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Some new results on the flow of waxy crude oils in a loop

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Waxy crude oils are characterized by a large concentration of paraffin (a mixture of heavy hydrocarbons). For sufficiently low temperatures paraffin is partly crystallized and crystals tend to form a structure that is stable below some critical stress. The system can be modelled as a Bingham fluid with rheological parameters dependent on: (i) the fraction of crystallized paraffin, (ii) the aggregation degree of crystals. In previous papers various situations have been analysed for the flow of waxy crude crude oils in a loop in conditions such that all relevant quantities do not depend on the longitudinal coordinate along the pipe. The new factors here analysed are the paraffin deposition rate on the walls influencing the paraffin content in the flow and the boundary conditions, and the possibility of heat exchange with the medium that surrounds the pipe. This problem has two free boundaries: the boundary of the inner plug of the Bingham flow and the boundary of the solid paraffin layer at the wall. The mathematical model is formulated and well posedness is proved.

Keywords: Waxy crude oils; Bingham fluids; free boundary problems; diffraction problems.

1. Introduction

Waxy crude oils are mineral oils with high concentration of *paraffin* (a mixture of heavy hydrocarbons) which at low temperatures may precipitate as a wax phase. They are known to cause difficulties in handling and pipelining, especially when they are transported across arctic regions and cold oceans. At high temperatures they behave like Newtonian fluids, but if the temperature is lowered below some critical value their flow properties become distinctly non-Newtonian. As shown in [1], paraffin begins to crystallize when the equilibrium pressure and temperature is reached (cloud point), while, at a lower temperature (pour point), crystals begin to agglomerate entrapping the oil in a gel-like structure, changing radically the rheological parameters of the flow. Below the cloud point the presence of a yield stress can be detected, so that we are led to consider the system as an incompressible Bingham fluid (although with variable parameters). In this paper we will treat the solid part of the Bingham fluid as a rigid body, i.e. we suppose it undergoes no deformation. Actually, it would be better to consider the solid zone as an elastic solid with the appropriate material symmetry associated with the crystalline structure, even though this is a very difficult task. Indeed, treating the solid part of the fluid as a rigid body does not take into account the discontinuous change in the symmetry group associated with the material (see [13, 14]).

Many simplified situations, based on the available experimental investigations, have been studied in previous works [6, 8], [3–5]. Here we investigate the flow in a cylindrical pipe taking into account that solid paraffin may adhere to the walls (also an experimentally observed fact) and that there can be heat exchange with the surrounding medium.

The problem of paraffin deposition is quite complex. There are two important mechanisms for transporting precipitated paraffin to the pipewalls, namely molecolar diffusion and shear dispersion [9, 15]. It has been shown [12] that molecolar diffusion is the dominant process at the higher temperatures, whereas shear dispersion dominates at the lower ones. Here we deal only with the shear dispersion process in a loop.

Particles, such as paraffin crystals, suspended in the fluid and transported by the flow, are also subjected to radial displacement because of the presence of a velocity gradient (shear dispersion). The latter phenomenon leads to the formation of a solid paraffin layer at the pipe wall. Therefore, the parameters affecting the deposition rate by shear dispersion are the shear stress on the walls and the precipitated paraffin content. Being in a loop situation, the formation of a solid paraffin layer implies a loss in the concentration of paraffin in the fluid, a phenomenon not occurring in pipelines where the concentration can be considered always the same, even if a deposition process is taking place. We will develop a mathematical model in a general form and study the stationary flow both for the isothermal and non-isothermal case.

2. The physical model

As a first step we have to consider some physical assumptions:

- *Temperature below the pour point.* The temperature T is always below the pour point and it depends only on the radial coordinate (we have no convection)
- *Incompressible fluid.* We consider $\rho = \text{constant}$, ρ being the oil density, in the range of temperature and pressure we are going to consider.
- *Laminar Flow.* We assume, in agreement with the experimental data, that the Reynolds number is less than the threshold value of the turbulent flow.

Following [3] we introduce two nondimensional parameters which can describe the state of the crystalline component. They are crucial in determining the basic rheological parameters of the system. We denote by C_w the concentration of paraffin in the oil, i.e C_w represents the quantity (mass) of paraffin per unit volume. If the temperature is sufficiently low the concentration C_{sat} of dissolved paraffin is less than C_w. So we define $C_p = C_w - C_{\text{sat}}$ representing the mass of precipitated paraffin per unit volume. We suppose that $C_{\text{sat}} = C_{\text{sat}}(\overline{T})$, where $\overline{T}$ is the mean temperature over a cross section of the fluid, and we introduce the nondimensional parameter β

$$\beta = \frac{C_p}{C_w} \tag{1}$$

which is called the crystallization degree and expresses the fraction of precipitated paraffin. Obviously β takes values between 0 and 1. For temperatures below the pour point, we will have aggregated crystals. We introduce C_a for the concentration of aggregated crystals and we define

$$\alpha = \frac{C_a}{C_p} \tag{2}$$

α, called the aggregation degree, i.e. the fraction of aggregated crystals, taking values between 0 and 1. We suppose that

$$\alpha = \alpha(t). \tag{3}$$

Assuming α space-independent is quite reasonable in experimental loops where pumps mix the fluid over time intervals much smaller that the evolution time scale of α. From (1) we see that

