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On the stability of nonsymmetric equilibrium figures of a rotating viscous incompressible liquid

V. A. SOLONNIKOV†

St. Petersburg Branch of V. A. Steklov Mathematical Institute of the Russian Academy of Sciences, Fontanka 27, 191023, St. Petersburg, Russia

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We consider a classical problem of stability of equilibrium figures of a liquid rotating uniformly as a rigid body about a fixed axis. We connect the problem of stability with the behavior for large t of solutions of an evolution problem governing the motion of an isolated liquid mass whose initial data are slight perturbations of the regime of a rigid rotation. The main attention is given to the case when the figure is not rotationally symmetric; in this case the regime of a rigid rotation defines a periodic solution of the above-mentioned nonstationary problem. It is proved that a sufficient condition of stability is the positivity of the second variation of the energy functional in an appropriate function space.

1. Introduction

The problem of the shape and stability of equilibrium figures of a uniformly rotating isolated liquid mass has drawn attention of many generations of mathematicians, beginning with I. Newton. A review of results obtained in the past and of some recent contributions can be found in [1, 7]. We recall that if the liquid rotating with constant angular velocity ω_0 about the x_3-axis is subjected to capillary forces at the boundary (which is assumed to be free) and to the forces of self-gravitation, then the equilibrium figure $\mathcal{F}$ is defined by the equation

$$\sigma\mathcal{H}(x) + \frac{\omega_0^2}{2}(x_1^2 + x_2^2) + \kappa\mathcal{U}(x) + p_0 = 0, \qquad x \in \mathcal{G} \equiv \partial\mathcal{F}, \tag{1.1}$$

which should be satisfied at the boundary $\mathcal{G}$ of the domain $\mathcal{F}$. Here $p_0 = \text{const}$, $\mathcal{H}$ is twice the mean curvature of $\mathcal{G}$, negative for convex domains, $\mathcal{U}(x) = \int_{\mathcal{F}} |x - y|^{-1}\,\mathrm{d}y$ is the Newtonian potential, and σ and κ are the constant coefficient of surface tension and the gravitational constant, respectively. The case $\kappa = 0$ corresponding to the absence of self-gravitation is not excluded but σ should be positive. The density of the liquid is assumed to equal one.

Equation (1.1) is the Euler equation for the functional

$$R = \sigma|\Gamma| + \frac{\beta^2}{2\int_\Omega (x_1^2 + x_2^2)\,\mathrm{d}x} - \frac{\kappa}{2}\int_\Omega\int_\Omega \frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|} - p_0|\Omega|, \qquad \Gamma = \partial\Omega, \tag{1.2}$$

where Ω is a domain in $\mathbb{R}^3$ close to $\mathcal{F}$ with the same volume $|\Omega|$ and the same position of the barycenter as $\mathcal{F}$, $\Gamma = \partial\Omega$, $|\Gamma| = \text{mes}\,\Gamma$, and

$$\beta = \omega_0 \int_{\mathcal{F}} (x_1^2 + x_2^2)\,\mathrm{d}x.$$

†Email: slk@dns.unife.it

is the magnitude of the total angular momentum of the rotating liquid. We assume that the barycenter of $\mathcal{F}$ coincides with the origin, and hence

$$|\Omega| = |\mathcal{F}|, \qquad \int_\Omega x_k \,\mathrm{d}x = \int_{\mathcal{F}} x_k \,\mathrm{d}x = 0, \qquad k = 1, 2, 3. \tag{1.3}$$

The fact that Ω is close to $\mathcal{F}$ means that Γ can be determined by the equation

$$x = y + N(y)\rho(y), \qquad y \in \mathcal{G}, \tag{1.4}$$

where $N(y)$ is the exterior normal to $\mathcal{G}$ and $\rho(y)$ is a certain small function; we assume that

$$|\rho|_{C^1(\mathcal{G})} = \delta \ll 1. \tag{1.5}$$

The restrictions (1.3) can be expressed in terms of ρ in the form

$$\int_{\mathcal{G}} \varphi(y;\rho)\,\mathrm{d}S_y = 0, \qquad \int_{\mathcal{G}} \psi_k(y;\rho)\,\mathrm{d}S_y = 0, \qquad k = 1, 2, 3, \tag{1.6}$$

where

$$\varphi(y;\rho) = \rho(y) - \frac{\rho^2(y)}{2}\mathcal{H}(y) + \frac{\rho^3(y)}{3}\mathcal{K}(y), \tag{1.7}$$

$$\psi_k(y;\rho) = y_k\varphi(y;\rho) + N_k(y)\left(\frac{\rho^2(y)}{2} - \frac{\rho^3(y)}{3}\mathcal{H}(y) + \frac{\rho^4(y)}{4}\mathcal{K}(y)\right), \tag{1.8}$$

and $\mathcal{K}(y)$ is the Gaussian curvature of $\mathcal{G}$.

Hence, R can be regarded as a functional defined on the set of small functions $\rho(y)$ described above, and it can be shown that its first variation vanishes:

$$\begin{aligned}\delta_0 R[\rho] &\equiv \frac{\partial}{\partial\lambda} R[\lambda\rho]\bigg|_{\lambda=0} \\ &= -\int_{\mathcal{G}}\left(\sigma\mathcal{H}(x) + \frac{\omega_0^2}{2}(x_1^2 + x_2^2) + \kappa\mathcal{U}(x) + p_0\right)\rho(x)\,\mathrm{d}S_x = 0,\end{aligned}$$

by (1.1).

It was conjectured in the papers of H. Poincaré and A. M. Lyapunov that a sufficient condition of the stability of the equilibrium figure is the positivity of the second variation of the energy functional. For the functional (1.2) the second variation is given by the formula

$$\begin{aligned}\delta_0^2 R[\rho] \equiv \frac{\partial^2}{\partial\lambda^2} R[\lambda\rho]\bigg|_{\lambda=0} &= \int_{\mathcal{G}}(\sigma|\nabla_{\mathcal{G}}\rho(y)|^2 - b(y)\rho^2(y))\,\mathrm{d}S_y \\ &+ \frac{\omega_0^2}{\mathcal{I}}\left(\int_{\mathcal{G}}\rho(y)(y_1^2 + y_2^2)\,\mathrm{d}S_y\right)^2 - \kappa\int_{\mathcal{G}}\int_{\mathcal{G}}\rho(y)\rho(z)\frac{\mathrm{d}S_y\,\mathrm{d}S_z}{|y-z|},\end{aligned} \tag{1.9}$$

where

$$b(y) = \sigma(\mathcal{H}^2 - 2\mathcal{K}) + \frac{\omega_0^2}{2}\frac{\partial}{\partial N}(y_1^2 + y_2^2) + \kappa\frac{\partial\mathcal{U}}{\partial N}, \qquad \mathcal{I} = \int_{\mathcal{F}}(y_1^2 + y_2^2)\,\mathrm{d}y$$

(see [1, 5–7, 18]). This criterion is now generally accepted but its justification given in [6, 1] is far from being complete because it is made under some a-priori assumptions concerning the perturbed free boundary of the liquid. Moreover, the corresponding evolution free boundary problem for the perturbation has not even been formulated. It was pointed out in [6] that a more careful justification of the principle of minimum of the energy functional based on the study of a perturbed motion of the liquid is highly desirable.

Our conclusion about stability of the equilibrium figures is based on the analysis of the above-mentioned evolution problem that consists in the determination of the bounded domain Ω_t, $t > 0$, the velocity vector field $\vec{v}(x, t)$ and the pressure function $p(x, t)$, $x \in \Omega_t$, satisfying the Navier–Stokes equations

$$\vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} - \nu \nabla^2 \vec{v} + \nabla p = 0, \quad \nabla \cdot \vec{v}(x, t) = 0, \quad x \in \Omega_t,\ t > 0, \tag{1.10}$$

as well as the dynamic and kinematic boundary conditions on the free surface $\Gamma_t = \partial\Omega_t$, namely,

$$T(\vec{v}, p)\vec{n} = (\sigma H + \kappa U(x, t))\vec{n}, \quad V_n = \vec{v} \cdot \vec{n}. \tag{1.11}$$

Here ν is a constant positive viscosity coefficient, $T(\vec{v}, p) = -pI + \nu S(\vec{v})$ is the stress tensor, $S(\vec{v}) = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)_{i,j=1,2,3}$ is the doubled rate-of-strain tensor, H is twice the mean curvature of Γ_t, V_n is the velocity of motion of Γ_t in the direction of the exterior normal $\vec{n}$, and

$$U(x, t) = \int_{\Omega_t} \frac{\mathrm{d}y}{|x - y|}$$

is the Newtonian potential calculated in the unknown domain Ω_t. Finally, the initial condition

$$\vec{v}(x, 0) = \vec{v}_0(x), \quad x \in \Omega_0, \tag{1.12}$$

is prescribed with a given Ω_0 whose boundary Γ_0 is defined by equation (1.4) with a given small $\rho = \rho_0(y)$ satisfying (1.5), (1.6). Concerning $\vec{v}_0$ it is assumed that it is close to the velocity vector field of a rigid rotation about the x_3-axis

$$\vec{\mathcal{V}}(x) = \omega_0(-x_2, x_1, 0) = \omega_0(\vec{e}_3 \times \vec{x}),$$

and that it satisfies the conditions

$$\int_{\Omega_0} \vec{v}_0(x)\,\mathrm{d}x = 0, \quad \int_{\Omega_0} (\vec{x} \times \vec{v}_0(x))\,\mathrm{d}x = \beta\vec{e}_3, \tag{1.13}$$

like $\vec{\mathcal{V}}$, and some natural compatibility conditions.

We say that the figure $\mathcal{F}$ is *stable* when the problem (1.10)–(1.12) is solvable in an infinite time interval $t > 0$ and the solution tends to the regime of a rigid rotation as $t \to \infty$.

The fact that a rigid rotation can be a limiting regime for the solutions of (1.10)–(1.12) as $t \to \infty$ was discovered in the papers [12, 13] in the case when β is small and $\mathcal{F}$ is close to a ball. In [9] it was shown that the convergence of the solution of (1.10)–(1.12) to this limiting regime is exponential. In [10, 14–16] the condition of smallness of β was replaced with the condition of the positivity of the second variation of the functional

$$G = \sigma|\Gamma| - \frac{\omega_0^2}{2}\int_{\Omega}(x_1^2 + x_2^2)\,\mathrm{d}x - \frac{\kappa}{2}\int_{\Omega}\int_{\Omega}\frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|} - p_0|\Omega|, \quad \Gamma = \partial\Omega, \tag{1.14}$$

also considered in the theory of equilibrium figures. In [18] a more natural functional R for the free motion of the liquid was invoked, which required certain modifications in the proofs. Concerning $\mathcal{F}$ the axial symmetry was always assumed. Under this assumption,

$$\vec{\mathcal{V}}(x) = \omega_0(\vec{e}_3 \times \vec{x}), \qquad \mathcal{P}(x) = \omega_0^2(x_1^2 + x_2^2)/2 + p_0, \qquad x \in \mathcal{F},$$

is a stationary solution of the problem (1.10), (1.11), because $(\vec{\mathcal{V}}(x), \mathcal{P}(x))$ satisfy (1.10), and the boundary conditions (1.11) reduce to (1.1).

Here we consider the case when $\mathcal{F}$ is nonsymmetric. For $\sigma = 0$, the existence of nonsymmetric equilibrium figures was known long ago; these are the Jacobi ellipsoids, the pear-formed figures of H. Poincaré etc. (see [1, 6]). In the case $\sigma > 0$, $\kappa = 0$ such figures were found in [11] (see also [7]) and computed numerically in [5]. If $\mathcal{F}$ is not axially symmetric, then along with $\mathcal{F} \equiv \mathcal{F}_0$ equation (1.1) determines a one-parameter family of equilibrium figures, $\mathcal{F}_\theta$, $\theta \in [0, 2\pi)$, obtained by rotating $\mathcal{F}_0$ about the x_3-axis through angle θ. It is natural to assume that θ is arbitrary and $\mathcal{F}_{\theta+2\pi} = \mathcal{F}_\theta$. It is easily seen that $(\vec{\mathcal{V}}(x), \mathcal{P}(x), x \in \mathcal{F}_{\omega_0 t+\varphi})$ is a periodic solution of (1.10), (1.11) for every constant φ, and that the velocity V_n of evolution of the free boundary in the normal direction equals $\omega_0 h(x)$, where

$$h(x) = \vec{N}(x) \cdot (\vec{e}_3 \times \vec{x}) = x_1 N_2 - x_2 N_1, \qquad x \in \mathcal{G}.$$

It is clear that $h(x) = 0$ for axially symmetric $\mathcal{G}$.

Since the functional R takes the same value for all $\mathcal{F}_\theta$, its second variation cannot be positive for $\rho(y)$ satisfying (1.6). As shown in [6] for the case $\sigma = 0$, we have

$$\delta_0^2 R[h] = 0, \tag{1.15}$$

which will be proved in Section 3 also in the case $\sigma > 0$ (this follows also from (4.43) with $b_1 = b_2 = 0$, $b_3 = 1$). Our main assumption concerning R is as follows: there exist two positive constants, c_1 and c_2, such that

$$c_1 \|\rho\|^2_{W_2^1(\mathcal{G})} \leqslant \delta_0^2 R[\rho] \leqslant c_2 \|\rho\|^2_{W_2^1(\mathcal{G})} \tag{1.16}$$

for all $\rho(x)$, $x \in \mathcal{G}$, satisfying the orthogonality conditions

$$\int_{\mathcal{G}} \rho(x)\, \mathrm{d}S_x = 0, \qquad \int_{\mathcal{G}} x_k \rho(x)\, \mathrm{d}S_x = 0, \qquad k = 1, 2, 3, \tag{1.17}$$

(a linearized variant of (1.6)) and the additional condition

$$\int_{\mathcal{G}} \rho(x) h(x)\, \mathrm{d}S_x = 0. \tag{1.18}$$

By the Gauss formula and (1.3),

$$\int_{\mathcal{G}} h(x)\, \mathrm{d}S_x = \int_{\mathcal{G}} h(x) x_k\, \mathrm{d}S_x = 0, \qquad k = 1, 2, 3,$$

so the functions $1, x_1, x_2, x_3, h(x)$, $x \in \mathcal{G}$, are linearly independent.

Inequalities (1.16) imply that the functional R takes its minimal value R_0 for $\Omega = \mathcal{F}_\theta$ and that $R > R_0$ if $\Omega \neq \mathcal{F}_\theta$, as required in [6]. The additional orthogonality condition (1.18) serves for "identifying" all the figures $\mathcal{F}_\theta$. It is clear that this can be done in many ways.

If inequalities (1.16) hold for every function ρ satisfying (1.17), (1.18), then they are also true, with other constants, for every small $\rho(x)$ satisfying (1.5), (1.6), (1.18). This can be easily verified by representing ρ in the form $\rho(x) = \rho_1(x) + \sum_{k=1}^4 \lambda_k f_k(x)$ with $f_i = x_i$, $i = 1, 2, 3$, $f_4 = 1$, and $\int_{\mathcal{G}} \rho_1 f_k \,\mathrm{d}S = 0$, which implies $|\lambda_k| \leqslant c\delta \int_{\mathcal{G}} |\rho| \,\mathrm{d}S$, by (1.5), (1.6). The converse assertion is also true.

The main result of the paper is the proof of stability of nonsymmetric equilibrium figures under the above assumptions. The precise formulation of the result will be given in the next section.

Another evolution free boundary problem for a viscous capillary liquid filling a layer-like domain over a rigid bottom is considered in [2, 3, 8].

