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Evolution of characteristic functions of convex sets in the plane by the minimizing total variation flow

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In this paper we compute the explicit evolution of the minimizing total variation flow when the initial condition is the characteristic function of a convex set in $\mathbb{R}^2$, or a finite number of them which are sufficiently separated. We also obtain some explicit solutions of the total variation formulation of the denoising problem in image processing. We illustrate these results with some experiments.

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1. Introduction

The purpose of this paper is to compute the explicit solution of the minimizing total variation flow in $\mathbb{R}^2$ given by the equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{|Du|}\right) \quad \text{in } Q_T =]0, T[\, \times \mathbb{R}^2, \tag{1.1}$$

when the initial datum

$$u(0, x) = \chi_C(x), \qquad x \in \mathbb{R}^2, \tag{1.2}$$

is the characteristic function of a bounded convex set $C \subseteq \mathbb{R}^2$. More generally, we shall compute the evolution corresponding to initial data $u_0(x) = \sum_{i=1}^m b_i \chi_{C_i}$, where $b_i \in \mathbb{R}$ and C_i are bounded convex sets in $\mathbb{R}^2$ which satisfy some additional condition (condition (c) in Theorem 5) which amounts to saying that the sets C_i are sufficiently far apart.

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The study of the explicit solutions of (1.1) was initiated in [5, 2], where the authors considered the evolution of characteristic functions of bounded sets C of finite perimeter in $\mathbb{R}^2$ which evolve at constant speed without distortion of the boundary. In the case that $C \subseteq \mathbb{R}^2$ is connected, those sets were characterized in [5] by the following result. For notational convenience, we set

$$\lambda_C := \frac{P(C)}{|C|},$$

where $P(C)$ denotes the perimeter of C and $|C|$ its area.

THEOREM 1 ([5]) Let $C \subset \mathbb{R}^2$ be a bounded set of finite perimeter, and assume that C is connected. Let $\lambda > 0$. The following conditions are equivalent:

(i) C decreases at speed λ, i.e., $u(t,x) := (1-\lambda t)^+ \chi_C(x)$ is the solution of (1.1) corresponding to $u(0,x) = \chi_C(x)$.

(ii) C is convex, $\lambda = \lambda_C$ and minimizes the functional

$$\mathcal{G}_{\lambda_C}(D) := P(D) - \lambda_C |D|, \qquad D \subseteq C,\ D \text{ of finite perimeter.}$$

(iii) C is convex, ∂C is of class $C^{1,1}$, $\lambda = \lambda_C$, and

$$\operatorname*{ess\,sup}_{p \in \partial C} \kappa_{\partial C}(p) \leqslant \lambda_C, \tag{1.3}$$

where $\kappa_{\partial C}(p)$ denotes the curvature of ∂C at the point p.

The main conclusion of this theorem is that the bounded connected subsets $C \subseteq \mathbb{R}^2$ which decrease at constant speed are convex sets whose curvature is bounded by (1.3). In particular, the evolution of polygons in $\mathbb{R}^2$ is not described by Theorem 1. Our purpose in this paper will be to describe the evolution of general convex sets in $\mathbb{R}^2$. In this case, and depending on the curvature of the boundary, some distortions may occur. To describe them let us recall some basic concepts useful in integral geometry.

As usual, if $x \in \mathbb{R}^2$ and $r > 0$, we let $B(x,r) := \{y \in \mathbb{R}^2 : |y - x| \leqslant r\}$. The Minkowski addition and subtraction of a set $A \subseteq \mathbb{R}^2$ and the ball $B(0,r)$ will be denoted by $A \oplus B(0,r) := \bigcup_{x \in A} B(x,r)$ and $A \ominus B(0,r) := \{x \in C : B(x,r) \subseteq A\}$.

Let C be a compact convex set in $\mathbb{R}^2$, and $r > 0$. We shall use the notations

$$C_r := C \ominus B(0,r),$$
$$C^r := (C \ominus B(0,r)) \oplus B(0,r) = \bigcup_{B(x,r) \subseteq C} B(x,r).$$

The family of the sets C^r, $r > 0$, is ordered by inclusion, i.e., $C^r \subseteq C^s \subseteq C$ if $0 < s < r$, $C = \overline{\bigcup_{r>0} C^r}$, and we shall find a value $R > 0$ of r characterized by

$$\frac{1}{R} = \frac{P(C^R)}{|C^R|}$$

for which the set C^R decreases at speed $1/R$. As we shall prove in Lemma 4, the set $C \setminus C^R$ is foliated by the boundaries $\partial C^r \setminus \partial C$, $0 < r < R$, which are a family of circular arcs of radius r tangent to ∂C. If $p \in C \setminus C^R$, we define $r(p)$ as the radius of the arc of $\{\partial C^r \setminus \partial C : r \in [0,R]\}$ passing through p, and if $p \in C^R$, we define $r(p) = R$; that is, we set

$$r(p) = \sup\{r \in [0,R] : p \in C^r\} = \inf\{r \in [0,R] : p \notin C^r\}, \qquad p \in C. \tag{1.4}$$

The evolution of χ_C can now be easily described: the points in C^R decrease at speed λ_{C^R}, the points $p \in \partial C^r \setminus \partial C$, $0 < r < R$, decrease at speed $1/r$, until the height reaches the value 0. This is the main result of the paper.