$\beta = \beta(\overline{T}, C_w)$. This is clearly a simplifying approach (otherwise a crystallization kinetics should be considered). Still following [3] we will assume that the evolution of α is governed by

$$\dot{\alpha} = K_1(\overline{T})(1-\alpha) - K_2(\overline{T})\alpha\overline{W} \tag{4}$$

where K_1, K_2 are two known positive functions of $\overline{T}$ and $\overline{W}$ is the average power dissipated by the viscous forces (note that $\alpha = 1$ is the value at rest). Suppose that the radius of the pipe is R and take a section of length one. Denoting by $\sigma(t)$ the thickness of the solid paraffin layer adhering to the wall ($\delta(t) = R - \sigma(t)$ will be the reduced pipe radius), by ρ_p the paraffin density (we suppose $\rho = \rho_p$) and supposing that the concentration C_w is uniform in the flow we can write the simple equation

$$C_w \pi \delta^2(t) + \rho\pi \left[R^2 - \delta^2(t)\right] = C_{w0}\pi R^2 \tag{5}$$

which states that the sum of the paraffin mass in the solid layer on the wall and the paraffin mass in solution must be constant in time (C_{w0} is the initial concentration, corresponding to $\delta(0) = R$). From (5) we deduce the concentration as a function of δ

$$C_w = C_w(\delta) = \frac{C_{w0}R^2}{\delta^2} - \rho\left[\frac{R^2}{\delta^2} - 1\right]. \tag{6}$$

From (6) we see that if we suppose $0 < C_{\text{sat}} < C_{w0} < \rho$ we have $C_p = \beta C_w = 0$ when $\delta = R((\rho - C_{w0})/(\rho - C_{\text{sat}}))^{1/2}$. Bearing in mind the shear dispersion phenomenon in the loop, we can write the equation for the evolution of δ in the following way

$$\begin{cases} \dot{\delta} = -\lambda(|\tau_w|)C_w(\delta)\beta \\ \delta(0) = R \end{cases} \tag{7}$$

where τ_w is the stress at the boundary $\delta(t)$ and $\lambda(\xi)$ is a positive bounded nondecreasing function for $\xi > 0$ and $\lambda(0) = 0$.

2.1 *The mathematical problem*

We want to write the equations for the flow through a cylindrical pipe. As we said, due to the presence of a yield stress, the simplest scheme we can consider for the fluid is the one of a Bingham fluid in which the relation between the shear stress and the strain rate is

$$(\tau - \tau_0)_+ = \eta\dot{\gamma} \tag{8}$$

$\tau_0 > 0$ being the yield stress, $\eta > 0$ the Bingham viscosity and $\dot{\gamma}$ the strain rate (see [2] for more details). We will assume $\eta = \eta(\alpha, \beta)$ and $\tau_0 = \tau_0(\alpha, \beta)$. Denoting the radial coordinate by r, we can divide the pipe into three regions: the so-called plug $0 < r < s(t)$ where the fluid undergoes no deformation, the intermediate one $s(t) < r < \delta(t)$ in which $\dot{\gamma} > 0$, and the paraffin layer $\delta(t) < r < R$. Considering the laminar axial flow $\vec{v} = v(r,t)\vec{e}_z$ in the fluid region we have (see [4])

$$\rho\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\tau_0 + \eta\frac{\partial v}{\partial r}\right)\right] \qquad s(t) < r < \delta(t); \quad t > 0 \tag{9}$$

where ρ is the density, $\tau = -\tau_0 + \eta v_r$ is the shear stress and $-\partial p/\partial z = f_0$ is the driving pressure gradient which will be considered known, constant and positive. For what concerns the initial and boundary data, we have

$$s(0) = s_0 \qquad 0 < s_0 < R \tag{10}$$

$$v(r, 0) = v_0(r) \qquad s_0 < r < R \tag{11}$$

$$v(\delta(t), t) = 0 \qquad t > 0 \tag{12}$$

$$\frac{\partial v}{\partial r}(s(t), t) = 0 \qquad t > 0 \tag{13}$$

$$\frac{\partial v}{\partial t}(s(t), t) = \frac{1}{\rho}\left[f_0 - \frac{2\tau_0}{s(t)}\right] \qquad t > 0 \tag{14}$$

where (12) and (13) express respectively the no-slip condition at $r = \delta(t)$ and the absence of strain rate at $r = s(t)$, s_0 is the initial position of the inner core plug, $v_0(r)$ is the initial velocity of the fluid phase, while (14) represents the momentum balance for a unit length portion of the rigid core. The average power density dissipated by the viscous force over a cross section is (see [4])

$$\overline{W}(t) = \frac{2}{\delta^2(t)} \int_{s(t)}^{\delta(t)} r \frac{\partial v}{\partial r}\left[-\tau_0 + \eta \frac{\partial v}{\partial r}\right] \mathrm{d}r.$$

Hence (4) becomes

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = K_1(1-\alpha) - K_2 \frac{2\alpha}{\delta^2(t)} \int_{s(t)}^{\delta(t)} r \frac{\partial v}{\partial r}\left[-\tau_0 + \eta \frac{\partial v}{\partial r}\right] \mathrm{d}r \tag{15}$$

with the initial condition $\alpha(0) = \alpha_0 \in (0, 1)$. The equation for δ is still

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\lambda(|\tau_w|) C_w(\delta)\beta \tag{16}$$

with the initial condition $\delta(0) = R$. If heat exchange with the surrounding medium is taking place we have also

$$\rho c_o \frac{\partial T}{\partial t} - k_o\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] = 0 \qquad t > 0 \quad 0 < r < \delta(t). \tag{17}$$

$$\rho c_p \frac{\partial T}{\partial t} - k_p\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] = 0 \qquad t > 0 \quad \delta(t) < r < R. \tag{18}$$

Here ρ_i, c_i and k_i are the density, the specific heat and the thermal conductivity of the oil ($i = o$) and of the paraffin ($i = p$). Typical values for the thermal moduli of the oil are $k_o = 0.134 \times 10^{-3}$ (W K^{-1}/cm^{-1}), $c_o = 1920 \times 10^{-3}$ (J K^{-1} gr^{-1}). In (17) we are tacitly assuming that the internal energy depends only on temperature. The initial, boundary and interface conditions are

$$[T]_{|r=\delta} = 0 \qquad \left[k\frac{\partial T}{\partial r}\right]_{|r=\delta} = 0 \tag{19}$$

$$T(R,t) = T_e \qquad\qquad T(r,0) = T_0(r) \tag{20}$$

where

$$k = \begin{cases} k_o & 0 \leqslant r < \delta(t) \\ k_w & \delta(t) < r < R \end{cases} \tag{21}$$

with [.] denoting jumps across the interface, and $T(R,0) = T_0(R)$. Here T_e is the wall temperature and $T_0(r)$ is the initial temperature of the system (both below the pour point)