2. Transformation of problem (1.10)–(1.12) and formulation of the main result

We start with the proof of some useful relations for an equilibrium figure $\mathcal{F}$ that is always assumed to be a bounded domain in $\mathbb{R}^3$ with a connected smooth boundary. Let us show, following A. M. Lyapunov [6], that the vector of total angular momentum of the rotating liquid,

$$\vec{\beta} = \int_{\mathcal{F}} \vec{x} \times \vec{\mathcal{V}}(x) \,\mathrm{d}x,$$

is directed along the x_3-axis. When we multiply (1.1) by $N_j x_3 - N_3 x_j$, $j = 1, 2$, integrate over $\mathcal{G}$ and take account of the relations

$$\begin{aligned}\int_{\mathcal{G}} \mathcal{U}(N_j x_3 - N_3 x_j) \,\mathrm{d}S_x &= \int_{\mathcal{F}} \int_{\mathcal{F}} \left(x_3 \frac{z_j - x_j}{|x-z|^3} - x_j \frac{z_3 - x_3}{|x-z|^3} \right) \mathrm{d}x \,\mathrm{d}z \\ &= \int_{\mathcal{F}} \int_{\mathcal{F}} \left(z_3 \frac{z_j - x_j}{|x-z|^3} - z_j \frac{z_3 - x_3}{|x-z|^3} \right) \mathrm{d}x \,\mathrm{d}z = 0\end{aligned}$$

and

$$\int_{\mathcal{G}} \mathcal{H}(N_j x_3 - N_3 x_j) \,\mathrm{d}S_x = \int_{\mathcal{G}} (x_3 \Delta_{\mathcal{G}} x_j - x_j \Delta_{\mathcal{G}} x_3) \,\mathrm{d}S_x = 0,$$

where $\Delta_{\mathcal{G}}$ is the Laplace–Beltrami operator on $\mathcal{G}$, we obtain

$$\frac{\omega_0^2}{2} \int_{\mathcal{F}} \frac{\partial}{\partial x_j} (x_1^2 + x_2^2) x_3 \,\mathrm{d}x = \omega_0^2 \int_{\mathcal{F}} x_3 x_j \,\mathrm{d}x = 0, \qquad j = 1, 2.$$

Hence,

$$\int_{\mathcal{F}} \vec{x} \times \vec{\mathcal{V}}(x) \,\mathrm{d}x = \beta \vec{e}_3, \tag{2.1}$$

where

$$\beta = \omega_0 \mathcal{I}, \quad \mathcal{I} = \int_{\mathcal{F}} (x_1^2 + x_2^2) \,\mathrm{d}x. \tag{2.2}$$

Similarly, multiplying (1.1) by N_j, $j = 1, 2$, and integrating we obtain the equation

$$\omega_0^2 \int_{\mathcal{F}} x_j \,\mathrm{d}x = 0, \tag{2.3}$$

which shows that the barycenter of $\mathcal{F}$ is located on the axis of rotation; hence, the first two relations (1.3) follow from (1.1). Finally, multiplication of (1.1) by $\vec{x}\cdot\vec{N}$ and integration leads to an expression for p_0:

$$p_0 = \frac{2\sigma|\mathcal{G}|}{3|\mathcal{F}|} - \frac{5}{6|\mathcal{F}|}\left(\omega_0^2\mathcal{I} + \kappa\int_{\mathcal{F}}\mathcal{U}(x)\,\mathrm{d}x\right). \tag{2.4}$$

For $\sigma = 0$, it is obtained in [6].

In fact, p_0 is the Lagrange multiplier corresponding to the constraint $|\Omega| = |\mathcal{F}|$; the multipliers corresponding to the restriction on the position of the barycenter vanish (see [15]).

Let us turn to problem (1.10)–(1.12) and recall that for the solution of this problem the following conservation laws for the mass, total and angular momenta hold:

$$\begin{aligned} &|\Omega_t| = |\Omega_0|,\\ &\int_{\Omega_t}\vec{v}(x,t)\,\mathrm{d}x = \int_{\Omega_0}\vec{v}_0(x)\,\mathrm{d}x,\\ &\int_{\Omega_t}(\vec{v}(x,t)\times\vec{x})\,\mathrm{d}x = \int_{\Omega_0}(\vec{v}_0(x)\times\vec{x})\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

By assumptions (1.13), we have

$$\int_{\Omega_t}\vec{v}(x,t)\,\mathrm{d}x = 0, \tag{2.6}$$

$$\int_{\Omega_t}(\vec{x}\times\vec{v}(x,t))\,\mathrm{d}x = \beta\vec{e}_3, \tag{2.7}$$

and, as a consequence,

$$\int_{\Omega_t}x_k\,\mathrm{d}x = \int_{\Omega_0}x_k\,\mathrm{d}x = 0, \qquad k = 1,2,3. \tag{2.8}$$

As in [10, 15, 16], we work with the problem for the perturbations

$$\vec{v}_r(x,t) = \vec{v}(x,t) - \vec{\mathcal{V}}(x), \qquad p_r(x,t) = p(x,t) - \mathcal{P}(x)$$

written in a coordinate system uniformly rotating with angular velocity ω_0. We make the change of variables

$$x = \mathcal{Z}(\omega_0 t)y \tag{2.9}$$

and introduce new unknown functions

$$\vec{w}(y,t) = \mathcal{Z}^{-1}(\omega_0 t)\vec{v}_r(\mathcal{Z}(\omega_0 t)y,t), \qquad s(y,t) = p_r(\mathcal{Z}(\omega_0 t)y,t), \tag{2.10}$$

where

$$\mathcal{Z}(\lambda) = \begin{pmatrix}\cos\lambda & -\sin\lambda & 0\\ \sin\lambda & \cos\lambda & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{2.11}$$

Then (1.10)–(1.12) is transformed into the following free boundary problem for $(\vec{w}, s)$:

$$\begin{aligned} &\vec{w}_t + (\vec{w}\cdot\nabla)\vec{w} + 2\omega_0(\vec{e}_3\times\vec{w}) - \nu\nabla^2\vec{w} + \nabla s = 0,\\ &\nabla\cdot\vec{w} = 0, \qquad y\in\Omega_t',\ t>0,\\ &T(\vec{w},s)\vec{n}' = (\sigma H' + \mathcal{P}(y) + \kappa U'(y,t))\vec{n}', \qquad V_n' = \vec{w}\cdot\vec{n}', \qquad y\in\Gamma_t',\\ &\vec{w}(y,0) = \vec{v}_0(y) - \vec{\mathcal{V}}(y) \equiv \vec{w}_0(y), \qquad y\in\Omega_0. \end{aligned} \tag{2.12}$$

Here $\Omega_t' = \mathcal{Z}^{-1}(\omega_0 t)\Omega_t$, $\Gamma_t' = \partial\Omega_t'$, $\vec{n}' = \mathcal{Z}^{-1}(\omega_0 t)\vec{n}$ is the exterior normal to Γ_t', V_n' is the velocity of motion of Γ_t' in the direction $\vec{n}'$, $H'(y)$ is the doubled mean curvature of Γ_t', and

$$U'(y,t) = \int_{\Omega_t'} \frac{\mathrm{d}z}{|y-z|}.$$

Now, we can present the main result of the paper.

THEOREM 2.1 Let the following conditions be satisfied:

(i) Γ_0 is given by equation (1.4) with $\mathcal{G} = \mathcal{G}_0$ and $\rho = \rho_0 \in C^{3+\alpha}(\mathcal{G}_0)$, $\alpha \in (0,1)$, satisfying (1.5), (1.6), (1.18);

(ii) $\vec{v}_0 \in C^{2+\alpha}(\Omega_0)$ satisfies conditions (1.13) and the compatibility conditions

$$\nabla \cdot \vec{v}_0(y) = 0, \qquad S(\vec{v}_0)\vec{n}_0 - \vec{n}_0(\vec{n}_0 \cdot S(\vec{v}_0)\vec{n}_0) = 0, \qquad y \in \Omega_0; \tag{2.13}$$

(iii) the functional $R[\rho]$ of (1.2) satisfies inequality (1.16) for every $\rho(y)$ subject to (1.17), (1.18).

If, in addition,

$$\|\vec{w}_0\|_{L_2(\Omega_0)} + \|\rho_0\|_{W_2^1(\mathcal{G}_0)} \leqslant \epsilon \tag{2.14}$$

with sufficiently small $\epsilon > 0$, then problem (2.12) has a unique solution defined for $t \geqslant 0$ and such that

(a) Γ_t' is given by (1.4) with $\mathcal{G} = \mathcal{G}_{\theta(t)}$, $\rho = \widehat{\rho}(\cdot,t) \in C^{3+\alpha}(\mathcal{G}_{\theta(t)})$, $\widehat{\rho}_t(\cdot,t) \in C^{2+\alpha}(\mathcal{G}_{\theta(t)})$, $\widehat{\rho}_{tt}(\cdot,t) \in C^{\alpha}(\mathcal{G}_{\theta(t)})$, for all $t > 0$; the function $\theta(t)$ is twice continuously differentiable; $\widehat{\rho}(x,t)$ satisfies (1.18), i.e.

$$\int_{\mathcal{G}_{\theta(t)}} \widehat{\rho}(x)h(x)\,\mathrm{d}S_x = 0; \tag{2.15}$$

(b) $\vec{w}(\cdot,t) \in C^{2+\alpha}(\Omega_t')$, $\vec{w}_t(\cdot,t) \in C^{\alpha}(\Omega_t')$, $s(\cdot,t) \in C^{1+\alpha}(\Omega_t')$, for all $t > 0$, and

$$\begin{aligned} |\vec{w}_t(\cdot,t)|_{C^{\alpha}(\Omega_t')} &+ |\vec{w}(\cdot,t)|_{C^{2+\alpha}(\Omega_t')} + |\nabla s(\cdot,t)|_{C^{1+\alpha}(\Omega_t')} \\ &+ |\widehat{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G}_0)} + |\widehat{\rho}_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{G}_0)} + |\widehat{\rho}_{tt}(\cdot,t)|_{C^{\alpha}(\mathcal{G}_0)} \\ &\leqslant c e^{-bt/2}(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}), \qquad b > 0, \end{aligned} \tag{2.16}$$

$$|\theta_t(t)| + |\theta_{tt}(t)| \leqslant c e^{-bt/2}(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}). \tag{2.17}$$

By $C^l(\Omega_t)$, $C^l(\Gamma_t)$ we mean the standard Hölder spaces of functions (or vector fields); $\widehat{\rho}_t(y,t)$, $\widehat{\rho}_{tt}(y,t)$ are derivatives calculated for a fixed argument $y \in \mathcal{G}_{\theta(t)}$; in other words, if $y = \mathcal{Z}(\theta(t))y'$, $y' \in \mathcal{G}_0$, then

$$\frac{\partial^i}{\partial t^i}\widehat{\rho}(y,t) = \frac{\partial^i}{\partial t^i}\widehat{\rho}(\mathcal{Z}(\theta(t'))y',t)\bigg|_{t'=t}, \qquad i = 1,2. \tag{2.18}$$

Estimates (2.16), (2.17) imply exponential stability of the periodic solution $(\vec{\mathcal{V}}, \mathcal{P}, \mathcal{F}_{\omega_0 t+\varphi_0})$. The decay of $\widehat{\rho}(y,t)$ to zero means that $\Gamma_t' \to \mathcal{G}_{\varphi_0}$, where $\varphi_0 = \lim_{t\to\infty}\theta(t) < \infty$. The existence of this limit follows from (2.17).

3. Auxiliary propositions

This section is devoted to calculations aimed at the determination of the function $\theta(t)$. We begin with some auxiliary constructions. It is well known that for every point $x \in \mathbb{R}^3$ with $\mathrm{dist}(x, \mathcal{G}) \leqslant \delta_1$, where $\mathcal{G} \equiv \mathcal{G}_0$, $\delta_1 \ll 1$, we have

$$x = y + N(y)r, \qquad y \in \mathcal{G}, \tag{3.1}$$

with $|r| \leqslant \delta_1$. Let us consider this relation more closely. Assume that $y \in G \subset \mathcal{G}$, where G is a subset of $\mathcal{G}$ given by

$$y = y(s), \qquad s = (s_1, s_2) \in \omega \subset \mathbb{R}^2$$

(s_1, s_2 are local coordinates on G). The transformation

$$E(s_1, s_2, r) = y(s_1, s_2) + N(s_1, s_2)r \equiv y(s) + N(s)r \tag{3.2}$$

makes the set $U = \{s \in \omega : |r| \leqslant \delta\}$ correspond to the set V of the points (3.1) with $y \in G$, $|\rho| \leqslant \delta$.

Let $\mathcal{J}$ be the Jacobi matrix of $E(s_1, s_2, r)$, i.e.

$$\mathcal{J} = \begin{pmatrix} y_{1,s_1}(s) + N_{1,s_1}(s)r & y_{1,s_2}(s) + N_{1,s_2}(s)r & N_1(s) \\ y_{2,s_1}(s) + N_{2,s_1}(s)r & y_{2,s_2}(s) + N_{2,s_2}(s)r & N_2(s) \\ y_{3,s_1}(s) + N_{3,s_1}(s)r & y_{3,s_2}(s) + N_{3,s_2}(s)r & N_3(s) \end{pmatrix}, \tag{3.3}$$

where $N_i(s) = N_i(y(s))$, $y_{k,s_j} = \partial y_k(s)/\partial s_j$, $N_{k,s_j} = \partial N_k(s)/\partial s_j$. The vectors $\vec{y}_{,s_j} = (y_{k,s_j})_{k=1,2,3} \equiv \vec{\tau}_j$, $j = 1, 2$, are linearly independent and tangent to $\mathcal{G}$, hence, $\det \mathcal{J}|_{r=0} \neq 0$ and $\det \mathcal{J}(s, r) \neq 0$, since δ_1 is small. Therefore we have the inverse transformation

$$E^{-1}(x) = \{s = \Sigma(x),\ r = R(x)\},$$

so that $U = E^{-1}V$. We denote by J_{km} the elements of $\mathcal{J}$ and by J^{km} the elements of $\mathcal{J}^{-1}$. It is clear that

$$x_{m,s_\alpha} \equiv \frac{\partial x_m}{\partial s_\alpha} = J_{m\alpha}, \qquad \frac{\partial x_m}{\partial r} = J_{m3}, \qquad \frac{\partial \Sigma_\alpha}{\partial x_k} = J^{\alpha k}, \qquad \frac{\partial R}{\partial x_k} = J^{3k},$$

where $\alpha = 1, 2$, $k = 1, 2, 3$. The elements J^{3k} are the components of the vector

$$\frac{\vec{x}_{,s_1} \times \vec{x}_{,s_2}}{\det \mathcal{J}} = \frac{\vec{x}_{,s_1} \times \vec{x}_{,s_2}}{\vec{N} \cdot (\vec{x}_{,s_1} \times \vec{x}_{,s_2})}.$$

Since the surface $\mathcal{G}$ and the parallel surface $\mathcal{G}^{(r)} = \{x = y + N(y)r,\ y \in \mathcal{G}\}$ have a common normal $\vec{N}(y)$, and $\vec{x}_{,s_j}$ are linearly independent tangent vectors to $\mathcal{G}^{(r)}$, we have

$$\frac{\vec{x}_{,s_1} \times \vec{x}_{,s_2}}{\vec{N} \cdot (\vec{x}_{,s_1} \times \vec{x}_{,s_2})} = \frac{\vec{N}|\vec{x}_{,s_1} \times \vec{x}_{,s_2}|}{|\vec{x}_{,s_1} \times \vec{x}_{,s_2}|} = \vec{N}$$

if the triple of vectors $\vec{y}_{,s_1}, \vec{y}_{,s_2}, \vec{N}$ has a right orientation. Hence, R is a function defined in the δ_1-neighborhood of $\mathcal{G}$, and

$$\frac{\partial R}{\partial x_k} = J^{3k} = N_k(y) \tag{3.4}$$

(this also follows from the fact that $R(x) = \operatorname{dist}(x, \mathcal{G})$). In what follows we also consider the matrix (3.3) with a variable $r = r(s)$; in this case the relation $J^{3k}(s) = N_k(s)$ remains valid. Indeed, fix an arbitrary $s' \in \omega$ and consider the matrix (3.3) with $r = r(s')$. Clearly, the relation considered holds for arbitrary $s \in \omega$, also for $s = s'$, which proves our assertion.

The second derivatives of Σ_α and R with respect to x_q are furnished by the equations

$$\frac{\partial J^{km}}{\partial x_q} = \sum_{\alpha=1}^{2} \frac{\partial J^{km}}{\partial s_\alpha} J^{\alpha q} + \frac{\partial J^{km}}{\partial r} J^{3q}.$$

From this formula higher order derivatives of Σ_α and R can be calculated.

