$$

It will be enough to describe the behavior of this positive solution. Indeed, as we shall see, one derives from it the behavior of all other solutions making use of symmetries and scalings in the problem.

The proof of Theorem 2 will be decomposed into a series of properties.

PROPOSITION 5.1 The positive solution f_ε satisfies $f_\varepsilon'(x) \leqslant 1$, $0 < f(x) \leqslant x$, $f_\varepsilon''(x) \leqslant 0$ for x in $(0, 1)$.

Proof. Since any linear function $h(x) = \mu x$ with $\mu \geqslant 1$ is a supersolution, it follows from the uniqueness of the positive solution that $f_\varepsilon(x) \leqslant x$, $\forall x \in [0, 1]$. In particular, this implies that $f_\varepsilon'(0) \leqslant 1$; we also know that $f_\varepsilon'(1) < 0 < 1$. If f_ε' reaches an interior maximum > 1 at a point say $x_0 \in (0, 1)$, then $f_\varepsilon'(x_0) > 1$, and it would follow from the equation that

$$\varepsilon f_\varepsilon''(x_0) = f_\varepsilon(x_0)(f_\varepsilon'(x_0) - 1) > 0$$

which is impossible. Hence, $f_\varepsilon' \leqslant 1$ in $(0, 1)$. From the equation it then follows that $f_\varepsilon'' \leqslant 0$ in $(0, 1)$. □

PROPOSITION 5.2 The solution f_ε depends in a monotonic non-increasing fashion on $\varepsilon > 0$. That is, if $0 < \varepsilon < \varepsilon_0$, then

$$0 < f_{\varepsilon_0} \leqslant f_\varepsilon \quad \text{in} \quad (0, 1).$$

When $\varepsilon \to 0$, f_ε converges to x in $(0, 1)$ pointwise.

Proof. Since $f_{\varepsilon_0}'' \leqslant 0$, we see that if $0 < \varepsilon < \varepsilon_0$, then f_{ε_0} is a subsolution of (5.1). Since there is a larger supersolution (e.g. $h(x) = x$), it follows from uniqueness that $f_{\varepsilon_0} < f_\varepsilon$. Therefore, when $\varepsilon \searrow 0$, $f_\varepsilon(x)$ converges pointwise to some limit $f(x)$ satisfying $f(x) \leqslant x$ in $(0, 1)$. Let us now prove that $f(x) = x$. In fact, we will show that $f_\varepsilon(x)$ converges to x as $\varepsilon \searrow 0$, uniformly on compact sets of $[0, 1)$, hence with a boundary layer behavior at 1.

To this end, let us construct a subsolution g of (5.1) for small enough $\varepsilon > 0$. Let a and λ be given numbers in $(0, 1)$—which can be arbitrarily close to 1. Let $b = (1 - a)/2$. Define a function $g_{\lambda,a}$ by setting

$$g_{\lambda,a}(x) = \begin{cases} \lambda x & \text{for } x \in [0, a] \\ \gamma(x) & \text{for } a \leqslant x \leqslant b \\ \frac{\lambda a}{1-b}(1 - x) & \text{for } x \in [b, 1] \end{cases}$$

where γ is a C^2 function on $[1, b]$ such that $\gamma(a) = \gamma(b) = \lambda a$, $\gamma'(a) = \lambda$, $\gamma'(b) = -\frac{\lambda a}{-b}$, $\gamma''(a) = \gamma''(b) = 0$ and $\gamma''(x) \leqslant 0$, $\quad \forall x \in [a, b]$.

Then, the function g is of class C^2 and it is straightforward to see that for given $\lambda < 1$ and $a < 1$, $g_{\lambda,a}$ is a subsolution of (5.1) provided $\varepsilon > 0$ is small enough.

Therefore, since x is a supersolution and $g_{\lambda,a}(x) \leqslant x$ it follows from uniqueness of the positive solution that

$$g_{\lambda,a}(x) \leqslant f_\varepsilon(x) \leqslant x \quad \text{for small enough } \varepsilon > 0. \tag{5.2}$$

Since λ and a can be chosen arbitrarily close to 1, (5.2) implies that $f_\varepsilon(x) \nearrow x$ uniformly on compact sets of $[0, 1)$. □

In order to derive further properties, it is convenient to make a classical transformation of (5.1) which reduces it to a first-order equation in the same way as in Section 3.

In the interval $[0, 1]$, u_ε has a unique maximum at some point a_ε, $0 < a_\varepsilon < 1$. Denote $m_\varepsilon = f_\varepsilon(a_\varepsilon)$. We take as a new unknown the function $p = f_\varepsilon'$, with the new variable $s = f_\varepsilon$. The equation then reads for $p = p(s)$:

$$-\varepsilon p' p + s(p - 1) = 0 \tag{5.3}$$

with $p(0) = f_\varepsilon'(0)$ and $p(m_\varepsilon) = 0$.

PROPOSITION 5.3 For some positive constants α, A, β and β', the solution f_ε satisfies for small enough ε,

$$0 < 1 - f_\varepsilon'(x) \leqslant A\mathrm{e}^{-\frac{\alpha}{\varepsilon}}, \qquad \forall x \in \left[0, \tfrac{1}{2}\right] \quad \text{and} \quad -\tfrac{\beta'}{\varepsilon} \leqslant f_\varepsilon'(1) \leqslant -\tfrac{\beta}{\varepsilon}.$$

Proof. A simple integration of (5.3) from 0 to m_ε yields

$$-\varepsilon p(0) + \varepsilon \ln \frac{1}{1 - p(0)} = \frac{m_\varepsilon^2}{2}.$$

Since $m_\varepsilon \nearrow 1$, we see that $0 < 1 - f_\varepsilon'(0) = 1 - p(0) \leqslant A\mathrm{e}^{-\alpha/\varepsilon}$ for some positive A and α (α can be chosen close to $\frac{1}{2}$). Actually, the same proof shows that for any $\eta \in (0, 1)$, there are positive constants A and α such that

$$0 < 1 - f_\varepsilon'(x) \leqslant A\mathrm{e}^{-\frac{\alpha}{\varepsilon}}, \qquad \forall\, x \in [0, 1 - \eta].$$

Consequently, we see that for all $\eta \in (0, 1)$,

$$0 < x - f_\varepsilon(x) \leqslant A\mathrm{e}^{-\frac{\alpha}{\varepsilon}}, \qquad \forall\, x \in [0, 1 - \eta].$$

Next, we can also invert the function f from $[0, m_\varepsilon]$ to $[a_\varepsilon, 1]$ this time. Let us denote t the new variable. The same equation (5.3) holds for $q = f_\varepsilon'$ as a function of t. An integration of (5.3) for q from m_ε to 0 yields

$$-\varepsilon q(0) + \varepsilon \ln\left(\frac{1}{1 - q(0)}\right) = \frac{m_\varepsilon^2}{2}. \tag{5.4}$$

But, now, $q(0) = f_\varepsilon'(1) < 0$. From (5.4) we infer that $\varepsilon q(0) \to -\frac{1}{2}$ that is $\lim_{\varepsilon \searrow 0} \varepsilon f_\varepsilon'(1) = -\frac{1}{2}$. Notice that since $f_\varepsilon'' \leqslant 0$, this implies that there is some constant $C > 0$ such that

$$\mathrm{Sup}_{x \in [0,1]} \left| f_\varepsilon'(x) \right| \leqslant \frac{C}{\varepsilon}. \tag{5.5}$$

As a consequence of Proposition 5.2 one gets that if $\varepsilon_0 > \varepsilon$, then $f_{\varepsilon_0}'(0) < f_\varepsilon'(0)$. □

From the previous study we can now infer the behavior of other types of solutions.

Consider the negative solution f_ε^-. Clearly, it is obtained from the positive solution of f_ε^+ by the following change of variables $f_\varepsilon^-(x) = -f_\varepsilon^+(1-x)$. Therefore f_ε^- converges to $x-1$ uniformly on compact sets of $(0, 1)$.

Let us now consider the positive solution $f_\varepsilon^+(x; a, b)$ of the problem

$$\begin{cases} -\varepsilon f'' + ff' - f = 0 & x \in (a, b) \\ f(a) = f(b) = 0 \end{cases} \tag{5.6}$$

for all $a < b$. This solution is obtained from f_ε^+ by scaling and shifting. Namely

$$f_\varepsilon^+(x; 0, b) = b f_{\tilde{\varepsilon}}^+\left(\frac{x}{b}\right), \qquad \tilde{\varepsilon} = \frac{\varepsilon}{b^2}, \qquad x \in [0, b] \tag{5.7}$$

$$f_\varepsilon^+(x; a, b) = f_\varepsilon^+(x-a; 0, b-a), \qquad x \in [a, b]. \tag{5.8}$$

Similarly we define $f_\varepsilon^-(x; a, b)$ to be the negative solution of (5.6) in the interval (a, b).

PROPOSITION 5.4 The positive solution $f_\varepsilon(x; 0, b)$ depends in a monotonic increasing fashion on b.