System (9)–(20), referred to as problem (P), is the mathematical formulation of the flow in its general form. This is a free boundary problem where the unknown boundaries are $s(t)$ and $\delta(t)$ and the other unknowns are v, α and T. We will study a stationary motion both for the isothermal and non-isothermal case, showing that both problems are well posed. Of course the isothermal situation is much easier because the thermal problem disappears and the problem is reduced to equations (9)–(16).

3. Quasisteady approximation: isothermal case

In this first simplified situation T is supposed constant and below the pour point. The thermal problem is no longer involved in our model and problem (P) is reduced to equations (9)–(16). We can consider K_1, K_2 and C_{sat} constant. What we intend to do is to perform a quasisteady approximation for v. This means that, under appropriate hypotheses, the velocity field v will be given by the steady solutions of problem (9), (12)–(14) with α and δ obtained by solving the system of ODEs (15) and (16). Of course this is possible only if the time scale of the evolution of v is considerably larger than the ones of α and δ. Let us make some assumptions on the data:

1. $C_{\text{sat}} = \text{constant}, \qquad 0 < C_{\text{sat}} < C_{w0} < \rho$
2. $K_1, K_2 > 0, \quad \tau_0 = \tau_0(\alpha,\beta), \ \eta = \eta(\alpha,\beta)$
3. $\tau_0 \in C^1([0,1]\times[0,1]), \ 0 < \tau_{0m} \leqslant \tau_0(\alpha,\beta) \leqslant \tau_{0M}$
4. $\eta \in C^1([0,1]\times[0,1]), \quad 0 < \eta_m \leqslant \eta(\alpha,\beta) \leqslant \eta_M$
5. $0 \leqslant \tau_{0\alpha}, \qquad 0 \leqslant \tau_{0\beta}$
6. $0 \leqslant \eta_\alpha, \qquad 0 \leqslant \eta_\beta$
7. $\lambda(|\tau_w|)$ Lipschitz continuous and nondecreasing, $\ 0 \leqslant \lambda(|\tau_w|) \leqslant N; \quad \lambda = 0 \Longleftrightarrow \tau_w = 0$
8. $f_0 > 2\tau_{0M}/R$.

Assumptions 5 and 6 state that the rheological parameters η and τ_0 are nondecreasing functions of the parameters α and β. Physically this means that the viscosity and the yield stress increase if the quantity of crystallized or aggregated paraffin is increasing. Assumption 8 ensures that the system never comes to a stop. From equation (6) we see that $C_p(\delta) = C_w(\delta) - C_{\text{sat}} \in C^\infty[\xi, R]$ (where $\xi = R[(\rho - C_{w0})/(\rho - C_{\text{sat}})]^{1/2}$), $0 \leqslant C_p \leqslant \rho - C_{\text{sat}}$ and $(2\rho - 2C_{\text{sat}})/R \leqslant (dC_p/d\delta) \leqslant (2\rho - 2C_{\text{sat}})/\xi$. In order to justify the quasisteady approximation, we trasform problem (P) in a nondimensional form by putting

$$\begin{aligned} v &= \tilde{v}v^*, \qquad r = \tilde{r}R, \qquad t = \tilde{t}t_\delta, \qquad \eta = \tilde{\eta}\eta^*; \\ \tau &= \tilde{\tau}\tau_0^*, \qquad \tau_0 = \tilde{\tau_0}\tau_0^*, \qquad \lambda(|\tau_w|) = \tilde{\lambda}(|\tilde{\tau}_w|)N \end{aligned} \tag{22}$$

where

$$\eta^* = \eta_M, \qquad v^* = \frac{f_0 R^2}{\eta^*}, \qquad t_\delta = \frac{R}{N\rho}, \qquad \tau_0^* = f_0 R \tag{23}$$

and we define

$$t_\alpha = \frac{1}{K_1}, \qquad t_v = \frac{R^2\rho}{\eta_M}. \tag{24}$$

Note that in choosing t_δ as in (23) we have selected the time scale of the evolution of δ. In the following, since we will always work with nondimensional variables, we omit the tilda. Problem (P) becomes

$$\frac{t_v}{t_\delta}\cdot\frac{\partial v}{\partial t} = 1 + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\tau_0 + \eta\frac{\partial v}{\partial r}\right)\right] \quad s(t) < r < \delta(t) \quad t > 0 \tag{25}$$

$$s(0) = s_0 \qquad 0 < s_0 < 1 \tag{26}$$

$$v(r,0) = v_0(r) \qquad s_0 < r < 1 \tag{27}$$

$$v(\delta(t),t) = 0 \qquad t > 0 \tag{28}$$

$$\frac{\partial v}{\partial r}(s(t),t) = 0 \qquad t > 0 \tag{29}$$

$$\frac{t_v}{t_\delta}\cdot\frac{\partial v}{\partial t}(s(t),t) = \left[1 - \frac{2\tau_0}{s(t)}\right] \qquad t > 0 \tag{30}$$

$$\frac{t_\alpha}{t_\delta}\cdot\frac{\mathrm{d}\alpha}{\mathrm{d}t} = (1-\alpha) - K\frac{2\alpha}{\delta^2(t)}\int_{s(t)}^{\delta(t)} r\frac{\partial v}{\partial r}\left[-\tau_0 + \eta\frac{\partial v}{\partial r}\right]\mathrm{d}r \tag{31}$$

$$\alpha(0) = \alpha_0 \qquad 0 < \alpha_0 < 1 \tag{32}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\lambda\left(|\tau_w|\right)F_0(\delta) \tag{33}$$

$$\delta(0) = 1 \tag{34}$$

where

$$K = \frac{K_2 f_0^2 R^2}{K_1\eta_M}; \qquad F_0(\delta) = \frac{1}{\rho}\left[\frac{(C_{w_0} - \rho) + \delta^2(\rho - C_{\mathrm{sat}})}{\delta^2}\right]. \tag{35}$$

