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A filtration problem through a heterogeneous porous medium

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The flow of a fluid through a heterogeneous porous medium is studied, assuming it is governed by a nonlinear Darcy law and Dirichlet boundary conditions. Under a general condition on the permeability we prove that the free boundary is locally a continuous curve in some local coordinates. We also prove the uniqueness of the reservoirs-connected solution.

Introduction

The dam problem has attracted the attention of many researchers over the last thirty years. However there still exist a number of unsolved questions related to this challenging problem, including the regularity of the free boundary and the uniqueness of the solution for flows in general heterogeneous porous media.

To begin with we would like to say a few words about the history of the problem; for brevity, we restrict ourselves to the steady state case with Dirichlet boundary conditions on the bottoms of the reservoirs.

First Baiocchi solved in [6] (see also [7] and [31]) the case of rectangular dams by using variational inequalities. For dams with general geometry a new approach was introduced by H. W. Alt in [3] for the heterogeneous case and by H. Brezis, D. Kinderlehrer, and G. Stampacchia in [12] for the homogeneous case. The two formulations are equivalent to

$$(\mathrm{P}_1)\begin{cases} \text{Find } (p,\chi)\in H^1(\Omega)\times L^\infty(\Omega) \text{ such that:}\\ \text{(i)}\quad p\geqslant 0,\quad 0\leqslant\chi\leqslant 1,\quad p(1-\chi)=0\quad \text{a.e. in } \Omega,\\ \text{(ii)}\quad p=\varphi\quad \text{on } S_2\cup S_3,\\ \text{(iii)}\quad \int_\Omega a(X)(\nabla p+\chi e)\cdot\nabla\xi\,\mathrm{d}X\leqslant 0,\quad e=(0,1),\\ \qquad \text{for all } \xi\in H^1(\Omega) \text{ with } \xi=0 \text{ on } S_3 \text{ and } \xi\geqslant 0 \text{ on } S_2,\end{cases}$$

where p is the fluid pressure, χ a function characterizing the wet part of the dam, $a(X)=(a_{ij}(X))$ is the permeability matrix of the medium and $X=(x,y)$. The existence of a solution (p,χ) was proved. Concerning the regularity of the free boundary, H. W. Alt proved in [4] that in the homogeneous case it is an analytic curve $y=\Phi(x)$. Uniqueness of the so-called S_3-connected solution was proved by J. Carrillo and M. Chipot in [14] and also by H.W. Alt and G. Gilardi in [5].

In [15], J. Carrillo and A. Lyaghfouri considered this problem, assuming the flow governed by the following nonlinear Darcy law (see [22]):

$$|v|^{m-1}v=-\nabla(p+y),\qquad m>0.$$

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They formulated the problem in terms of the hydrostatic head $u = p+y$ and were led to an extension of (P_1) corresponding to the general problem (P) given in the next section, with $\mathcal{A}(X, \xi) = |\xi|^{q-2}\xi$ and $q = 1 + 1/m$. Despite the nonlinearity, the authors showed that this problem is well posed and proved the existence of a solution, the continuity of the free boundary $y = \Phi(x)$ and the uniqueness of the S_3-connected solution in the case $n = 2$. For $n \geqslant \max(2, q)$, they proved existence and uniqueness of a minimal solution.

The case of a general heterogeneous dam of general geometry was formulated first in [3] by H. W. Alt who proved the existence of a solution and local Lipschitz continuity of the pressure. Moreover he gave a counterexample showing that χ may not be the characteristic function of the wet set $[p > 0]$. He also proved that

$$\operatorname{div}(a_{12}, a_{22}) \geqslant 0 \text{ in } \mathcal{D}'(\Omega) \;\Rightarrow\; \nabla\chi \cdot a(X)(e) \leqslant 0 \text{ in } \mathcal{D}'(\Omega).$$

In [21] and [32], the authors showed that if $a(X) = k(x, y)I_2$ with $\partial k/\partial y \geqslant 0$ in $\mathcal{D}'(\Omega)$, then the free boundary is a continuous curve $y = \Phi(x)$ and the S_3-connected solution is unique. These results were generalized by the second author in [27] to the case where

$$a(X) = \begin{pmatrix} a_{11}(X) & 0 \\ a_{21}(X) & a_{22}(X) \end{pmatrix} \quad \text{and} \quad \frac{\partial a_{22}}{\partial y} \geqslant 0 \text{ in } \mathcal{D}'(\Omega).$$

From the description of the heterogeneous case, the following natural question arises: Can we always describe the free boundary globally or at least locally as the graph of a continuous function, that is, is it necessarily of the form $y = \Phi(x)$ or $x = \Psi(y)$?

It is our purpose in this paper to address this issue in the more general case where the flow is governed by the nonlinear law

$$v = -\mathcal{A}(X, \nabla(p + y)).$$

Then by using a similar formulation to [15], i.e. $u = p + y$ and $g = 1 - \chi$, and by assuming that

$$\operatorname{div}(\mathcal{A}(X, e)) \geqslant 0 \text{ in } \mathcal{D}'(\Omega) \quad \text{and} \quad \mathcal{A}(X, e) \in C^1(\overline{\Omega}),$$

we give a positive answer to the above question.

The main new idea is the following: we remark that under the above assumption, the function g is nondecreasing along the orbits of the ordinary differential equation

$$X'(t) = \mathcal{A}(X(t), e),$$

which generalizes the fact that χ is nonincreasing with respect to the second variable y when $a_{12} = 0$ and a_{22} is nondecreasing with respect to y (see [27]). It follows that if the pressure is positive at some point $X_0 = X(t_0)$ of the porous medium, where $X(\cdot)$ is the orbit containing X_0, then

$$p(X(t)) > 0 \quad \forall t \leqslant t_0.$$

This important property is then exploited to prove that the free boundary is represented locally by continuous graphs. This is done essentially by introducing two C^1-diffeomorphisms related to the above ordinary differential equation. As a consequence we deduce that g is the characteristic function of the dry part $[p = 0]$. This helps to show the uniqueness of the S_3-connected solution which we prefer here to call the reservoirs-connected solution.

We would like to point out that in all previous studies, the dams considered are enclosed between two curves $y = s_-(x)$ and $y = s_+(x)$ which represent respectively the bottom and top of the dam. This implicitly assumes that the dam is vertically convex. In this study we do not assume this constraint and allow a wide variety of geometrical forms for our dam. We recall that for the existence of a solution it is only required that Ω is locally Lipschitz. However for the study of the free boundary we will assume that Ω is locally of class C^1. Finally we have chosen to introduce various hypotheses gradually into the text as the need arises.

The paper is organized as follows: in Section 1, we give the weak formulation of the problem and some of its properties. In Section 2, we prove a monotonicity property for the function g. In Section 3, we define a family $(\Phi_h)_h$ of functions representing locally the free boundary and prove they are lower semicontinuous. In Section 4, we prove some useful lemmas. In Section 5, we prove the continuity of the functions Φ_h. Finally in Section 6, we prove the uniqueness of the reservoirs-connected solution.

1. Formulation of the problem

A porous medium that we denote by Ω is supplied by several reservoirs of a fluid which infiltrates through Ω. We assume that Ω is a bounded locally Lipschitz domain of $\mathbb{R}^2$ with boundary $\partial\Omega = S_1 \cup S_2 \cup S_3$, where S_1 is the impervious part, S_2 is the part in contact with air and $S_3 = \bigcup_{i=1}^{N} S_{3,i}$ with $S_{3,i}$ $(i = 1, \ldots, N)$ the part in contact with the bottom of the i^{th} reservoir. We assume that the flow in Ω has reached a steady state and we look for the fluid pressure p and the saturated region S of the porous medium. The boundary ∂S of S is divided into four parts (see Figure 1):

- $\Gamma_1 \subset S_1$: the impervious part,
- $\Gamma_2 \subset \Omega$: the free boundary,
- $\Gamma_3 \subset S_3$: the part covered by fluid,
- $\Gamma_4 \subset S_2$: the part where the fluid flows outside Ω.

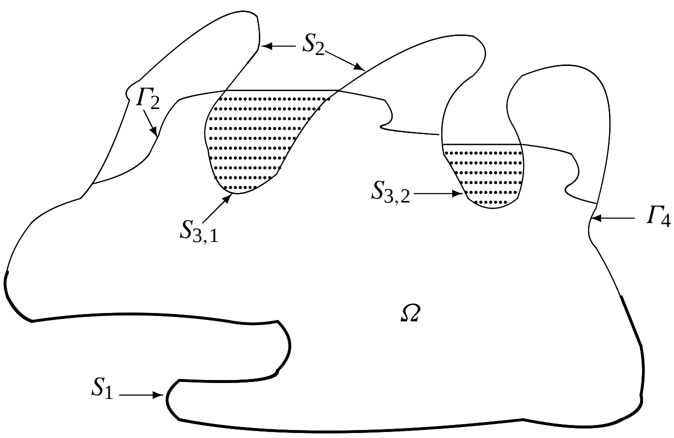

FIG. 1

The flow is governed by the following nonlinear Darcy law:

$$v = -\mathcal{A}(X, \nabla(p + y)) = -\mathcal{A}(X, \nabla u), \tag{1.1}$$

where v is the fluid velocity, $u = p + y$ is the hydrostatic head and $\mathcal{A} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping that satisfies the following assumptions for some constants $q > 1$ and $0 < \lambda \leqslant M < \infty$:

$$\begin{cases} \text{(i)} & X \mapsto \mathcal{A}(X, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^2, \\ \text{(ii)} & \xi \mapsto \mathcal{A}(X, \xi) \text{ is continuous for a.e. } X \in \Omega, \\ \text{(iii)} & \text{for all } \xi \in \mathbb{R}^2 \text{ and for a.e. } X \in \Omega, \\ & \quad \mathcal{A}(X, \xi) \cdot \xi \geqslant \lambda |\xi|^q \quad \text{and} \quad |\mathcal{A}(X, \xi)| \leqslant M |\xi|^{q-1}, \\ \text{(iv)} & \text{for all } \xi, \zeta \in \mathbb{R}^2 \text{ and for a.e. } X \in \Omega, \\ & \quad (\mathcal{A}(X, \xi) - \mathcal{A}(X, \zeta)) \cdot (\xi - \zeta) \geqslant 0. \end{cases} \tag{1.2}$$

Moreover we have the following boundary conditions:

$$\begin{cases} p = 0 \text{ on } S_2, \quad p = \varphi \text{ on } S_3, \quad v \cdot \nu = 0 \text{ on } \Gamma_1, \\ p = 0 \text{ and } v \cdot \nu = 0 \text{ on } \Gamma_2, \qquad v \cdot \nu \geqslant 0 \text{ on } \Gamma_4, \end{cases} \tag{1.3}$$

where φ is a nonnegative Lipschitz continuous function which represents the fluid pressure at the bottoms of the reservoirs. For convenience we assume that S_3 is open relatively to $\partial\Omega$.

Assuming the flow to be incompressible and taking into account (1.1) and (1.3), we are led (see [15]) to the following problem:

$$\text{(P)} \begin{cases} \text{Find } (u, g) \in W^{1,q}(\Omega) \times L^\infty(\Omega) \text{ such that:} \\ \quad \text{(i)} \quad u \geqslant y, \quad 0 \leqslant g \leqslant 1, \quad g(u - y) = 0 \quad \text{a.e. in } \Omega, \\ \quad \text{(ii)} \quad u = \varphi + y \quad \text{on } S_2 \cup S_3, \\ \quad \text{(iii)} \quad \int_\Omega (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) \cdot \nabla \xi \, dX \leqslant 0 \\ \qquad\quad \text{for all } \xi \in W^{1,q}(\Omega) \text{ with } \xi = 0 \text{ on } S_3 \text{ and } \xi \geqslant 0 \text{ on } S_2. \end{cases}$$

For the existence of a solution of (P) under the assumptions (1.2), we refer the reader to [29] where an existence result is given for more general boundary conditions. The reader can also adapt the proof in [15] obtained for the case $\mathcal{A}(X, \xi) = |\xi|^{q-2}\xi$.