Now, let Γ be a surface that is close to $\mathcal{G}_0 \equiv \mathcal{G}$ and is given by equation (1.4) with $\rho(y)$ satisfying (1.5), where $\delta \leqslant \delta_1/2$. We consider other representation formulas for Γ of the type (1.4),

$$x = y + N_\theta \rho_\theta(y), \qquad y \in \mathcal{G}_\theta, \tag{3.5}$$

where N_θ is the exterior normal to $\mathcal{G}_\theta$, in order to find the value of θ such that $\int_{\mathcal{G}_\theta} \rho_\theta(y) h(y)\, \mathrm{d}S_y = 0$. Instead of rotating the equilibrium figure, we can rotate Γ and try to satisfy the equation

$$f(\lambda) = \int_{\mathcal{G}} \widetilde{\rho}(z,\lambda) h(z)\, \mathrm{d}S_z = 0, \tag{3.6}$$

where $\widetilde{\rho}(z,\lambda)$ is the function that defines the surface $\Gamma(\lambda) = \mathcal{Z}(\lambda)\Gamma$ by equation (1.4) with $\mathcal{G} = \mathcal{G}_0$, $\rho = \widetilde{\rho}$, i.e.

$$x = z + N(z)\widetilde{\rho}(z,\lambda), \qquad z \in \mathcal{G}_0$$

(we assume that λ is so small that $\Gamma(\lambda)$ is contained in the δ_1-neighborhood of $\mathcal{G}$). It is clear that the point $x = y + N(y)\rho(y) \in \Gamma$ and the corresponding point $X = \mathcal{Z}(\lambda)x \in \Gamma(\lambda)$ are related to each other by

$$\mathcal{Z}(\lambda)(y + N(y)\rho(y)) = z + N(z)\widetilde{\rho}(z,\lambda), \quad z \in \mathcal{G}. \tag{3.7}$$

If, in addition, $z, y \in G$, $y = y(s)$, $z = y(\sigma)$, $\sigma = (\sigma_1, \sigma_2) \in \omega$, then

$$y(\sigma) + N(\sigma)\widetilde{\rho}(\sigma,\lambda) = \mathcal{Z}(\lambda)(y(s) + N(s)\rho(s)), \tag{3.8}$$

where $\rho(s) \equiv \rho(y(s))$, $\widetilde{\rho}(\sigma,\lambda) = \widetilde{\rho}(y(\sigma),\lambda)$, $N(s) = N(y(s))$. Hence, for a given λ we have

$$\widetilde{\rho}(\sigma,\lambda) = R(X) = R(\mathcal{Z}(\lambda)x(s)), \tag{3.9}$$

$$\sigma = \Sigma(X) = \Sigma(\mathcal{Z}(\lambda)x(s)) = \mathcal{S}(s,\lambda). \tag{3.10}$$

The difference

$$\widetilde{\rho}(z,\lambda) - \rho(y) = R(\mathcal{Z}x) - R(x)$$

satisfies the inequality

$$|\widetilde{\rho}(z,\lambda) - \rho(y)| \leqslant |\mathcal{Z}x - x| \leqslant c|\lambda|. \tag{3.11}$$

Let us show that the transformation (3.10) is invertible. We have

$$\frac{\partial \mathcal{S}_\alpha(s,\lambda)}{\partial s_\beta} = \sum_{k,m=1}^{3} \frac{\partial \Sigma_\alpha}{\partial X_k} Z_{km}(\lambda) \frac{\mathrm{d}x_m(s)}{\mathrm{d}s_\beta} \equiv B_{\alpha\beta}(s,\lambda), \tag{3.12}$$

where
$$\frac{\mathrm{d}x_m(s)}{\mathrm{d}s_\beta} = \frac{\partial y_m(s)}{\partial s_\beta} + \frac{\partial N_m(s)\rho(s)}{\partial s_\beta} = \frac{\partial x_m}{\partial s_\beta} + N_m \frac{\partial \rho}{\partial s_\beta}$$
(i.e. here the dependence of ρ on s is taken into account). In particular,
$$\frac{\partial \mathcal{S}_\alpha}{\partial s_\beta}\bigg|_{\lambda=0} = \sum_{m=1}^{3} \frac{\partial \Sigma_\alpha}{\partial x_m}\left(\frac{\partial x_m}{\partial s_\beta} + N_m \frac{\partial \rho}{\partial s_\beta}\right) = \sum_{m=1}^{3} J^{\alpha m}\left(J_{m\beta} + J_{m3}\frac{\partial \rho}{\partial s_\beta}\right) = \delta_{\alpha\beta},$$
hence, $\mathcal{S}^{-1}(\sigma, \lambda)$ exists for small values of λ, and
$$\left|\frac{\partial \mathcal{S}_\alpha}{\partial s_\beta} - \delta_{\alpha\beta}\right| \leqslant c|\lambda|.$$

Let us compute the derivatives $\partial \mathcal{S}^{-1}/\partial\lambda$. When we differentiate the identities
$$\sigma_\alpha = \Sigma_\alpha(\mathcal{Z}(\lambda)x(\mathcal{S}^{-1}(\sigma,\lambda))) \equiv \mathcal{S}_\alpha(\mathcal{S}^{-1}(\sigma,\lambda),\lambda), \qquad \alpha = 1, 2,$$
with respect to λ and take account of (3.10), we obtain
$$0 = \sum_{\beta=1}^{2} B_{\alpha\beta}\frac{\partial \mathcal{S}_\beta^{-1}}{\partial \lambda} + \sum_{k,m=1}^{3} \frac{\partial \Sigma_\alpha}{\partial X_k}\frac{\mathrm{d}Z_{km}}{\mathrm{d}\lambda} x_m(s)\bigg|_{s=\mathcal{S}^{-1}(\sigma,\lambda)},$$
and, as a consequence,
$$\frac{\partial \mathcal{S}_\alpha^{-1}(\sigma,\lambda)}{\partial\lambda} = -\sum_{\beta=1}^{2}\sum_{k,m=1}^{3} B^{\alpha\beta}\frac{\partial \Sigma_\beta}{\partial X_k}\frac{\mathrm{d}Z_{km}}{\mathrm{d}\lambda}x_m(s)\bigg|_{s=\mathcal{S}^{-1}(\sigma,\lambda)}, \tag{3.13}$$
where $B^{\alpha\beta} = \partial \mathcal{S}_\alpha^{-1}/\partial\sigma_\beta$ are the elements of $\mathcal{B}^{-1}$.

Next, we calculate the derivative of $\widetilde{\rho}(\sigma,\lambda) = R(X)$ with respect to λ. Differentiation of (3.9) gives
$$\frac{\widetilde{\partial}\rho(\sigma,\lambda)}{\partial\lambda} = \sum_{k=1}^{3}\frac{\partial R}{\partial X_k}\frac{\partial X_k}{\partial\lambda} = \sum_{k,m=1}^{3}\frac{\partial R}{\partial X_m}\left(\frac{\mathrm{d}Z_{mk}}{\mathrm{d}\lambda}x_k + \sum_{\alpha=1}^{2} Z_{mk}\frac{\mathrm{d}x_k}{\mathrm{d}s_\alpha}\frac{\partial \mathcal{S}_\alpha^{-1}}{\partial\lambda}\right).$$
From (3.13) and
$$\frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}\lambda}\vec{x} = \frac{\mathrm{d}\mathcal{Z}}{\mathrm{d}\lambda}\mathcal{Z}^{-1}\vec{X} = \vec{e}_3 \times \vec{X},$$
we have
$$\frac{\partial \vec{X}}{\partial\lambda} = (I - \mathcal{D})[\vec{e}_3 \times \vec{X}],$$
and
$$\frac{\widetilde{\partial}\rho}{\partial\lambda} = \nabla_X R(X)\cdot(I-\mathcal{D})[\vec{e}_3\times\vec{X}], \tag{3.14}$$
where $\mathcal{D}$ is the matrix with elements
$$D_{mk} = \sum_{j=1}^{3}\sum_{\alpha,\beta=1}^{2} Z_{mj}(\lambda)\frac{\mathrm{d}x_j}{\mathrm{d}s_\alpha}\frac{\partial s_\alpha}{\partial\sigma_\beta}\frac{\partial\sigma_\beta}{\partial X_k} = \sum_{\beta=1}^{2}\frac{\mathrm{d}X_m(\sigma)}{\mathrm{d}\sigma_\beta}\frac{\partial\sigma_\beta}{\partial X_k}. \tag{3.15}$$

It is easily seen that D_{mk} can be expressed in terms of the elements of the matrix (3.3) calculated for $s = \sigma$, $r = \widetilde{\rho}(\sigma, \lambda)$ (we denote it by $\mathcal{J}(\sigma, \lambda)$) and of the inverse matrix $\mathcal{J}^{-1}(\sigma, \lambda)$. Indeed,

$$D_{mk} = \sum_{\beta=1}^{2} \left(\frac{\partial X_m}{\partial \sigma_\beta} + N_m(\sigma) \frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} \right) \frac{\partial \sigma_\beta}{\partial X_k}$$
$$= \sum_{\beta=1}^{2} \left(J_{m\beta}(\sigma, \lambda) + J_{m3}(\sigma, \lambda) \frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} \right) J^{\beta k}(\sigma, \lambda),$$

hence,

$$\delta_{mk} - D_{mk} = J_{m3}(\sigma, \lambda) J^{3k}(\sigma, \lambda) - J_{m3}(\sigma, \lambda) \sum_{\beta=1}^{2} \frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} J^{\beta k}(\sigma, \lambda)$$
$$= N_m(\sigma) \left(N_k(\sigma) - \sum_{\beta=1}^{2} \frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} J^{\beta k}(\sigma, \lambda) \right) \tag{3.16}$$

and

$$\frac{\partial \widetilde{\rho}}{\partial \lambda} = \sum_{k=1}^{3} \left(N_k(\sigma) - \sum_{\beta=1}^{2} \frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} J^{\beta k}(\sigma, \lambda) \right) (\vec{e}_3 \times \vec{X})_k, \tag{3.17}$$

where

$$\frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} = \sum_{m=1}^{3} \sum_{\alpha=1}^{2} \frac{\partial R(X)}{\partial X_m} \frac{dX_m}{d\sigma_\beta} = \sum_{m,j=1}^{3} \sum_{\alpha=1}^{2} \frac{\partial R}{\partial X_m} Z_{mj} \frac{\mathrm{d}x_j(s)}{ds_\alpha} B^{\alpha\beta} \Big|_{s=\mathcal{S}^{-1}(\sigma,\lambda)}. \tag{3.18}$$

Finally, taking into account

$$\vec{N}(\sigma) \cdot (\vec{e}_3 \times \vec{X}) = \vec{N}(\sigma) \cdot (\vec{e}_3 \times \vec{y}(\sigma))$$

we obtain

$$\frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \lambda} = \vec{N}(\sigma) \cdot (\vec{e}_3 \times \vec{y}(\sigma)) - \sum_{\beta=1}^{2} \frac{\partial \widetilde{\rho}(\sigma, \lambda)}{\partial \sigma_\beta} \sum_{k=1}^{3} J^{\beta k}(\sigma, \lambda)(\vec{e}_3 \times \vec{X})_k. \tag{3.19}$$

Computation of $J^{\beta k}$ shows that the last term in (3.19) is equal to

$$(\vec{e}_3 \times \vec{X}) \cdot \frac{(\widetilde{\rho}_{,\sigma_1} \vec{X}_{,\sigma_2} - \widetilde{\rho}_{,\sigma_2} \vec{X}_{,\sigma_1}) \times \vec{N}(\sigma)}{\det \mathcal{J}(\sigma, \lambda)},$$

where $\widetilde{\rho}_{,\sigma_j} = \partial \widetilde{\rho}(\sigma, \lambda)/\partial \sigma_j$ and

$$\vec{X}_{,\sigma_j} = \frac{\partial \vec{y}(\sigma)}{\partial \sigma_j} + \frac{\partial \vec{N}(\sigma)}{\partial \sigma_j} \widetilde{\rho}(\sigma, \lambda).$$

The above term is independent of the choice of local coordinates, since both the numerator and the denominator are multiplied by $\det(\partial\sigma'/\partial\sigma)$ when the transformation $\sigma' = F(\sigma)$ is made. Hence, (3.19) can be written in the form

$$\frac{\widetilde{\partial}\rho(z,\lambda)}{\partial\lambda} = h(z) + \vec{h}_1(z, \widetilde{\rho}(z,\lambda)) \cdot \nabla_{\mathcal{G}}\widetilde{\rho}(z,\lambda), \tag{3.20}$$

where $\vec{h}_1$ is a differentiable vector-valued function depending on $\widetilde{\rho}$ but not on the derivatives of $\widetilde{\rho}$.

One of the consequences of (3.19) is the formula (1.15). To prove it, we compute $\partial\widetilde{\rho}(\sigma,\lambda)/\partial\lambda$ for $\Gamma(\lambda) = \mathcal{G}_\lambda$. It is clear that in this case $\widetilde{\rho}(\sigma,\lambda)$ is a smooth function of both arguments and that $\rho(s) = 0$. Passing to the limit in (3.19), (3.20) we obtain

$$\left.\frac{\partial\widetilde{\rho}(z,\lambda)}{\partial\lambda}\right|_{\lambda=0} = h(z),$$

because, by (3.18),

$$\left.\frac{\partial\widetilde{\rho}}{\partial\sigma_\beta}\right|_{\lambda=0,\rho=0} = \sum_{j=1}^{3} \frac{\partial R(x)}{\partial x_j}\frac{\partial y_j}{\partial s_\beta} = 0.$$

Hence,

$$\begin{aligned}\widetilde{\rho}(z,\lambda) - \lambda h(z) &= \int_0^\lambda (\widetilde{\rho}'_\mu(z,\mu) - h(z))\,\mathrm{d}\mu = \int_0^\lambda \mathrm{d}\mu \int_0^\mu \widetilde{\rho}''_{\mu'\mu'}(z,\mu')\,\mathrm{d}\mu' \\ &= \lambda^2 \int_0^1 \mathrm{d}t \int_0^{t'} \widetilde{\rho}''(z,\lambda t')\,\mathrm{d}t' \equiv \lambda^2 \rho_1(z,\lambda).\end{aligned}$$

Now, since $R[\widetilde{\rho}(\cdot,\lambda)] = R_0$ does not depend on λ, we have

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} R[\widetilde{\rho}(\cdot,\lambda)] = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} R[\lambda h] + \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \int_0^{\lambda^2} \frac{\mathrm{d}}{\mathrm{d}t} R[\lambda h + t\rho_1(\cdot,\lambda)]\,\mathrm{d}t. \tag{3.21}$$

The last term equals

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(2\lambda \frac{\mathrm{d}}{\mathrm{d}t} R[\lambda h + t\rho_1(\cdot,\lambda)]\right)\Big|_{t=\lambda^2} + \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^{\lambda^2} \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}\lambda} R[\lambda h + t\rho_1]\,\mathrm{d}t \\ &\qquad = 2\frac{\mathrm{d}}{\mathrm{d}t} R[\lambda h + t\rho_1(\cdot,\lambda)]\Big|_{t=\lambda^2} + 2\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}}{\mathrm{d}t} R[\lambda h + t\rho_1]\Big|_{t=\lambda^2}\right) \\ &\qquad\quad + \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^{\lambda^2} \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}\lambda} R[\lambda h + t\rho_1]\,\mathrm{d}t,\end{aligned}$$

and it tends to zero as $\lambda \to 0$, since $\delta_0 R = 0$. Hence, (3.21) implies

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} R[\lambda h]\right|_{\lambda=0} = \delta_0^2 R[h] = 0,$$

and (1.15) is proved.

Let us turn to our original problem of finding the value of $\lambda = \lambda_0$ for which equation (3.6) is satisfied. It is equivalent to

$$f(0) = -\int_0^{\lambda_0} f'(\lambda)\,\mathrm{d}\lambda. \tag{3.22}$$

We have the following simple lemma.

LEMMA 3.1 There exist positive constants ϵ_1 and l_0 depending only on $\mathcal{G}$ and such that if $|\lambda| \leqslant l_0$ and

$$\int_{\mathcal{G}} |\rho(y)|\,\mathrm{d}S_y \leqslant \epsilon_1, \tag{3.23}$$

then

$$f'(\lambda) \geqslant \frac{1}{2}\int_{\mathcal{G}} h^2(y)\,\mathrm{d}S_y. \tag{3.24}$$

If

$$\max_{\mathcal{G}} |h(y)|\epsilon_1 \leqslant \frac{l_0}{2}\int_{\mathcal{G}} h^2(y)\,\mathrm{d}S_y, \tag{3.25}$$

then equation (3.22) has a unique solution.