Let us refer to problem (25)–(34) as problem (P_d). When

$$\frac{t_v}{t_\delta} = \frac{\rho^2 RN}{\eta_M} \ll 1 \tag{36}$$

it makes sense to neglect the terms containing $\partial v/\partial t$ in (25) and (30). If we consider typical values $C_p = 0.1\ \ \mathrm{g\,cm^{-3}}$, $|\dot{\delta}| = 0.8\times 10^{-4}$ and $\eta_M = 10$ Po (Jackson-Hutton oil) we have

$$N = 0.8\times 10^{-3}\quad \frac{\mathrm{cm}^4}{\mathrm{g}\cdot\mathrm{s}} \tag{37}$$

and consequently, for $R = 25$ cm and $\rho = 0.8$ g cm^{-3},

$$\frac{t_v}{t_\delta} = 2.56 \times 10^{-3} \tag{38}$$

which confirms the validity of the quasisteady flow approximation. By simple calculations, we obtain

$$v(r,t) = -\frac{1}{4\eta}\left[r^2 - \delta^2(t)\right] + \frac{\tau_0}{\eta}\left[r - \delta(t)\right] \tag{39}$$

$$s(t) = 2\tau_0 \tag{40}$$

with $\tau_0 = \tau_0(\alpha, \beta)$, $\eta = \eta(\alpha, \beta)$, and the pair $(\alpha(t), \delta(t))$ given by the solution of the system of ODEs

$$\dot{\alpha}\frac{t_\alpha}{t_\delta} = (1-\alpha) - K\alpha(24\eta\delta^2)^{-1}\left(3\delta^4 - 4s\delta^3 + s^4\right); \qquad \alpha(0) = \alpha_0 \tag{41}$$

$$\dot{\delta} = -\lambda(\delta/2)F_0(\delta); \qquad \delta(0) = 1. \tag{42}$$

Note that in (41) W is obtained from (31) with v given by (39) and in (42) τ_w is obtained by $\tau_w = -\tau_0 + \eta v_{r|r=\delta}$ with v given by (39). In order to show well posedness of the quasistationary problem it remains to prove that the system (41)–(42) has a unique global solution $(\alpha(t), \delta(t))$. ODE (42) can be integrated separately. Since we are working with nondimensional variables we have

$$0 < \frac{\tau_{0m}}{f_0 R} \leqslant \tau_0 \leqslant \frac{\tau_{0M}}{f_0 R} \qquad\qquad 0 < \frac{2\tau_{0m}}{f_0 R} \leqslant s \leqslant s_1 := \frac{2\tau_{0M}}{f_0 R}.$$

Define

$$-\lambda(\delta/2)F_0(\delta) = \Lambda(\delta).$$

We have

- $\dot{\delta} = \Lambda(\delta)$
- $\Lambda \in C[\delta', 1]$; $\qquad \delta' = ((\rho - C_{w0})/(\rho - C_{\text{sat}}))^{1/2}$
- $\Lambda(\delta') = 0$
- $0 \leqslant |\Lambda| \leqslant 1$
- Λ Lipschitz continuous in $[\delta', 1]$.

As a consequence of these properties ODE (42) has a unique solution $\delta(t)$ that belongs to $C^1[0, \infty)$ with $\delta' < \delta(t) \leqslant 1$. If we now put it in (41), we see that the right-hand side of the ODE is a bounded Lipschitz-continuous function in α. Thus we have a unique solution $\alpha(t) \in C^1[0, \infty)$, with $0 < \alpha(t) < 1$. A condition that guarantees that the system does not come to a stop in a finite time is the following:

$$f_0 \geqslant \frac{2\tau_{0M}}{R\delta'}. \tag{43}$$

Indeed, this condition ensures that the solid layer will never touch the inner core of the Bingham fluid.

3.1 *Continuous dependence upon the data*

Consider the system (41)–(42). We introduce

- $\epsilon \in [0, 1)$
- $\hat{\Lambda}(\delta)$ Lipschitz continuous and bounded in $[\delta', 1]$
- $\hat{\tau}_0(\alpha, \beta) \in C^1([0, 1] \times [0, 1])$ with the same properties as τ_0
- $\hat{\eta}(\alpha, \beta) \in C^1([0, 1] \times [0, 1])$ with the same properties as η
- $\hat{K} > 0; \quad \hat{t}_\alpha > 0; \quad \hat{t}_\delta > 0$
- $\hat{\alpha}_0 \in (0, 1)$.