Proof. Let $b \in (0, 1)$. The function $h(x)$ defined by $h(x) = f_\varepsilon(x; 0, b)$ if $x \in [0, b]$ and $h(x) = 0$ if $x \in [b, 1]$ is a subsolution of the problem (3.1), (3.2). Since $h(x) \leqslant x$ it follows from the uniqueness of f_ε that $h(x) \leqslant f_\varepsilon$, hence $f_\varepsilon(x; 0, b) \leqslant f_\varepsilon(x)$ for all $x \in (0, b)$. By the same reason $f_\varepsilon(x; 0, b_1) < f_\varepsilon(x; 0, b_2)$ if $b_1 < b_2$. □

COROLLARY 5.5 If $0 < a < b < 1$ then

$$\left.\frac{\mathrm{d}f_\varepsilon^+(x; 0, a)}{\mathrm{d}x}\right|_{x=0} < \left.\frac{\mathrm{d}f_\varepsilon^+(x; 0, b)}{\mathrm{d}x}\right|_{x=0} \tag{5.9}$$

and if $0 < b - a < c - b$ then

$$\left.\frac{\mathrm{d}f_\varepsilon^-(x; a, b)}{\mathrm{d}x}\right|_{x=b} < \left.\frac{\mathrm{d}f^+(x; b, c)}{\mathrm{d}x}\right|_{x=b}. \tag{5.10}$$

Next we define the solutions $f_{1,\varepsilon}^+$ and $f_{1,\varepsilon}^-$. Let

$$f_{1,\varepsilon}^+(x) = \begin{cases} f_\varepsilon^+\left(x; 0, \frac{1}{2}\right) & x \in [0, \frac{1}{2}] \\ f_\varepsilon^-\left(x; \frac{1}{2}, 1\right) & x \in [\frac{1}{2}, 1]) \end{cases}$$

and

$$f_{1,\varepsilon}^-(x) = \begin{cases} f_\varepsilon^-\left(x; 0, \frac{1}{2}\right) & x \in [0, \frac{1}{2}] \\ f_\varepsilon^+\left(x; \frac{1}{2}, 1\right) & x \in [\frac{1}{2}, 1]. \end{cases}$$

From Proposition 5.2 we infer the limiting behaviors of $f_{1,\varepsilon}^+$ and $f_{1,\varepsilon}^-$. The function $f_{1,\varepsilon}^+$ converges to the function

$$\varphi_1^+(x) = \begin{cases} x & \text{if } 0 \leqslant x < \frac{1}{2} \\ x - 1 & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases}$$

The convergence is uniform in $[0, 1]\backslash\{1/2\}$. Likewise, $f^-_{1,\varepsilon}$ converges to the function $\varphi^-_1(x) = x - \frac{1}{2}$, uniformly on compact sets of $(0, 1)$.

Theorem 1 now follows from Theorems 3.1, 4.1 and the definitions of $f^+_{1,\varepsilon}$ and $f^-_{1,\varepsilon}$. Theorem 2 follows from Propositions 5.2, 5.3 and these definitions. □

Similar statements are straightforward to derive for all other stationary solutions.

6. Nonlinear stability

We now study the stability of stationary solutions of (2.1), (2.2). In Section 8, we will further study the asymptotic behavior of the solutions of (2.1)–(2.3), with general initial data. To these ends we construct here various weak sub- and supersolutions of problem (2.4).

The concept of supersolution was already used in the proof of Proposition 3.1. Let us recall the definition of weak sub- and supersolution in the space $H^1(0, 1)$.

DEFINITION 6.1 A function $v \in H^1[0, 1]$ is a subsolution (respectively supersolution) of problem (2.4) if $v(0), v(1) \leqslant 0$ (respectively $v(0), v(1) \geqslant 0$) and

$$\int_0^1 [\varepsilon v'\varphi' + (vv' - v)\varphi] \leqslant 0 \quad (\text{resp. } \geqslant 0) \tag{6.1}$$

for all test functions $\varphi \in C^1[0, 1]$ such that $\varphi \geqslant 0$ in [0,1] and $\varphi(0) = \varphi(1) = 0$.

We now use the solutions $f^+_\varepsilon(x; a, b)$ and $f^-_\varepsilon(x; a, b)$ which have been defined and discussed in Section 5. Recall that $f^+_\varepsilon(x; a, b)$ is the solution of (3.1) on (a, b) which vanishes at a and b.

Let us now define some functions which will serve as sub- and supersolutions:

$$v_1(x) = Cx + \delta, \quad \text{with parameters } C > 1, \qquad \delta > 0, \tag{6.2}$$

$$v_2(x) = \begin{cases} f^+_\varepsilon(x; a, b) & \text{if } x \in [a, b] \\ 0 & \text{if } x \in [0, 1]\backslash(a, b) \end{cases} \tag{6.3}$$

$$v_3(x) = \begin{cases} f^-_\varepsilon(x; a, b) & \text{if } x \in [a, b] \\ 0 & \text{if } x \in [0, 1]\backslash(a, b) \end{cases} \tag{6.4}$$

for some parameters a, b such that $0 < a < b < 1$;

$$v_4(x) = \begin{cases} f^-_\varepsilon(x; a, b) & \text{if } x \in [a, b] \\ f^+_\varepsilon(x; b, c) & \text{if } x \in [b, c] \\ 0 & \text{if } x \in [c, 1] \end{cases} \tag{6.5}$$

for v_4, the parameters are such that $a \leqslant 0,\ 0 < b < c \leqslant 1,\ b - a \leqslant c - b$;

$$v_5(x) = \begin{cases} f^-_\varepsilon(x; 0, a) & \text{if } x \in [0, a) \\ f^+_\varepsilon(x; a, b) & \text{if } x \in [a, b) \\ f^-_\varepsilon(x; b, 1) & \text{if } x \in [b, 1] \end{cases} \tag{6.6}$$

for v_5, we take parameters such that $0 < a, b < 1$ and $0 < a < b - a,\ 1 - b < b - a$.

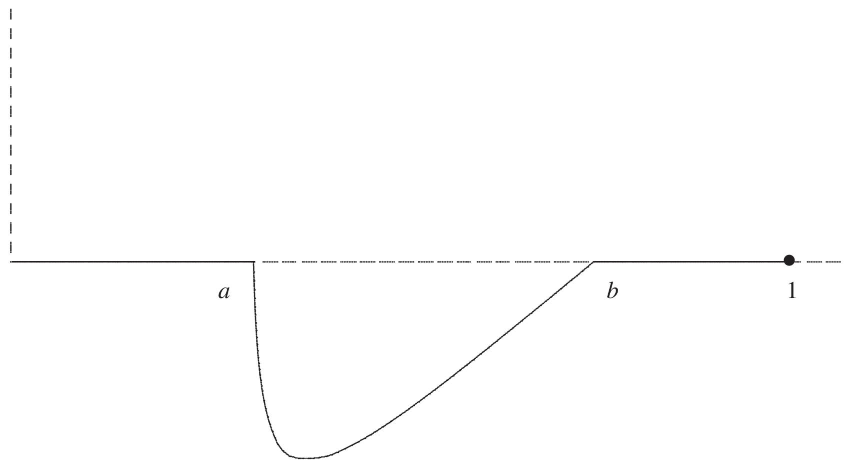

FIG. 4. Profile of $v_3(x)$.

PROPOSITION 6.2 The functions $v_1(x)$ and $v_3(x)$ are supersolutions, $v_2(x)$, $v_4(x)$ and $v_5(x)$ are subsolutions of (2.4).

Proof. Consider the operator $Nv = -\varepsilon v'' + vv' - v$. Then, $Nv_1 = (Cx + \delta)(C - 1) > 0$ and $v_1(0), v_1(1) \geqslant 0$. Thus $v_1(x)$ is a (classical) supersolution.

We only detail the proof for the function $v_3(x)$; the remaining cases are very similar. Note that v_3 is actually a solution of the equation $Lv = 0$ separately in each of the three intervals: $(0, a)$, (a, b) and (b, a). The derivative v_3' is discontinuous at the points $x = a$ and $x = b$. At these points the following inequalities hold: $v_3'(a + 0) - v_3'(a - 0) = v_3'(a + 0) < 0$ and $v_3'(b + 0) - v_3'(b - 0) = -v_3'(b - 0) < 0$ (Fig. 4).

Therefore, for any test function $\varphi \geqslant 0$ in (a, b) with $\varphi(a) = \varphi(b) = 0$, we find that

$$\int_0^1 \left[\varepsilon v_3'\varphi' + (v_3 v_3' - v_3)\varphi\right] \mathrm{d}x = -\varepsilon\varphi'(a)v_3'(a + 0) + \varepsilon\varphi'(b)v_3'(b - 0) \geqslant 0 \tag{6.7}$$

and hence $v_3(x)$ is a supersolution. □

Next we state a useful result about the evolution problem when starting from a sub- or a supersolution.

PROPOSITION 6.3 Suppose that $v(x)$ is a subsolution (respectively, supersolution) of (2.4) in the weak sense of Definition 6.1. Let $u(t, x)$ be the solution of (2.1)–(2.3) with $u_0 = v$. Then for $t \nearrow \infty$ the solution converges monotonically to a stationary solution $f(x)$ of (2.1), (2.2), $u(t, x) \nearrow f(x)$ ($u(t, x) \searrow f(x)$) and $f(x)$ is a solution of (2.4).

This result is well known for classical subsolutions (supersolutions). A similar proof for $v \in H^1$ follows the lines of [2, 19].

Let us now turn to the question of nonlinear stability of stationary solutions. By stability we mean that the solution $u(t, x)$ of the parabolic equation (2.1) is stable with respect to perturbations of initial data. Actually, we will prove a weak form of stability. Our main result is the following theorem.

THEOREM 6.4 The stationary solutions f_ε^+ and f_ε^- are stable and $f_{1,\varepsilon}^+$ and $f_{1,\varepsilon}^-$ are unstable.