Arguing as in [17] or [27], we obtain

PROPOSITION 1.1 For each solution (u, g) of (P), we have

$$\operatorname{div}(\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) = 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{1.4}$$

Moreover if $\operatorname{div}(\mathcal{A}(X, e)) \geqslant 0$ in $\mathcal{D}'(\Omega)$, we obtain

$$\operatorname{div}(\mathcal{A}(X, \nabla u)) = \operatorname{div}(g\mathcal{A}(X, e)) \geqslant 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{1.5}$$

2. A monotonicity property of g

From now on, we shall assume that

$$\mathcal{A}(\cdot, e) = (a^1(\cdot), a^2(\cdot)) \in C^1(\overline{\Omega}), \tag{2.1}$$

$$\operatorname{div}(\mathcal{A}(X, e)) \geqslant 0 \quad \text{in } C^0(\Omega), \tag{2.2}$$

$$\Gamma = \partial\Omega \quad \text{is of class } C^1, \tag{2.3}$$

$$\mathcal{A}(X, e) \cdot \nu \neq 0 \quad \forall X \in \partial\Omega. \tag{2.4}$$

Then we consider the following differential system:

$$(E(\omega, h)) \quad \begin{cases} X'(t, \omega, h) = \mathcal{A}(X(t, \omega, h), e) \\ X(0, \omega, h) = (\omega, h) \end{cases}$$

where $h \in \pi_y(\Omega)$ and $\omega \in \pi_x(\Omega \cap [y = h])$ and where π_x and π_y are respectively the orthogonal projections on the x and y axes.

By the classical theory of ordinary differential equations there exists a unique maximal solution $X(\cdot, \omega, h)$ of $E(\omega, h)$ which is defined on $[\alpha_-(\omega, h), \alpha_+(\omega, h)]$ with $X(\alpha_-(\omega, h), \omega, h) \in \partial\Omega \cap [y < h]$, $X(\alpha_+(\omega, h), \omega, h) \in \partial\Omega \cap [y > h]$ (see Figure 2).

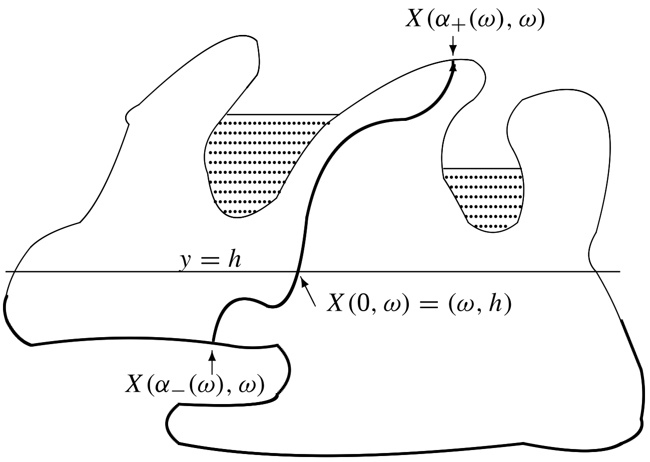

FIG. 2

For simplicity we will denote $X(t, \omega, h)$, $\alpha_-(\omega, h)$ and $\alpha_+(\omega, h)$ respectively by $X(t, \omega)$, $\alpha_-(\omega)$ and $\alpha_+(\omega)$. We note that (2.4) means that the orbits of $E(\omega, h)$ do not meet $\partial\Omega$ tangentially. Moreover under the assumptions (2.1), (2.3) and (2.4), one has

PROPOSITION 2.1 $\alpha_-, \alpha_+ \in C^1(\pi_x(\Omega \cap [y = h]))$.

Proof. Let $h \in \pi_y(\Omega)$ and $\omega_0 \in \pi_x(\Omega \cap [y = h])$. By (2.3) there exists a C^1 function σ and $\eta > 0$ small enough such that one of the following situations holds:

$$\text{(i)} \quad \sigma(X_1(\alpha_-(\omega), \omega)) = X_2(\alpha_-(\omega), \omega) \quad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta),$$

$$\text{(ii)} \quad \sigma(X_2(\alpha_-(\omega), \omega)) = X_1(\alpha_-(\omega), \omega) \quad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta).$$

Assume for example that (i) holds. This means that $\alpha_-(\omega)$ satisfies

$$F(\alpha_-(\omega), \omega) = 0 \quad \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta), \quad \text{with} \quad F = \sigma \circ X_1 - X_2.$$

Taking into account (2.1) there exists an open set Ω^* containing $\overline{\Omega}$ such that $\mathcal{A}(\cdot, e) \in C^1(\Omega^*)$. Then for each $\omega \in \pi_x(\Omega^* \cap [y = h])$, there exists a unique maximal solution $X^*(\cdot, \omega)$ of the differential system $E(\omega, h)$ defined on $[\alpha_-^*(\omega), \alpha_+^*(\omega)]$. Obviously we have $X^*_{|(\alpha_-(\omega),\alpha_+(\omega))} = X$ when $\omega \in \pi_x(\Omega \cap [y = h])$.

Let $F^* = \sigma \circ X_1^* - X_2^*$ defined on $D^* = \{(t, \omega) \mid \omega \in (\omega_0 - \eta, \omega_0 + \eta),\ t \in (\alpha_-^*(\omega), \alpha_+^*(\omega))\}$. We have $F^* \in C^1(D^*)$ since $X_i^* \in C^1(D^*)$ and σ is C^1. In addition F^* is a C^1 extension of F to D^* and by (2.1) we have

$$\begin{aligned}\frac{\partial F^*}{\partial t}(t, \omega) &= \sigma'(X_1^*(t, \omega)) \cdot \frac{\partial X_1^*}{\partial t}(t, \omega) - \frac{\partial X_2^*}{\partial t}(t, \omega)\\ &= \sigma'(X_1^*(t, \omega)) \cdot a^1(X^*(t, \omega)) - a^2(X^*(t, \omega)).\end{aligned}$$

In particular by (2.4) we obtain

$$\frac{\partial F^*}{\partial t}(\alpha_-(\omega_0), \omega_0) = \sigma'(X_1(\alpha_-(\omega_0), \omega_0)) \cdot a^1(X(\alpha_-(\omega_0), \omega_0)) - a^2(X(\alpha_-(\omega_0), \omega_0)) \neq 0.$$

Therefore by the implicit function theorem, we deduce that there exists $\delta \in (0, \eta)$ and a unique function $f : (\omega_0 - \delta, \omega_0 + \delta) \to \mathbb{R}$ such that

$$F^*(t, \omega) = 0 \ \Leftrightarrow\ t = f(\omega), \qquad f(\omega_0) = \alpha_-(\omega_0), \qquad f \in C^1(\omega_0 - \delta, \omega_0 + \delta).$$

As $F^*(\alpha_-(\omega), \omega) = F(\alpha_-(\omega), \omega) = 0$, it follows that $\alpha_-(\omega) = f(\omega)$ and $\alpha_- \in C^1(\omega_0 - \delta, \omega_0 + \delta)$.

If (ii) holds, the proof is similar. Thus $\alpha_- \in C^1(\pi_x(\Omega \cap [y = h]))$. In the same way we prove that $\alpha_+ \in C^1(\pi_x(\Omega \cap [y = h]))$. □

DEFINITION 2.1 For each $h \in \pi_y(\Omega)$ we define the set

$$D_h = \{(t, \omega) \mid \omega \in \pi_x(\Omega \cap [y = h]),\ t \in (\alpha_-(\omega), \alpha_+(\omega))\}$$

and consider the mappings $T_h : D_h \to T_h(D_h)$ and $S_h : D_h \to S_h(D_h)$ defined by

$$T_h(t, \omega) = X(t, \omega) = (T_h^1, T_h^2)(t, \omega), \qquad S_h(t, \omega) = (\omega, L_h(t, \omega)) = (\omega, \tau),$$

where

$$L_h(t, \omega) = \int_{\alpha_-(\omega)}^t |\mathcal{A}(X(s, \omega), e)|\, \mathrm{d}s = \int_{\alpha_-(\omega)}^t |X'(s, \omega)|\, \mathrm{d}s$$

represents the arc length of the curve $X(\cdot, \omega)$ from the point $X(\alpha_-(\omega), \omega)$ to $X(t, \omega)$.

Then we have

PROPOSITION 2.2

$$\Omega = \bigsqcup_{h \in \pi_y(\Omega)} T_h(D_h), \qquad T_h \text{ and } S_h \text{ are } C^1 \text{ diffeomorphisms.}$$

Proof. First for each $(x, y) \in \Omega$ we have $(x, y) = X(0, \omega) = T_h(0, \omega)$ with $\omega = x$ and $h = y$. Next thanks to (2.1) we have $T_h \in C^1(D_h)$. By Proposition 2.1, S_h is also in $C^1(D_h)$. To see that they are diffeomorphisms, it suffices to verify that $\det(\mathcal{J}T_h)$ and $\det(\mathcal{J}S_h)$ do not vanish; here we denote by $\mathcal{J}F$ the Jacobian matrix of the transformation F.

One can easily check that

$$\det \mathcal{J} S_h = -|\mathcal{A}(X(t,\omega), e)| < 0,$$
$$Y_h(t,\omega) = \det(\mathcal{J} T_h) = a^1(X(t,\omega))\frac{\partial X_2}{\partial \omega} - a^2(X(t,\omega))\frac{\partial X_1}{\partial \omega},$$
$$\frac{\partial Y_h}{\partial t}(t,\omega) = Y_h(t,\omega)\cdot\{\mathrm{div}(\mathcal{A}(\cdot,e))\}(X(t,\omega)).$$

Therefore

$$Y_h(t,\omega) = Y_h(0,\omega)\exp\left(\int_0^t \{\mathrm{div}(\mathcal{A}(\cdot,e))\}(X(s,\omega))\,\mathrm{d}s\right). \tag{2.5}$$

Since $Y_h(0,\omega) = -a^2(X(0,\omega)) < 0$, we get $Y_h(t,\omega) < 0$ for all $t \in (\alpha_-(\omega), \alpha_+(\omega))$ and all $\omega \in \pi_x(\Omega \cap [y = h])$. □

The following key theorem generalizes the fact that $g_y \geqslant 0$ in $\mathcal{D}'(\Omega)$ when $a_1 = 0$ and a_2 is nondecreasing with respect to y (see [15], [17], and [27]). It will play a major role for the definition of the free boundary and the proof of its continuity.

THEOREM 2.1 Let (u, g) be a solution of (P). For each $h \in \pi_y(\Omega)$,

$$\frac{\partial}{\partial \tau}(\widetilde{g}\cdot(-Y_h \circ S_h^{-1})) \geqslant 0 \qquad \text{in } \mathcal{D}'(S_h(D_h)),$$

where Y_h is given by (2.5) and $\widetilde{g} = g \circ T_h \circ S_h^{-1}$.