We write system (41)–(42) for the new data and find a new solution $(\hat{\alpha}, \hat{\delta})$ that satisfies

$$\frac{\hat{t}_\alpha}{\hat{t}_\alpha} \cdot \dot{\hat{\alpha}} = (1 - \hat{\alpha}) - \hat{K}\hat{\alpha}(24\hat{\eta}\hat{\delta}^2)^{-1}(3\hat{\delta}^4 - 8\hat{\tau}_0\hat{\delta}^3 + 16\hat{\tau}_0^4); \quad \hat{\alpha}(0) = \hat{\alpha}_0 \tag{44}$$

$$\dot{\hat{\delta}} = \hat{\Lambda}(\hat{\delta}); \qquad \hat{\delta}(0) = 1 - \epsilon. \tag{45}$$

Note that the choice $\hat{\delta}(0) = 1 - \epsilon$ means that, in the dimensional problem, the initial datum for the reduced pipe radius is given by some $\hat{R}$ with $\hat{R} \in (0, R]$. Let us fix a time $\bar{t} > 0$. We have

$$|\delta(t) - \hat{\delta}(t)| \leqslant \epsilon + t\|\Lambda - \hat{\Lambda}\|_{[\delta',1]} + \int_0^t \hat{L}|\delta(\tau) - \hat{\delta}(\tau)|\,\mathrm{d}\tau \quad t \in [0, \bar{t}] \tag{46}$$

where $\hat{L}$ is the Lipschitz constant of $\hat{\Lambda}$ and $\|.\|$ is the sup norm defined by

$$\|f\|_A = \sup_{x \in A} |f(x)| \tag{47}$$

where A is the domain of definition of f. Applying Gronwall's lemma we have

$$\|\delta - \hat{\delta}\|_{\bar{t}} \leqslant C(\bar{t})\{\|\Lambda - \hat{\Lambda}\|_{[\delta',1]} + \epsilon\} \tag{48}$$

where $C(\bar{t})$ is a constant depending on $\bar{t}$. Proceeeding in the same way for α and $\hat{\alpha}$ we obtain

$$\begin{aligned} \|\alpha - \hat{\alpha}\|_{\bar{t}} \leqslant C(\bar{t}) \Big\{ &\|\eta - \hat{\eta}\|_{[0,1]\times[0,1]} + \|\tau_0 - \hat{\tau}_0\|_{[0,1]\times[0,1]} \\ &+ |K - \hat{K}| + \|\delta - \hat{\delta}\|_{\bar{t}} + |\alpha_0 - \hat{\alpha}_0| + \left|\frac{\hat{t}_\alpha}{\hat{t}_\delta} - \frac{t_\alpha}{t_\delta}\right| \Big\}. \end{aligned} \tag{49}$$

Inequalities (48) and (49) ensure continuous dependence for every fixed time interval $[0, \bar{t}]$.

4. Quasisteady approximation: non-isothermal case

Let us come back to problem (P), i.e. equations (9)–(20). Now we assume that there is heat exchange with the outer medium. The rheological parameters are $\eta = \eta(\alpha, \beta)$ and $\tau_0 = \tau_0(\alpha, \beta)$ with $\alpha = \alpha(t)$ and β given by (1). We suppose that the assumptions 3–7 of Section 3.1 are still valid and assume

- C_{sat} Lipschitz continuous in $\overline{T}$ with Lipschitz constant L_s
- $0 < C_{\min} \leqslant C_{\text{sat}}(\overline{T}) \leqslant C_{\max} < \rho$ in the range of $\overline{T}$.

We assume again hypotesis (36) and consider the stationary motion of the fluid. We write the explicit form for the velocity field and for the inner core boundary, now in a dimensional form:

$$v(r,t) = -\frac{f_0}{4\eta}\left[r^2 - \delta^2(t)\right] + \frac{\tau_0}{\eta}\left[r - \delta(t)\right] \tag{50}$$

$$s(t) = \frac{2\tau_0}{f_0} \tag{51}$$

with $\eta = \eta(\alpha, \beta(\overline{T}, C_w))$ and $\tau_0 = \tau_0(\alpha, \beta(\overline{T}, C_w))$. For what concerns α we have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = K_1(\overline{T})(1-\alpha) - K_2(\overline{T})\frac{\alpha f_0^2}{24\eta\delta^2}\left[3\delta^4 - 4s\delta^3 + s^4\right];\ \alpha(0) = \alpha_0 \tag{52}$$

where we suppose that K_1, K_2 are positive bounded smooth functions of $\overline{T}$. The equations for δ and T are coupled:

$$\begin{cases} \dot{\delta} = -\lambda(f_0\delta/2)[C_w(\delta) - C_{\text{sat}}(\overline{T})] =: \Psi(\delta, \overline{T}) & \\ \delta(0) = R_1 & 0 < R_1 < R \end{cases} \tag{53}$$

$$\begin{cases} \rho c_o T_t - k_o\left(T_{rr} + r^{-1}T_r\right) = 0 & t > 0 \quad 0 < r < \delta(t) \\ \rho c_p T_t - k_p\left(T_{rr} + r^{-1}T_r\right) = 0 & t > 0 \quad \delta(t) < r < R \\ [T]_{|r=\delta} = 0 & t > 0 \\ \left[k\partial T/\partial \vec{n}\right]_{|r=\delta} = 0 & t > 0 \\ T(R,t) = 0 & t > 0 \\ T(r,0) = T_0(r) & 0 \leqslant r \leqslant R \end{cases} \tag{54}$$

with

$$\overline{T}(t) = \frac{2}{\delta^2(t)}\int_0^{\delta(t)} T(r,t)r\,\mathrm{d}r \tag{55}$$

Here we put $T_e = 0$. For simplicity we have supposed that we have an initial layer of paraffin, i.e. $\delta(0) = R_1 < R$ with $s(0) < R_1 < R$. The latter inequality ensures that the system can move. Problem (54) is a parabolic diffraction problem (see [11], page 224).

4.1 *Existence and uniqueness*

We will prove the following theorem.

THEOREM 1 Suppose $s_M = (2\tau_{0_M}/f_0) < R_1 < R$. For every time $t_0 > 0$ there exist two functions $\delta(t)$, $T(r,t)$ such that:

1. $\delta(t) \in C^1[0, t_0]$ with $s_M < \delta(t) \leqslant R_1$ in $[0, t_0]$

2. $T(r,t) \in C(\bar{\Omega} \times [0, t_0])$, $\Omega = \{0 \leqslant r < R\}$
3. $T(r,t) \in C^{2,1}(Q_o)$, $Q_o = \{0 \leqslant r < \delta(t), 0 < t < t_0\}$
4. $T(r,t) \in C^{2,1}(Q_p)$, $Q_p = \{\delta(t) < r < R, 0 < t < t_0\}$
5. T is continuously differentiable with respect to r up to δ
6. $\delta(t)$ and $T(r,t)$ satisfy all the equations of the system (53)–(54).