Proof. By (3.20),

$$f'(\lambda) = \int_{\mathcal{G}} h^2(z)\,\mathrm{d}S_z + \int_{\mathcal{G}} h(z)\vec{h}_1(z,\widetilde{\rho})\cdot\nabla_{\mathcal{G}}\widetilde{\rho}(z,\lambda)\,\mathrm{d}S_z.$$

Integrating by parts in the second term and making use of (3.11), we obtain

$$f'(\lambda) \geqslant \int_{\mathcal{G}} h^2(z)\,\mathrm{d}S_z - c_1\int_{\mathcal{G}} |\rho(y)|\,\mathrm{d}S_y - c_2|\lambda| \geqslant \int_{\mathcal{G}} h^2(z)\,\mathrm{d}S_z - c_1\epsilon_1 - c_2 l_0$$

from which the estimate (3.24) follows. Since

$$|f(0)| \leqslant \max|h(y)|\epsilon_1$$

and $\int_0^\lambda f'(\mu)\,\mathrm{d}\mu$ is a monotone function for $|\lambda| \leqslant l_0$, the existence of solution of (3.22) is evident. The lemma is proved.

Now, let us assume that there is given a one-parameter family of surfaces Γ_t, $t \in [0, t_0]$ (e.g. $\Gamma_t = \Gamma_t'$ in the problem (2.12)), that each Γ_t is given by equation (1.4), where $\rho = \rho(y,t)$ satisfies (1.5), and is differentiable with respect to t. As above, we consider the surfaces $\Gamma_t(\lambda) = \mathcal{Z}(\lambda)\Gamma_t$ given by the same equation with $\rho = \widetilde{\rho}(y,t,\lambda)$, $y \in \mathcal{G} \equiv \mathcal{G}_0$, and we look for the value $\lambda(t)$ of the angle λ such that

$$f(\lambda,t) \equiv \int_{\mathcal{G}} \widetilde{\rho}(z,t,\lambda)h(z)\,\mathrm{d}S_z = 0 \tag{3.26}$$

for $\lambda = \lambda(t)$. The following proposition is a consequence of Lemma 3.1.

LEMMA 3.2 If $\rho(y,t)$ satisfies (3.23) and if (3.25) holds, then equation (3.26) defines a function $\lambda(t)$ such that $f(\lambda(t),t) = 0$. This function is continuously differentiable with respect to t and

$$|\lambda'(t)| \leqslant c\int_{\mathcal{G}} |\rho_t(y,t)|\,\mathrm{d}S_y. \tag{3.27}$$

Proof. We observe, first of all, that the above calculations, in particular, formulas (3.13), (3.20), hold true also in the case when ρ depends on t (t enters these formulas as a parameter). Lemma 3.1 is also true if (3.23) is replaced with

$$\int_{\mathcal{G}} |\rho(y,t)|\,\mathrm{d}S_y \leqslant \epsilon_1.$$

Therefore, under this condition equation (3.26) has a unique solution $\lambda = \lambda(t) \in [-l_0, l_0]$. Now, let us show that $f(\lambda, t)$ is continuously differentiable with respect to t. Since

$$\frac{\partial X}{\partial t} = \mathcal{Z}(\lambda) N \rho_t(s,t), \tag{3.28}$$

we have formulas similar to (3.13), (3.14), namely,

$$\frac{\partial \mathcal{S}_\alpha^{-1}(\sigma,t,\lambda)}{\partial t} = -\sum_{\beta=1}^{2}\sum_{k,m=1}^{3} B^{\alpha\beta}\frac{\partial \Sigma_\beta}{\partial X_k} Z_{km}(\lambda) N_m(s)\rho_t(s,t)\bigg|_{s=\mathcal{S}^{-1}(\sigma,t,\lambda)}, \tag{3.29}$$

$$\frac{\partial \widetilde{\rho}(z,t,\lambda)}{\partial t} = \nabla_X R \cdot (I - \mathcal{D})\mathcal{Z}(\lambda) N(y)\rho_t(y,t), \tag{3.30}$$

where y is the point of $\mathcal{G}$ related to z as in (3.7). Hence,

$$f_t(\lambda,t) = \int_{\mathcal{G}} h(z)\nabla_X R \cdot (I - \mathcal{D})\mathcal{Z}(\lambda) N(y)\rho_t(y,t)\,\mathrm{d}S_z. \tag{3.31}$$

On splitting $\mathcal{G}$ into submanifolds where local coordinates can be introduced and on making use of (3.12) (we omit the details), one can write the last integral as an integral with respect to $\mathrm{d}S_y$ and obtain the estimate

$$|f_t(\lambda,t)| \leqslant c\int_{\mathcal{G}} |\rho_t(y,t)|\,\mathrm{d}S_y. \tag{3.32}$$

It follows that $\lambda(t)$ is also continuously differentiable, and

$$\lambda'(t) = -\frac{f_t(\lambda,t)}{f_\lambda(\lambda,t)}\bigg|_{\lambda=\lambda(t)}.$$

Inequality (3.27) is a consequence of (3.32), (3.24). The lemma is proved.

If $\rho(y,t)$ is twice continuously differentiable with respect to t, then we can evaluate the second derivative

$$\lambda_{tt}(t) = -\frac{f_{tt}}{f_\lambda} + \frac{f_t f_{\lambda t}}{f_\lambda^2}\bigg|_{\lambda=\lambda(t)}. \tag{3.33}$$

For this we should compute f_{tt} and $f_{\lambda t}$. Differentiation of (3.14) leads to

$$\begin{aligned}\frac{\partial^2\widetilde{\rho}}{\partial\lambda\partial t} = {}& \sum_{k=1}^{3}\frac{\partial\nabla_X R}{\partial X_k}\frac{\partial X_k}{\partial t}(1-\mathcal{D})[\vec{e}_3\times\vec{X}] - \nabla_X R\cdot\frac{\partial\mathcal{D}}{\partial t}[e_3\times X]\\ &+ \nabla_X R\cdot(I-\mathcal{D})\left[\vec{e}_3\times\frac{\partial\vec{X}}{\partial t}\right].\end{aligned} \tag{3.34}$$

The derivatives $\partial D_{mk}/\partial t$ can be computed by differentiating (3.16) (we recall that $J^{\beta k}$ depends on $\widetilde{\rho}(\sigma, t, \lambda)$):

$$\frac{\partial D_{mk}}{\partial t} = N_m(\sigma)\frac{\partial}{\partial t}\sum_{\beta=1}^{2}\frac{\partial\widetilde{\rho}(\sigma, t, \lambda)}{\partial\sigma_\beta}J^{\beta k}(\sigma, t, \lambda). \tag{3.35}$$

Taking also (3.18), (3.28) and (3.30) into account, it is not hard to see that (3.34) can be written in the form

$$\frac{\partial^2\widetilde{\rho}(\sigma, t, \lambda)}{\partial\lambda\partial t} = a(s, t, \lambda)\rho_t(s, t) + \sum_{\beta=1}^{2} a_\beta(s, t, \lambda)\frac{\partial\rho_t(s, t)}{\partial s_\beta}\bigg|_{s=\mathcal{S}^{-1}(\sigma,t,\lambda)}, \tag{3.36}$$

where a and a_β are functions with the same regularity properties as the second and the first derivatives of ρ, respectively. Hence, on integrating by parts one obtains

$$f_{t\lambda}(\lambda, t) = \int_{\mathcal{G}} F(y, \lambda, t)\rho_t(y, t)\,\mathrm{d}S_y, \tag{3.37}$$

where F is as smooth as $\partial^2\rho/\partial s_\alpha\partial s_\beta$.

The derivatives $\widetilde{\rho}_{tt}$ and $f_{tt}(\lambda, t)$ can be computed in a similar way. We have

$$\begin{aligned}\frac{\partial^2\widetilde{\rho}(\sigma, t, \lambda)}{\partial t^2} = &\sum_{k,m=1}^{3}\frac{\partial\nabla_X R}{\partial X_k}\frac{\partial X_k}{\partial t}(1-\mathcal{D})\mathcal{Z}(\lambda)\vec{N}(s)\rho_t(s, t)\\ &+ \nabla_X R\cdot(I-\mathcal{D})\mathcal{Z}(\lambda)\bigg(\sum_{\gamma=1}^{2}\frac{\partial}{\partial s_\gamma}(\vec{N}(s)\rho_t(s, t))\frac{\partial\mathcal{S}_\gamma^{-1}}{\partial t} + \vec{N}(s)\rho_{tt}(s, t)\bigg)\\ &- \nabla_X R\cdot\frac{\partial\mathcal{D}}{\partial t}\mathcal{Z}(\lambda)\vec{N}(s)\rho_t(s, t)\bigg|_{s=\mathcal{S}^{-1}(\sigma,t,\lambda)}.\end{aligned}$$

From this formula, as well as from (3.28), (3.35), (3.20) it follows that $\widetilde{\rho}_{tt}$ can be represented in the form (3.36) with an additional term $a'\rho_{tt}$ on the right hand side, which implies

$$f_{tt}(\lambda, t) = \int_{\mathcal{G}}(F_1(y, \lambda, t)\rho_t(y, t) + F_2(y, \lambda, t)\rho_{tt}(y, t))\,\mathrm{d}S_y. \tag{3.38}$$

Hence,

$$|\lambda_{tt}(t)| \leqslant c\int_{\mathcal{G}}(|\rho_t(y, t)| + |\rho_{tt}(y, t)|)\,\mathrm{d}S_y, \tag{3.39}$$

by (3.33), (3.37), (3.38).

In the same way higher order derivatives of $f(\lambda, t)$ and $\lambda(t)$ can be computed and estimated.

If $\lambda(0) = 0$, then

$$|\lambda(t)| \leqslant c\int_0^t\int_{\mathcal{G}}|\rho_\tau(y, \tau)|\,\mathrm{d}S_y\,\mathrm{d}\tau, \tag{3.40}$$

$$|\widetilde{\rho}(z, t, \lambda(t))| \leqslant |\rho(y, t)| + c\int_0^t\int_{\mathcal{G}}|\rho_\tau(y, \tau)|\,\mathrm{d}S_y\,\mathrm{d}\tau. \tag{3.41}$$

An estimate of the gradient of $\widetilde{\rho}(z, t, \lambda(t))$ can be deduced from (3.18). Since

$$\frac{\partial\rho(s)}{\partial s_\beta} = \sum_{m=1}^{3} \frac{\partial R(x)}{\partial x_m} \frac{\mathrm{d}x_m}{\mathrm{d}s_\beta},$$

we have

$$\begin{aligned} \left|\frac{\widetilde{\partial}\rho(\sigma,\lambda)}{\partial\sigma_\beta} - \frac{\partial\rho(s)}{\partial s_\beta}\right| &\leqslant \sum_{j=1}^{3}\sum_{\alpha=1}^{2} \left|\sum_{m=1}^{3} \frac{\partial R}{\partial X_m} Z_{mj} - \frac{\partial R(x)}{\partial x_j}\right| \left|\frac{\mathrm{d}x_j(s)}{\mathrm{d}s_\alpha}\right| |B^{\alpha\beta}| \\ &\quad + \sum_{\alpha=1}^{2}\left|\sum_{j=1}^{3} \frac{\partial R(x)}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}s_\alpha}\right| |B^{\alpha\beta} - \delta_{\alpha\beta}| \leqslant c|\lambda|, \end{aligned}$$

and, as a consequence,

$$|\widetilde{\rho}(\cdot, t, \lambda)|_{C^1(\mathcal{G})} \leqslant |\rho(\cdot, t)|_{C^1(\mathcal{G})} + c|\lambda| \leqslant |\rho(\cdot, t)|_{C^1(\mathcal{G})} + c\int_0^t\int_{\mathcal{G}} |\rho_\tau(y, \tau)|\,\mathrm{d}S_y\,\mathrm{d}\tau. \tag{3.42}$$

4. Proof of Theorem 2.1

As in the case of axisymmetric $\mathcal{F}$ (see [18]), Theorem 2.1 reduces to the proof of the solvability of problem (2.12) in a finite time interval and of uniform estimates for the solution. Additional attention should be given to the construction of the function $\theta(t)$.

In what follows we work only with problem (2.12) without addressing (1.10)–(1.12) any more. Changing notations slightly, we write (2.12) in the form

$$\begin{aligned} &\vec{w}_t + (\vec{w}\cdot\nabla)\vec{w} + 2\omega_0(\vec{e}_3\times\vec{w}) - \nu\nabla^2\vec{w} + \nabla s = 0, \\ &\nabla\cdot\vec{w}(x,t) = 0, \quad x\in\Omega_t,\ t>0, \\ &T(\vec{w}, s)\vec{n} = \left(\sigma H + \frac{\omega_0^2}{2}(x_1^2+x_2^2) + \kappa U(x,t) + p_0\right)\vec{n}, \\ &V_n = \vec{w}\cdot\vec{n}, \quad x\in\Gamma_t\equiv\partial\Omega_t, \\ &\vec{w}(x,0) = \vec{v}_0(x) - \vec{\mathcal{V}}(x) \equiv \vec{w}_0(x), \quad x\in\Omega_0, \end{aligned} \tag{4.1}$$

where $\vec{n}$ is the exterior normal to Γ_t, V_n is the velocity of evolution of Γ_t in the direction $\vec{n}$, and $U(x,t) = \int_{\Omega_t} |x-y|^{-1}\,\mathrm{d}y$. We recall that $\vec{v}_0(x)$ satisfies conditions (1.13).

Let us verify directly that $\vec{w}(x,t)$ satisfies the orthogonality conditions

$$\int_{\Omega_t} \vec{w}(x,t)\,\mathrm{d}x = 0, \tag{4.2}$$

$$\int_{\Omega_t} \vec{w}(x,t)\cdot\vec{\eta}_j(x)\,\mathrm{d}x = -\omega_0\int_{\Omega_t}\vec{\eta}_j(x)\cdot\vec{\eta}_3(x)\,\mathrm{d}x + \beta\delta_{j3}, \quad j=1,2,3, \tag{4.3}$$

where $\vec{\eta}_i(x) = \vec{e}_i\times\vec{x}$. Integration of the first equation in (4.1) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\vec{w}(x,t)\,\mathrm{d}x + 2\omega_0\int_{\Omega_t}(\vec{e}_3\times\vec{w})\,\mathrm{d}x - \int_{\Gamma_t}\left(\sigma H + \frac{\omega_0^2}{2}(x_1^2+x_2^2) + \kappa U(x,t) + p_0\right)\vec{n}\,\mathrm{d}S_x = 0.$$

Since

$$\int_{\Gamma_t} H\vec{n}\,\mathrm{d}S_x = 0, \qquad \int_{\Gamma_t} U\vec{n}\,\mathrm{d}S_x = \int_{\Omega_t}\int_{\Omega_t} \frac{\vec{y}-\vec{x}}{|x-y|^3}\,\mathrm{d}y\,\mathrm{d}x = 0,$$

the surface integral reduces to

$$\frac{\omega_0^2}{2}\int_{\Gamma_t}(x_1^2+x_2^2)\vec{n}\,\mathrm{d}S_x = \omega_0^2\int_{\Omega_t}\vec{x}'\,\mathrm{d}x, \qquad \vec{x}' = (x_1, x_2, 0),$$

and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\vec{w}\,\mathrm{d}x + 2\omega_0\int_{\Omega_t}(\vec{e}_3\times\vec{w})\,\mathrm{d}x - \omega_0^2\int_{\Omega_t}\vec{x}'\,\mathrm{d}x = 0,$$

i.e.

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} - \omega_0 I_2(t) = 0, \qquad \frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + \omega_0 I_1(t) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}I_3(t) = 0,$$

where

$$I_j(t) = \vec{e}_j\cdot\left(\int_{\Omega_t}\vec{w}\,\mathrm{d}x + \omega_0\int_{\Omega_t}\vec{\eta}_3(x)\,\mathrm{d}x\right), \qquad j = 1,2,3.$$

It follows that $I_j(t) = I_j(0) = 0$; for $j = 1, 2$ this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} x_1\,\mathrm{d}x - \omega_0\int_{\Omega_t} x_2\,\mathrm{d}x = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} x_2\,\mathrm{d}x + \omega_0\int_{\Omega_t} x_1\,\mathrm{d}x = 0;$$

hence,

$$\int_{\Omega_t} x_j\,\mathrm{d}x = \int_{\Omega_0} x_j\,\mathrm{d}x = 0, \qquad j = 1, 2, \tag{4.4}$$

and, as a consequence,

$$\int_{\Omega_t} w_k\,\mathrm{d}x = I_k(t) = 0, \qquad k = 1, 2, 3.$$

Thus, (4.2) is verified.