Proof. Let $\phi \in \mathcal{D}(S_h(D_h))$, $\phi \geqslant 0$. Then $\phi \circ S_h \circ T_h^{-1} \in C_0^1(T_h(D_h))$ and by (1.5) and (2.2), we have

$$\int_{T_h(D_h)} g\mathcal{A}(X,e)\nabla(\phi \circ S_h \circ T_h^{-1})\,\mathrm{d}X \leqslant 0.$$

Using the change of variables $T_h(t,\omega) = (x,y)$ and the fact that

$$\mathcal{A}(X(t,\omega),e)(\nabla(\phi\circ S_h \circ T_h^{-1}))\circ T_h \cdot(-Y_h(t,\omega)) = -Y_h(t,\omega)\frac{\partial}{\partial t}(\phi\circ S_h)$$

we get

$$\int_{D_h} g\circ T_h(t,\omega)\cdot(-Y_h(t,\omega))\cdot\frac{\partial}{\partial t}(\phi\circ S_h)\,\mathrm{d}t\,\mathrm{d}\omega \leqslant 0,$$

which becomes, after using the change of variables S_h^{-1},

$$\int_{S_h(D_h)} g\circ T_h\circ S_h^{-1}(\omega,\tau)\cdot(-Y_h\circ S_h^{-1}(\omega,\tau))\cdot\left(\frac{\partial}{\partial t}(\phi\circ S_h)\right)\circ S_h^{-1}\cdot|\det(\mathcal{J}S_h^{-1})|\,\mathrm{d}\omega\,\mathrm{d}\tau \leqslant 0.$$

Taking into account that

$$\left(\frac{\partial}{\partial t}(\phi\circ S_h)\right)\circ S_h^{-1} = \frac{\partial \phi}{\partial \tau}\cdot|\mathcal{A}(\cdot,e)|\circ T_h\circ S_h^{-1}(\omega,\tau) = \frac{\partial\phi}{\partial\tau}\cdot|\det(\mathcal{J}S_h)|,$$

we obtain

$$\int_{S_h(D_h)} \widetilde{g}(\omega,\tau)\cdot(-Y_h\circ S_h^{-1}(\omega,\tau))\cdot\frac{\partial\phi}{\partial\tau}\,\mathrm{d}\omega\,\mathrm{d}\tau \leqslant 0.$$ □

REMARK 2.1 In order to avoid complicated notations we will write $\widetilde{f}$ to denote the function $f \circ T_h \circ S_h^{-1}$ for any function f defined on $T_h(D_h)$. We will also denote by $\mathcal{T}_h$ and $\mathcal{Y}_h$ the functions $T_h \circ S_h^{-1}$ and $-Y_h \circ S_h^{-1}$ respectively.

3. Lower semicontinuity of the free boundary

In what follows we assume that there exist nonnegative constants κ, σ and positive constants λ_0, λ_1 with $\sigma \leqslant 1$ and $\lambda_1 \geqslant \lambda_0$ such that for all $X, Y \in \overline{\Omega}$, $\zeta, \xi \in \mathbb{R}^2$,

$$\sum_{i,j} \frac{\partial \mathcal{A}^i}{\partial \zeta_j}(X,\zeta)\xi_i\xi_j \geqslant \lambda_0(\kappa + |\zeta|^{q-2})|\xi|^2, \tag{3.1}$$

$$\left|\frac{\partial \mathcal{A}^i}{\partial \zeta_j}(X,\zeta)\right| \leqslant \lambda_1(\kappa + |\zeta|^{q-2}), \tag{3.2}$$

$$|\mathcal{A}(X,\zeta) - \mathcal{A}(Y,\zeta)| \leqslant \lambda_1(1 + |\zeta|^{q-1})(|X-Y|^{\sigma}). \tag{3.3}$$

REMARK 3.1 (i) If $\mathcal{A}(X,\zeta) = a(X)\zeta$ with $a(X)$ a bounded 2-by-2 matrix, then (3.1) and (3.2) are satisfied and (3.3) is not needed. In this case $u \in C^{0,\gamma}_{\text{loc}}(\Omega \cup S_2 \cup S_3)$ for some $\gamma \in (0,1)$.

(ii) Assumptions (3.1)–(3.3) are satisfied in the case where $\mathcal{A}(X,\zeta) = |\zeta|^{q-2}\zeta$. Moreover under these assumptions, we deduce from (1.4) (see [20], [30]) that $u \in C^{0,\gamma}_{\text{loc}}(\Omega \cup S_2 \cup S_3)$ for some $\gamma \in (0,1)$. Also by (1.4) and (P)(i) we have $\text{div}(\mathcal{A}(X,u)) = 0$ in $\mathcal{D}'([u > y])$. It then follows by (3.1)–(3.3) (see [19], [25] for example) that $u \in C^{1,\delta}_{\text{loc}}([u > y])$ for some $\delta \in (0,1)$.

The following strong maximum principle (see [18], [1]) will be needed:

LEMMA 3.1 (Strong maximum principle) Let u_1 and u_2 be two functions defined on a domain D of $\mathbb{R}^2$ such that $u_1, u_2 \in C^1(D)$, $u_1 \geqslant u_2$ in D, the set $\{X \in D \mid \nabla u_1(X) = \nabla u_2(X) = 0\}$ is empty and $\text{div}(\mathcal{A}(X,\nabla u_1) - \mathcal{A}(X,\nabla u_2)) \leqslant 0$. Then either

$$u_1 = u_2 \text{ in } D \quad \text{or} \quad u_1 > u_2 \text{ in } D.$$

The theorem below will allow us to define the free boundary $\partial([u > y]) \cap \Omega$ locally as a curve.

THEOREM 3.1 Let (u,g) be a solution of (P) and $X_0 = \mathcal{T}_h(\omega_0,\tau_0) = (x_0,y_0) \in \Omega$.

(i) If $p(X_0) = \widetilde{p}(\omega_0,\tau_0) > 0$, then there exists $\epsilon > 0$ such that

$$\widetilde{p}(\omega,\tau) > 0 \quad \forall(\omega,\tau) \in C_\epsilon = \{(\omega,\tau) \in S_h(D_h) \mid |\omega - \omega_0| < \epsilon,\ \tau < \tau_0 + \epsilon\},$$

(ii) If $p(X_0) = \widetilde{p}(\omega_0,\tau_0) = 0$, then $\widetilde{p}(\omega_0,\tau) = 0$ for all $\tau \geqslant \tau_0$.

Proof. (i) By continuity, there exists $\epsilon > 0$ such that

$$\widetilde{p}(\omega,\tau) > 0 \quad \forall(\omega,\tau) \in (\omega_0 - \epsilon, \omega_0 + \epsilon) \times (\tau_0 - \epsilon, \tau_0 + \epsilon) = Q_\epsilon.$$

Then $\widetilde{g}(\omega,\tau) = 0$ for a.e. $(\omega,\tau) \in Q_\epsilon$. By Theorem 2.1 and since $\mathcal{Y}_h > 0$, $\widetilde{g} \geqslant 0$, we get $\widetilde{g} = 0$ a.e. in C_ϵ, i.e. $g = 0$ a.e. in $\mathcal{T}_h(C_\epsilon)$ (see Figure 3).

By (1.4) we have $\text{div}(\mathcal{A}(X,\nabla u)) = \text{div}(g\mathcal{A}(X,e)) = 0$ in $\mathcal{D}'(\mathcal{T}_h(C_\epsilon))$. Since $\text{div}(\mathcal{A}(X,\nabla y)) \geqslant 0$ in $\mathcal{D}'(\Omega)$, $\nabla y = e \neq 0$, $u \geqslant y$ in $\mathcal{T}_h(C_\epsilon)$ and $u > y$ in $\mathcal{T}_h(Q_\epsilon)$, it follows by Lemma 3.1 that $u > y$ in $\mathcal{T}_h(C_\epsilon)$.

(ii) This is a consequence of (i). □

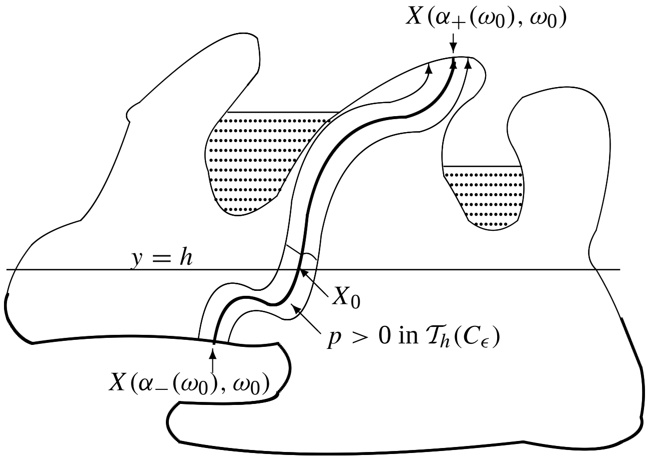

FIG. 3

REMARK 3.2 (i) The result of Theorem 3.1 means that if a point X_0 is in the wet region, then the part of the curve $X(\cdot, \omega)$ passing through X_0 at t_0 remains in the wet region for all $t \leqslant t_0$.

(ii) In [17] and [27] we assumed that $\mathcal{A}(X, e) = k(X)e$, which leads to $X_1'(t) = 0$ for all t and the curve $X(\cdot, \omega)$ is a vertical segment. Therefore the free boundary is represented by a curve of the form $y = \Phi(x)$.

(iii) We have $u = p + y = \varphi + y > y$ on $S_{3,i}$ ($\varphi > 0$ on S_3), $i = 1, \ldots, N$, and $u \in C^0(\Omega \cup S_3)$. So $p > 0$ below S_3 in the following sense:

$$p(X(t, \omega)) > 0 \quad \forall t \in [\alpha_-(\omega), \alpha_+(\omega)] \text{ such that } X(\alpha_+(\omega), \omega) \in S_3.$$

DEFINITION 3.1 For each $h \in \pi_y(\Omega)$ we define the function Φ_h on $\pi_x(\Omega \cap [y = h])$ by

$$\Phi_h(\omega) = \begin{cases} \sup\{\tau \mid (\omega, \tau) \in S_h(D_h),\ \widetilde{p}(\omega, \tau) > 0\} & \text{if this set is not empty,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Thanks to Theorem 3.1, Φ_h is well defined and we have

PROPOSITION 3.1 Φ_h is lower semicontinuous on $\pi_x(\Omega \cap [y = h])$. Moreover

$$[\widetilde{p}(\omega, \tau) > 0] = [\tau < \Phi_h(\omega)]. \tag{3.5}$$

Proof. First we show the lower semicontinuity of Φ_h. Let $\omega_0 \in \pi_x(\Omega \cap [y = h])$.

If $\Phi_h(\omega_0) = 0$ then for each $\epsilon > 0$, $\Phi_h(\omega) \geqslant 0 > \Phi_h(\omega_0) - \epsilon$ for all ω.

If $\Phi_h(\omega_0) > 0$ then for each $\epsilon > 0$, there is $\tau_\epsilon = L_h(t_\epsilon, \omega_0) > 0$ with $t_\epsilon \in (\alpha_-(\omega_0), \alpha_+(\omega_0))$ such that

$$\Phi_h(\omega_0) \geqslant \tau_\epsilon > \Phi_h(\omega_0) - \epsilon/2, \qquad \widetilde{p}(\omega_0, \tau_\epsilon) > 0.$$

Moreover (see [23, Theorem 3.4, p. 24]) one can find $\eta_1 > 0$ such that $X(t, \omega)$ exists for all $(t, \omega) \in [\alpha_-(\omega_0), \alpha_+(\omega_0)] \times (\omega_0 - \eta_1, \omega_0 + \eta_1)$ and $(t, \omega) \mapsto X(t, \omega)$ is continuous. Then by continuity, there exists $0 < \eta_2 < \eta_1$ such that

$$p(X(t, \omega)) > 0 \quad \forall (t, \omega) \in (t_\epsilon - \eta_2, t_\epsilon + \eta_2) \times (\omega_0 - \eta_2, \omega_0 + \eta_2).$$

By Theorem 3.1, we deduce that

$$p(X(t,\omega)) > 0 \quad \forall (t,\omega) \in (\alpha_-(\omega), t_\epsilon + \eta_2) \times (\omega_0 - \eta_2, \omega_0 + \eta_2).$$

Using the definition of Φ_h we get, for $\omega \in (\omega_0 - \eta_2, \omega_0 + \eta_2)$,

$$\begin{aligned}\Phi_h(\omega) &= \sup_{t\in(\alpha_-(\omega),\alpha_+(\omega))} \{L_h(t,\omega) \mid (t,\omega) \in D_h \text{ and } p(X(t,\omega)) > 0\}\\ &\geqslant L_h(t_\epsilon, \omega).\end{aligned}$$

Since $L_h(t_\epsilon,\omega) = \int_{\alpha_-(\omega)}^{t_\epsilon} |\mathcal{A}(X(s,\omega),e)|\,\mathrm{d}s$ is continuous with respect to ω, there exists $0 < \eta < \eta_2$ such that for all $\omega \in (\omega_0 - \eta, \omega_0 + \eta)$,

$$\begin{aligned}L_h(t_\epsilon,\omega) &= \int_{\alpha_-(\omega)}^{t_\epsilon} |\mathcal{A}(X(s,\omega),e)|\,\mathrm{d}s \geqslant \int_{\alpha_-(\omega_0)}^{t_\epsilon} |\mathcal{A}(X(s,\omega_0),e)|\,\mathrm{d}s - \frac{\epsilon}{2}\\ &= L_h(t_\epsilon,\omega_0) - \frac{\epsilon}{2} = \tau_\epsilon - \frac{\epsilon}{2} > \Phi_h(\omega_0) - \frac{\epsilon}{2} - \frac{\epsilon}{2}.\end{aligned}$$

Thus $\Phi_h(\omega) \geqslant \Phi_h(\omega_0) - \epsilon$ for all $\omega \in (\omega_0 - \eta, \omega_0 + \eta)$.