This theorem ensures global existence and uniqueness of the quasisteady non-isothermal problem. The solution will be given by (50), (51), where $\alpha(t)$ is obtained integrating (52) with $\delta(t)$ and $T(r,t)$ given by Theorem 1. The hypothesis $s_M < R_1$ guarantees that the inner core and the solid layer will never touch. If we remove this hypothesis we can no longer assert that the system cannot come to a stop in a finite time. Let us prove Theorem 1. We choose

$$\bar{t} = \frac{R_1 - s_M}{A} \tag{56}$$

with $A = N(\rho - C_{\min}) = \sup|\lambda C_w \beta|$ and $s_M = (2\tau_{0_M}/f_0)$ and consider the space $C^1[0,\bar{t}]$ endowed with the norm

$$\|\delta\|_{1,\bar{t}} = \sup_{t\in[0,\bar{t}]} |\delta(t)| + \sup_{t\in[0,\bar{t}]} |\dot{\delta}(t)|. \tag{57}$$

Let us fix a $\delta \in C^1[0,\bar{t}]$ with $\dot{\delta}(t) \in [-A, 0]$ for all $t \in [0,\bar{t}]$. With such a choice, following [10], we know we have a unique classical solution $T(r,t)$ of problem (54), where by classical we mean that $T(r,t)$ has the properties 2–5 of Theorem 1 and satisfies all the equations of (54). Further, we have

$$|\Psi(\delta(t'), \overline{T}(t')) - \Psi(\delta(t''), \overline{T}(t''))| \leqslant L|t' - t''| \quad \forall\ t',\ t'' \in [0,\bar{t}] \tag{58}$$

where Ψ is defined in (53) and L is a constant that depends only upon A, ρ, R, R_1, N, C_{w0}, $C_{\min}$, L_s. We denote by Γ the subset of $C^1[0,\bar{t}]$ made by functions δ with the properties:

1. $\delta(0) = R_1$
2. $-A \leqslant \dot{\delta}(t) \leqslant 0$ forall $t \in [0,\bar{t}]$
3. $|\dot{\delta}(t') - \dot{\delta}(t'')| \leqslant L|t' - t''|$ forall $t',\ t'' \in [0,\bar{t}]$.

The subset Γ is closed, convex and compact with respect to the norm (57). Let us take a $\delta \in \Gamma$. We define

$$\tilde{\delta}(t) = R_1 + \int_0^t \Psi(\delta(\tau), \overline{T}(\tau))\, \mathrm{d}\tau \qquad t \in [0,\bar{t}]. \tag{59}$$

It is easy to see that $\tilde{\delta}(t) \in \Gamma$; thus the operator φ such that $\varphi(\delta) = \tilde{\delta}$ maps Γ into itself. If we show the continuity of φ in the norm (57), then, by the Schauder fixed-point theorem, we have the existence of a fixed point δ in the time interval $[0,\bar{t}]$. Let us fix $R_0 \in (0, s_M)$. We search for a coordinate transformation $y = y(\delta, r)$ with the following properties:

1. $y(\delta, r) \quad \in \quad C^\infty([s_M, R_1] \times [0, R])$

2. $y(\delta, r) = r \qquad r \in [0, R_0]$
3. $y(\delta, \delta) = R_1$
4. $y(\delta, R) = R$
5. $y_r(\delta, r) > 0$ in $[s_M, R_1] \times [0, R]$
6. The inverse $r = r(\delta, y) \in C^{\infty}([s_M, R_1] \times [0, R])$.

When $\delta(t)$ is fixed in Γ such a transformation "rectifies" the surface $r = \delta(t)$ into $y = R_1$ and in the inner cylinder of radius R_0 the transformation is simply the identity. We take δ_1 and δ_2 in Γ and a smooth function $y = y(\delta, r)$ that has the above properties. In the following C will represent a constant that depends only upon the data of the problem. We have

$$\begin{aligned} \|\tilde{\delta}_1 - \tilde{\delta}_2\|_t &\leqslant Ct\|\delta_1 - \delta_2\|_t \\ &+ C\int_0^t \left| \int_0^{\delta_1} T_1(r, \tau) r \, \mathrm{d}r - \int_0^{\delta_2} T_2(r, \tau) r \, \mathrm{d}r \right| \mathrm{d}\tau \end{aligned} \tag{60}$$

where $\|.\|_t = \|.\|_{[0,t]}$. We transform the integral for T_1 in the right-hand side of inequality (60) by means of $y = y(\delta_1, r)$ and the integral for T_2 by means of $y = y(\delta_2, r)$. We obtain

$$\begin{aligned} \|\tilde{\delta}_1 - \tilde{\delta}_2\|_t &\leqslant Ct\|\delta_1 - \delta_2\|_t \\ &+ C\int_0^t \int_0^{R_1} \left| \hat{T}_1(y, \tau) - \hat{T}_2(y, \tau) \right| \mathrm{d}y \end{aligned} \tag{61}$$

where $\hat{T}_i(y, t) = T_i(r(\delta_i, y), t)$, $i = 1, 2$. We put $\hat{W} = \hat{T}_1 - \hat{T}_2$:

$$\begin{aligned} \frac{\partial \hat{W}}{\partial t} &= \sigma G(\delta_1, y) \frac{\partial^2 \hat{W}}{\partial y^2} + \left[\dot{\delta}_1 H(\delta_1, y) + \sigma F(\delta_1, y)\right] \frac{\partial \hat{W}}{\partial y} \\ &+ \left\{ \sigma \frac{\partial^2 \hat{T}_2}{\partial y^2} [G(\delta_1, y) - G(\delta_2, y)] + \sigma \frac{\partial \hat{T}_2}{\partial y} [F(\delta_1, y) - F(\delta_2, y)] \right. \\ &\left. + \frac{\partial \hat{T}_2}{\partial y} \dot{\delta}_1 [H(\delta_1, y) - H(\delta_2, y)] + \frac{\partial \hat{T}_2}{\partial y} H(\delta_2, y) \left[\dot{\delta}_1 - \dot{\delta}_2\right] \right\} \end{aligned}$$

where

$$\sigma = \begin{cases} [k_o/\rho c_o] & 0 < y < R_1 \\ \left[k_p/\rho c_p\right] & R_1 < y < R \end{cases}$$

$$H(\delta, y) = \frac{\partial r}{\partial \delta}(\delta, y) \frac{\partial y}{\partial r}(\delta, r(\delta, y))$$

$$G(\delta, y) = \left[\frac{\partial y}{\partial r}(\delta, r(\delta, y))\right]^2$$

$$F(\delta, y) = \left[\frac{\partial^2 y}{\partial r^2}(\delta, r(\delta, y)) + \frac{1}{r(\delta, y)} \frac{\partial y}{\partial r}(\delta, r(\delta, y))\right].$$

The problem satisfied by $\hat{W}(y,t)$ is

$$\frac{\partial \hat{W}}{\partial t} = \sigma G(\delta_1, y)\frac{\partial^2 \hat{W}}{\partial y^2} + \left[\dot{\delta}_1 H(\delta_1, y) + \sigma F(\delta_1, y)\right]\frac{\partial \hat{W}}{\partial y}$$
$$+E_1(y,t)\left[\delta_1 - \delta_2\right] + E_2(y,t)\left[\dot{\delta}_1 - \dot{\delta}_2\right] \tag{62}$$

with the initial, boundary and interface data given by

$$\hat{W}(y,0) = 0 \qquad \hat{W}(R,t) = 0 \qquad \left[k\frac{\partial \hat{W}}{\partial y}\right]_{y=R_1} = 0 \qquad \left[\hat{W}\right]_{y=R_1} = 0 \tag{63}$$

where

$$E_1(y,t) = \sigma\frac{\partial^2 \hat{T}_2}{\partial y^2}\frac{\partial G}{\partial \delta}(\xi_1, y) + \sigma\frac{\partial \hat{T}_2}{\partial y}\frac{\partial F}{\partial \delta}(\xi_2, y) + \frac{\partial \hat{T}_2}{\partial y}\frac{\partial H}{\partial \delta}(\xi_3, y)\dot{\delta}_1$$
$$E_2(y,t) = \frac{\partial \hat{T}_2}{\partial y}H(\delta_2, y);$$
$$\inf\{\delta_1(t), \delta_2(t)\} < \xi_j(t) < \sup\{\delta_1(t), \delta_2(t)\} \qquad t \in [0, \bar{t}].$$

By the hypotheses 1–6 on $y(\delta, r)$ and by Lemma 7 of [10], E_1 and E_2 are two bounded functions with a jump discontinuity on the interface $y = R_1$. We write problem (62)–(63) in Cartesian coordinates, i.e. we use the transformation $y = \sqrt{x_1^2 + x_2^2}$ and the new function $W(x,t) = W(x_1, x_2, t) = \hat{W}(y,t)$, and come to

$$\frac{\partial W}{\partial t} = \sum_{i,j=1}^{2} a_{ij}(x,t)\frac{\partial^2 W}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x,t)\frac{\partial W}{\partial x_i}$$
$$+f_1(x,t)(\delta_1 - \delta_2) + f_2(x,t)(\dot{\delta}_1 - \dot{\delta}_2) = \mathcal{L}(W) + f(x,t) \tag{64}$$

with

$$\mathcal{L} = \sum_{i,j=1}^{2} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i(x,t)\frac{\partial}{\partial x_i} \tag{65}$$

$$f(x,t) = f_1(x,t)(\delta_1 - \delta_2) + f_2(x,t)(\dot{\delta}_1 - \dot{\delta}_2) \tag{66}$$

where a_{ij}, b_i, f_i are bounded functions in $\{0 \leqslant y < R\} \times (0, \bar{t})$ and have a jump discontinuity in $y = R_1$ (note that $\mathcal{L}$ is simply the Laplacian when $0 \leqslant y \leqslant R_0$). Let us consider the sets

- $Q_{\bar{t}} = \{(x,t): \quad 0 \leqslant y < R, \quad t \in (0, \bar{t})\}$
- $Q_{o\bar{t}} = \{(x,t): \quad 0 \leqslant y < R_1, \quad t \in (0, \bar{t})\}$
- $Q_{p\bar{t}} = \{(x,t): \quad R_1 < y < R, \quad t \in (0, \bar{t})\}$
- $\Omega_o = \{(x,t): \quad 0 \leqslant y < R_1, \quad t = 0\}$
- $\Omega_p = \{(x,t): \quad R_1 < y < R, \quad t = 0\}$
- $\Gamma_{\bar{t}} = \{(x,t): \quad y = R_1, \quad t \in (0, \bar{t}]\}$.