Next, we multiply the first equation in (4.1) by $\vec{\eta}_i(x)$ and integrate over Ω_t. On integrating by parts we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\vec{w}\cdot\vec{\eta}_i\,\mathrm{d}x + 2\omega_0\int_{\Omega_t}(\vec{e}_3\cdot[\vec{w}\times\vec{\eta}_i])\,\mathrm{d}x - \omega_0^2\int_{\Omega_t}\vec{x}'\cdot\vec{\eta}_i\,\mathrm{d}x = 0.$$

For $i = 3$ this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\vec{w}\cdot\vec{\eta}_3\,\mathrm{d}x + 2\omega_0\int_{\Omega_t}(\vec{x}'\cdot\vec{w})\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}(\vec{w}+\omega_0\vec{\eta}_3(x))\cdot\vec{\eta}_3(x)\,\mathrm{d}x = 0,$$

and for $i = 1, 2$ we obtain the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}(\vec{w}+\omega_0\vec{\eta}_3)\cdot\vec{\eta}_1\,\mathrm{d}x - \omega_0\int_{\Omega_t}(\vec{w}+\omega_0\vec{\eta}_3)\cdot\vec{\eta}_2\,\mathrm{d}x = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}(\vec{w}+\omega_0\vec{\eta}_3)\cdot\vec{\eta}_2\,\mathrm{d}x + \omega_0\int_{\Omega_t}(\vec{w}+\omega_0\vec{\eta}_3)\cdot\vec{\eta}_1\,\mathrm{d}x = 0.$$

Since $\int_{\Omega_0}(\vec{w}_0+\omega_0\vec{\eta}_3)\cdot\vec{\eta}_k\,\mathrm{d}x=\delta_{k3}\beta$, $k=1,2,3$, we conclude from the last three equations that (4.3) holds.

We also need to introduce the part of $\vec{w}$ orthogonal to all rigid rotations $\vec{\eta}_i$, i.e.

$$\begin{aligned}&\vec{w}^{\perp}(x,t)=\vec{w}(x,t)-\sum_{i=1}^{3}\gamma_i(t)\vec{\eta}_i(x),\\&\int_{\Omega_t}\vec{w}^{\perp}\cdot\vec{\eta}_k\,\mathrm{d}x=0,\qquad k=1,2,3.\end{aligned}\tag{4.5}$$

The latter equations yield an algebraic system for γ_i:

$$\sum_{i=1}^{3}S_{ki}(t)\gamma_i(t)=\int_{\Omega_t}\vec{w}\cdot\vec{\eta}_k\,\mathrm{d}x=-\omega_0S_{k3}(t)+\beta\delta_{k3},\qquad k=1,2,3,$$

where

$$S_{ki}(t)=\int_{\Omega_t}\vec{\eta}_k\cdot\vec{\eta}_i\,\mathrm{d}x=\int_{\Omega_t}(\delta_{ki}|x|^2-x_ix_k)\,\mathrm{d}x$$

are elements of a nonsingular matrix $\mathcal{S}(t)$. Hence,

$$\gamma_i(t)=\sum_{j=1}^{3}S^{ij}(\beta\delta_{j3}-\omega_0S_{j3})=\alpha_i(t)-\omega_0\delta_{i3},\qquad i=1,2,3,$$

where S^{ik} are the elements of $\mathcal{S}^{-1}$ and $\alpha_i(t)=S^{i3}(t)\beta$. It follows that the vector fields $\vec{\mathcal{V}}(x,t)=\omega_0\vec{\eta}_3(x)$,

$$\vec{w}'(x,t)=\sum_{i=1}^{3}\gamma_i(t)\vec{\eta}_i(x)=\vec{\gamma}(t)\times\vec{x}$$

and

$$\vec{w}''(x,t)=\sum_{i=1}^{3}\alpha_i(t)\vec{\eta}_i(x)=\vec{\alpha}(t)\times\vec{x}$$

are related to each other by

$$\vec{w}''(x,t)=\vec{w}'(x,t)+\vec{\mathcal{V}}(x,t),$$

and that

$$\|\vec{w}\|^2_{L_2(\Omega_t)}=\|\vec{w}^{\perp}\|^2_{L_2(\Omega_t)}+\|\vec{w}'\|^2_{L_2(\Omega_t)},\tag{4.6}$$

$$\|\vec{w}'\|^2_{L_2(\Omega_t)}=\sum_{i,k=1}^{3}\gamma_i(t)\gamma_k(t)S_{ik}(t)=S^{33}(t)\beta^2+S_{33}(t)\omega_0^2-2\beta\omega_0.\tag{4.7}$$

Now, we pass to the proof of the solvability of problem (4.1). For this we need some estimates of the solution of a linear problem

$$\begin{aligned}&\vec{v}_t-\nu\nabla^2\vec{v}+\nabla p=\vec{f}(\xi,t),\qquad\nabla\cdot\vec{v}=g(\xi,t),\qquad\xi\in\Omega,\\&\vec{v}(\xi,0)=\vec{v}_0(\xi),\\&\Pi S(\vec{v})\vec{n}=\vec{b}(\xi,t),\qquad\xi\in\Gamma\equiv\partial\Omega,\\&\vec{n}\cdot T(\vec{v},p)\vec{n}-\sigma\vec{n}\cdot\int_0^t\Delta\vec{v}(\xi,\tau)\,\mathrm{d}\tau=b(\xi,t)+\int_0^t B(\xi,\tau)\,\mathrm{d}\tau,\end{aligned}\tag{4.8}$$

in a given bounded domain Ω with a smooth boundary Γ. Here $\vec{n}$ is the exterior normal to Γ and

$$\Pi\vec{\phi}(\xi) = \vec{\phi}(\xi) - \vec{n}(\xi)(\vec{n}(\xi)\cdot\vec{\phi}(\xi))$$

is the projection of the vector $\vec{\phi}(\xi)$ given on Γ to the tangent plane to Γ at the point ξ. Finally, Δ denotes the Laplace–Beltrami operator on Γ.

THEOREM 4.1 ([14, 17]) Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{2+\alpha}$, $\alpha \in (0,1)$, and let $\vec{f}(\cdot,t) \in C^{\alpha}(\Omega)$, $g(\cdot,t) \in C^{1+\alpha}(\Omega)$, $\vec{b} \in C^{1+\alpha,(1+\alpha)/2}(\Gamma\times(0,T))$, $b(\cdot,t) \in C^{1+\alpha}(\Gamma)$, $B(\cdot,t) \in C^{\alpha}(\Gamma)$, $\forall t \in (0,T)$, satisfy the compatibility conditions

$$\Pi S(\vec{v}_0)\vec{n}|_{\Gamma} = b(\xi,0), \qquad \nabla\cdot\vec{v}_0(\xi) = g(\xi,0), \qquad \vec{b}(\xi,t)\cdot\vec{n}(\xi) = 0,$$

and the condition

$$g(\xi,t) = \nabla\cdot\vec{h}(\xi,t)$$

with $\vec{h}_t(\cdot,t) \in C^{\alpha}(\Omega)$, $\forall t \in (0,T)$. Then problem (4.8) has a unique solution $\vec{v} \in C^{2+\alpha}(\Omega)$, $p \in C^{1+\alpha}(\Omega)$ with $\vec{v}_t \in C^{\alpha}(\Omega)$, $\forall t < T$, and the solution satisfies the inequality

$$\begin{aligned}\sup_{t<T}|\vec{v}_t(\cdot,t)|_{C^{\alpha}(\Omega)} + \sup_{t<T}|\vec{v}(\cdot,t)|_{C^{2+\alpha}(\Omega)} + \sup_{t<T}|p(\cdot,t)|_{C^{1+\alpha}(\Omega)}&\\ \leqslant c(T)(|\vec{f}(\cdot,t)|_{C^{\alpha}(\Omega)} + \sup_{t<T}|g(\cdot,t)|_{C^{1+\alpha}(\Omega)} + \sup_{t<T}|\vec{h}_t(\cdot,t)|_{C^{\alpha}(\Omega)}&\\ + |\vec{b}|_{C^{1+\alpha,(1+\alpha)/2}(\Gamma\times(0,T))} + \sup_{t<T}|b(\cdot,t)|_{C^{1+\alpha}(\Gamma)} + \sup_{t<T}|B(\cdot,t)|_{C^{\alpha}(\Gamma)}).&\end{aligned}$$

The local existence theorem for problem (4.1) reads as follows.

THEOREM 4.2 Under the hypotheses of Theorem 2.1, problem (4.1) has a unique solution defined in a certain finite time interval $(0,t_0)$ and possessing the following properties:

(i) Γ_t is given by equation (1.4) with $\mathcal{G} = \mathcal{G}_0$, $\rho = \rho(\cdot,t) \in C^{3+\alpha}(\mathcal{G}_0)$, $t \in (0,t_0)$, $\rho_t(\cdot,t) \in C^{2+\alpha}(\mathcal{G}_0)$, $\rho_{tt}(\cdot,t) \in C^{\alpha}(\mathcal{G}_0)$;
(ii) $\vec{w}(\cdot,t) \in C^{2+\alpha}(\Omega_t)$, $\vec{w}_t(\cdot,t) \in C^{\alpha}(\Omega_t)$, $s(\cdot,t) \in C^{1+\alpha}(\Omega_t)$;
(iii) we have the inequality

$$\begin{aligned}\sup_{t<t_0}|\vec{w}_t(\cdot,t)|_{C^{\alpha}(\Omega_t)} + \sup_{t<t_0}|\vec{w}(\cdot,t)|_{C^{2+\alpha}(\Omega_t)} + \sup_{t<t_0}|\nabla s(\cdot,t)|_{C^{1+\alpha}(\Omega_t)}&\\ + \sup_{t<t_0}|\rho(\cdot,t)|_{C^{3+\alpha}(\mathcal{G}_0)} + \sup_{t<t_0}|\rho(\cdot,t)|_{C^{2+\alpha}(\mathcal{G}_0)} + \sup_{t<t_0}|\rho_{tt}(\cdot,t)|_{C^{\alpha}(\mathcal{G}_0)}&\\ \leqslant c(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)});&\end{aligned} \tag{4.9}$$

(iv) there exists a twice continuously differentiable function $\theta(t)$ such that $\theta(0) = 0$ and that Γ_t can also be given by the equation

$$x = y + N_{\theta(t)}(y)\widehat{\rho}(y,t), \qquad y \in \mathcal{G}_{\theta(t)}, \tag{4.10}$$

with $\widehat{\rho}$ possessing the same regularity properties as ρ and, in addition, the property (2.15). The functions θ and $\widehat{\rho}$ satisfy the inequalities

$$\begin{aligned}|\theta_t(t)| &\leqslant c\int_{\mathcal{G}_{\theta(t)}}|\rho_t(y,t)|\,\mathrm{d}S_y \leqslant c\int_{\Gamma_t}|\vec{w}\cdot\vec{n}|\,\mathrm{d}S_y,\\ |\theta_{tt}(t)| &\leqslant c\int_{\mathcal{G}_{\theta(t)}}(|\rho_{tt}(y,t)| + |\rho_t(y,t)|)\,\mathrm{d}S_y \leqslant c\int_{\Gamma_t}(|\vec{w}| + |\vec{w}_t|)\,\mathrm{d}S_y\end{aligned} \tag{4.11}$$

and

$$\sup_{t<t_0} |\widehat{\rho}(\cdot, t)|_{C^{3+\alpha}(\mathcal{G}_{\theta(t)})} + \sup_{t<t_0} |\widehat{\rho}_t(\cdot, t)|_{C^{2+\alpha}(\mathcal{G}_{\theta(t)})} + \sup_{t<t_0} |\widehat{\rho}_{tt}(\cdot, t)|_{C^{\alpha}(\mathcal{G}_{\theta(t)})} \leqslant c(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}), \quad (4.12)$$

where $\widehat{\rho}_t(y, t)$ and $\widehat{\rho}_{tt}(y, t)$ are understood as in Section 2 (see (2.18)).

Proof. The proof of the solvability of problem (4.1) and of estimate (4.9) is based on the passage to the Lagrangean coordinates and on the use of Theorem 4.1. It is identical with the corresponding arguments in [18, Theorem 3.2], and we only give a very rough idea of it. The Lagrangean coordinates $\xi \in \Omega_0$ are related to the Eulerian coordinates $x \in \Omega_t$ by

$$\vec{x} = \vec{\xi} + \int_0^t \vec{u}(\xi, \tau)\,\mathrm{d}\tau = \vec{X}(\xi, t), \quad (4.13)$$

where $\vec{u}(\xi, t) = \vec{w}(X(\xi, t), t)$ is the velocity vector field written as a function of ξ, t. Together with $q(\xi, t) = s(X(\xi, t), t)$, $\vec{u}$ satisfies the relations

$$\vec{u}_t - \nu\nabla_u^2\vec{u} + 2\omega_0\vec{e}_3 \times \vec{u} + \nabla_u q = 0, \quad \nabla_u \cdot \vec{u} = 0, \quad \xi \in \Omega_0, \quad (4.14)$$

$$\vec{u}(\xi, 0) = \vec{w}_0(\xi), \quad (4.15)$$

$$T_u(\vec{u}, q)\vec{n} - \sigma H\vec{n} = \left(\frac{\omega_0^2}{2}|X'(\xi, t)|^2 + p_0 + \kappa U(X, t)\right)\vec{n}, \quad \xi \in \Gamma_0, \quad (4.16)$$

where $\nabla_u = A\nabla$ is the transformed gradient, $A = (A_{ij})_{i,j=1,2,3}$ is the matrix of cofactors of the Jacobi matrix of the transformation (4.13) (the Jacobian of this transformation equals one), $|X'|^2 = X_1^2 + X_2^2$, and finally $T_u(\vec{u}, q) = -qI + \nu S_u(\vec{u})$ and

$$S_u(\vec{u}) = \left(\sum_{k=1}^3 \left(A_{ik}\frac{\partial u_j}{\partial \xi_k} + A_{jk}\frac{\partial u_i}{\partial \xi_k}\right)\right)_{i,j=1,2,3}$$

are the transformed stress and rate-of-strain tensors, respectively. Using the well known formula $H\vec{n} = \Delta(t)\vec{X}$, where $\Delta(t)$ is the Laplace–Beltrami operator on Γ_t, one can easily show that under the condition $\vec{n} \cdot \vec{n}_0 > 0$, (4.16) is equivalent to two equations

$$\begin{aligned} &\Pi_0 \Pi S_u(\vec{u})\vec{n} = 0, \\ &\vec{n}_0 \cdot T_u(\vec{u}, q)\vec{n} - \sigma\vec{n}_0 \cdot \Delta(t)\left(\vec{\xi} + \int_0^t \vec{u}(\xi, \tau)\,\mathrm{d}\tau\right) \\ &\qquad = \left(\frac{\omega^2}{2}|X'(\xi, t)|^2 + p_0 + \kappa U(X, t)\right)\vec{n} \cdot \vec{n}_0, \quad \xi \in \Gamma_0, \end{aligned} \quad (4.17)$$

where $\vec{n}_0$ is the exterior normal to Γ_0 and

$$\Pi\vec{\phi} = \vec{\phi} - \vec{n}(\vec{n} \cdot \vec{\phi}), \quad \Pi_0\vec{\phi} = \vec{\phi} - \vec{n}_0(\vec{n}_0 \cdot \vec{\phi}).$$

The first statement of Theorem 4.2 is obtained by linearizing problem (4.14)–(4.16) and using Theorem 4.1 (see [16, 17] for more details). From the interpolation inequalities and from (2.14) it follows that

$$\sup_{\mathcal{G}_0} |\rho_0(y)| \leqslant c\epsilon^{(3+\alpha)/(4+\alpha)} |\rho_0|^{1/(4+\alpha)}_{C^{3+\alpha}(\mathcal{G}_0)},$$

$$\sup_{\mathcal{G}_0} |\nabla\rho_0(y)| \leqslant c\epsilon^{(2+\alpha)/(4+\alpha)} |\rho_0|^{2/(4+\alpha)}_{C^{3+\alpha}(\mathcal{G}_0)}.$$

Since

$$|\rho(y,t)| \leqslant |\rho_0(y)| + \int_0^t |\rho_\tau(y,\tau)|\,\mathrm{d}\tau,$$

$$|\nabla_{\mathcal{G}_0}\rho(y,t)| \leqslant |\nabla_{\mathcal{G}_0}\rho_0(y)| + \int_0^t |\nabla_{\mathcal{G}_0}\rho_\tau(y,\tau)|\,\mathrm{d}\tau,$$

condition (1.5) holds for $\rho(y,t)$, $t \leqslant t_0$, if ε and t_0 are sufficiently small.