Now we prove (3.5). Let $(\omega_0, \tau_0) \in [\tau < \Phi_h(\omega)]$. Assume that $\widetilde{p}(\omega_0, \tau_0) = 0$. Then by Theorem 3.1, $\widetilde{p}(\omega_0, \tau) = 0$ for all $\tau \geqslant \tau_0$. So $\Phi_h(\omega_0) = \sup\{\tau \mid (\omega_0,\tau) \in S_h(D_h)$ and $\widetilde{p}(\omega_0,\tau) > 0\} \leqslant \tau_0$, which is a contradiction.

Now let $(\omega_0, \tau_0) \in [\widetilde{p}(\omega,\tau) > 0]$. By continuity, there exists $\eta > 0$ such that $\widetilde{p}(\omega_0,\tau) > 0$ for all $\tau \in (\tau_0 - \eta, \tau_0 + \eta)$. By Theorem 3.1, we deduce that $\widetilde{p}(\omega_0,\tau) > 0$ for all $\tau < \tau_0 + \eta$ such that $(\omega_0, \tau) \in S_h(D_h)$. Hence $\Phi_h(\omega_0) \geqslant \tau_0 + \eta > \tau_0$ and $(\omega_0, \tau_0) \in [\tau < \Phi_h(\omega)]$. □

4. Some technical lemmas

The following lemma plays an important role in the proof of the continuity of the free boundary.

LEMMA 4.1 Let (u, g) be a solution of (P). Let $(\omega_1, \tau_0), (\omega_2, \tau_0) \in S_h(D_h)$ with $\omega_1 < \omega_2$ and

$$\widetilde{p}(\omega_i, \tau) = 0 \quad \forall (\omega_i, \tau) \in S_h(D_h),\ \tau > \tau_0 \geqslant 0.$$

Set $Z_{\tau_0} = \mathcal{T}_h((\omega_1,\omega_2) \times (\tau_0, +\infty) \cap S_h(D_h))$ and assume that $\overline{Z}_{\tau_0} \cap S_3 = \emptyset$. Let $y_0 \in \mathbb{R}$ be such that $D_{y_0,\tau_0} = [y > y_0] \cap Z_{\tau_0} \neq \emptyset$ (see Figure 4). Then

$$\int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot e \,\mathrm{d}X \leqslant 0.$$

To prove this lemma, we need another lemma:

LEMMA 4.2 Under the assumptions of Lemma 4.1, we have

$$\int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - \chi([u=y])\mathcal{A}(X,e)) \cdot \nabla\zeta \,\mathrm{d}X \leqslant \int_{\omega_1}^{\omega_2} \mathcal{Y}_h(s,\Phi_h(s)) \cdot \widetilde{\zeta}(s,\Phi_h(s))\,\mathrm{d}s$$

for all $\zeta \in W^{1,q}(D_{y_0,\tau_0}) \cap C^0(\overline{D}_{y_0,\tau_0})$ such that $\zeta \geqslant 0$ and $\zeta(x,y_0) = 0$ for all $(x,y_0) \in \overline{D}_{y_0,\tau_0}$.

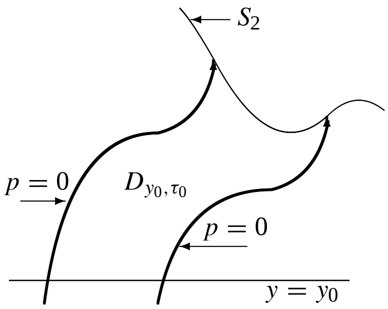

FIG. 4

Proof of Lemma 4.2. For $\epsilon > 0$, $\xi = \chi(D_{y_0,\tau_0})\min((u-y)/\epsilon, \zeta)$ is a test function for (P). So since $g \cdot (u-y) = 0$ a.e. in Ω, we have

$$\int_{D_{y_0,\tau_0}} \mathcal{A}(X, \nabla u) \cdot \nabla\left(\frac{u-y}{\epsilon} \wedge \zeta\right) \mathrm{d}X \leqslant 0.$$

Using the monotonicity of $\mathcal{A}$, we get

$$\begin{aligned}\int_{D_{y_0,\tau_0} \cap [u-y \geqslant \epsilon\zeta]} (\mathcal{A}(X, \nabla u) - \mathcal{A}(X, e)) \cdot \nabla\zeta \, \mathrm{d}X &\leqslant -\int_{D_{y_0,\tau_0}} \mathcal{A}(X, e) \cdot \nabla\left(\frac{u-y}{\epsilon} \wedge \zeta\right) \mathrm{d}X \\ &= -I_\epsilon.\end{aligned}$$

Moreover

$$\begin{aligned} I_\epsilon &= \int_{D_{y_0,\tau_0}} \chi([u > y])\mathcal{A}(X, e) \cdot \nabla\left(\frac{u-y}{\epsilon} \wedge \zeta\right) \mathrm{d}X \\ &= \int_{D_{y_0,\tau_0}} \chi([u > y])\mathcal{A}(X, e) \cdot \nabla\zeta \, \mathrm{d}X \\ &\quad - \int_{D_{y_0,\tau_0}} \mathcal{A}(X, e) \cdot \nabla\left(\zeta - \frac{u-y}{\epsilon}\right)^+ \chi([u > y]) \, \mathrm{d}X = I^1 - I_\epsilon^2. \end{aligned}$$

Now using the change of variables T_h, we obtain

$$\begin{aligned} I_\epsilon^2 &= \int_{T_h^{-1}(D_{y_0,\tau_0})} \chi([p \circ T_h(t, \omega) > 0])\mathcal{A}(X(t, \omega), e) \cdot \left(\nabla\left(\zeta - \frac{p}{\epsilon}\right)^+\right) \circ T_h |Y_h(t, \omega)| \, \mathrm{d}t \, \mathrm{d}\omega \\ &= -\int_{T_h^{-1}(D_{y_0,\tau_0})} \chi([p \circ T_h(t, \omega) > 0]) \cdot Y_h(t, \omega) \cdot \frac{\partial}{\partial t}\left(\left(\zeta - \frac{p}{\epsilon}\right)^+ \circ T_h\right) \mathrm{d}t \, \mathrm{d}\omega \end{aligned}$$

since

$$\mathcal{A}(X(t, \omega), e) \cdot \left(\nabla\left(\zeta - \frac{p}{\epsilon}\right)^+\right) \circ T_h \cdot |Y_h(t, \omega)| = -Y_h(t, \omega) \cdot \frac{\partial}{\partial t}\left(\left(\zeta - \frac{p}{\epsilon}\right)^+ \circ T_h\right).$$

Next using the change of variables S_h^{-1}, we get

$$I_\epsilon^2 = -\int_{S_h\circ T_h^{-1}(D_{y_0,\tau_0})} \chi([\widetilde{p}(\omega,\tau) > 0]) \cdot Y_h \circ S_h^{-1}(\omega,\tau) \cdot \left(\frac{\partial}{\partial t}\left(\left(\zeta - \frac{p}{\epsilon}\right)^+ \circ T_h\right)\right) \circ S_h^{-1}(\omega,\tau) \cdot |\det(\mathcal{J} S_h^{-1})| \,d\omega\, d\tau$$

$$= \int_{\mathcal{T}_h^{-1}(D_{y_0,\tau_0})} \chi([\widetilde{p}(\omega,\tau) > 0]) \cdot \mathcal{Y}_h(\omega,\tau) \cdot \frac{\partial}{\partial \tau}\widetilde{\left(\zeta - \frac{p}{\epsilon}\right)}^+(\omega,\tau)\, d\omega\, d\tau$$

since

$$\left(\frac{\partial}{\partial t}\left(\left(\zeta - \frac{p}{\epsilon}\right)^+ \circ T_h\right)\right) \circ S_h^{-1}(\omega,\tau) = \frac{\partial}{\partial \tau}\left(\left(\zeta - \frac{p}{\epsilon}\right)^+ \circ T_h \circ S_h^{-1}\right)(\omega,\tau) \cdot |\mathcal{A}(T_h \circ S_h^{-1}(\omega,\tau), e)|.$$

Therefore

$$I_\epsilon^2 = \int_{(\omega_1,\omega_2)\times(\tau_0,+\infty)\cap\mathcal{T}_h^{-1}([y>y_0])} \chi([\tau < \Phi_h(\omega)]) \cdot \mathcal{Y}_h \cdot \frac{\partial}{\partial \tau}\widetilde{\left(\zeta - \frac{p}{\epsilon}\right)}^+ d\omega\, d\tau.$$

Note that for every $\omega \in (\omega_1,\omega_2)$ there is a unique $t_{y_0}(\omega)$ such that

$$X_2(t_{y_0}(\omega),\omega) = y_0, \quad \tau_{y_0}(\omega) = \int_{\alpha_-(\omega)}^{t_{y_0}(\omega)} |\mathcal{A}(X(s,\omega), e)|\, ds,$$

and one can check that

$$((\omega_1,\omega_2)\times(\tau_0,+\infty))\cap\mathcal{T}_h^{-1}([y > y_0]) = \{(\omega,\tau) \in S_h(D_h) \mid \omega \in (\omega_1,\omega_2),\ \tau > \sup(\tau_0, \tau_{y_0}(\omega))\}.$$

It follows by the second mean value theorem that

$$I_\epsilon^2 = \int_{\omega_1}^{\omega_2}\int_{\sup(\tau_0,\tau_{y_0}(\omega))}^{\Phi_h(\omega)} \mathcal{Y}_h(\omega,\tau) \cdot \frac{\partial}{\partial \tau}\widetilde{\left(\zeta - \frac{p}{\epsilon}\right)}^+(\omega,\tau)\, d\omega\, d\tau$$

$$= \int_{\omega_1}^{\omega_2} \mathcal{Y}_h(\omega,\Phi_h(\omega)) \int_{\tau^*(y_0,\omega)}^{\Phi_h(\omega)} \frac{\partial}{\partial \tau}\widetilde{\left(\zeta - \frac{p}{\epsilon}\right)}^+(\omega,\tau)\, d\tau$$

$$\leqslant \int_{\omega_1}^{\omega_2} \mathcal{Y}_h(\omega,\Phi_h(\omega))\widetilde{\zeta}(\omega,\Phi_h(\omega))\, d\omega,$$

where $\tau^*(y_0,\omega) \in [\sup(\tau_0,\tau_{y_0}(\omega)), \Phi_h(\omega)]$. Thus

$$\int_{D_{y_0,\tau_0}\cap[u-y\geqslant\epsilon\zeta]} (\mathcal{A}(X,\nabla u) - \mathcal{A}(X,e)) \cdot \nabla\zeta\, dX + \int_{D_{y_0,\tau_0}} \chi([u > y])\mathcal{A}(X,e)\cdot\nabla\zeta\, dX \leqslant \int_{\omega_1}^{\omega_2} \mathcal{Y}_h(\omega,\Phi(\omega)) \cdot \widetilde{\zeta}(\omega,\Phi_h(\omega))\, d\omega$$

and the lemma follows by letting ϵ go to 0. □

Proof of Lemma 4.1. Let $\epsilon > 0$ and $h_\epsilon = \theta_\epsilon \circ S_h \circ T_h^{-1}$, where

$$\theta_\epsilon(\omega) = \min\left(\frac{(\omega-\omega_1)^+}{\epsilon}, 1\right) \cdot \min\left(\frac{(\omega_2-\omega)^+}{\epsilon}, 1\right).$$