Proof. Let $v_1(t,x)$ be the solution of (2.1), (2.2) and $v_1(0,x) = v_1(x)$, where $v_1(x)$ is defined by (6.2). By Propositions 6.2 and 6.3 the solution $v_1(t,x)$ decreases with respect to t. The limit function when $t \nearrow \infty$ obviously exists and is equal to the unique positive solution f_ε^+ of (2.4). On the other hand, the function $v_5(x)$, defined by (6.6), is a subsolution of (2.4). Let $v_5(t,x)$ be the solution of (2.1), (2.2) with that initial condition, i.e. $v_5(0,x) = v_5(x)$. Then $v_5(t,x)$ increases as $t \nearrow \infty$ and converges to some limit function $f(x)$. This function is a stationary solution and if a and $1-b$ are small enough $f(x) = f_\varepsilon^+$ (as there is no other stationary solution satisfying $f \geqslant v_5(x)$).

Let $u(t,x)$ be the solution of (2.1), (2.2) with $v_5(x) \leqslant u(0,x) \leqslant v_1(x)$ for some constants a, b, C and δ. By the maximum principle, $v_5(t,x) \leqslant u(t,x) \leqslant v_1(t,x)$, thus $u(t,x) \to f_\varepsilon^+(x)$ as $t \to \infty$ and the stability of f_ε^+ follows. The stability of f_ε^- is proved in a similar way.

To prove the instability of f_1^- we note that if $a = 0$, $0 < b < \frac{1}{2}$, $c = 1$, then by Proposition 5.4, $f_{1,\varepsilon}^-(x) \leqslant v_4(x)$, where $v_4(x)$ is defined in (6.5). Let $v_4(t,x)$ be the solution of (2.1), (2.2) and $v_4(0,x) = v_4(x)$. Then $v_4(t,x) \nearrow f_\varepsilon^+(x)$ as $t \to \infty$. This proves the instability of $f_{1,\varepsilon}^-$, because $v_4(x) \to f_{1,\varepsilon}^-(x)$ as $b \to \frac{1}{2}$. The proof of instability of $f_{1,\varepsilon}^+(x)$ is similar. □

7. Finite-time Behavior of solution for small ε

Let $Q = \{(x,t) : 0 < x < 1,\ t \in \mathbb{R}^+\}$ and u_ε be the solution of (2.1)–(2.3). It turns out to be convenient to formulate the problem in the real line so as to use some known results. Hence, we consider the Cauchy problem in $S = \{(x,t) : x \in \mathbb{R},\quad t \in \mathbb{R}^+\}$. Extend the function $u_0(x)$ to the whole real line by requiring it to be odd about $x = 0$ and periodic with period 2. Namely, u_0 is first extended to $(-1,1)$ by setting $u_0(x) = -u_0(-x)$ for $x \in [-1,0]$, and then to all of $\mathbb{R}$ by letting $u_0(x) = u_0(x+2)$, $\forall x \in \mathbb{R}$.

Let $\overline{u}_\varepsilon$ be the solution of

$$u_t - \varepsilon u_{xx} + u u_x - u = 0 \quad \text{in} \quad S \tag{7.1}$$

and

$$\overline{u}_\varepsilon(0,x) = u_0(x), \qquad x \in \mathbb{R}. \tag{7.2}$$

By the uniqueness of the solution of the Cauchy problem (7.1), (7.2) we have $\overline{u}_\varepsilon(t,0) = \overline{u}_\varepsilon(t,1) = 0$. Therefore $u_\varepsilon(t,x) = \overline{u}_\varepsilon(t,x)$ for $x \in [0,1]$. In this section we use u_ε for $\overline{u}_\varepsilon$. As is known [10, 16], in the limit of vanishing viscosity, when $\varepsilon \to 0$,

$$u_\varepsilon(t,x) \to U(t,x) \quad \text{as} \quad \varepsilon \to 0, \tag{7.3}$$

where $U(t,x)$ is the entropy solution of the first-order equation

$$U_t + U U_x - U = 0 \tag{7.4}$$

satisfying

$$U(0,x) = u_0(x). \tag{7.5}$$

Note that, in addition, here U (as well as u_ε) satisfies the boundary condition

$$U(t,0) = U(t,1) = 0.$$

Therefore it is a solution of the boundary value problem. This might appear surprising at first sight as such boundary value problems are usually not well posed in the framework of first-order equations. But here, due to the special structure of the equation and the fact that u takes on 0 boundary conditions, we are able to obtain a solution of this problem.

The convergence in (7.3) is in L^1_{loc}. It was recently proved in the series of papers [14, 15, 24] that in fact the convergence in (7.3) is uniform away from shock waves. More precisely, the following result holds.

PROPOSITION 7.1 [14]. Suppose $u_0(x) \in L^\infty(\mathbb{R})$ and there exists $M > 0$ such that, for all x, y, $x \neq y$,

$$\frac{u_0(x) - u_0(y)}{x - y} \leqslant M < \infty.$$

Let $u_\varepsilon(t,x)$ be the solution of (7.1) in S with initial condition $u_\varepsilon(0,x) = u_0(x)$, $x \in \mathbb{R}$, and $U(t,x)$ be the entropy solution of (7.4), (7.5) in S. Then, as $\varepsilon \to 0$, and away from the shock waves, $u_\varepsilon \to U$ in C^0_{loc}. That is, the convergence is uniform on any compact set K in S which does not contain shock waves.

As already defined in Section 2, we distinguish four types of initial data:

Type A: $u_0(x) > 0 \quad x \in (0,1)$
Type A′: $u_0(x) < 0 \quad x \in (0,1)$
Type B: $u_0(x) < 0$ for $x \in (0,a)$, $u_0(x) > 0$ for $x \in (a,1)$, for some a, $0 < a < 1$
Type C: $u_0(x) > 0$ for $x \in (0,a)$, $u_0(x) < 0$ for $x \in (a,1)$, for some a, $0 < a < 1$.

Compare the pictures for cases A, B and C in Section 2 (Fig. 3).

The asymptotic behavior of solutions $U(t,x)$ for some classes of first-order hyperbolic equations has been studied by Lyberopoulos [11]. We now precisely state the version of the result of [11] which we will use here. It concerns the special class of initial data $u_0(x)$ which has at most one sign change.

PROPOSITION 7.2 [11]. Suppose that $u_0(x)$ changes sign at most once in (0,1). Then, the limiting behavior of $U(t,x)$ as $t \to \infty$ is given by either one of the following four kinds.

(i) $U(t,x) = x - a + \mathrm{o}(1) \quad 0 < x < 1$, for some $a \in (0,1)$,

(ii) $U(t,x) = \begin{cases} x + \mathrm{o}(1) & 0 \leqslant x < \frac{1}{2} \\ x - 1 + \mathrm{o}(1) & \frac{1}{2} < x \leqslant 1 \end{cases}$

(iii) $U(t,x) = x + \mathrm{o}(1) \quad 0 \leqslant x < 1$

(iv) $U(t,x) = x - 1 + \mathrm{o}(1) \quad 0 < x \leqslant 1$.

Using Proposition 7.2 we prove

THEOREM 7.3 Let $u_0(x)$ be an initial datum which changes sign at most once. Then, depending on the type of initial data, the asymptotic behavior of $U(t,x)$ is given by (iii) in case A, (iv) in case A′, (i) in case B, and lastly, (ii), (iii) or (iv) in case C. The limits are uniform respectively on compact sets of [0, 1) in case A, of (0, 1] in case A′, of (0, 1) in case B and of $[0,1]\backslash\{\frac{1}{2}\}$ in case (ii).

Proof. The cases A or A′ are straightforward, using the maximum principle. Let us now consider case B.

CASE B. We first require the following result.

LEMMA 7.4 Suppose $U_0(x)$ satisfies condition B. Then, $U(t, a) = 0$ for all $t \geqslant 0$. Furthermore, for some small $\gamma > 0$, $U(t, x)$ is continuous for $x \in (a - \gamma, a + \gamma)$, $t > 0$.

For the proof we use the construction of the entropy solution by characteristic curves. Solving the system

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = 1, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = U, \qquad \frac{\mathrm{d}U}{\mathrm{d}\tau} = U,$$

we find that the characteristic curve which starts at $t = 0$, $x = a$ is vertical, $U(t, a) = 0$, and the characteristic curves which are near this one are going out. Actually, the only reason for $U(t, a)$ to be different from zero may be the appearance of a shock wave at $x = a$. But the velocity of such a shock is equal to $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{u_1+u_2}{2}$ where u_1 and u_2 are the limits of U from both sides of the shock (compare [20]). Thus the shock waves may move only to the right if $a < x < 1$ and to the left if $0 < x < a$ and no shock wave can reach $x = a$. Therefore $U(t, x)$ is continuous for $|x - a| < \gamma$ for some small enough $\gamma > 0$ and $U(t, a) = 0$, for all $t > 0$. From Lemma 7.4 and Proposition 7.2, case B then follows.