For the construction of $\theta(t)$, all the necessary calculations are carried out in Section 3. Due to Lemma 3.1, we may assume without loss of generality that $\rho_0 = \widehat{\rho}_0$, i.e., $\mathcal{F}_0$ is chosen in such a way that ρ_0 satisfies (1.18). Then we make use of Lemma 3.2 and set $\theta(t) = -\lambda(t)$; we assume that $\theta(t)$ is defined in the same time interval $[0, t_0]$ as $\vec{w}, s, \rho$. The estimates (4.11) follow from (3.27), (3.39) and from the kinematic boundary condition $V_n = \vec{w} \cdot \vec{n}$ that can also be written in an equivalent form

$$\rho_t(y,t) = \frac{\vec{w}(x,t)\cdot\vec{n}(x)}{\vec{n}(x)\cdot\vec{N}_0(y)}, \qquad x = y + N_0(y)\rho(y,t) \in \Gamma_t,\ y \in \mathcal{G}_0.$$

Finally, we set $\widehat{\rho}(y,t) = \widetilde{\rho}(\mathcal{Z}^{-1}(\theta(t))y,t) = \widetilde{\rho}(\mathcal{Z}(\lambda(t))y,t)$, $y \in \mathcal{G}_{\theta(t)}$, where $\widetilde{\rho}(z,t) = \widetilde{\rho}(z,\lambda(t),t)$, $z \in \mathcal{G}_0$. The $C^1(\mathcal{G}_{\theta(t)})$-norm of $\widehat{\rho}$ can be estimated with the help of (3.41), (3.42). It is easily seen that $\widehat{\rho}$ satisfies (1.5) if t_0 and ϵ are sufficiently small. An estimate of the $C^{3+\alpha}(\mathcal{G}_{\theta(t)})$-norm of $\widehat{\rho}$ can be derived from the equation

$$\sigma(H(x) - \widehat{\mathcal{H}}(y)) + \frac{\omega_0^2}{2}(x_1^2 + x_2^2) - \frac{\omega_0^2}{2}(y_1^2 + y_2^2) + \kappa(U(x,t) - \widehat{\mathcal{U}}(y)) = \vec{n}\cdot T(\vec{w},s)\vec{n}, \qquad y \in \mathcal{G}_{\theta(t)}, \tag{4.18}$$

which is a consequence of the boundary conditions. Here $x = y + N_{\theta(t)}\widehat{\rho}(y,t) \in \Gamma_t$, $\widehat{\mathcal{H}}(y)$ is the doubled mean curvature of $\mathcal{G}_{\theta(t)}$ at the point y, and $\widehat{\mathcal{U}}(y) = \int_{\mathcal{F}_{\theta(t)}} |y - z|^{-1}\,\mathrm{d}z$. By Proposition 3.1 in [16] and (4.8), equation (4.18) implies

$$\begin{aligned}|\widehat{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G}_{\theta(t)})} &\leqslant c(|\vec{n}\cdot T(\vec{w},s)\vec{n}|_{C^{1+\alpha}(\Gamma_t)} + \|\widehat{\rho}(\cdot,t)\|_{L_2(\mathcal{G}_{\theta(t)})}) \\ &\leqslant c(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}).\end{aligned} \tag{4.19}$$

The simplest way to estimate the norms of the derivatives $\widehat{\rho}_t$, $\widehat{\rho}_{tt}$ is to make use of the kinematic boundary condition $V_n = \vec{w}\cdot\vec{n}$. Let us show that the velocity $\widetilde{V}_n$ of evolution of the surface $\Gamma_t(\lambda(t)) = \mathcal{Z}(\lambda(t))\Gamma_t$ in the direction of the exterior normal can be expressed in terms of V_n as follows:

$$\widetilde{V}_n = V_n - \theta_t'(t)(\vec{e}_3 \times \vec{x})\cdot\vec{n} = (\vec{w} - \theta_t'(t)(\vec{e}_3 \times \vec{x}))\cdot\vec{n}(x), \qquad x \in \Gamma_t.$$

We recall that $\Gamma_t(\lambda(t))$ and Γ_t are given by the equations

$$\widetilde{x} = z + N_0(z)\widetilde{\rho}(z,t) \equiv X(z,t), \qquad z \in \mathcal{G}_0, \tag{4.20}$$

and

$$x = \mathcal{Z}(\theta(t))X(z,t) \equiv Y(z,t), \qquad z \in \mathcal{G}_0,$$

respectively, and that the normal $\vec{\tilde{n}}$ at the point $\tilde{x}$ is related to $\vec{n}(x)$ by

$$\vec{n} = \mathcal{Z}(\theta(t))\vec{\tilde{n}}.$$

Hence,

$$V_n = Y_t' \cdot \vec{n} = (\mathcal{Z}X)_t' \cdot \mathcal{Z}\vec{\tilde{n}} = X_t' \cdot \vec{\tilde{n}} + \mathcal{Z}_t'\mathcal{Z}^{-1}Y \cdot \vec{n} = \widetilde{V}_n + \theta_t'(\vec{e}_3 \times \vec{x}) \cdot \vec{n},$$

as claimed. On the other hand, $\widetilde{V}_n = \widetilde{\rho}_t(\vec{N} \cdot \vec{\tilde{n}})$, so

$$\widetilde{\rho}_t(z,t) = (\vec{N} \cdot \vec{\tilde{n}})^{-1}(\vec{w}(x,t) - \theta_t'(t)(\vec{e}_3 \times \vec{x})) \cdot \vec{n}(x). \tag{4.21}$$

From this relation and from (4.9) it is easy to deduce estimate (4.12) for the time derivatives of $\widetilde{\rho}$ and, as a consequence, of $\widehat{\rho}$. The theorem is proved.

Let us turn to uniform estimates of the solution of problem (4.1). One of them is an estimate of a generalized energy.

THEOREM 4.3 Assume that problem (4.1) has a classical solution defined for $t \in [0,T]$, $T \leqslant \infty$, and that Γ_t is given by equation (4.10) with $\widehat{\rho}(y,t)$ satisfying (2.15). If (1.16) holds, then there exists a function $E(t)$ such that

$$c_3(\|\vec{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\widehat{\rho}(\cdot,t)\|^2_{W_2^1(\mathcal{G}_{\theta(t)})}) \leqslant E(t) \leqslant c_4(\|\vec{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\widehat{\rho}(\cdot,t)\|^2_{W_2^1(\mathcal{G}_{\theta(t)})}) \tag{4.22}$$

and

$$E(t) \leqslant c^{-bt}E(0), \qquad b > 0, \tag{4.23}$$

for $t \leqslant T$. The constants c_1, c_2, b are independent of T.

Proof. First of all, we have the energy relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\vec{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + G(t)\right) + \frac{\nu}{2}\|S(\vec{w})\|^2_{L_2(\Omega_t)} = 0, \tag{4.24}$$

where $G(t)$ is the functional (1.14) with $\Omega = \Omega_t$. This relation is obtained by multiplying the first equation in (4.1) by $\vec{w}$ and integrating over Ω_t (cf. [9, 10]). By (4.6) and (4.7), relation (4.24) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\vec{w}^{\perp}(\cdot,t)\|^2_{L_2(\Omega_t)} + R_1(t) - R_0\right) + \frac{\nu}{2}\|S(\vec{w}^{\perp})\|^2_{L_2(\Omega_t)} = 0, \tag{4.25}$$

where

$$R_1(t) = \frac{\beta^2}{2}S^{33}(t) + \frac{\omega_0^2}{2}S_{33}(t) + G(t) = \frac{1}{2}\beta^2\left(S^{33}(t) - \frac{1}{S_{33}(t)}\right) + R(t),$$

and $R(t)$, R_0 are defined by (1.2) with $\Omega = \Omega_t$ and $\Omega = \mathcal{F}_{\theta(t)}$, respectively (it is clear that R_0 is independent of t). The expression

$$\begin{aligned} S^{33}(t) - \frac{1}{S_{33}(t)} &= -\frac{1}{S_{33}(t)}\sum_{j=1}^{2} S^{j3}(t)S_{j3}(t) \\ &= \frac{S_{22}(t)S_{13}^2(t) + S_{11}(t)S_{23}^2(t) - 2S_{12}(t)S_{13}(t)S_{23}(t)}{S_{33}(t)\det\mathcal{S}(t)} \end{aligned} \tag{4.26}$$

is a positive definite quadratic form with respect to $S_{13}(t), S_{23}(t)$ (this follows from $S_{12}^2(t) \leqslant S_{11}(t)S_{22}(t)$). By our main hypothesis concerning R, the difference $R(t) - R_0$ is equivalent to the square of the norm $\|\widehat{\rho}(\cdot,t)\|_{W_2^1(\mathcal{G}_{\theta(t)})}$. Indeed,

$$R(t)-R(0) = R[\widehat{\rho}]-R[0] = \delta_0 R[\widehat{\rho}]+\int_0^1 \mathrm{d}\lambda \int_0^\lambda \frac{\mathrm{d}^2}{\mathrm{d}\mu^2}R[\mu\widehat{\rho}]\,\mathrm{d}\mu = \delta_0 R[\widehat{\rho}]+\frac{1}{2}\delta_0^2 R[\widehat{\rho}]+R_1[\widehat{\rho}],$$

where

$$\delta_0 R[\widehat{\rho}] = \frac{\mathrm{d}}{\mathrm{d}\lambda}R[\lambda\widehat{\rho}]\Big|_{\lambda=0} = 0$$

and

$$R_1[\widehat{\rho}] = \int_0^1 (1-\mu)\left(\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}R[\mu\widehat{\rho}] - \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}R[\lambda\widehat{\rho}]\Big|_{\lambda=0}\right)\mathrm{d}\mu$$

is a remainder not exceeding $c\delta\|\widehat{\rho}\|^2_{W_2^1(\mathcal{G}_{\theta(t)})}$, since $\widehat{\rho}$ satisfies (1.5). In addition, $\widehat{\rho}$ satisfies (1.6), (1.18) (with $\mathcal{G} = \mathcal{G}_{\theta(t)}$), hence,

$$c_1'\|\widehat{\rho}\|^2_{W_2^1(\mathcal{G}_{\theta(t)})} \leqslant R[\widehat{\rho}] - R[0] \leqslant c_2'\|\widehat{\rho}\|^2_{W_2^1(\mathcal{G}_{\theta(t)})} \tag{4.27}$$

if δ is small enough.

To complete the proof of (4.23), we need to obtain an additional estimate for $\|\widehat{\rho}\|_{\mathcal{G}_{\theta(t)}}$. According to Lemma 4.1 in [18], in the domain $\widetilde{\Omega}(t) = \mathcal{Z}(-\theta(t))\Omega_t$ whose boundary $\widetilde{\Gamma}_t = \mathcal{Z}(-\theta(t))\Gamma_t$ is given by equation (4.20) there exists a solenoidal vector field $\vec{U}(x,t)$ with the following properties:

(1) $\vec{U}$ satisfies the boundary conditions

$$\vec{U}(x,t)\cdot\vec{n}(x) = m(y,\widetilde{\rho}(y,t))\varphi(y;\widetilde{\rho}(y,t)), \qquad x \in \widetilde{\Gamma}_t,$$

where $\varphi(y,\rho)$ is defined in (1.7), y is the point of $\mathcal{G}_0$ such that $x = y + N_0(y)\widetilde{\rho}(y,t)$, and $m(y,\widetilde{\rho}(y,t))$ is a positive function satisfying

$$\int_{\widetilde{\Gamma}_t} f(x)m(y;\widetilde{\rho})\,\mathrm{d}S_x = \int_{\mathcal{G}_0} f(y+N_0(y)\widetilde{\rho}(y,t))\,\mathrm{d}S_y$$

for any $f(x)$, $x \in \widetilde{\Gamma}_t$;

(2) $\vec{U}$ is orthogonal to all vectors of rigid rotation:

$$\int_{\widetilde{\Omega}_t} \vec{U}(x,t)\cdot\vec{\eta}_i(x)\,\mathrm{d}x = 0, \qquad i = 1,2,3;$$

(3) we have the estimates

$$\|\vec{U}(\cdot,t)\|_{W_2^1(\widetilde{\Omega}_t)} \leqslant c\|\widetilde{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_0)},$$
$$\|\vec{U}(\cdot,t)\|_{L_2(\widetilde{\Omega}_t)} \leqslant c\|\widetilde{\rho}(\cdot,t)\|_{L_2(\mathcal{G}_0)},$$
$$\|\vec{U}_t(\cdot,t)\|_{L_2(\widetilde{\Omega}_t)} \leqslant c(\|\widetilde{\rho}_t(\cdot,t)\|_{L_2(\mathcal{G}_0)} + \|\widetilde{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_0)}).$$

It is easy to verify that the vector field $\vec{W}(x,t) = \mathcal{Z}(\theta(t))\vec{U}(\mathcal{Z}^{-1}(\theta(t))x,t)$, $x \in \Omega_t$, has the same properties in Ω_t, in particular,

$$\|\vec{W}(\cdot,t)\|_{W_2^1(\Omega_t)} \leqslant c\|\widetilde{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_0)}, \tag{4.28}$$

and, moreover, the derivative

$$\vec{W}_t(x,t) = \mathcal{Z}'(\theta(t))\theta'(t)\vec{U}(\mathcal{Z}^{-1}x,t) + \mathcal{Z}(\theta(t))(\vec{U}_{,t}(\mathcal{Z}^{-1}(\theta(t))x,t) + \sum_{k=1}^{3}\vec{U}_{,k}(\mathcal{Z}^{-1}(\theta(t))x,t)((\mathcal{Z}^{-1})'(\theta(t))x)_k\theta'(t)),$$

where $\vec{U}_{,t}(z,t) = \partial\vec{U}(z,t)/\partial t$ and $\vec{U}_{,k}(z,t) = \partial\vec{U}(z,t)/\partial z_k$, satisfies the inequality

$$\|\vec{W}_t(\cdot,t)\|_{L_2(\Omega_t)} \leqslant c(\|\widetilde{\rho}_t(\cdot,t)\|_{L_2(\mathcal{G}_0)} + \|\widetilde{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_0)}). \tag{4.29}$$

We write the first equation in (4.1) in the form

$$\vec{w}_t^{\perp} + (\vec{w}\cdot\nabla)\vec{w}^{\perp} + \omega_0(\vec{e}_3\times\vec{w}) + (\vec{w}\cdot\nabla)\vec{w}'' - \nu\nabla^2\vec{w} + \nabla s = -\vec{w}_t',$$

multiply it by $\vec{W}$ and integrate over Ω_t. After integration by parts we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\vec{w}^{\perp}\cdot\vec{W}\,\mathrm{d}x - \int_{\Omega_t}\vec{w}^{\perp}\cdot(\vec{W}_t + (\vec{w}\cdot\nabla)\vec{W})\,\mathrm{d}x + \int_{\Omega_t}(\vec{w}\cdot\nabla)\vec{w}''\cdot\vec{W}\,\mathrm{d}x + \int_{\Omega_t}\omega_0(\vec{e}_3\times\vec{w})\cdot\vec{W}\,\mathrm{d}x + \frac{\nu}{2}\int_{\Omega_t}S(\vec{w}):S(\vec{W})\,\mathrm{d}x - \int_{\Gamma_t}\left(\sigma H(x) + \frac{1}{2}|\vec{\mathcal{V}}(x,t)|^2 + \kappa U(x,t) + p_0\right)\vec{W}\cdot\vec{n}\,\mathrm{d}S_x = 0. \tag{4.30}$$