Since $\chi(D_{y_0,\tau_0})h_\epsilon(y-y_0)$ is a test function for (P) we have

$$\begin{aligned}
&\int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot e \,\mathrm{d}X \\
&\quad = \int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot \nabla(y-y_0)\,\mathrm{d}X \\
&\quad = \int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot \nabla(h_\epsilon(y-y_0))\,\mathrm{d}X \\
&\qquad + \int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot \nabla((1-h_\epsilon)(y-y_0))\,\mathrm{d}X \\
&\quad \leqslant \int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot \nabla((1-h_\epsilon)(y-y_0))\,\mathrm{d}X \\
&\quad = \int_{D_{y_0,\tau_0}} (\mathcal{A}(X,\nabla u) - \chi([u=y])\mathcal{A}(X,e)) \cdot \nabla((1-h_\epsilon)(y-y_0))\,\mathrm{d}X \\
&\qquad + \int_{D_{y_0,\tau_0}} (\chi([u=y]) - g)\mathcal{A}(X,e) \cdot \nabla((1-h_\epsilon)(y-y_0))\,\mathrm{d}X = J_\epsilon^1 + J_\epsilon^2.
\end{aligned}$$

By Lemma 4.2, with $\zeta = (1-h_\epsilon)\cdot(y-y_0) = (1-\theta_\epsilon \circ S_h \circ T_h^{-1})\cdot(y-y_0)$, we have

$$J_\epsilon^1 \leqslant \int_{\omega_1}^{\omega_2} \mathcal{Y}_h(\omega, \Phi_h(\omega)) \cdot (1-\theta_\epsilon(\omega)) \cdot \widetilde{(y-y_0)}(\omega, \Phi_h(\omega))\,\mathrm{d}\omega.$$

Moreover

$$\begin{aligned}
J_\epsilon^2 &= \int_{\mathcal{T}_h^{-1}(D_{y_0,\tau_0})} (\chi([\widetilde{p}(\omega,\tau)=0]) - \widetilde{g}(\omega,\tau)) \\
&\qquad \cdot \mathcal{Y}_h(\omega,\tau). \frac{\partial}{\partial\tau}((1-\theta_\epsilon(\omega)) \cdot \widetilde{(y-y_0)}(\omega,\tau))\,\mathrm{d}\omega\,\mathrm{d}\tau \\
&= \int_{\mathcal{T}_h^{-1}(D_{y_0,\tau_0})} (\chi([\widetilde{p}(\omega,\tau)=0]) - \widetilde{g}(\omega,\tau)) \\
&\qquad \cdot \mathcal{Y}_h(\omega,\tau) \cdot (1-\theta_\epsilon(\omega)) \frac{\partial}{\partial\tau}(T_h^2 \circ S_h^{-1}(\omega,\tau))\,\mathrm{d}\omega\,\mathrm{d}\tau.
\end{aligned}$$

Since $\theta_\epsilon \to 1$ as $\epsilon \to 0$, we conclude that $J_\epsilon^1 + J_\epsilon^2 \to 0$. This completes the proof. □

LEMMA 4.3 Let (u, g) be a solution of (P). Let $\mathcal{C}_h$ be the connected component of $[\tau < \Phi_h(\omega)]$ such that $\overline{\mathcal{T}_h(\mathcal{C}_h)} \cap S_3 = \emptyset$. Then for $C_h = \mathcal{T}_h(\mathcal{C}_h)$ we have

$$\int_{C_h} (\mathcal{A}(X,\nabla u) - g\mathcal{A}(X,e)) \cdot e\,\mathrm{d}X \leqslant 0.$$

Proof. *Step 1.* Arguing as in the proof of Lemma 4.2, we get, for all nonnegative $\zeta \in W^{1,q}(\mathcal{C}_h) \cap C^0(\overline{\mathcal{C}}_h)$,

$$\int_{C_h} (\mathcal{A}(X, \nabla u) - \chi([u = y])\mathcal{A}(X, e)) \cdot \nabla\zeta \, \mathrm{d}X \leqslant \int_{\pi_\omega(\mathcal{C}_h)} \mathcal{Y}_h(s, \Phi_h(s)) \cdot \widetilde{\zeta}(s, \Phi_h(s)) \, \mathrm{d}s.$$

Step 2. Let $\epsilon > 0$ and $A = \mathbb{R} \setminus \pi_\omega(\mathcal{C}_h)$. We consider $\alpha_\epsilon(\omega) = \min(1, d(\omega, A)/\epsilon)$, $h_\epsilon = \alpha_\epsilon \circ \mathcal{T}_h^{-1}$ and we argue as in the proof of Lemma 4.1. □

LEMMA 4.4 Let (u, g) be a solution of (P). Let $X_0 = (x_0, y_0) = \mathcal{T}_h(\omega_0, \tau_0)$ be a point in Ω, $(\omega_0, \tau_0) \in S_h(D_h)$. Denote by $B_r(\omega_0, \tau_0)$ a ball with center (ω_0, τ_0) and radius r contained in $S_h(D_h)$. If $\widetilde{p} = 0$ in $B_r(\omega_0, \tau_0)$, then (see Figure 5)

$$\widetilde{p} = 0 \quad \text{in } C_r, \qquad \widetilde{g} = 1 \quad \text{a.e. in } C_r,$$

where $C_r = \{(\omega, \tau) \in S_h(D_h) \mid |\omega - \omega_0| < r, \ \tau > \tau_0\} \cup B_r(\omega_0, \tau_0)$, i.e. if $p = 0$ in $\mathcal{T}_h(B_r(\omega_0, \tau_0))$, then $p = 0$, $g = 1$ a.e. in $\mathcal{T}_h(C_r)$.

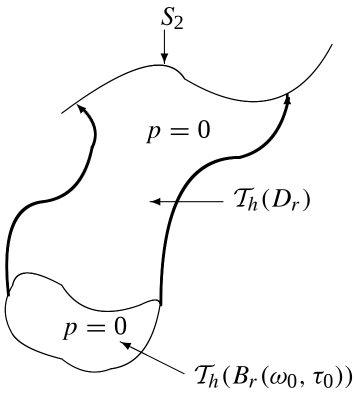

FIG. 5

Proof. Note that by Remark 3.2, we necessarily have

$$X(\alpha_+(\omega), \omega) \in S_2 \quad \forall \omega \in (\omega_0 - r, \omega_0 + r).$$

By Theorem 3.1(ii), we have $\widetilde{p} = 0$ in C_r. Applying Lemma 4.1 with $Z_{\tau_0} = \mathcal{T}_h((\omega_1, \omega_2) \times (\tau_0, +\infty) \cap S_h(D_h)) \subset \mathcal{T}_h(C_r)$ we obtain

$$\int_{[y > y_0] \cap Z_{\tau_0}} (1 - g)\mathcal{A}(X, e) \cdot e \, \mathrm{d}X \leqslant 0 \quad \forall y_0 \in \mathbb{R} \text{ such that } [y > y_0] \cap Z_{\tau_0} \neq \emptyset.$$

So $g = 1$ a.e. in Z_{τ_0}. This holds for all domains Z_{τ_0} in $\mathcal{T}_h(C_r)$ and we get $g = 1$ a.e. in $\mathcal{T}_h(C_r)$. □

The following result is a sort of maximum principle.

LEMMA 4.5 Let (u, g) be a solution of (P), $X_0 = (x_0, y_0) = \mathcal{T}_h(\omega_0, \tau_0)$ be a point of Ω and B_r be an open ball in $S_h(D_h)$ with center (ω_0, τ_0) and radius r. Then the following situations are impossible (see Figures 6, 7 and 8):

$$\text{(i)} \quad \begin{cases} \widetilde{p}(\omega_0, \tau) = 0 & \forall \tau \in (\tau_0 - r, \tau_0 + r), \\ \widetilde{p}(\omega, \tau) > 0 & \forall (\omega, \tau) \in B_r \setminus S, \ S = \{\omega_0\} \times (\tau_0 - r, \tau_0 + r), \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \widetilde{p}(\omega, \tau) = 0 & \forall (\omega, \tau) \in B_r \cap [\omega \leqslant \omega_0], \\ \widetilde{p}(\omega, \tau) > 0 & \forall (\omega, \tau) \in B_r \cap [\omega > \omega_0], \end{cases}$$

$$\text{(iii)} \quad \begin{cases} \widetilde{p}(\omega, \tau) = 0 & \forall (\omega, \tau) \in B_r \cap [\omega \geqslant \omega_0], \\ \widetilde{p}(\omega, \tau) > 0 & \forall (\omega, \tau) \in B_r \cap [\omega < \omega_0]. \end{cases}$$

Proof. (i) Since $u > y$ a.e. in $\mathcal{T}_h(B_r)$, we have $g = 0$ a.e. in $\mathcal{T}_h(B_r)$ and then by (1.4), u is $\mathcal{A}$-harmonic in this domain. It follows by Lemma 3.1 that $u > y$ or $u = y$ in $\mathcal{T}_h(B_r)$, which contradicts (i).

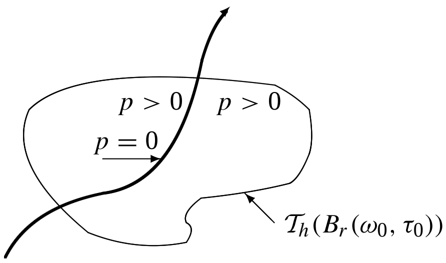

FIG. 6

(ii) Let $\xi \in \mathcal{D}(\mathcal{T}_h(B_r))$, $\xi \geqslant 0$. Using the fact that $\pm\xi$ are test functions for (P) and applying the changes of variables T_h and S_h, we obtain, as in the proof of Theorem 2.1,

$$\int_{\mathcal{T}_h(B_r)} \mathcal{A}(X, \nabla u) \cdot \nabla\xi \, \mathrm{d}X = \int_{\mathcal{T}_h(B_r)} g\mathcal{A}(X, e) \cdot \nabla\xi \, \mathrm{d}X = \int_{B_r} \widetilde{g} \cdot \mathcal{Y}_h \cdot \frac{\partial\widetilde{\xi}}{\partial\tau} \, \mathrm{d}\omega \, \mathrm{d}\tau.$$

Now by Lemma 4.4, we have $\widetilde{g} = 1$ a.e. in $B_r \cap [\omega < \omega_0]$ and then

$$\int_{T_h \circ S_h^{-1}(B_r)} \mathcal{A}(X, \nabla u) \cdot \nabla\xi \, \mathrm{d}X = \int_{B_r \cap [\omega < \omega_0]} \mathcal{Y}_h \cdot \frac{\partial\widetilde{\xi}}{\partial\tau} \, \mathrm{d}\omega \, \mathrm{d}\tau$$
$$= \int_{B_r \cap [\omega < \omega_0]} -\widetilde{\xi} \, \frac{\partial\mathcal{Y}_h}{\partial\tau} \, \mathrm{d}\omega \, \mathrm{d}\tau + \int_S \mathcal{Y}_h \widetilde{\xi} \nu_\tau = \int_{B_r \cap [\omega < \omega_0]} -\widetilde{\xi} \, \frac{\partial\mathcal{Y}_h}{\partial\tau} \, \mathrm{d}\omega \, \mathrm{d}\tau.$$