CASE C. We show that (i) is not possible. By contradiction, suppose that $U(t, x)$ satisfies (i). Then, as it is proved in [11], $U(t, x)$ should be continuous at the point $x = a$ for all t and $U(t, a) = 0$. Using the construction of the entropy solution as in Lemma 7.4, we obtain that the characteristic curves that begin at $x \neq a$, $t = 0$ do not cross the line $x = a$. Therefore $U(t, x_0)$ has the same sign as $u_0(x_0)$ for x_0 close to a, $x_0 \neq a$. Thus (i) cannot occur. Next (ii) can appear only under the condition $\int_0^1 u_0(x)\,\mathrm{d}x = 0$ (see [11]). Actually, one can construct examples in which all these cases occur. □

REMARK. Suppose that $u_0(x)$ satisfies condition B and in addition the condition

$$u_0(x) \text{ is monotone increasing near } x = a. \tag{7.6}$$

Then, for some small enough $\gamma \in (0, 1)$, $U(t, x)$ is a continuous function for all t large enough and $x \in (\gamma, 1 - \gamma)$. The proof is the same as for Lemma 7.4. Construction of the entropy solution by characteristic curves shows that for large t the shocks may concentrate only near $x = 0$ and $x = 1$.

Next we consider the behavior of $u_\varepsilon(t, x)$ for large t. The most interesting is case B in which metastable behavior appears.

THEOREM 7.5 Suppose $u_0(x)$ satisfies condition B. Then for any arbitrary small numbers γ and δ there exist $T = T(\gamma, \delta)$ and $\tilde{\varepsilon} = \tilde{\varepsilon}(T)$ such that for any $\varepsilon \leqslant \tilde{\varepsilon}$,

$$u_\varepsilon(T, x) < x - a + \delta \quad \text{for} \quad x \in [\gamma, 1] \tag{7.7}$$

$$u_\varepsilon(T, x) > x - a - \delta \quad \text{for} \quad x \in [0, 1 - \gamma]. \tag{7.8}$$

Proof. Let γ and δ be some small fixed numbers. First suppose $u_0(x)$ satisfies (7.6). Then by the previous Remark and Proposition 7.1 there exists T_1 such that $u_\varepsilon(t, x) \to U(t, x)$ as $\varepsilon \to 0$ and the convergence is uniform on any compact $K \subset [\gamma, 1 - \gamma] \times [T_1, \infty)$.

Choose now $T_2 \geqslant T_1$ such that $|U(t,x) - (x-a)| < \delta/2$ for $x \in [\gamma, 1-\gamma]$ and $t \geqslant T_2$. The existence of T_2 is ensured by Theorem 7.3 for the case B. From this, it follows that for any $T \geqslant T_2$ there exists $\varepsilon = \varepsilon_0(T)$ such that for $x \in [\gamma, 1-\gamma]$

$$|u_\varepsilon(T,x) - (x-a)| < \delta$$

for $\varepsilon \leqslant \varepsilon_0(T)$.

Next we require some refined estimates on the behavior of u_ε near the boundary points $x = 0$ and $x = 1$: that is, we consider the behavior of $u_\varepsilon(t,x)$ for $x \in [0,\gamma] \cup [1-\gamma, 1]$. For this purpose we use as a barrier the function

$$V(t,x) = \frac{\alpha(x-a-\delta/2)\mathrm{e}^t}{\alpha \mathrm{e}^t + 1 - \alpha}, \quad 0 < x < a,$$

where α is a constant such that $\alpha > 1$. Indeed, this function $V(t,x)$ is a solution of (2.1). Furthermore,

$$V(0,x) = \alpha(x-a-\delta/2) < u_0(x), \quad \text{for} \quad x \in [0,a], \ \text{if } \alpha \text{ is large enough},$$

$$V(t,0) < 0 = u_\varepsilon(t,0) \quad \text{and} \quad V(t,a) = -\frac{\alpha\delta/2}{\alpha + (1-\alpha)\mathrm{e}^{-t}} < -\delta/2 \quad \text{for} \quad \alpha > 1.$$

Let $t \to \infty$. Then $V(t,x) \to x - a - \delta/2$. Let $T_3 \geqslant T_2$ be sufficiently large so that for all $t \geqslant T_3$, $V(T_3,x) > x - a - \delta$.

Next choose $\varepsilon_1 = \varepsilon_1(T_3)$ so small that $u_\varepsilon(t,a) > -\delta/2$ for $0 \leqslant t \leqslant T_3$ and $\varepsilon \leqslant \varepsilon_1$. This is possible because $U(t,a) = 0$ and $U(t,x)$ is continuous at $x = 0$. Then by comparing u_ε and V, both solutions of (2.1), in the strip $x \in [0,a]$, $0 \leqslant t \leqslant T_3$ we get

$$u_\varepsilon(T_3,x) \geqslant V(T_3,x) > x - a - \delta, \qquad x \in [0,a].$$

Finally, we choose $T(\gamma,\delta) = T_3$ and $\tilde{\varepsilon}(T) = \min\{\varepsilon_1(T_3), \varepsilon_0(T_3)\}$. Thus (7.7) is proved under the assumption (7.6).

To prove (7.8) we use

$$V_1(t,x) = \frac{\beta(x-a+\delta/2)\mathrm{e}^t}{\beta \mathrm{e}^t + 1 - \beta}, \qquad a < x < 1$$

with β large enough.

To remove condition (7.6) we construct the function $\tilde{u}_0(x)$ such that $\tilde{u}_0(a) = 0$, $\tilde{u}_0(x) \geqslant u_0(x)$, $\tilde{u}_0(0) = \tilde{u}_0(1) = 0$, $\tilde{u}_0(x)$ satisfies (7.6) and condition B. Then the corresponding solution of (2.1), $\tilde{u}_\varepsilon(t,x)$ satisfies (7.7). By the comparison principle $u_\varepsilon(t,x) \leqslant \tilde{u}_\varepsilon(t,x)$ and therefore (7.7) also holds for $u_\varepsilon(t,x)$. Similarly, it can be shown that (7.8) holds for some T. As a corollary of Theorem 7.5 one gets Theorem 3 of Section 2. □

Next, for case B, we fix T such that inequalities (7.7), (7.8) are satisfied. Let ε_1 be fixed. In the next section we show that, for $\varepsilon \leqslant \varepsilon_1$ small enough, $u_\varepsilon(t,x) \to f_\varepsilon^+(x)$ as $t \to \infty$. But this convergence happens exponentially slow. This will be described in more detail below.

Case C will not be considered here. We just mention that the limit behavior (ii) of Proposition 7.2 is not stable.

8. Large-time metastable behavior

In this section we mainly study the behavior of the solution u_ε for the case B. As we proved in the previous section there exists T such that for all ε small enough $u_\varepsilon(T,x)$ satisfies the estimates (7.10) and (7.11). Let $a_\varepsilon(t)$ be the curve defined by

$$u_\varepsilon(t, a_\varepsilon(t)) = 0. \tag{8.1}$$

We now prove that the curve $a_\varepsilon(t)$ is almost vertical for an exponentially long interval of time. It will also be proved below that if $a_\varepsilon(0) < \frac{1}{2}$, then for ε small enough the separation point $x = a_\varepsilon(t)$ eventually moves to the left, that is $\lim_{t\to\infty} a_\varepsilon(t) = 0$ and $u_\varepsilon(t,x) \to f_\varepsilon^+(x)$, for all $x \in [0,1)$ as $t \to \infty$. However, the separation remains near the point $a = a_\varepsilon(0)$ for an exponentially long period of time, hence giving rise to a metastable behavior. If $a_\varepsilon(0) > \frac{1}{2}$ then $u_\varepsilon(t,x) \to f_\varepsilon^-(x)$ and $a_\varepsilon(t) \to 1$ as $\varepsilon \to 0$. We begin with the next proposition.

PROPOSITION 8.1 Let $0 < \gamma < a$, and suppose that

$$\begin{cases} u_0(x) < 0 & \text{on } (0,\gamma), \\ u_0(x) \leqslant x - a & \text{on } [\gamma, 1]. \end{cases} \tag{8.2}$$

Then for any fixed $\delta \in (0, a-\gamma)$ there exists $\alpha > 0$ such that for all ε small enough

$$a_\varepsilon(t) > a - \delta \quad \text{for} \quad 0 \leqslant t \leqslant T_\varepsilon = \mathrm{O}\big(\mathrm{e}^{\frac{\alpha}{\varepsilon}}\big).$$

The proof will be divided into several lemmas. First we introduce some notation. Let δ be some given number $\delta \in (0, a-\gamma)$. Let $a_1 = a - \delta > \gamma$ and $a_2 = a - \frac{\delta}{2}$. The first step is to construct a supersolution.

Let $f_{2\varepsilon}^-(x; a_1, a_2)$ be the negative solution of the problem

$$\begin{cases} 2\varepsilon f'' = f(f'-1) & \text{in} \quad (a_1, a_2) \\ f(a_1) = f(a_2) = 0. \end{cases} \tag{8.3}$$

We also define the functions

$$F_\varepsilon(x) = \begin{cases} f_{2\varepsilon}^-(x; a_1, a_2) & \text{on} \quad [a_1, a_2] \\ (x - a_2) f_{2\varepsilon}'(a_2; a_1, a_2) & \text{on} \quad (a_2, 1] \end{cases} \tag{8.4}$$

and

$$v(t,x) = v_\varepsilon(t,x) = F_\varepsilon(x) + Kt \qquad x \in [a_1, 1] \tag{8.5}$$

where $K = A\mathrm{e}^{-\frac{\beta}{\varepsilon}}$ and the constants $A > 0$ and $\beta > 0$ will be chosen later.

It follows from the properties of $f_{2\varepsilon}^-$ described in Section 5 that under the assumptions (8.2) for ε_0 small enough and $\varepsilon \leqslant \varepsilon_0$

$$u_0(x) \leqslant v_\varepsilon(0,x) = F_\varepsilon(x) \quad \text{on} \quad [a_1, 1] \tag{8.6}$$

(compare Fig. 5).