It is easily verified that

$$(\vec{w}\cdot\nabla)\vec{w}'' + \omega_0(\vec{e}_3\times\vec{w}) = (\vec{w}^{\perp}\cdot\nabla)(\vec{\mathcal{V}}+\vec{w}'') + (\vec{w}''\cdot\nabla)\vec{w}'' - (\vec{\mathcal{V}}\cdot\nabla)\vec{\mathcal{V}} + [(\vec{w}''\cdot\nabla)\vec{\mathcal{V}} - (\vec{\mathcal{V}}\cdot\nabla)\vec{w}'']$$

and that the last term is a rigid rotation:

$$(\vec{w}''\cdot\nabla)\vec{\mathcal{V}} - (\vec{\mathcal{V}}\cdot\nabla)\vec{w}'' = \omega_0\sum_{j=1}^{2}\alpha_j(t)[(\eta_j\cdot\nabla)\eta_3 - (\eta_3\cdot\nabla)\eta_j] = \omega_0(\alpha_1\vec{\eta}_2 - \alpha_2\vec{\eta}_1).$$

Hence, it is orthogonal to $\vec{W}$. We also observe that

$$(\vec{w}''\cdot\nabla)\vec{w}'' - (\vec{\mathcal{V}}\cdot\nabla)\vec{\mathcal{V}} = \frac{1}{2}\nabla(|\vec{\mathcal{V}}|^2 - |\vec{w}''|^2),$$

so (4.30) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\vec{w}^{\perp}\cdot\vec{W}\,\mathrm{d}x - \int_{\Omega_t}\vec{w}^{\perp}\cdot(\vec{W}_t + (\vec{w}\cdot\nabla)\vec{W})\,\mathrm{d}x + \int_{\Omega_t}(\vec{w}^{\perp}\cdot\nabla)(\vec{w}''+\vec{\mathcal{V}})\cdot\vec{W}\,\mathrm{d}x + \frac{\nu}{2}\int_{\Omega_t}S(\vec{w}^{\perp}):S(\vec{W})\,\mathrm{d}x - \int_{\Gamma_t}\left(\sigma H(x) + \frac{1}{2}|\vec{w}''(x,t)|^2 + p_0 + \kappa U(x,t)\right)\vec{W}\cdot\vec{n}\,\mathrm{d}S_x = 0. \tag{4.31}$$

Finally, we multiply (4.31) by a small positive γ and add to (4.25), which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + E_1(t) = 0,$$

where

$$\begin{aligned} E(t) &= \|\vec{w}^{\perp}(\cdot,t)\|^2_{L_2(\Omega_t)} + R_1(t) - R_0 + \gamma \int_{\Omega_t} \vec{w}^{\perp} \cdot \vec{W}\,\mathrm{d}x, \\ E_1(t) &= \frac{\nu}{2}\|S(\vec{w}^{\perp})\|^2_{L_2(\Omega)} - \gamma \int_{\Omega_t} \vec{w}^{\perp} \cdot (\vec{W}_t + (\vec{w}\cdot\nabla)\vec{W})\,\mathrm{d}x \\ &\quad + \gamma \int_{\Omega_t} (\vec{w}^{\perp}\cdot\nabla)(\vec{w}'' + \vec{w}')\cdot \vec{W}\,\mathrm{d}x + \frac{\gamma\nu}{2}\int_{\Omega_t} S(\vec{w}^{\perp}) : S(\vec{W})\,\mathrm{d}x - \gamma I_{\mathcal{G}} \end{aligned} \tag{4.32}$$

and $I_{\mathcal{G}}$ is the last integral in (4.31).

It is clear that (4.22) follows from (1.16) if γ is a sufficiently small (but fixed) constant.

In order to obtain (4.23), we estimate the function $E_1(t)$ from below (in the same way as in [18, Theorem 4.1]). By (4.28), (4.29) and the Korn inequality

$$\|\vec{w}^{\perp}(\cdot,t)\|_{W_2^1(\Omega_t)} \leqslant c\|S(\vec{w}^{\perp})\|_{L_2(\Omega_t)},$$

we have

$$\begin{aligned} \left|\int_{\Omega_t} \vec{w}^{\perp}\cdot(\vec{W}_t + (\vec{w}\cdot\nabla)\vec{W})\,\mathrm{d}x\right| &\leqslant c\|\vec{w}^{\perp}(\cdot,t)\|_{L_2(\Omega_t)}(\|\widetilde{\rho}_t(\cdot,t)\|_{L_2(\mathcal{G}_0)} + \|\widetilde{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_0)}) \\ &\leqslant c\|S(\vec{w}^{\perp}(\cdot,t))\|_{L_2(\Omega_t)}(\|\vec{w}(\cdot,t)\|_{L_2(\Gamma_t)} + \|\widehat{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_{\theta(t)})}), \end{aligned}$$

so the first four integrals in (4.30) are not less than

$$\frac{\nu}{2}\|S(\vec{w}^{\perp})\|^2_{L_2(\Omega_t)} - c\gamma\|S(\vec{w}^{\perp})\|_{L_2(\Omega_t)}(\|\vec{w}(\cdot,t)\|_{L_2(\Gamma_t)} + \|\widehat{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_{\theta(t)})}). \tag{4.33}$$

Now, we estimate the $L_2(\Gamma_t)$-norm of $\vec{w} = \vec{w}^{\perp} + \vec{w}'' - \vec{\mathcal{V}}$. Analysis of the difference $\vec{w}'' - \vec{\mathcal{V}}$ (see [18, proof of Theorem 4.1]) shows that

$$\|\vec{w}'' - \vec{\mathcal{V}}\|_{L_2(\Gamma_t)} \leqslant c\|\widehat{\rho}(\cdot,t)\|_{L_2(\mathcal{G}_{\theta(t)})},$$

hence,

$$\|\vec{w}\|_{L_2(\Gamma_t)} \leqslant \|\vec{w}^{\perp}\|_{L_2(\Gamma_t)} + c\|\widehat{\rho}\|_{L_2(\mathcal{G}_{\theta(t)})}.$$

Using again the Korn inequality we conclude that the difference (4.33) is not less than

$$(\nu/2 - c\gamma)\|S(\vec{w}^{\perp})\|^2_{L_2(\Omega_t)} - c\gamma\|S(\vec{w}^{\perp})\|_{L_2(\Omega_t)}\|\widehat{\rho}(\cdot,t)\|_{W_2^{1/2}(\mathcal{G}_{\theta(t)})}.$$

The surface integral $I_{\mathcal{G}}$ can be written in the form

$$\begin{aligned} I_{\mathcal{G}} = \int_{\mathcal{G}_{\theta(t)}} \Big[&\sigma(H(x) - \widehat{\mathcal{H}}(y)) + \frac{1}{2}(|\vec{w}''(y,t)|^2 - \omega_0^2(y_1^2 + y_2^2)) \\ &+ \kappa(U(x) - \widehat{\mathcal{U}}(y))\Big]\varphi(y;\widehat{\rho}(y,t))\,\mathrm{d}S_y, \qquad x = y + N_{\theta(t)}\widehat{\rho}(y,t) \in \Gamma_t. \end{aligned}$$

Repeating the calculations carried out in [15, 16, 18] for symmetric $\mathcal{F}$, one easily shows that

$$-I_{\mathcal{G}} = \mathcal{Q}[\widehat{\rho}] + \delta_0^2 R[\widehat{\rho}] + R'[\widehat{\rho}], \tag{4.34}$$

where $\mathcal{Q}$ is the quadratic form

$$\mathcal{Q}[\widehat{\rho}] = \frac{\omega_0^2}{S_{11}^{(0)} S_{22}^{(0)} - S_{12}^{(0)2}} (S_{22}^{(0)} \Sigma_{13}^2[\widehat{\rho}] + S_{11}^{(0)} \Sigma_{23}^2[\widehat{\rho}] - 2S_{12}^{(0)} \Sigma_{13}[\widehat{\rho}] \Sigma_{23}[\widehat{\rho}]), \tag{4.35}$$

$S_{jk}^{(0)} = \int_{\mathcal{F}_{\theta(t)}} \vec{\eta}_j(x) \cdot \vec{\eta}_k(x)\, \mathrm{d}x$ and

$$\Sigma_{j3}[\widehat{\rho}] = \delta_0 S_{j3} = -\int_{\mathcal{G}_\theta(t)} x_3 x_j \widehat{\rho}(x,t)\, \mathrm{d}S_x, \qquad j = 1, 2.$$

Since $S_{12}^{(0)2} \leqslant S_{11}^{(0)} S_{22}^{(0)}$, $\mathcal{Q}[\widehat{\rho}]$ is nonnegative. The last expression R' in (4.34) is the sum of the terms in $-I_{\mathcal{G}}$ of degree higher than 2; it satisfies the inequality

$$|R'[\widehat{\rho}]| \leqslant c\delta \|\widehat{\rho}\|^2_{W_2^1(\mathcal{G}_{\theta(t)})},$$

because $\widehat{\rho}$ satisfies (1.5). From the above estimates it follows that

$$\begin{aligned} E_1(t) \geqslant (\nu/2 - c\gamma) \|S(\vec{w}^\perp)\|^2_{L_2(\Omega_t)} - c\gamma \|S(\vec{w}^\perp)\|_{L_2(\Omega_t)} \|\widehat{\rho}(\cdot,t)\|_{W_2^1(\mathcal{G}_{\theta(t)})} \\ + \gamma \delta_0^2 R[\widehat{\rho}] - c\delta \|\widehat{\rho}\|^2_{W_2^1(\mathcal{G}_{\theta(t)})}. \end{aligned} \tag{4.36}$$

By (1.16), this inequality implies $E_1(t) \geqslant bE(t)$ for some $b > 0$, and, as a consequence, (4.23), for appropriate sufficiently small γ and δ. The theorem is proved.

The next theorem concerns uniform estimates of the Hölder norms of the solution.

THEOREM 4.4 Assume that the solution of problem (4.1) is defined for $t \in (0, T)$ and that it has properties (ii)–(iv) of Theorem 4.2. Then

$$\begin{aligned} |\vec{w}_t(\cdot,t)|_{C^\alpha(\Omega_t)} + |\vec{w}(\cdot,t)|_{C^{2+\alpha}(\Omega_t)} + |s(\cdot,t)|_{C^{1+\alpha}(\Omega_t)} + |\widehat{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G}_{\theta(t)})} + |\widehat{\rho}_t(\cdot,t)|_{C^{2+\alpha}(\mathcal{G}_{\theta(t)})} \\ + |\widehat{\rho}_{tt}(\cdot,t)|_{C^\alpha(\mathcal{G}_{\theta(t)})} \leqslant c\Big(\sup_{t-2\tau_0 \leqslant t' \leqslant t} \|\vec{w}(\cdot,t)\|_{L_2(\Omega'_{t'})} + \sup_{t-2\tau_0 \leqslant t' \leqslant t} \|\widehat{\rho}(\cdot,t)\|_{W_2^1(\mathcal{G}_{\theta(t')})}\Big), \end{aligned} \tag{4.37}$$

where τ_0 is a certain small number. The constant c is independent of t.

For completeness, we give the main ideas of the proof that is practically identical with the proof of Theorem 4.1 in [16]. Let $t_0 > 2\tau_0$, $t_1 = t_0 - 2\tau_0$, $\lambda \in (0, \tau_0)$, and let $\zeta_\lambda(t)$ be a smooth function equal to one for $t > t_1 + \lambda$, to zero for $t < t_1 + \lambda/2$, and satisfying the inequalities $0 \leqslant \zeta_\lambda(t) \leqslant 1$ and

$$\left| \frac{\partial^k \zeta_\lambda}{\partial t^k} \right| \leqslant c\lambda^{-k}, \qquad k = 1, 2.$$

We pass to the Lagrangean coordinates $\xi \in \Omega_{t_1}$:

$$\vec{x} = \vec{\xi} + \int_{t_1}^t \vec{u}(\xi,\tau)\, \mathrm{d}\tau \equiv X(\xi,t), \qquad \vec{u}(\xi,t) = \vec{v}(X(\xi,t),t),$$

and we introduce the functions $q(\xi,t) = p(X(\xi,t),t)$, $\vec{u}_\lambda(\xi,t) = \vec{u}(\xi,t)\zeta_\lambda(t)$, $q_\lambda(\xi,t) = q(\xi,t)\zeta_\lambda(t)$. They satisfy the relations

$$\vec{u}_{\lambda t} - \nu\nabla_u^2\vec{u}_\lambda + 2\omega_0\vec{e}_3 \times \vec{u}_\lambda + \nabla_u q_\lambda = \vec{u}\zeta_\lambda'(t),$$
$$\nabla_u \cdot \vec{u}_\lambda = 0, \quad \xi \in \Omega_\lambda,\ t \in (t_1, T),$$
$$\vec{u}_\lambda(\xi, t_1) = 0,$$
$$\Pi_1 \Pi S_u(\vec{u}_\lambda)\vec{n} = 0, \quad \xi \in \Gamma_{t_1},$$
$$\vec{n}_1 \cdot T_u(\vec{u}_\lambda, q_\lambda)\vec{n} - \sigma\int_{t_1}^t \vec{n}_1 \cdot \Delta(\tau)\vec{u}_\lambda(\xi,\tau)\,\mathrm{d}\tau = b_\lambda + \int_{t_1}^t B_\lambda(\xi,\tau)\,\mathrm{d}\tau,$$

where $\vec{n}_1$ is the exterior normal to Γ_{t_1}, $\Pi_1\vec{\phi} = \vec{\phi} - \vec{n}_1(\vec{n}_1 \cdot \vec{\phi})$, and

$$b_\lambda(\xi,t) = \sigma\vec{n}_1 \cdot \int_{t_1}^t \zeta_\lambda(\tau)\frac{\mathrm{d}\Delta(\tau)}{\mathrm{d}\tau}\vec{\xi}\,\mathrm{d}\tau,$$
$$B_\lambda(\xi,t) = \vec{n}_1 \cdot T_u(\vec{u}, q)\vec{n}\zeta_\lambda'(t) + \sigma\vec{n}_1(\xi)\cdot\zeta_\lambda(t)\frac{\mathrm{d}\Delta(t)}{\mathrm{d}t}\int_{t_1}^t \vec{u}(\xi,\tau)\,\mathrm{d}\tau$$
$$+ \zeta_\lambda(t)\frac{\partial}{\partial t}\left[\left(\frac{\omega_0^2}{2}(X_1^2(\xi,t) + X_2^2(\xi,t)) + p_1 + \kappa U(X)\right)(\vec{n}\cdot\vec{n}_1)\right]$$

(see [16] for more details). By Theorem 4.1, one obtains (for τ_0 sufficiently small)

$$\sup_{t_1<t<t_0} |\vec{u}_{\lambda t}(\cdot,t)|_{C^\alpha(\Omega_{t_1})} + \sup_{t_1<t<t_0} |\vec{u}_\lambda(\cdot,t)|_{C^{2+\alpha}(\Omega_{t_1})} + \sup_{t_1<t<t_0} |q_\lambda(\cdot,t)|_{C^{1+\alpha}(\Omega_{t_1})}$$
$$\leqslant c\lambda^{-1}\Big(\sup_{t_1+\lambda/2<t<t_0} |\vec{u}(\cdot,t)|_{C^{1+\alpha}(\Omega_{t_1})} + \sup_{t_1+\lambda/2<t<t_0} |q(\cdot,t)|_{C^\alpha(\Gamma_{t_1})}\Big). \tag{4.38}$$

The norm of q on the right hand side is estimated by using the boundary condition (4.18). We have

$$|s(\cdot,t)|_{C^\alpha(\Gamma_t)} \leqslant c(|\vec{w}(\cdot,t)|_{C^{1+\alpha}(\Gamma_t)} + |\widehat{\rho}(\cdot,t)|_{C^{2+\alpha}(\mathcal{G}_{\theta(t)})}).$$

Now, we use the interpolation inequalities

$$|\vec{u}(\cdot,t)|_{C^{1+\alpha}(\Omega_{t_1})} \leqslant (\theta|\vec{u}(\cdot,t)|_{C^{2+\alpha}(\Omega_{t_1})} + \theta^{-5/2-\alpha}\|\vec{u}(\cdot,t)\|_{L_2(\Gamma_{t_1})}),$$
$$|\widehat{\rho}(\cdot,t)|_{C^{2+\alpha}(\mathcal{G}_{\theta(t)})} \leqslant (\theta|\widehat{\rho}(\cdot,t)|_{C^{3+\alpha}(\mathcal{G}_{\theta(t)})} + \theta^{-5/2-\alpha}\|\widehat{\rho}(\cdot,t)\|_{W_2^1(\mathcal{G}_{\theta(t)})})$$

with $\theta = \lambda\epsilon_1$ and estimate (4.19). We multiply (4.38) by $\lambda^{\alpha+7/2}$ and arrive after easy calculations at

$$f(\lambda) \leqslant c\epsilon_1 f(\lambda/2) + K,$$

where

$$f(\lambda) = \lambda^{\alpha+7/2}\Big(\sup_{t_1+\lambda<t<t_0} |\vec{u}_t(\cdot,t)|_{C^\alpha(\Omega_{t_1})} + \sup_{t_1+\lambda<t<t_0} |\vec{u}(\cdot,t)|_{C^{2+\alpha}(\Omega_{t_1})}$$
$$+ \sup_{t_1+\lambda<t<t_0} |q(\cdot,t)|_{C^{1+\alpha}(\Omega_{t_1})}\Big),$$
$$K = c(\epsilon)\Big(\sup_{t_1<t<t_0} \|\vec{w}(\cdot,t)\|_{L_2(\Omega_t)} + \sup_{t_1<t<t_0} \|\rho(\cdot,t)\|_{W_2^1(\mathcal{G}_{\theta(t)})}\Big).$$

Setting $\epsilon_1 = 1/2c$ we easily obtain

$$f(\lambda) \leqslant 2K;$$

taking here $\lambda = \tau_0$ we arrive at the estimate (4.37) for $\vec{w}$ and s. It follows from (4.19), (4.21) that $\widehat{\rho}$ also satisfies (4.37). This completes the proof of the theorem.