$W(x,t)$ is a classical solution of problem (64) in both the domains $Q_{i\bar{t}}$ (i.e $W(x,t) \in C^{2,1}(Q_{i\bar{t}}) \cap C(\overline{Q}_{i\bar{t}})$, $i = o, p$) and $W = 0$ in the parabolic boundary of $Q_{\bar{t}}$. Following [11: p. 13], we put

$$W(x,t) = v(x,t)\mathrm{e}^{\lambda t}$$

for a certain $\lambda > 0$. The new function $v(x,t)$ satisfies

$$v_t - a_{ij}(x,t)v_{x_i x_j} - b_i(x,t)v_{x_i} + \lambda v = f(x,t)\mathrm{e}^{-\lambda t}. \tag{67}$$

Select $t_1 \in (0, \bar{t})$. There are three possible cases:

1. $v(x,t) \leqslant 0$ in $\overline{Q}_{t_1}$
2. $0 < \sup_{Q_{t_1}} v(x,t) \leqslant \max_{\Gamma_{t_1}} v(x,t) = v(x_0,t_0) \qquad (x_0,t_0) \in \Gamma_{t_1}$
3. $0 < \sup_{Q_{t_1}} v(x,t) \leqslant v(x_0,t_0) \qquad (x_0,t_0) \in (\Omega_o \cup \Omega_p) \times (0, t_1]$.

In case 3, in (x_0,t_0), we have $v_t \geqslant 0$, $v_{x_i} = 0$ and $-a_{ij}v_{x_i x_j} \geqslant 0$, i.e.

$$\lambda v(x_0,t_0) \leqslant f(x_0,t_0)\mathrm{e}^{-\lambda t_0}$$

which becomes

$$W(x,t_1) \leqslant \sup_{Q_{t_1}} \left\{ \frac{f(x,t)\mathrm{e}^{\lambda(t_1 - t)}}{\lambda} \right\}. \tag{68}$$

In case 2 we have that $v_{x_i}(x_0,t_0) = 0$ ($\partial v/\partial y$ keeps its sign in passing through $y = R_1$) and (68) still holds. In the end we get

$$W(x,t_1) \leqslant \max \left\{ 0; \sup_{Q_{t_1}} \left[\frac{f(x,t)\mathrm{e}^{\lambda(t_1 - t)}}{\lambda} \right] \right\} \qquad \forall t_1 \in [0, \bar{t}] \tag{69}$$

and analogously, considering a point of least nonpositive value,

$$W(x,t_1) \geqslant \min \left\{ 0; \inf_{Q_{t_1}} \left[\frac{f(x,t)\mathrm{e}^{\lambda(t_1 - t)}}{\lambda} \right] \right\} \qquad \forall t_1 \in [0, \bar{t}]. \tag{70}$$

Inequalities (69) and (70) can be coupled:

$$|W(x,t_1)| \leqslant \sup_{Q_{t_1}} \frac{|f(x,t)|}{\lambda}\mathrm{e}^{\lambda t_1} \qquad \forall t_1 \in [0, \bar{t}] \tag{71}$$

which is the estimate we need to conclude our demonstration. Inequality (71) states that in $\{0 \leqslant y < R\} \times (0,t)$, with $t \in (0, \bar{t}]$:

$$|W(x,t)| = |\hat{T}_1(y,t) - \hat{T}_2(y,t)| \leqslant \frac{C\mathrm{e}^{\lambda t}}{\lambda} \|\delta_1 - \delta_2\|_{1,t}. \tag{72}$$

If we put (72) in (61) we have

$$\|\tilde{\delta}_1 - \tilde{\delta}_2\|_t \leqslant Ct\|\delta_1 - \delta_2\|_t + \frac{Cte^{\lambda t}}{\lambda}\|\delta_1 - \delta_2\|_{1,t} \tag{73}$$

and proceeding in the same way for the derivatives of δ_1 and δ_2,

$$\|\dot{\tilde{\delta}}_1 - \dot{\tilde{\delta}}_2\|_t \leqslant C\|\delta_1 - \delta_2\|_t + \frac{Ce^{\lambda t}}{\lambda}\|\delta_1 - \delta_2\|_{1,t}. \tag{74}$$

Now, coupling (73) and (74) and setting $\lambda = t^{-1/2}$, we finally get

$$\|\tilde{\delta}_1 - \tilde{\delta}_2\|_{1,t} \leqslant f(t)\|\delta_1 - \delta_2\|_{1,t} \tag{75}$$

where $f(t)$ is a nondecreasing function of time. It is easy to see that for t sufficiently small $f(t) < 1$. Inequality (75) shows that φ is continuous in the norm (57) and that φ is a contraction in a sufficiently small time interval. Our argument can be repeated up to time $\bar{t}$, showing existence and uniqueness in $[0, \bar{t}]$, hence the proof of Theorem 1 up to time $\bar{t}$. We have

$$\delta(\bar{t}) = R_1 - \int_0^{\bar{t}} \Psi(\delta(\tau), \overline{T}(\tau))\,d\tau > R_1 - A\bar{t} = s_M. \tag{76}$$

This means that we can find another time interval $[\bar{t}, \bar{\bar{t}}]$ in which the solution exists and is unique. If there were a time t_0 such that

$$\lim_{t\to t_0} \delta(t) = s_M \tag{77}$$

then

$$At_0 = R_1 - s_M = \int_0^{t_0} \Psi(\delta(\tau), \overline{T}(\tau))\,d\tau > At_0 \tag{78}$$

which is a contradiction that ensures that $\delta(t)$ cannot reach s_M in a finite time, hence the proof of Theorem 1.

5. Conclusions

We have proved the well posedness of the problem related to the flow of a waxy crude oil in a cylindrical pipe of radius R (laboratory loop). We have modelled the oil as an incompressible Bingham fluid, considering that deposition on the pipe walls can occur. We have considered both the isothermal and the non-isothermal case, showing conditions that guarantee that the system starts to move and never comes to a stop in a finite time.

Even though many simplifying hypotheses have been made, the mathematical model we have formulated is quite complex, especially in the non-isothermal case, where we have considered different thermal coefficients for the layer and for the oil.

Many interesting questions still remain open. A problem we have not treated here is the mechanism of molecular diffusion, which occurs in presence of large thermal gradients and affects, together with shear dispersion, the growth of the solid layer. Another interesting problem is the study of a model in which the aggergation degree and the crystallization degree depend on the radial coordinate. All these problems will be the subject of further research.

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