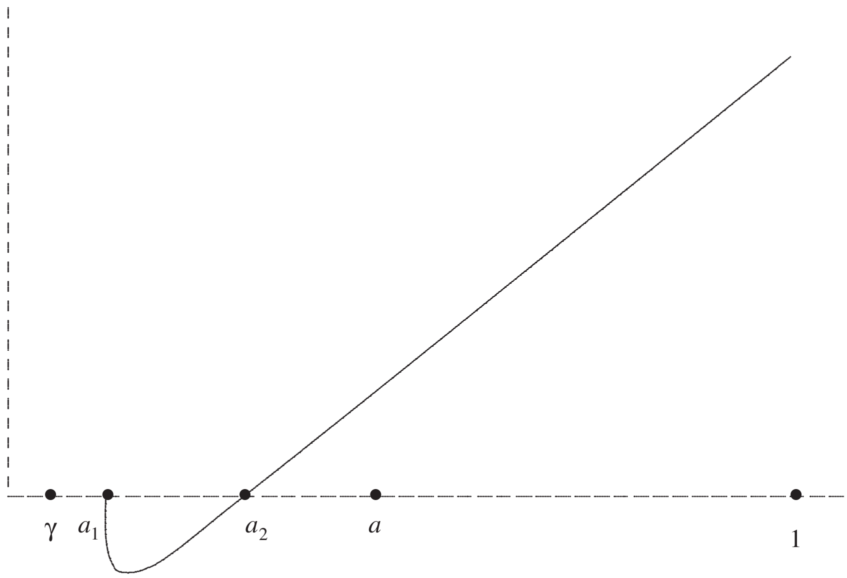

FIG. 5. Profile of $F_\varepsilon(x)$.

We suppose now that $\varepsilon \leqslant \varepsilon_0$ and use the notation $f = f^-_{2\varepsilon}(x; a_1, a_2)$ and $F = F_\varepsilon$.

Next we consider the continuous curve $s(t)$ defined on some time interval $[0, t^*]$ by the conditions.

$$F(s(t)) = -2Kt, \qquad s(0) = a_1, \qquad s(t) > a_1. \tag{8.7}$$

Note that $F_\varepsilon(\frac{a_1+a_2}{2}) \to \frac{a_1+a_2}{2} - a_2 = \frac{a_1-a_2}{2}$ as $\varepsilon \to 0$. Therefore for $t < K^{-\frac{1}{2}}$, we have

$$F_\varepsilon\left(\frac{a_1 + a_2}{2}\right) + 2Kt < \frac{a_1 - a_2}{2} + \eta + 2\sqrt{K} < 0$$

for some positive η and for ε small enough. On the other hand $F_\varepsilon(a_1) + 2Kt = 2Kt > 0$ for $t > 0$. Therefore $s(t) < \frac{a_1+a_2}{2}$ for all ε small enough and $t \leqslant K^{-\frac{1}{2}}$. Hence, s is defined at least on the time interval $[0, K^{-\frac{1}{2}}]$.

Note that for $x \in (s(t), \frac{a_1+a_2}{2})$, $t \in (0, K^{-\frac{1}{2}})$,

$$f(x) + 2Kt = F_\varepsilon(x) + 2Kt < 0. \tag{8.8}$$

Finally, we define the domain D (Fig. 6):

$$D = \left\{(t, x); \quad 0 < t < K^{-\frac{1}{2}}, \quad s(t) < x < 1\right\}. \tag{8.9}$$

LEMMA 8.2 There exist $A > 0$ and $\beta > 0$ such that $Lv_\varepsilon \geqslant 0$ in D, where v_ε is defined by (8.5) and D by (8.9).

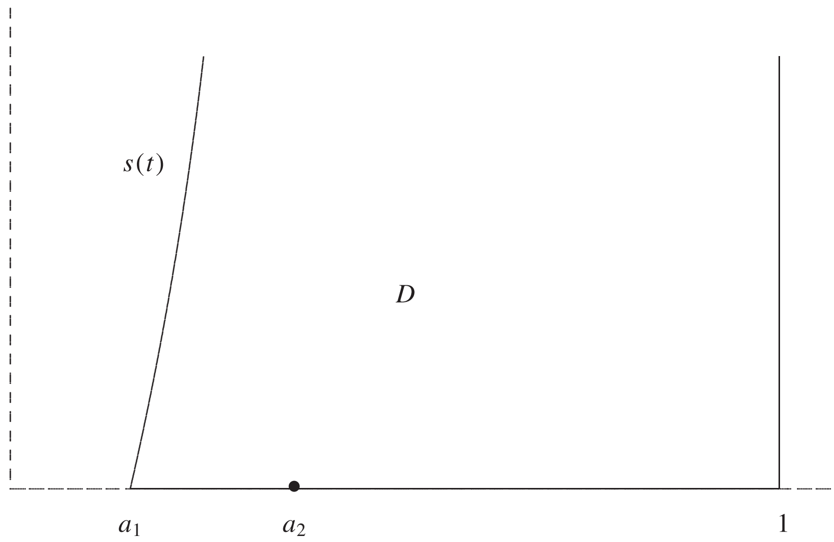

FIG. 6. Domain D.

Proof. For $v = v_\varepsilon$, we have

$$Lv = v_t - \varepsilon v_{xx} + vv_x - v = K - \varepsilon F'' + (F + Kt)(F' - 1). \tag{8.10}$$

Applying Proposition 5.3 to $f = f^-_{2\varepsilon}(x; a_1, a_2)$ we get for $x \in (\frac{a_1+a_2}{2}, a_2)$ and small enough ε

$$0 < 1 - f' \leqslant B\mathrm{e}^{-\frac{\beta}{\varepsilon}}, \qquad 0 \leqslant f'' \leqslant B\mathrm{e}^{-\frac{\beta}{\varepsilon}} \tag{8.11}$$

for some positive constants B and β. In view of (8.4) and (8.11), there exists A so large that for $x \in [\frac{a_1+a_2}{2},\ 1]$

$$|\varepsilon F''| + (|F| + 1)(1 - F') \leqslant A\mathrm{e}^{-\frac{\beta}{\varepsilon}}. \tag{8.12}$$

Assume ε is small enough so that $A\mathrm{e}^{-\frac{\beta}{\varepsilon}} < 1$. Now set

$$K = A\mathrm{e}^{-\frac{\beta}{\varepsilon}} < 1. \tag{8.13}$$

Then by (8.10) and (8.12) we obtain that for $0 < t < K^{-\frac{1}{2}}$ and $x \in (\frac{a_1+a_2}{2}\ , 1)$

$$Lv \geqslant 0. \tag{8.14}$$

It remains to show that (8.14) is also satisfied for all $x \in (s(t),\ \frac{a_1+a_2}{2})$. Using (8.3) and (8.10) we obtain that for $x \in (a_1,\ \frac{a_1+a_2}{2})$, $Lv = K - \frac{1}{2}f(f'-1) + (f+Kt)(f'-1) = K + (\frac{1}{2}f + Kt)(f'-1)$. Hence in order to show that $Lv \geqslant 0$ for $x \in (s(t),\ \frac{a_1+a_2}{2})$ it is enough to observe that $f' < 1$ and by (8.8), $\frac{1}{2}f + Kt < 0$. Thus Lemma 8.2 is proved. □

LEMMA 8.3 Suppose

$$u_0(x) \leqslant -\frac{hx}{b} \quad \text{for} \quad 0 \leqslant x \leqslant b < a \tag{8.15}$$

with some positive constants h and b. Moreover, suppose that for $0 \leqslant t \leqslant T$

$$u(t, b) \leqslant -h, \qquad u(t, 0) = 0. \tag{8.16}$$

Then for $0 \leqslant x \leqslant b$ and $0 \leqslant t \leqslant T$

$$u(t, x) \leqslant -h\frac{x}{b}. \tag{8.17}$$

The proof follows from the fact that if $w(x) = -hx/b$, then $Lw = \frac{hx}{b}(1 + \frac{h}{b}) > 0$ and $w(x) \geqslant u_0(x)$, $w(0) = u(t, 0)$, $w(b) \geqslant u(t, b)$.

LEMMA 8.4 Assume that conditions (8.2) are satisfied. Then, $u(t, s(t)) < 0$, for all ε small enough and $t \leqslant \frac{1}{\sqrt{A}}e^{\beta/2\varepsilon}$, where A and β are defined by (8.11), (8.12) and $s(t)$ by (8.7).

Proof. Without loss of generality we may assume that $u_0'(0) < 0$. Otherwise we may consider a smaller domain and use the comparison principle.