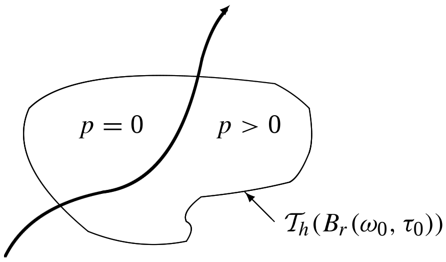

FIG. 7

It follows that

$$\int_{\mathcal{T}_h(B_r)} (\mathcal{A}(X, \nabla u) - \mathcal{A}(X, \nabla y)) \cdot \nabla\xi \, \mathrm{d}X = \int_{B_r} \widetilde{\xi} \, \frac{\partial \mathcal{Y}_h}{\partial \tau} \, \mathrm{d}\omega \, \mathrm{d}\tau - \int_{B_r \cap [\omega < \omega_0]} \widetilde{\xi} \, \frac{\partial \mathcal{Y}_h}{\partial \tau} \, \mathrm{d}\omega \, \mathrm{d}\tau$$
$$= \int_{B_r \cap [\omega > \omega_0]} \widetilde{\xi} \, \frac{\partial \mathcal{Y}_h}{\partial \tau} \, \mathrm{d}\omega \, \mathrm{d}\tau \geqslant 0.$$

Since $u \geqslant y$ in $\mathcal{T}_h(B_r)$, $u = y$ in $\mathcal{T}_h(B_r \cap [\omega \leqslant \omega_0])$, $\nabla y = e \neq 0$, we deduce from Lemma 3.1 that $u = y$ in $\mathcal{T}_h(B_r)$, which contradicts the fact that $u > y$ in $\mathcal{T}_h(B_r \cap [\omega > \omega_0])$.

(iii) Argue as in (ii). □

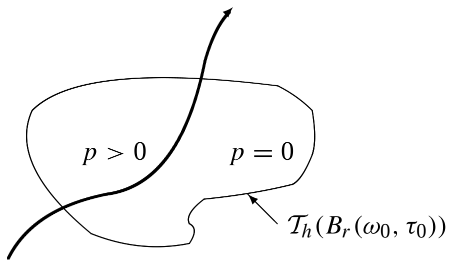

FIG. 8

5. Continuity of the free boundary

In this section we assume that $\mathcal{A}$ is strictly monotone in the following sense:

$$(\mathcal{A}(X, \xi) - \mathcal{A}(X, \zeta)) \cdot (\xi - \zeta) > 0 \quad \forall \xi \neq \zeta, \ \forall X \in \Omega. \tag{5.1}$$

Then we have

THEOREM 5.1 For each $h \in \pi_y(\Omega)$, the function Φ_h defined in (3.4) is continuous on $\pi_x(\Omega \cap [y = h])$.

Proof. It suffices to prove that Φ_h is upper semicontinuous. Let $X_0 = \mathcal{T}_h(\omega_0, \tau_0) \in \Omega \cap \partial[p > 0]$ and let $\epsilon > 0$ be small enough. Thanks to Theorem 3.1 and Remark 3.2 we have necessarily $X(\alpha_+(\omega_0), \omega_0) \notin S_3$. Two cases are to be distinguished:

(i) First we assume that $X(\alpha_+(\omega_0), \omega_0) \notin \overline{S}_3$, where $\overline{S}_3$ denotes the closure of S_3 relative to $\partial\Omega$. Since $p(X_0) = \widetilde{p}(\omega_0, \tau_0) = 0$ and $p, \ \alpha_+$ are continuous, there exists a ball $B_{\epsilon'}(\omega_0, \tau_0)$ $(0 < \epsilon' < \epsilon)$ such that

$$\begin{cases} \widetilde{p}(\omega, \tau) \leqslant \epsilon \quad \forall (\omega, \tau) \in B_{\epsilon'}(\omega_0, \tau_0), \\ X(\alpha_+(\omega), \omega) \notin \overline{S}_3 \quad \forall \omega \in (\omega_0 - \epsilon', \omega_0 + \epsilon'). \end{cases}$$

By Lemma 4.5, one of the following situations occurs:

$$\text{(a)} \quad \exists (\omega_1, \tau_1) \in B_{\epsilon'}(\omega_0, \tau_0) \quad \text{such that} \quad \omega_1 < \omega_0, \quad \widetilde{p}(\omega_1, \tau_1) = 0,$$
$$\text{(b)} \quad \exists (\omega_2, \tau_2) \in B_{\epsilon'}(\omega_0, \tau_0) \quad \text{such that} \quad \omega_2 > \omega_0, \quad \widetilde{p}(\omega_2, \tau_2) = 0.$$

Assume that for example (a) holds and set $X_1 = \mathcal{T}_h(\omega_1, \tau_1)$ and $\tau_M = \max(\tau_0, \tau_1)$ (see Figure 9).

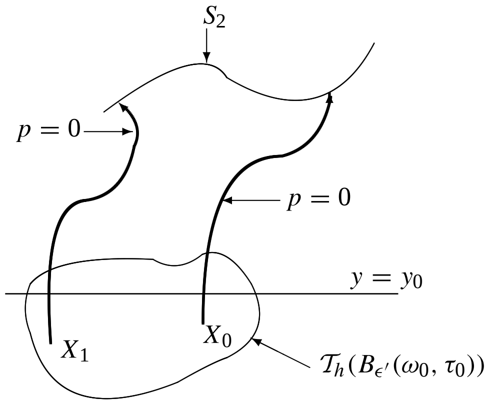

FIG. 9

Then by Theorem 3.1 we have

$$\widetilde{p}(\omega_i, \tau) = 0 \quad \forall(\omega_i, \tau) \in S_h(D_h) \text{ such that } \tau > \tau_M \ (i = 0, 1).$$

Set $Z_{\tau_M} = \mathcal{T}_h((\omega_1, \omega_0) \times (\tau_M, +\infty) \cap S_h(D_h))$ and let $y_0 \in \mathbb{R}$ be such that $\mathcal{T}_h^{-1}([y = y_0]) \cap B_{\epsilon'}(\omega_0, \tau_0) \cap [\tau > \tau_M] \neq \emptyset$. Note that

$$\mathcal{T}_h^{-1}([y = y_0]) = \left\{(\omega, \tau_{y_0}(\omega)) \in S_h(D_h) \;\middle|\; \tau_{y_0}(\omega) = \int_{\alpha_-(\omega)}^{t_{y_0}(\omega)} |\mathcal{A}(X(s, \omega), e)| \, \mathrm{d}s, \ X_2(t_{y_0}(\omega), \omega) = y_0\right\}.$$

Set

$$\begin{cases} D_{y_0, \tau_M} = [y > y_0] \cap Z_{\tau_M} \neq \emptyset, \\ v(y) = (\epsilon + y_0 - y)^+ + y, \\ \xi(x, y) = \chi(D_{y_0, \tau_M})(u - v)^+. \end{cases}$$

Since $v \geqslant y = u$ on $\partial D_{y_0, \tau_M} \setminus ([y = y_0])$, we have $\xi = 0$ on $\partial D_{y_0, \tau_M} \setminus ([y = y_0])$. Moreover $v(y_0) = \epsilon + y_0 \geqslant u(x, y_0)$ and then $\xi(x, y_0) = 0$. It follows that $\xi = 0$ on $\partial D_{y_0, \tau_M}$. So $\pm\xi$ are test functions for (P) and we have

$$\int_{D_{y_0, \tau_M}} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) \cdot \nabla(u - v)^+ \, \mathrm{d}X = 0. \tag{5.2}$$

We also have

$$\int_{D_{y_0, \tau_M}} (\mathcal{A}(X, \nabla v) - \chi([v = y])\mathcal{A}(X, e)) \cdot \nabla(u - v)^+ \, \mathrm{d}X = 0. \tag{5.3}$$

Subtracting (5.3) from (5.2), we obtain

$$\int_{D_{y_0, \tau_M} \cap [v > y]} (\mathcal{A}(X, \nabla u) - \mathcal{A}(X, \nabla v)) \cdot \nabla(u - v)^+ \, \mathrm{d}X + \int_{D_{y_0, \tau_M} \cap [v = y]} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) \cdot \nabla(u - y) \, \mathrm{d}X = 0. \tag{5.4}$$

By Lemma 4.1, for $[y > y_0 + \epsilon] \cap Z_{\tau_M} = D_{y_0,\tau_M} \cap [v = y]$ we have

$$\int_{D_{y_0,\tau_M}\cap[v=y]} (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) \cdot e \,\mathrm{d}X \leqslant 0. \tag{5.5}$$

Adding (5.4) and (5.5), by taking into account (P)(i) we get

$$\begin{aligned}&\int_{D_{y_0,\tau_M}\cap[v>y]} (\mathcal{A}(X, \nabla u) - \mathcal{A}(X, \nabla v)) \cdot \nabla(u - v)^+ \,\mathrm{d}X \\ &+ \int_{D_{y_0,\tau_M}\cap[v=y]\cap[u>y]} \mathcal{A}(X, \nabla u) \cdot \nabla u \,\mathrm{d}X + \int_{D_{y_0,\tau_M}\cap[v=y]\cap[u=y]} (1 - g)\mathcal{A}(X, e) \cdot e \,\mathrm{d}X \leqslant 0.\end{aligned}$$

Since the three integrals on the left hand side of the above inequality are all nonnegative, we conclude by (5.1) that $\nabla(u - v)^+ = 0$ a.e. in $D_{y_0,\tau_M} \cap [v > y]$ and then, since $(u - v)^+ = 0$ on $\partial D_{y_0,\tau_M}$, we get $u \leqslant v$ in $D_{y_0,\tau_M} \cap [v > y]$. This leads to $p(x, y_0 + \epsilon) = 0$ for all $x \in \pi_x(D_{y_0,\tau_M})$. Now for each $\omega \in (\omega_1, \omega_0)$, there exists a unique $t_{y_0,\epsilon}(\omega) \in (\alpha_-(\omega), \alpha_+(\omega))$ such that

$$X_2(t_{y_0,\epsilon}(\omega), \omega) = y_0 + \epsilon, \qquad p(X_1(t_{y_0,\epsilon}(\omega), \omega), y_0 + \epsilon) = 0,$$

and if $\tau_{y_0,\epsilon}(\omega) = L_{y_0+\epsilon}(t_{y_0,\epsilon}(\omega), \omega)$, then we obtain $\widetilde{p}(\omega, \tau_{y_0,\epsilon}(\omega)) = p \circ T_h(t_{y_0,\epsilon}(\omega), \omega) = p(X_1(t_{y_0,\epsilon}(\omega), \omega), y_0 + \epsilon) = 0$ for all $\omega \in (\omega_1, \omega_0)$. Therefore $\Phi_h(\omega) \leqslant \tau_{y_0,\epsilon}(\omega)$. But since X_2 is increasing with respect to t, and $X_2(t_{y_0}(\omega), \omega) = y_0$ and $X_2(t_{y_0,\epsilon}(\omega), \omega) = y_0 + \epsilon$, it follows that $t_{y_0}(\omega) < t_{y_0,\epsilon}(\omega)$ and then

$$\begin{aligned}\epsilon &= X_2(t_{y_0,\epsilon}(\omega), \omega) - X_2(t_{y_0}(\omega), \omega) = \int_{t_{y_0}(\omega)}^{t_{y_0,\epsilon}(\omega)} a^2(X(s, \omega)) \,\mathrm{d}s \\ &\geqslant \lambda(t_{y_0,\epsilon}(\omega) - t_{y_0}(\omega))\end{aligned}$$

and

$$\begin{aligned}\tau_{y_0,\epsilon}(\omega) &= \tau_{y_0}(\omega) + \int_{t_{y_0}(\omega)}^{t_{y_0,\epsilon}(\omega)} |\mathcal{A}(X(s, \omega), e)| \,\mathrm{d}s \\ &\leqslant \tau_{y_0}(\omega) + M(t_{y_0,\epsilon}(\omega) - t_{y_0}(\omega)) \leqslant \tau_{y_0}(\omega) + \frac{M}{\lambda}\epsilon.\end{aligned}$$

So

$$\Phi_h(\omega) \leqslant \tau_{y_0}(\omega) + \frac{M}{\lambda}\epsilon < \tau_0 + \epsilon' + \frac{M}{\lambda}\epsilon < \Phi_h(\omega_0) + \left(1 + \frac{M}{\lambda}\right)\epsilon \quad \forall \omega \in (\omega_1, \omega_0).$$

Hence Φ_h is u.s.c. at ω_0 from the left. Using Lemma 4.5 and arguing as above, one can prove that Φ_h is u.s.c. at ω_0 from the right. Thus Φ_h is continuous at ω_0.