Proof of Theorem 2.1. By Theorem 4.2, the solution of problem (4.1) exists, the function $\theta(t)$ is defined and inequalities (4.9), (4.11)–(4.12) hold for $t \in [0, t_0]$, where t_0 is determined by

$$L_0 = |\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G}_0)}.$$

It follows from (4.9)–(4.12) that

$$L \leqslant c_0 L_0,$$

where

$$L = \sup_{t<t_0} |\vec{w}|_{C^{2+\alpha}(\Omega_t)} + \sup_{t<t_0} |\widetilde{\rho}|_{C^{3+\alpha}(\mathcal{G}_{\theta(t)})}.$$

In addition, we have estimates (4.23), (4.37), i.e.

$$E(t) \leqslant e^{-bt} E(0) \leqslant c_1(L) e^{-bt} \epsilon, \tag{4.39}$$

$$\begin{aligned} &|\vec{w}_t(\cdot, t)|_{C^{\alpha}(\Omega_t)} + |\vec{w}(\cdot, t)|_{C^{2+\alpha}(\Omega_t)} + |s(\cdot, t)|_{C^{1+\alpha}(\Omega_t)} + |\widehat{\rho}(\cdot, t)|_{C^{3+\alpha}(\mathcal{G}_{\theta(t)})} \\ &\quad + |\widehat{\rho}_t(\cdot, t)|_{C^{2+\alpha}(\mathcal{G}_{\theta(t)})} + |\widehat{\rho}_{tt}(\cdot, t)|_{C^{\alpha}(\mathcal{G}_{\theta(t)})} \leqslant c_2 e^{-bt/2} E^{1/2}(0) \leqslant c_3(L) e^{-bt/2} \epsilon. \end{aligned} \tag{4.40}$$

They are satisfied for $t \in [2\tau_0, t_0]$ (we choose $\tau_0 < t_0/2$). In particular, the last inequality holds for $t = t_0$, and we assume ϵ to be so small that

$$c_3(c_0 L_0) e^{-bt_0/2} \epsilon \leqslant L_0,$$

and that the smallness conditions (1.5), (3.23) for $\widehat{\rho}$ are satisfied when ϵ is replaced with $\epsilon' = c_3(c_0 L_0)\epsilon$. Then we can apply the local existence theorem once more and extend the solution of our problem to the interval $[t_0, 2t_0]$. By the same procedure as above we find the function $\theta(t)$ in this interval (but the role of $\mathcal{G}_0$ is played this time by the surface $\mathcal{G}_{t_0}$). The fact that the constants in (4.23) and (4.37) are independent of T allows us to repeat this procedure again and again and extend the solution to the intervals $[kt_0, (k+1)t_0]$, $k = 1, 2, \ldots$. In all these intervals, inequalities (4.39), (4.40) hold with the same constants. It is clear that estimates (2.16), (2.17) are satisfied. The theorem is proved.

REMARK In fact, Theorem 2.1 was proved under the apparently weaker (than (1.16)) hypothesis of the positivity of the second variation of the functional

$$R_1 = \frac{\beta^2}{2}\left(S^{33} - \frac{1}{S_{33}}\right) + R,$$

where $S^{33} - 1/S_{33}$ is expressed as in (4.26) in terms of $S_{jk} = \int_{\Omega} \vec{\eta}_j(x) \cdot \vec{\eta}_k(x)\, \mathrm{d}x$. This functional appears in the crucial relations (4.25) and (4.34) leading to (4.23), since

$$\mathcal{Q}[\widehat{\rho}] + \delta_0^2 R[\widehat{\rho}] = \delta_0^2 R_1[\widehat{\rho}].$$

As shown by A. M. Lyapunov [6], in the case $\sigma = 0$ the hypotheses of positivity of $\delta_0^2 R$ and $\delta_0^2 R_1$ are equivalent to each other. Let us prove that the same is true for $\sigma > 0$.

THEOREM 4.5 If $\delta_0^2 R_1$ has property (1.16) for arbitrary $\rho(y)$ satisfying (1.17), (1.18), then

$$\delta_0^2 R_1[\rho] \leqslant c\delta_0^2 R[\rho], \tag{4.41}$$

so $\delta_0^2 R$ has the same property (with other constants c_1, c_2).

Proof. Without restriction of generality we can assume that

$$\int_{\mathcal{F}} x_1 x_2 \, \mathrm{d}x = 0$$

(this condition can be satisfied by appropriate choice of the axes x_1, x_2). Then, according to (4.35),

$$\mathcal{Q}[\rho] = \frac{\omega_0^2}{S_{11}^{(0)}} \Sigma_{13}^2[\rho] + \frac{\omega_0^2}{S_{22}^{(0)}} \Sigma_{23}^2[\rho].$$

Let us calculate $\delta_0^2 R[\rho_0]$, where $\rho_0(x) = \vec{N}(x) \cdot \vec{\eta}(x)$ and $\vec{\eta}(x) = \vec{b} \times \vec{x}$ is an arbitrary vector of rigid rotation. To this end, we write $\delta_0^2 R[\rho]$ in the form

$$\delta_0^2 R[\rho] = \int_{\mathcal{G}} \rho B[\rho] \, \mathrm{d}x,$$

where

$$B[\rho] = B_0[\rho] + \frac{\omega_0^2}{S_{33}^{(0)}} (x_1^2 + x_2^2) \int_{\mathcal{G}} (y_1^2 + y_2^2) \rho(y) \, \mathrm{d}S_y,$$

$$B_0[\rho] = -\Delta_{\mathcal{G}} \rho(x) - b(x)\rho(x) - \kappa \int_{\mathcal{G}} \frac{\rho(y) \, \mathrm{d}S_y}{|x-y|},$$

and compute $B_0[\rho_0]$. We take an arbitrary small smooth function $r(x)$, $x \in \mathcal{G}$, and consider the integral

$$I[r] = \int_{\Gamma} \left(\sigma H(x) + \frac{\omega_0^2}{2} (x_1^2 + x_2^2) + \kappa U(x) + p_0 \right) \rho_0(x) \, \mathrm{d}S_x,$$

where $U(x) = \int_{\Omega} |x-y|^{-1} \, \mathrm{d}y$ and

$$\Gamma = \partial\Omega = \{x = y + N(y)r(y) \equiv e_r(y), \; y \in \mathcal{G}\}.$$

It can be easily shown that only the term containing ω_0^2 is different from zero and that

$$I[r] = \omega_0^2 \int_{\Omega} \vec{\eta}(x) \cdot \vec{x}' \, \mathrm{d}x, \qquad x' = (x_1, x_2, 0).$$

Now, we write $I[r]$ as an integral over $\mathcal{G}$:

$$I[r] = \int_{\mathcal{G}} \left(\sigma H(x) + \frac{\omega_0^2}{2} (x_1^2 + x_2^2) + \kappa U(x) + p_0 \right) \rho_0(x) \Big|_{x=e_r(y)} m(y; r(y)) \, \mathrm{d}S_y,$$

where m is the function introduced above (see the proof of Theorem 4.3), and we calculate the first variation of $I[r]$. Taking account of (1.1), we obtain

$$\begin{aligned} \delta_0 I[r] &= \int_{\mathcal{G}} \delta_0 \left(\sigma H(x) + \frac{\omega_0^2}{2} (x_1^2 + x_2^2) + \kappa U(x) + p_0 \right) \Big|_{x=e_r(y)} \rho_0(y) \, \mathrm{d}S_y, \\ &= \omega_0^2 \delta_0 \int_{\Omega} \vec{\eta}(x) \cdot \vec{x}' \, \mathrm{d}x = \omega_0^2 \int_{\mathcal{G}} \vec{\eta}(y) \cdot \vec{y}' r(y) \, \mathrm{d}S_y. \end{aligned} \tag{4.42}$$

Since $\delta_0 H(e_r(y)) = \Delta_{\mathcal{G}} r(y) + (\mathcal{H}^2(y) - 2\mathcal{K}(y))$ (see [4]) and

$$\delta_0 U(e_r(y)) = r(y)\frac{\partial \mathcal{U}(y)}{\partial N} + \int_{\mathcal{G}} \frac{r(z)\,\mathrm{d}S_z}{|y-z|}$$

(see [16]), (4.42) implies

$$\int_{\mathcal{G}} \rho_0(y) B_0[r]\,\mathrm{d}S_y = -\omega_0^2 \int_{\mathcal{G}} \vec{\eta}(y)\cdot \vec{y}' r(y)\,\mathrm{d}S_y,$$

and, as a consequence,

$$B_0[\rho_0](y) = -\omega_0^2 \vec{\eta}(y)\cdot\vec{y}' = -\omega_0^2(b_2 y_1 - b_1 y_2) y_3.$$

From

$$\int_{\mathcal{G}} (y_1^2 + y_2^2)\rho_0(y)\,\mathrm{d}S_y = 2\omega_0^2 \int_{\mathcal{F}} \vec{\eta}(x)\cdot\vec{x}'\,\mathrm{d}x = \omega_0^2(\vec{b}\times\vec{e}_3)\cdot\int_{\mathcal{F}} x_3\vec{x}'\,\mathrm{d}x = 0$$

we conclude that also $B[\rho_0(y)] = -\omega_0^2\vec{\eta}(y)\cdot\vec{y}'$. Multiplying this equation by $\rho_0(y)$ and integrating we obtain the desired expression for $\delta_0^2 R[\rho_0]$:

$$\delta_0^2 R[\rho_0] = -\omega_0^2 \int_{\mathcal{F}} \vec{\eta}(x)\cdot\nabla[(b_2x_1 - b_1x_2)x_3]\,\mathrm{d}x$$
$$= \omega_0^2\left(b_2^2\int_{\mathcal{F}}(x_1^2 - x_3^2)\,\mathrm{d}x + b_1^2\int_{\mathcal{F}}(x_2^2 - x_3^2)\,\mathrm{d}x\right) = \omega_0^2[b_2^2(S_{33}^{(0)} - S_{11}^{(0)}) + b_1^2(S_{33}^{(0)} - S_{22}^{(0)})]. \quad (4.43)$$

Finally, since

$$\mathcal{Q}[\rho_0] = \frac{\omega_0^2}{S_{11}^{(0)}} b_2^2 (S_{33}^{(0)} - S_{11}^{(0)})^2 + \frac{\omega_0^2}{S_{22}^{(0)}} b_1^2 (S_{33}^{(0)} - S_{22}^{(0)})^2,$$

we have

$$\delta_0^2 R_1[\rho_0] = \delta_0^2 R[\rho_0] + \mathcal{Q}[\rho_0] = \omega_0^2\left[b_2^2\frac{S_{33}^{(0)}}{S_{11}^{(0)}}(S_{33}^{(0)} - S_{11}^{(0)}) + b_1^2\frac{S_{33}^{(0)}}{S_{22}^{(0)}}(S_{33}^{(0)} - S_{22}^{(0)})\right]$$

for arbitrary $\vec{b} = (b_1, b_2, b_3)$. It is easily verified that ρ_0 satisfies (1.17) and (1.18) (the latter with an appropriate choice of b_3). In this case, by our hypothesis concerning $\delta_0^2 R_1$, $\delta_0^2 R_1[\rho_0]$ should be positive, which means that

$$S_{33}^{(0)} > S_{11}^{(0)}, \qquad S_{33}^{(0)} > S_{22}^{(0)},$$

and

$$\delta_0^2 R[\rho_0] > c\delta_0^2 R_1[\rho_0]. \quad (4.44)$$

Now, let us show that every $\rho(y)$ satisfying (1.17), (1.18) can be represented in the form

$$\rho(y) = \rho_0(y) + \rho_1(y) = \vec{N}(y)\cdot(\vec{b}\times\vec{y}) + \rho_1(y)$$

with ρ_1 satisfying the additional orthogonality conditions

$$\int_{\mathcal{G}} y_1 y_3 \rho_1(y)\,\mathrm{d}S_y = \int_{\mathcal{G}} y_2 y_3 \rho_1(y)\,\mathrm{d}S_y = 0. \quad (4.45)$$

A simple computation shows that (4.45) holds if

$$b_1 = \frac{1}{S_{33}^{(0)} - S_{22}^{(0)}} \int_{\mathcal{G}} y_2 y_3 \rho(y)\, \mathrm{d}S_y, \qquad b_2 = -\frac{1}{S_{33}^{(0)} - S_{11}^{(0)}} \int_{\mathcal{G}} y_1 y_3 \rho(y)\, \mathrm{d}S_y,$$

and if

$$b_3 = -\frac{1}{\int_{\mathcal{G}} h^2(y)\, \mathrm{d}S_y} \sum_{i=1}^{2} b_i \int_{\mathcal{G}} h(y) \vec{N}(y) \cdot \vec{\eta}_i(y)\, \mathrm{d}S_y,$$

then both ρ and ρ_1 satisfy (1.17), (1.18).

By (4.45),

$$\delta_0^2 R[\rho_1] = \delta_0^2 R_1[\rho_1],$$

so taking account of (4.44) we obtain

$$\delta_0^2 R[\rho_0] + \delta_0^2 R[\rho_1] \geqslant c\delta_0^2 R_1[\rho_0] + \delta_0^2 R_1[\rho_1] \geqslant c(\delta_0^2 R_0[\rho_0] + \delta_0^2 R_1[\rho_1]). \tag{4.46}$$

Finally, it is easy to see that

$$\delta_0^2 R_1[\rho] = \delta_0^2 R_1[\rho_0] + \delta_0^2 R_1[\rho_1] + \mathcal{R}_1[\rho_0, \rho_1],$$
$$\delta_0^2 R[\rho] = \delta_0^2 R[\rho_0] + \delta_0^2 R[\rho_1] + \mathcal{R}[\rho_0, \rho_1],$$

where

$$\mathcal{R}[\rho_0, \rho_1] = \frac{\mathrm{d}}{\mathrm{d}s} \delta_0^2 R[\rho_0 + s\rho_1]\Big|_{s=0}$$
$$= 2\int_{\mathcal{G}} \rho_1(y) B_0[\rho_0]\, \mathrm{d}S_y + 2\frac{\omega_0^2}{S_{33}^{(0)}} \int_{\mathcal{G}} (y_1^2 + y_2^2) \rho_0(y)\, \mathrm{d}S_y \int_{\mathcal{G}} (y_1^2 + y_2^2) \rho_1(y)\, \mathrm{d}S_y = 0,$$

$$\mathcal{R}_1[\rho_0, \rho_1] = \mathcal{R}[\rho_0, \rho_1] + 2\frac{\omega_0^2}{S_{11}^{(0)}} \Sigma_{13}[\rho_0] \Sigma_{13}[\rho_1] + 2\frac{\omega_0^2}{S_{22}^{(0)}} \Sigma_{23}[\rho_0] \Sigma_{23}[\rho_1] = 0.$$

Hence, (4.46) coincides with (4.41) and the theorem is proved.

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