Let $v = v_\varepsilon$ be defined by (8.5). Obviously $u(0, s(0)) = u_0(a_1) < 0 = v(0, s(0))$. Let $\tilde{t}$ be the maximal t such that $t \leqslant K^{-\frac{1}{2}}$ and

$$u(t, s(t)) \leqslant v(t, s(t)) = -Kt. \tag{8.18}$$

We prove that $\tilde{t} = K^{-\frac{1}{2}}$. Suppose on the contrary that

$$\tilde{t} < K^{-\frac{1}{2}} \quad \text{and} \quad u(\tilde{t}, s(\tilde{t})) = v(\tilde{t}, s(\tilde{t})) = -K\tilde{t} > -\sqrt{K}. \tag{8.19}$$

We may now apply the comparison principle to u and v in the domain

$$D_0 = \{s(t) \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant \tilde{t}\} \subset D.$$

By Lemma 8.2 we have $Lv \geqslant 0$ in D_0. Moreover $u(t, 1) = 0 \leqslant v(t, 1)$. Therefore by (8.6) and (8.18)

$$u(t, x) \leqslant v(t, x) \quad \text{in} \quad D_0.$$

Hence for $t \leqslant \tilde{t}$ and all ε small enough

$$u\left(t, \frac{a_1 + a_2}{2}\right) \leqslant v\left(t, \frac{a_1 + a_2}{2}\right) = Kt + f\left(\frac{a_1 + a_2}{2}\right) < \sqrt{K} + f\left(\frac{a_1 + a_2}{2}\right) \leqslant -h$$

where $h > 0$ is some constant. Since $u_0 < 0$ on $(0, a)$ and $u_0'(0) < 0$, we may choose h small enough to get

$$u_0(x) < -h\frac{2x}{a_1 + a_2} \quad \text{for} \quad x \in \left(0, \frac{a_1 + a_2}{2}\right).$$

Now we apply Lemma 8.3 with $b = \frac{a_1+a_2}{2}$ and obtain

$$u(\tilde{t}, s(\tilde{t})) \leqslant -hs(\tilde{t})\frac{2}{a_1+a_2} \leqslant -\frac{2ha_1}{a_1+a_2}. \tag{8.20}$$

On the other hand, by (8.19)

$$u(\tilde{t}, s(\tilde{t})) > -\sqrt{K} = -\sqrt{A}\mathrm{e}^{-\frac{\beta}{2\varepsilon}}. \tag{8.21}$$

The contradiction between (8.20) and (8.21) for ε small enough proves that $u(t, s(t)) \leqslant v(t, s(t)) < 0$ for all $t \leqslant K^{-\frac{1}{2}}$. Thus Lemma 8.4 is proved. □

Proposition 8.1 now follows from (8.1), (8.7) and Lemma 8.4.

COROLLARY 8.5 Let $\gamma < 1 - a$ and suppose that

$$\begin{aligned} &u_0(x) > 0 \quad \text{on} \quad (1-\gamma, 1) \\ &u_0(x) \geqslant x - a \quad \text{on} \quad (0, 1-\gamma]. \end{aligned}$$

Then for any fixed $\delta \in (0, 1-\gamma-a)$ there exists $\alpha > 0$ such that for all ε small enough

$$a_\varepsilon(t) < a + \delta \quad \text{for} \quad 0 \leqslant t \leqslant T_\varepsilon = \mathrm{O}\big(\mathrm{e}^{\frac{\alpha}{\varepsilon}}\big).$$

Proof. Let $w(t, x) = -u(t, 1-x)$. Then w satisfies the same equation (2.1), $w(0, x) = -u(0, 1-x) \leqslant x - (1-a)$ for $x \in [\gamma, 1]$ and $w(0, x) < 0$ for $x \in (0, \gamma)$. Applying Proposition 8.5 to $w_\varepsilon(t, x)$ one gets $1 - a_\varepsilon(t) > 1 - a - \delta$ and hence $a_\varepsilon(t) < a + \delta$. □

Proof of Theorem 4. Let η, δ be fixed numbers satisfying the conditions of Theorem 4. By Theorem 7.5 there exist $T = T(\delta, \eta)$ and ε_0 such that for all $\varepsilon \leqslant \varepsilon_0$

$$\begin{aligned} &u_\varepsilon(T, x) < x - a_0 + \frac{\delta}{4}, \quad \text{if} \quad x \in \left(\frac{\eta}{4}, 1\right) \\ &u_\varepsilon(T, x) > x - a_0 - \frac{\delta}{4}, \quad \text{if} \quad x \in \left(0, 1-\frac{\eta}{4}\right). \end{aligned}$$

Moreover, the maximum principle yields

$$\begin{aligned} &u_\varepsilon(T, x) < 0, \qquad \text{if} \quad x \in \left(0, \frac{\eta}{4}\right) \\ &u_\varepsilon(T, x) > 0, \qquad \text{if} \quad x \in \left(1-\frac{\eta}{4}, 1\right). \end{aligned}$$

Now we introduce $\tau = t - T_1$ and apply Proposition 8.1 with $a = a_0 - \frac{\delta}{4}$, $\gamma = \frac{\eta}{4}$. From these inequalities, we obtain that there exists $\alpha > 0$ such that for all ε small enough,

$$a_\varepsilon(t) > a_0 - \frac{\delta}{2} \quad \text{for} \quad T \leqslant t \leqslant T + T_{1,\varepsilon} = \mathrm{O}\big(\mathrm{e}^{\frac{\alpha}{\varepsilon}}\big). \tag{8.22}$$

Similarly, and with the aid of Corollary 8.5, we get

$$a_\varepsilon(t) < a_0 + \frac{\delta}{2} \quad \text{for} \quad T \leqslant t \leqslant T + T_{2,\varepsilon} = \mathrm{O}\big(\mathrm{e}^{\frac{\alpha}{\varepsilon}}\big). \tag{8.23}$$

From (8.22) and (8.23) follows (2.11) with $T_\varepsilon = T + \min\{T_{1,\varepsilon}, T_{2,\varepsilon}\}$. It remains to prove (2.12). For this purpose we use sub- and supersolutions of initial boundary value problems with $u|_{t=T} = u_\varepsilon(T, x)$, $T \leqslant t \leqslant T_\varepsilon$ and with boundary conditions equal to those of u_ε:
$w_1(x) = x - (a_0 - \delta)$ is a supersolution for $x \in [a_0 - \delta, 1]$, $t \in [T, T_\varepsilon]$;
$w_2(x) = x - (a_0 + \delta)$ is a subsolution for $x \in [0, a_0 + \delta]$, $t \in [T, T_\varepsilon]$;
$w_3(x) = \begin{cases} f_\varepsilon^+\big(x; a_0 + \frac{\delta}{2}, 1 - \frac{\eta}{2}\big), & x \in \big[a_0 + \frac{\delta}{2}, 1 - \frac{\eta}{2}\big) \\ 0 & x \in \big[1 - \frac{\eta}{2}, 1\big] \end{cases}$ is a subsolution for $x \in \big[a_0 + \frac{\delta}{2}, 1\big]$, $t \in [T, T_\varepsilon]$;
$w_4(x) = \begin{cases} 0 & x \in \big[0, \frac{\eta}{2}\big) \\ f_\varepsilon^-\big(x; \frac{\eta}{2}, a_0 - \frac{\delta}{2}\big) & x \in \big[\frac{\eta}{2}, a_0 - \frac{\delta}{2}\big] \end{cases}$ is a supersolution for $x \in [0, a_0 - \frac{\delta}{2}]$, $t \in [T, T_\varepsilon]$.

By comparison of $u_\varepsilon(t, x)$ with w_1, w_2, w_3, w_4 for $t \in [T, T_\varepsilon]$ we get (2.12). □

PROPOSITION 8.6 Let $0 < \tilde{a} < \frac{1}{2}$, $0 < \tilde{\gamma} < 1 - 2\tilde{a}$, $u_0(x) > x - \tilde{a}$ for $x \in (0, 1 - \tilde{\gamma})$ and $u_0(x) > 0$ for $x \in (1 - \tilde{\gamma}, 1)$. Then for all ε small enough

$$\underline{\lim} u_\varepsilon(t, x) \geqslant f_\varepsilon^+(x) \quad \text{as} \quad t \to \infty.$$

Proof. Let $v_4(x)$ be defined in (6.5) with $a = -\delta$, $b = \tilde{a} + \delta$, $c = 1 - \tilde{\gamma}$ where δ is chosen small enough such that $b - a < c - b$. Function $v_4(x)$ is a subsolution. Let $V_\varepsilon(t, x)$ be the solution of (2.1) satisfying $V_\varepsilon(0, x) = v_4(x)$ with $V_\varepsilon(t, 0) = V_\varepsilon(t, 1) = 0$. For ε small enough, $V_\varepsilon(0, x) = v_4(x) \leqslant u_0(x)$ and $V_\varepsilon(t, 0) = u(t, 0)$, $V_\varepsilon(t, 1) = u(t, 1)$. By the comparison principle, $V_\varepsilon(t, x) \leqslant u_\varepsilon(t, x)$. On the other hand, by Proposition 6.3 and Theorem 4.1, $V_\varepsilon(t, x) \nearrow f^+(x)$; thus the assertion follows. □

Proof of Theorem 5. First assume $a_0 \in (0, \frac{1}{2})$. Function $v_1(x) = Cx + \delta$, $C > 1$, $\delta > 0$ is a supersolution. Moreover $v_1(x) > f_\varepsilon^+(x)$ for $x \in [0, 1]$. Let $V_\varepsilon^1(t, x)$ be the solution of (2.1) satisfying $V_\varepsilon^1(0, x) = v_1(x)$, with $V_\varepsilon^1(t, 0) = V_\varepsilon^1(t, 1) = 0$. As in the proof of Proposition 8.6, $V_\varepsilon^1(t, x) \geqslant u_\varepsilon(t, x)$ and $V_\varepsilon^1(t, x) \searrow f_\varepsilon^+(x)$. Therefore, $\overline{\lim} u_\varepsilon(t, x) \leqslant f_\varepsilon^+(x)$.