(ii) Now we assume that $X(\alpha_+(\omega_0), \omega_0) \in \overline{S}_3 \setminus S_3$. Then one of the following situations holds:

(a) $\exists \eta > 0\ \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta) \quad X(\alpha_+(\omega), \omega) \in S_3 \iff \omega \in (\omega_0, \omega_0 + \eta)$
(b) $\exists \eta > 0\ \forall \omega \in (\omega_0 - \eta, \omega_0 + \eta) \quad X(\alpha_+(\omega), \omega) \in S_3 \iff \omega \in (\omega_0 - \eta, \omega_0)$.

Assume for example that (a) holds. Then it is easy to see that

$$\Phi_h(\omega) = L_h(\alpha_+(\omega), \omega) \quad \forall \omega \in (\omega_0, \omega_0 + \eta). \tag{5.6}$$

Arguing as in (i) one can prove that Φ_h is continuous at ω_0 from the left. On the other hand, we deduce from (5.6) that $u > y$ in a right neighborhood of the curve $X(\cdot, \omega_0)$. Using the continuity from the left and Lemma 4.5, we have necessarily $\Phi_h(\omega_0) = L_h(\alpha_+(\omega_0), \omega_0)$. Therefore we now have

$$\Phi_h(\omega) = L_h(\alpha_+(\omega), \omega) \quad \forall \omega \in [\omega_0, \omega_0 + \eta),$$

which leads to the continuity of Φ_h from the right at ω_0.

We argue similarly if (b) holds. □

REMARK 5.1 (i) For each $X_0 \in \Omega \cap \partial[p > 0]$, there exists $h \in \pi_y(\Omega)$ such that $X_0 \in T_h(D_h) \cap \partial[p > 0]$ and then $X_0 = \mathcal{T}_h(\omega_0, \tau_0)$ with $\tau_0 = \Phi_h(\omega_0)$. Therefore from Theorem 5.1, the free boundary is represented in a neighborhood of X_0 by the graph of the continuous function Φ_h.

(ii) Since the free boundary is now represented locally by graphs of continuous functions, it follows in particular that $\partial[p > 0]$ is of Lebesgue measure zero.

The following result expresses the fact that g is the characteristic function of the dry region.

COROLLARY 5.1 Let (u, g) be a solution of (P). Then

$$g = \chi([p = 0]) = \chi([u = y]). \tag{5.7}$$

Proof. First by (P)(i), we have $g = 0$ in $[p > 0]$. Next if $(x_0, y_0) \in \Omega \setminus \overline{[p > 0]}$, then since $\Omega \setminus \overline{[p > 0]}$ is an open set there exists $\epsilon_0 > 0$ small enough and $h \in \pi_y(\Omega)$ such that $B_{\epsilon_0}(x_0, y_0) \subset (\Omega \setminus \overline{[p > 0]}) \cap T_h(D_h)$.

By Lemma 4.4, $\widetilde{g} = 1$ a.e. in $\mathcal{T}_h^{-1}(B_{\epsilon_0}(x_0, y_0))$ or equivalently $g = 1$ in $B_{\epsilon_0}(x_0, y_0)$. Therefore $g = 1$ in $\Omega \setminus \overline{[p > 0]}$.

Since the set $\partial[p > 0] \cap \Omega$ is of Lebesgue measure zero, we conclude that $g = \chi([p = 0]) = \chi([u = y])$. □

6. Uniqueness of reservoirs-connected solutions

DEFINITION 6.1 A solution (u, g) of (P) is a *reservoirs-connected solution* if for each connected component C of $[u > y]$, we have $\overline{C} \cap S_3 \neq \emptyset$.

REMARK 6.1 Thanks to Remark 3.2, if $\overline{C} \supset S_{3,i}$ $(i = 1, \dots, N)$, then C contains the strip of Ω below $S_{3,i}$ in the following sense:

$$\bigcup_{h \in \pi_y(\Omega)} (\mathcal{T}_h((\pi_\omega(\mathcal{S}_{3,i}^h) \times \mathbb{R}) \cap S_h(D_h)) \subset C, \quad \text{where} \quad \mathcal{S}_{3,i}^h = \mathcal{T}_h^{-1}(S_{3,i}).$$

THEOREM 6.1 Let (u, g) be a solution of (P) and C a connected component of $[u > y]$ such that $\overline{C} \cap S_3 = \emptyset$. Set $h_C = \sup\{y \mid (x, y) \in C\}$. Then

$$u(x, y) = (h_C - y)^+ \chi(C) + y, \quad g = 1 - \chi(C)$$

for

$$(x, y) \in \bigcup_{h\in\pi_y(\Omega)} (\mathcal{T}_h(\pi_\omega(\mathcal{C}_h) \times \mathbb{R} \cap S_h(D_h))), \quad \text{where} \quad \mathcal{C}_h = \mathcal{T}_h^{-1}(C \cap T_h(\Omega)).$$

Proof. Since $\pm\chi(C)(u - y)$ are test functions for (P), we have

$$\int_C (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) \cdot \nabla(u - y)\, \mathrm{d}X = 0. \tag{6.1}$$

By Lemma 4.3, we have

$$\int_C (\mathcal{A}(X, \nabla u) - g\mathcal{A}(X, e)) \cdot e\, \mathrm{d}X \leqslant 0. \tag{6.2}$$

Adding (6.1) and (6.2), we obtain

$$\int_{C\cap[u>y]} \mathcal{A}(X, \nabla u) \cdot \nabla u\, \mathrm{d}X + \int_{C\cap[u=y]} (1 - g)\mathcal{A}(X, e) \cdot e\, \mathrm{d}X \leqslant 0.$$

It follows that $\nabla u = 0$ a.e. in C and so u is equal to some positive constant k in C, which can be easily verified to be equal to h_C.

Using Theorem 3.1 and (5.7) we deduce that $u = (h_C - y)^+\chi(C) + y$ and $g = 1 - \chi(C)$ in $\bigcup_{h\in\pi_y(\Omega)}(\mathcal{T}_h(\pi_\omega(\mathcal{C}_h) \times \mathbb{R} \cap S_h(D_h)))$. □

DEFINITION 6.2 We define a *pool* in Ω to be a pair $(\overline{u}, \overline{g})$ of functions defined in Ω by

$$\overline{u} = (h_C - y)^+\chi(C) + y \quad \text{and} \quad \overline{g} = 1 - \chi(C) \quad \text{a.e. in } \Omega,$$

where C is a subdomain of Ω and $h_C = \max\{y \mid (x, y) \in C\}$.

THEOREM 6.2 Each solution of (P) can be written as the sum of a reservoirs-connected solution and pools.

Proof. See [14], [15], [17] and [27]. □

In order to prove the uniqueness of the reservoirs-connected solution we assume that S_3 is of class $C^{1,\alpha}$, and Ω is covered by a finite number of sets $T_h(D_h)$, that is, there are $h_1, \ldots, h_n \in \pi_y(\Omega)$ such that

$$\Omega = \bigcup_{k=1}^{n} T_{h_k}(D_{h_k}). \tag{6.3}$$

We also assume that

$$\mathcal{A}(X, \xi) = |a(X)\xi \cdot \xi|^{(q-2)/2} a(X)\xi, \quad \text{where} \quad a(X) \in C^1(\overline{\Omega}) \text{ is a 2-by-2 matrix.} \tag{6.4}$$

Then we can state our uniqueness theorem

THEOREM 6.3 Under assumptions (6.3)–(6.4), the reservoirs-connected solution is unique.

The proof of Theorem 6.3 requires three lemmas.

LEMMA 6.1 Let (u_1, g_1) and (u_2, g_2) be two solutions of (P). Then for $i = 1, 2$ and $\zeta \in W^{1,q}(\Omega)$ we have

$$\mathcal{F}_i(\zeta) = \int_\Omega ((\mathcal{A}(X, \nabla u_i) - \mathcal{A}(X, \nabla u_m)) - (g_i - g_M)\mathcal{A}(X, e)) \cdot \nabla\zeta \, \mathrm{d}X = 0,$$

where $u_m = \min(u_1, u_2)$ and $g_M = \max(g_1, g_2)$.

To prove Lemma 6.1 we need another lemma:

LEMMA 6.2 Let (u_1, g_1) and (u_2, g_2) be two solutions of (P) and $\Phi^i_{h_k}$ $(i = 1, 2$ and $k = 1, \ldots, n)$ be the corresponding free boundary functions. Then for $i = 1, 2$ and $\zeta \in W^{1,q}(\Omega) \cap C^0(\overline{\Omega})$ with $\zeta \geqslant 0$ we have

$$\mathcal{F}_i(\zeta) \leqslant \sum_{k=1}^n \int_{\mathcal{D}^i_{h_k}} \mathcal{Y}_{h_k}(\omega, \Phi^i_{h_k}(\omega)) \cdot \zeta \circ T_{h_k} \circ S^{-1}_{h_k}(\omega, \Phi^i_{h_k}(\omega)) \, \mathrm{d}\omega,$$

where $\mathcal{D}^i_{h_k} = \{\omega \in \pi_\omega(S_{h_k}(D_{h_k})) \mid \Phi^0_{h_k}(\omega) = \min(\Phi^1_{h_k}(\omega), \Phi^2_{h_k}(\omega)) < \Phi^i_{h_k}(\omega)\}$.

Proof. First thanks to (6.3) there exists a partition of unity $(\theta_k)^n_{k=1}$ corresponding to the open covering $(T_{h_k}(D_{h_k}))^n_{k=1}$ of Ω, that is,

$$\theta_k \in \mathcal{D}(T_{h_k}(D_{h_k})), \quad 0 \leqslant \theta_k \leqslant 1 \quad \forall k = 1, \ldots, n, \quad \sum_{k=1}^n \theta_k = 1 \quad \text{in } \Omega. \tag{6.5}$$

Let $\zeta \in W^{1,q}(\Omega) \cap C^0(\overline{\Omega})$, $\zeta \geqslant 0$ and let $\zeta_k = \theta_k \zeta$. Since $\mathcal{F}_i(\zeta) = \sum_{k=1}^n \mathcal{F}_i(\zeta_k)$, it suffices to show that

$$\mathcal{F}_i(\zeta_k) \leqslant \int_{\mathcal{D}^i_{h_k}} \mathcal{Y}_{h_k}(\omega, \Phi^i_{h_k}(\omega)) \cdot \zeta_k \circ T_{h_k} \circ S^{-1}_{h_k}(\omega, \Phi^i_{h_k}(\omega)) \, \mathrm{d}\omega. \tag{6.6}$$

So let $\epsilon > 0$ and $\xi_k = \min(\zeta_k, (u_1 - u_2)^+/\epsilon)$. Clearly $\pm\xi_k$ are test functions for (P) and we have

$$\int_{T_{h_k}(D_{h_k})} ((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_2)) - (g_1 - g_2)\mathcal{A}(X, e)) \cdot \nabla\xi_k \, \mathrm{d}X = 0. \tag{6.7}$$

Since we integrate only on the set $[u_1 > u_2]$ where $u_2 = u_m$, (6.7) is equivalent to $\mathcal{F}_1(\xi_k) = 0$, which can be written as

$$\int_{T_{h_k}(D_{h_k}) \cap [(u_1 - u_2)^+ > \epsilon\zeta_k]} ((\mathcal{A}(X, \nabla u_1) - \mathcal{A}(X, \nabla u_m)) \cdot \nabla\zeta_k \, \mathrm{d}X - \int_{T_{h_k}(D_{h_k})} (g_1 - g_M)\mathcal{A}(X, e) \cdot \nabla\zeta_k \, \mathrm{d}X$$
$$\leqslant -\int_{T_{h_k}(D_{h_k})} (g_1 - g_M)\mathcal{A}(X, e) \cdot \nabla\left(\zeta_k - \frac{(u_1 - u_2)^+}{\epsilon}\right)^+ \mathrm{d}X = I^k_\epsilon.$$

Using the C^1 diffeomorphisms T_{h_k} and S_{h_k}, for $\widetilde{g}_{1k} = g_1 \circ T_{h_k} \circ S^{-1}_{h_k}$ and $\widetilde{g}_{Mk} = g_M \circ T_{h_k} \circ S^{-1}_{h_k}$ we obtain

$$I^k_\epsilon = -\int_{S_{h_k}(D_{h_k})} (\widetilde{g}_{1k} - \widetilde{g}_{Mk}) \cdot \mathcal{Y}_{h_k}(\omega, \tau) \frac{\partial}{\partial\tau}\left(\widetilde{\zeta}_k - \frac{\widetilde{(u_1 - u_2)^+}}{\epsilon}\right)^+ \mathrm{d}\omega \, \mathrm{d}\tau$$

$$= \int_{[\widetilde{p}_{1k}>0=\widetilde{p}_{mk}=\widetilde{p}_{2k}]} \mathcal{Y}_{h_k}(\omega,\tau)\frac{\partial}{\partial\tau}\left(\widetilde{\zeta}_k - \frac{\widetilde{(u_1-u_2)^+}}{\epsilon}\right)^+ \mathrm{d}\omega\,\mathrm{d}\tau$$

$$= \int_{[\Phi^2_{h_k}(\omega)<\tau<\Phi^1_{h_k}(\omega)]} \mathcal{Y}_{h_k}(\omega,\tau)\frac{\partial}{\partial\tau}\left(\widetilde{\zeta}_k - \frac{\widetilde{(u_1-u_2)^+}}{\epsilon}\right)^+ \mathrm{d}\omega\,\mathrm{d}\tau$$

$$= \int_{\mathcal{D}^1_{h_k}} \int_{\Phi^2_{h_k}(\omega)}^{\Phi^1_{h_k}(\omega)} \mathcal{Y}_{h_k}(\omega,\tau)\frac{\partial}{\partial\tau}\left(\widetilde{\zeta}_k - \frac{\widetilde{(u_1-u_2)^+}}{\epsilon}\right)^+ \mathrm{d}\omega\,\mathrm{d}\tau.$$

By the second mean value theorem there exists $\Phi^*_k(\omega) \in [\Phi^2_{h_k}(\omega), \Phi^1_{h_k}(\omega)]$ such that

$$I^k_\epsilon = \int_{\mathcal{D}^1_{h_k}} \mathcal{Y}_{h_k}(\omega,\Phi^1_k(\omega)) \int_{\Phi^*_k(\omega)}^{\Phi^1_{h_k}(\omega)} \frac{\partial}{\partial\tau}\left(\widetilde{\zeta}_k - \frac{\widetilde{(u_1-u_2)^+}}{\epsilon}\right)^+ \mathrm{d}\omega\,\mathrm{d}\tau$$

$$\leqslant \int_{\mathcal{D}^1_{h_k}} \mathcal{Y}_{h_k}(\omega,\Phi^1_{h_k}(\omega))\cdot\widetilde{\zeta}_k(\omega,\Phi^1_{h_k}(\omega))\,\mathrm{d}\omega.$$

It follows that

$$\int_{T_{h_k}(D_{h_k})\cap[(u_1-u_2)^+>\epsilon\zeta_k]} (\mathcal{A}(X,\nabla u_1)-\mathcal{A}(X,\nabla u_m))\cdot\nabla\zeta_k\,\mathrm{d}X$$

$$- \int_{T_{h_k}(D_{h_k})} (g_1-g_M)\mathcal{A}(X,e)\cdot\nabla\zeta_k\,\mathrm{d}X \leqslant \int_{\mathcal{D}^1_{h_k}} \mathcal{Y}_{h_k}(\omega,\Phi^1_{h_k}(\omega))\cdot\widetilde{\zeta}_k(\omega,\Phi^1_{h_k}(\omega))\,\mathrm{d}\omega.$$

Letting ϵ go to 0, we get (6.6) for $i=1$. The proof for $i=2$ is similar. □

Proof of Lemma 6.1. Let $\zeta \in C^1(\overline{\Omega})$, $\zeta \geqslant 0$. For $\delta > 0$ we set

$$A_m = \bigcup_{k=1}^n [\widetilde{p}_{mk} > 0], \qquad \alpha_\delta(\omega,\tau) = \left(1 - \frac{d((\omega,\tau),A_m)}{\delta}\right)^+, \qquad h_\delta = \alpha_\delta \circ S_{h_k} \circ T^{-1}_{h_k}$$

and remark that $\mathcal{F}_1(\zeta) = \mathcal{F}_1(h_\delta\zeta) + \mathcal{F}_1((1-h_\delta)\zeta)$. By the previous lemma we have

$$\mathcal{F}_1(h_\delta\zeta) \leqslant \sum_{k=1}^n \int_{\mathcal{D}^1_{h_k}} \mathcal{Y}_{h_k}(\omega,\Phi^1_{h_k}(\omega))\widetilde{\zeta}_k(\omega,\Phi^1_{h_k}(\omega))\alpha_\delta(\omega,\Phi^1_{h_k}(\omega))\,\mathrm{d}\omega.$$

Since $(1-h_\delta)\zeta$ is a test function for (P), we have

$$\int_\Omega (\mathcal{A}(X,\nabla u_1) - g_1\mathcal{A}(X,e))\cdot\nabla((1-h_\delta)\zeta)\,\mathrm{d}X \leqslant 0.$$

Moreover the function $1-\alpha_\delta$ vanishes on $\overline{A}_m$ and $g_M = 1$ in $\Omega\setminus\overline{A}_m$. So

$$\int_\Omega (\mathcal{A}(X,\nabla u_m) - g_M\mathcal{A}(X,e))\cdot\nabla((1-h_\delta)\zeta)\,\mathrm{d}X = 0.$$

This leads to $\mathcal{F}_1((1-h_\delta)\zeta) \leqslant 0$ and

$$\mathcal{F}_1(\zeta) \leqslant \sum_{k=1}^{n} \int_{\mathcal{D}^1_{h_k}} \mathcal{Y}_{h_k}(\omega, \Phi^1_{h_k}(\omega))\widetilde{\zeta}_k(\omega, \Phi^1_{h_k}(\omega))\alpha_\delta(\omega, \Phi^1_{h_k}(\omega))\,\mathrm{d}\omega.$$

Letting $\delta \to 0$, we deduce $\mathcal{F}_1(\zeta) \leqslant 0$. It is then easy (see [15], [17], [27]) to show that $\mathcal{F}_1(\zeta) = 0$ for all $\zeta \in W^{1,q}(\Omega)$. Similarly we obtain $\mathcal{F}_2(\zeta) = 0$ for all $\zeta \in W^{1,q}(\Omega)$. □

LEMMA 6.3 Let Ω_0 be a domain of $\mathbb{R}^2$, $\Gamma_0 \subset \partial\Omega_0$ of class $C^{1,\alpha}$ and let $u_1, u_2 \in W^{1,q}_{\mathrm{loc}}(\Omega_0)$ be such that:

$$\begin{cases} \text{(i)} & \operatorname{div}(\mathcal{A}(X, \nabla u_1)) = \operatorname{div}(\mathcal{A}(X, \nabla u_2)) = 0 \quad \text{in } \mathcal{D}'(\Omega_0), \\ \text{(ii)} & u_1 \leqslant u_2 \quad \text{in } \Omega_0, \\ \text{(iii)} & u_1 = u_2 \text{ on } \Gamma_0, \quad u_1, u_2 \in C^1(\Omega_0 \cup \Gamma_0), \\ \text{(iv)} & \mathcal{A}(X, \nabla u_1) \cdot \nu = \mathcal{A}(X, \nabla u_2) \cdot \nu \quad \text{on } \Gamma_0, \\ \text{(v)} & \nabla u_1(X) \neq 0 \quad \forall X \in \Gamma_0 \quad \text{or} \quad \nabla u_2(X) \neq 0 \quad \forall X \in \Gamma_0. \end{cases}$$

Then $u_1 = u_2$ in Ω.

Proof. See [16]. □

Proof of Theorem 6.3. When $q = 2$ the proof is given in [27]. For the general case we use Lemma 6.3.

Let (u_1, g_1), (u_2, g_2) be two solutions of (P). By Lemma 6.1 one can see that (u_m, g_M) is also a solution of (P). Let $C_{m,i}$ be the connected component of the set $[u_m > y]$ that contains $S_{3,i}$ on its boundary. By Lemma 6.1 we deduce easily that u_1 and u_m satisfy the conditions (i) and (iv) of Lemma 6.3 with $\Omega_0 = C_{m,i}$ and $\Gamma_0 = S_{3,i}$.

The condition (ii) is obviously satisfied and the first part of (iii) is true since $u_1 = u_2 = \varphi + y$ on S_3. The second part of (iii) is also true (see [19] and [26]). So if we show that (v) is satisfied we will get $u_1 = u_m$ in $C_{m,i}$. For this purpose, we distinguish two cases:

- If ∇u_1 and ∇u_m are not identically zero on Γ_0 then by (iii) there exists $\Gamma_0' \subset \Gamma_0$ such that $\nabla u_1 \neq 0$ on Γ_0' or $\nabla u_m \neq 0$ on Γ_0'. Therefore (v) is satisfied on Γ_0'.
- If $\nabla u_1 = 0$ on Γ_0 and $\nabla u_m = 0$ on Γ_0, then u_1 and u_m are both constant along Γ_0. Since $u_1 = u_m$ on S_3, it follows that $u_1 = u_m = h_i$ on Γ_0 for some constant h_i. Therefore one can extend u_1 and u_m into $B \setminus \Omega$ by h_i where B is a ball centered at a point of $S_{3,i} = \Gamma_0$ in such a way that $u_j \in C^1(C_{m,i} \cup B)$ and $\operatorname{div}(\mathcal{A}(X, \nabla u_j)) = 0$ in $\mathcal{D}'(C_{m,i} \cup B)$. So ∇u_j has nonisolated zeros and then (see [4]) $u_j = h_i$ in $C_{m,i}$ $(j = 1, m)$. Thus $u_1 = u_m$ in $C_{m,i}$.

One can show as in [15], [17], [27] that $C_{1,i} = C_{m,i}$ and $u_1 = u_m$ in $C_{1,i}$. In the same way we prove that $u_2 = u_m$ in $C_{2,i} = C_{m,i}$. We conclude that $u_1 = u_2$ in $C_{1,i} = C_{2,i}$ for all $i = 1, \ldots, N$. This means that $u_1 = u_2$ in $[u_1 > y] = [u_2 > y]$, which leads to $u_1 = u_2$ in Ω. Finally, we deduce from (5.7) that $g_1 = g_2$ in Ω. □

REMARK 6.2 The assumption (6.4) is needed only to ensure the isolation of critical points of $\mathcal{A}$-harmonic functions (see [1]).

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