On the other hand, it follows from the previous lower bound on $u_\varepsilon(t, x)$ on $(0, 1 - \eta/4)$ and Proposition 8.6 that $\underline{\lim} u_\varepsilon(t, x) \geqslant f_\varepsilon^+ \quad \text{as} \quad t \to \infty$. The proof in this case is complete. The case $a_0 \in (\frac{1}{2}, 1)$ is similar and therefore we omit it. □

Proof of Theorem 6. As in the proof of Theorem 3.1 we use functions $v_1(x) = Cx + \delta$ and

$$w(x) = \begin{cases} \alpha \sin \frac{\pi}{1-2\delta}(x - \delta) & \text{if } x \in [\delta, 1 - \delta] \\ 0 & \text{otherwise} \end{cases}$$

where δ is some fixed small number. For C large enough and α small enough

$$w(x) < u_0(x) < v_1(x).$$

On the other hand, $w(x)$ is a subsolution and $v_1(x)$ is a supersolution of the problem (2.4). By Proposition 6.3 and Theorem 4.1, $u_\varepsilon(t, x) \to f_\varepsilon^+(x)$. □

9. Concluding remarks

In this paper, we considered a one-dimensional formulation in which the interface equation (1.4) is readily reduced to Burgers-type equation (2.1) for the interface slope. In this setting we were able to carry out a full analysis of the interface dynamics. We first classified stationary solutions. Then, in the limit of small Markstein diffusivity, we obtained the dynamic behavior of solutions by using the associated hyperbolic equations. We thus described and explained the metastable behavior of flames with parabolic shapes. In the present work, we gave a rigorous description of both stages of the metastable behavior whereas in most previous works, only the second stage had been considered.

There is no doubt that the basic metastability effect will also appear in the two-dimensional version of this problem, for flames propagating in vertical tubes. In this case, however, the representation of the problem in terms of the interface slope is obviously ruled out. Therefore, a completely different approach is required. We plan to address this problem in a forthcoming study.

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Appendix A. Uniqueness of the positive solution

In spite of the simplicity of the equation, the uniqueness result for the positive solution of (2.4) does not seem to follow directly from a straightforward calculation. Also, it seems that one cannot readily apply the methods of [3].

In this appendix, we prove Lemma A1 below which is used in the proof of uniqueness in Section 4. It relies on some explicit computations. Let us first state this result again.

Consider the two functions $x(t)$ and $y(t)$ defined implicitly for $0 < t < \infty$ by

$$\begin{cases} -x - \ln(1-x) = t \\ \quad y - \ln(1+y) = t \end{cases} \tag{A1}$$

with values $x(t) \in (0, 1)$ and $y(t) \in (0, \infty)$. Set

$$H(t) = \sqrt{t}\left(\frac{1}{x(t)} + \frac{1}{y(t)}\right).$$

The result is the following lemma.

LEMMA A1 The function $H(t)$ is increasing on $(0, \infty)$.

This fact is somewhat more delicate that one might first think because it can be seen (and it follows from the proof given below) that while $\frac{\sqrt{t}}{x(t)}$ is increasing, the function $\frac{\sqrt{t}}{y(t)}$ is *decreasing* and one has to carry out some very precise computations in order to see that $H(t)$ is increasing. In terms of the notation of Section 4, this means that $a(m)$ is increasing but that $L(m) - a(m)$ is decreasing.

Let us now turn to the proof. Note first that $x(t)$ and $y(t)$ are increasing, $x(0) = y(0) = 0$, $x'(0) = y'(0) = \infty$ and $x(\infty) = 1$, $y(\infty) = \infty$.

We set $X(t) = x(t^2/2)$, $Y(t) = y(t^2/2)$. It suffices to show that $t \mapsto H(t^2/2)$ is increasing on $(0, \infty)$. Hence, we set $h(t) = \sqrt{2}H(t^2/2)$ so that

$$h(t) = t\left(\frac{1}{X(t)} + \frac{1}{Y(t)}\right). \tag{A2}$$

We will prove that $h(t)$ is increasing.

The functions $X(t)$ and $Y(t)$ are defined implicitly by

$$\begin{cases} -X(t) - \ln(1 - X(t)) = t^2/2 \\ \quad Y(t) - \ln(1 + Y(t)) = t^2/2. \end{cases} \tag{A3}$$

Therefore, it follows that

$$\frac{X}{1-X}\,\mathrm{d}X = t\,\mathrm{d}t, \qquad \frac{Y}{1+Y}\,\mathrm{d}Y = t\,\mathrm{d}t. \tag{A4}$$

We start with the analysis of the behavior of $h(t)$ near $t = 0$.

LEMMA A2 Near $t = 0$, h satisfies $h(0) = 1/2$, $h'(0) = 0$ and $h'(t) > 0$ for small $t > 0$.

Proof. Observe that $X(0) = Y(0) = 0$ and that near $t = 0$, the following expansions hold:

$$\begin{cases} t^2 = X^2 + \frac{2}{3}X^3 + \frac{1}{2}X^4 + \mathrm{O}(X^5), \\ t^2 = Y^2 - \frac{2}{3}Y^3 + \frac{1}{2}Y^4 + \mathrm{O}(Y^5). \end{cases} \tag{A5}$$

Thus, as $t \to 0$, $X(t) \sim t$ and $Y(t) \sim t$ which shows that $h(t) \to 1/2$ as $t \to 0$.

An exact computation relying on (A2) and (A4) yields

$$h'(t) = \frac{1}{X} + \frac{1}{Y} - t^2\left(\frac{1-X}{X^3} + \frac{1+Y}{Y^3}\right).$$

Then, using the expansions (A5) to substitute the term t^2, we get

$$h'(t) = \tfrac{1}{6}(X + Y) + \mathrm{O}(t^2) \quad \text{as} \quad t \searrow 0^+. \tag{A6}$$

This completes the proof of Lemma A2. □

Let us now proceed with the proof of Lemma A1. Let us write $h'(t) = V + W$ with $V(t) = V(X) = \frac{1}{X} - t^2(\frac{1-X}{X^3})$, and $W(t) = W(Y) = \frac{1}{Y} - t^2(\frac{1+Y}{Y^3})$. We write indifferently $V(t)$ on $V(X)$ and likewise for W according to which variable—t or X—we choose as an independent variable.

The claim we will now prove is that $V + W > 0$ for $t > 0$. As said before, the source of the difficulty lies in that while $V > 0$, the term W on the contrary is negative. Since $V + W|_{t=0} = 0$ and $V + W > 0$ for small $t > 0$, it will be sufficient to show that $V' + W' > 0$. Now this we will show to hold separately for V and W—that is we will now show that $V' > 0$ and $W' > 0$.

LEMMA A3 For all positive $t > 0$, $V'(t) > 0$.

Proof. Since $\frac{\mathrm{d}X}{\mathrm{d}t} > 0$, it suffices to show that $V'(X) > 0$ for $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}t}$. (Here and in the following, $V'(t)$ means $\frac{\mathrm{d}V}{\mathrm{d}t}$ and $V'(X)$ refers to $\frac{\mathrm{d}V}{\mathrm{d}X}$.)

A straightforward computation which uses the first relation in (A4), $t\frac{\mathrm{d}t}{\mathrm{d}X} = \frac{X}{1-X}$, yields

$$V'(X) = -\frac{3}{X^2} + t^2\left(\frac{3}{X^4} - \frac{2}{X^3}\right). \tag{A7}$$

Use of the first expansion of (A5) in (A7) shows that $V'(X) = \frac{1}{6} + \mathrm{O}(X)$ as $t \to 0^+$ so that $V'(0) = \frac{1}{6}$.

Next, write $V'(X) = X^{-4}V_1(X)$ with $V_1(0) = 0$, $V_1(X) = -3X^2 + t^2(3 - 2X)$. Making use again of the expression (A4), some direct computations lead to

$$V_1'(X) = -2t^2 + \frac{2X^2}{1-X}, \qquad V_1''(X) = \frac{2X^2}{(1-X)^2}.$$

Therefore, $V_1'(0) = 0$, $V_1'' > 0$ and V_1' is increasing. Hence, $V_1'(X) > 0$ for all X and thus $V_1(X) > 0$ for all X which shows that $V'(X) > 0$. □

LEMMA A4 $W'(t)$ (or $W'(Y)$) is positive for all t.

Proof. The same type of computations as above (using $t\frac{\mathrm{d}t}{\mathrm{d}Y} = \frac{Y}{1+Y}$) allows one to write

$$W'(Y) = Y^{-4}W_1(Y)$$

with

$$W_1(Y) = -3Y^2 + t^2(3 + 2Y), \qquad W_1(0) = 0. \tag{A8}$$

As above, we have $W_1'(Y) = 2t^2 - \frac{2Y^2}{1+Y}$, $W_1''(Y) = \frac{2Y^2}{(1+Y)^2} > 0$. Therefore $W_1'(0) = 0$, $W_1'' > 0$ and W_1' is increasing. Hence, we derive first that $W_1' > 0$ and then that $W_1 > 0$ for all values of t. This shows that $W' > 0$ and the proof is complete. □

Appendix B. Derivation of the basic interface equation

In order to derive the flame evolution equation (1.4) we adopt a simple hydrodynamic model which considers the flame as a geometrical surface moving at a prescribed curvature-dependent velocity relative to the underlying flow field. Transport and chemical kinetics effects are ignored, but the change in gas density is taken into consideration. We consider an upward propagating flame in a vertical channel. The corresponding set of appropriately scaled Euler equations read

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial \hat{x}}, \tag{B1}$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{1}{\hat{\rho}}\frac{\partial \hat{p}}{\partial \hat{y}} - \hat{g}, \tag{B2}$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0. \tag{B3}$$

Here $\hat{\mathbf{v}} = (\hat{u}, \hat{v})$ is the scaled velocity of gas in units of U_b; $(\hat{x}, \hat{y}, \hat{t})$, scaled spatio-temporal coordinates in units of L, L/U_b, respectively; $\hat{\rho}$, scaled density in units of ρ_b; $\hat{p}$, scaled pressure in units of $\rho_b U_b^2$; $\hat{g} = gL/U_b^2$, scaled acceleration of gravity. L, U_b, ρ_b are defined in Section 1.

Equaitons (B1)–(B3) are considered in the frame of reference attached to the planar flame ($\hat{y} = 0$) pertinent to the zero-gravity condition.

For the general non-zero-gravity situation the following relations on the flame interface, $\hat{y} = \hat{F}(\hat{x}, \hat{t})$, must be held:

(i) continuity of mass flow

$$\left[\hat{\rho}(\hat{\mathbf{v}}\cdot\mathbf{n} - D)\right]_-^+ = 0, \tag{B4}$$

(ii) continuity of momentum flow

$$\left[\hat{\rho}\hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot\mathbf{n} - D) + p\mathbf{n}\right]_-^+ = 0, \tag{B5}$$

(iii) curvature-dependent mass flow through the flame interface

$$\hat{\rho}(\hat{\mathbf{v}}\cdot\mathbf{n} - D) = -1 - \mu/R. \tag{B6}$$

Here $1/R = \hat{F}_{\hat{x}\hat{x}}/(1 + (\hat{F}_x)^2)^{3/2}$ is the interface curvature, and $\mu = D_M/U_b L$ is the scaled Markstein diffusivity (compare Section 1)

$$\begin{aligned} \mathbf{n} &= \{1 + (\hat{F}_{\hat{x}})^2\}^{-1/2}\,(-\hat{F}_{\hat{x}}, 1) \\ D &= \{1 + (\hat{F}_{\hat{x}})^2\}^{-1/2}\,\hat{F}_{\hat{t}}. \end{aligned} \tag{B7}$$

At the channel walls we impose the impermeability conditions,

$$\hat{u} = 0, \quad \hat{F}_{\hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0, 1. \tag{B8}$$

Hydrodynamic quantities corresponding to the burned gas region ($\hat{y} < \hat{F}$) are assigned the index $(-)$; those corresponding to the fresh gas region ($\hat{y} > \hat{F}$), the index $(+)$. With this convention,

$$\rho_- = 1, \qquad \rho_+ = 1/(1-\gamma) \qquad \big(\gamma = (\rho_+ - \rho_-)/\rho_+\big). \tag{B9}$$

Using (B6) and (B9), conditions (B4) and (B5) are transformed to the following form, more convenient for further treatment:

$$[\mathbf{v}\cdot\mathbf{n}]_-^+ = \gamma(1 + \mu/R) \tag{B10}$$

$$[\mathbf{v}\cdot\boldsymbol{\tau}]_-^+ = 0, \qquad \boldsymbol{\tau} = \{1 + (\hat{F}_{\hat{x}})^2\}^{-1/2}\,(1, \hat{F}_{\hat{x}}) \tag{B11}$$

$$[p]_-^+ = \gamma(1 + \mu/R). \tag{B12}$$

For the sequel, it will be convenient to introduce the reduced pressure,

$$\hat{q} = \hat{p} + \hat{\rho}\hat{g}\hat{y}, \tag{B13}$$

eliminating the external acceleration from (B2). In terms of the reduced pressure (B11) becomes

$$[q]_-^+ = \left(\frac{\gamma\hat{g}}{1-\gamma}\right)\hat{F} + \gamma(1 + \mu/R)^2. \tag{B14}$$

At zero gravity ($\hat{g} = 0$) the above problem allows for the following time-independent, one-dimensional solution pertinent to the planar flame, $\hat{F} = 0$:

$$\hat{u}^- = 0, \qquad \hat{v}^- = -1, \quad \hat{q}^- = 1 \quad (\hat{y} < 0), \tag{B15}$$

$$\hat{u}^+ = 0, \qquad \hat{v}^+ = -(1-\gamma), \qquad \hat{q}^+ = 1+\gamma \quad (\hat{y} > 0). \tag{B16}$$

At non-zero gravity we consider the limit $\gamma \ll \gamma\hat{g} = \alpha \ll 1$, and introduce the scaled quantities, $v^{\pm}$, $V^{\pm}$, $Q^{\pm}$, s, defined as

$$\begin{aligned} &\hat{u}^+ = \alpha^2 U^+, \quad \hat{v}^+ = -(1-\gamma) + \alpha^2 V^+, \\ &\hat{q}^+ = (1+\gamma) + \alpha^2 Q^+ \\ &\ \hat{u} = \alpha^2 U^-, \qquad \hat{v}^- = -1 + \alpha^2 V^- \\ &\hat{q}^- = 1 + \alpha^2 Q^-, \qquad \alpha\hat{t} = s. \end{aligned} \tag{B17}$$

In terms of the scaled variables, for the leading-order asymptotics with respect to γ and α, (B1)–(B3) become

$$\begin{aligned} &\frac{\partial U^{\pm}}{\partial \hat{y}} = \frac{\partial Q^{\pm}}{\partial \hat{x}}, \qquad \frac{\partial V^{\pm}}{\partial \hat{y}} = \frac{\partial Q^{\pm}}{\partial \hat{y}}, \\ &\frac{\partial U^{\pm}}{\partial \hat{x}} + \frac{\partial V^{\pm}}{\partial \hat{y}} = 0. \end{aligned} \tag{B18}$$

Putting $\hat{F} = \alpha\Theta$, $\mu = \alpha\kappa$, for the leading-order asymptotics the conditions (B6), (B10), (B11), (B14) yield

$$\Theta_s = \kappa\Theta_{\hat{x}\hat{x}} + \tfrac{1}{2}(\Theta_{\hat{x}})^2 + V^+ \quad \text{at} \quad \hat{y} = 0, \tag{B19}$$

$$U^+ = U^-, \qquad V^+ = V^-, \qquad Q^+ = Q^- + \Theta \quad \text{at} \quad \hat{y} = 0. \tag{B20}$$

Conditions (B8) at the channel walls become,

$$U^+ = U^- = 0, \qquad \Theta_{\hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0, 1. \tag{B21}$$

We also assume that far ahead of the flame interface the hydrodynamic disturbances vanish, i.e.,

$$U^+ \to 0, \qquad V^+ \to 0, \qquad Q^+ \to 0 \quad \text{at} \quad \hat{y} \to \infty. \tag{B22}$$

In the adopted approximation the nonlinear term $(\Theta_{\hat{x}})^2$ figures only in (B19). Thus, the hydrodynamic quantities $U^{\pm}$, $V^{\pm}$, $Q^{\pm}$ may be expressed linearly in terms of the flame interface Θ, which appears in (B20).

For the sequel it is convenient to express the flame front configuration as a cosine-Fourier series (see (B21)),

$$\Theta(\hat{x}, s) = \Theta_0(s) + \sum_{n=1}^{\infty} \Theta_n(s)\cos(\pi n\hat{x}) \tag{B23}$$

where

$$\Theta_0(s) = \langle \Theta(\hat{x}, s) \rangle \tag{B24}$$

is the mean value of $\Theta(\hat{x}, s)$ over the channel cross-section.

The solution of system (B18) with boundary conditions (B21), (B22), expressed in terms of $\Theta_0(s)$, $\Theta_n(s)$, is written in complex notation (keeping in mind that the functions are real)

$$
\begin{aligned}
V_+ + \mathrm{i}U_+ &= \frac{1}{2}\sum_{n=1}^{\infty} \Theta_n(s) \exp\left[\pi n(-\hat{y} + \mathrm{i}\hat{x})\right] \\
-V_- + \mathrm{i}U_- &= \frac{1}{2}\sum_{n=1}^{\infty} \Theta_n(s) \exp\left[\pi n(\hat{y} + \mathrm{i}\hat{x})\right] - \sum_{n=1}^{\infty} \Theta_n(s)\cos(\pi n\hat{x}) \\
Q_\pm &= \pm\frac{1}{2}\sum_{n=1}^{\infty} \Theta_n(s)\cos(\pi n\hat{x})\exp(\mp\pi n\hat{y})
\end{aligned} \tag{B25}
$$

Hence,

$$
V^+(\hat{x}, 0, s) = \frac{1}{2}\sum_{n=1}^{\infty} \Theta_n(s)\cos(\pi n\hat{x}) = \tfrac{1}{2}\big(\Theta(\hat{x}, s) - \langle\Theta(\hat{x}, s)\rangle\big) \tag{B26}
$$

and (B19) yields the desired equation for the flame interface

$$
\Theta_s = \kappa\,\Theta_{\hat{x}\hat{x}} + \tfrac{1}{2}(\Theta_{\hat{x}})^2 + \tfrac{1}{2}(\Theta - \langle\Theta\rangle), \tag{B27}
$$

which should be considered jointly with the boundary conditions,

$$
\Theta_{\hat{x}}(0, s) = \Theta_{\hat{x}}(1, s) = 0. \tag{B28}
$$

Setting $2\kappa = \varepsilon$, $2\Theta = \Phi$, $2s = \tau$, $\hat{x} = \xi$, (B27) and (B28) reduce to equations (1.4). □