THEOREM 2 Let C be a nonempty bounded convex set in $\mathbb{R}^2$. Let $r(x)$, $x \in C$, be the function defined in (1.4). Extend $r(x)$ to $\mathbb{R}^2$ by defining $r(x) = 0$ if $x \in \mathbb{R}^2 \setminus C$, and write $(1/r(x))\chi_C(x) = 0$ if $x \in \mathbb{R}^2 \setminus C$. Let $u(t, x) = (1 - t/r(x))^+ \chi_C(x)$. Then u is the solution of (1.1) corresponding to the initial condition $u(0, x) = \chi_C(x)$.

We shall also consider the evolution of sets of the form $\Omega = C_1 \cup \cdots \cup C_m$, where the C_i are nonempty bounded convex sets which are sufficiently far from each other (see Theorem 5).

One of the main motivations for our analysis comes from image processing. Equation (1.1) corresponds to the gradient descent flow associated to total variation minimization, which was introduced by L. Rudin, S. Osher and E. Fatemi [19] in the context of image denoising and restoration. Denote by Ω the image domain which, for simplicity, we assume to be a rectangle in $\mathbb{R}^2$. When dealing with the restoration problem one minimizes the total variation functional

$$\int_\Omega |Du| \tag{1.5}$$

under some constraints which model the process of image acquisition, including blur and noise. The constraint can be written as $f = K * u + n$, where $f \in L^2(\Omega)$ is the observed image, K is a convolution operator whose kernel represents the point spread function of the optical system, n is the noise, and u is the ideal image, previous to distortion. The denoising problem corresponds to $K = I$ and, in this case, the constraint becomes

$$f = u + n. \tag{1.6}$$

In practice, the only information we have about the noise is statistical. Assuming that n is a Gaussian white noise of zero mean and standard deviation σ, the constraint (1.6) can be imposed in an integral form as

$$\int_\Omega (f - u)^2 \, dx = \sigma^2 |\Omega|. \tag{1.7}$$

Among all images satisfying this constraint, the denoised image is chosen as the one minimizing (1.5) (see [19]). As proved by A. Chambolle and P. L. Lions in [9], minimizing (1.5) under the constraint (1.7) amounts to minimizing

$$\min_{u \in \mathrm{BV}(\Omega)} \left\{ \int_\Omega |Du| + \frac{1}{2\lambda} \int_\Omega (u - f)^2 \, \mathrm{d}x \right\}, \tag{1.8}$$

for some Lagrange multiplier $\lambda > 0$. Notice that, as a by-product of our analysis, we shall obtain some explicit solutions of (1.8) (which, for simplicity, we state for $\Omega = \mathbb{R}^2$). These explicit solutions, together with other ones found in [5], contribute to display the qualitative behavior of total variation when denoising the data f according to (1.8). They also serve as tests for the numerical algorithms used to minimize (1.8).

It is important to recall that one of the main features of total variation denoising (1.8), confirmed by numerical experiments, is its ability to restore the discontinuities of the image [19], [9], [11], [12]. Indeed, the underlying functional model is the space of BV functions, i.e., functions of bounded variation, which admit a set of discontinuities which is countably rectifiable [1], [13], [21]. The total variation approach to denoising had a strong influence on the use of BV functions in image processing.

Finally, observe that the Euler–Lagrange equation corresponding to (1.8) coincides with the first step of an implicit Euler discretization of (1.1) with $\lambda = \Delta t$, also called the Crandall–Liggett scheme. Thus, studying each of these problems gives information about the other, and this is reflected by our results in Sections 3.3 and 4. For an account on existence, uniqueness, and qualitative behavior of (1.1) under different boundary conditions we refer to [2], [5].

Let us explain the plan of the paper. In Section 2 we recall some basic notions on functions of bounded variation, a generalized Green formula, and the notion of solution for problem (1.1), (1.2). Section 3 describes, after some technical preliminaries, the evolution of a general convex set in $\mathbb{R}^2$ and the evolution of sets which are unions of a finite number of convex sets which are sufficiently separated (condition (c) of Theorem 5). In Section 4 we construct further explicit solutions of the denoising problem (1.8) for some particular functions f. Finally, in Section 5 we present some numerical experiments which are in agreement with the results of previous sections and have been obtained using the numerical scheme of [10].

2. Preliminaries

2.1 BV *functions*

Let Q be an open subset of $\mathbb{R}^N$. A function $u \in L^1(Q)$ whose gradient Du in the sense of distributions is a (vector-valued) Radon measure with finite total variation in Q is called a *function of bounded variation*. The class of such functions will be denoted by $\mathrm{BV}(Q)$. The total variation of Du on Q turns out to be

$$\sup\left\{\int_Q u \operatorname{div} z \, \mathrm{d}x : z \in C_0^\infty(Q;\mathbb{R}^n),\ \|z\|_{L^\infty(Q)} := \operatorname*{ess\,sup}_{x\in Q} |z(x)| \leqslant 1\right\} \tag{2.1}$$

(where for a vector $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$), and will be denoted by $|Du|(Q)$ or $\int_Q |Du|$. It turns out that the map $u \mapsto |Du|(Q)$ is $L^1_{\mathrm{loc}}(Q)$-lower semicontinuous. $\mathrm{BV}(Q)$ is a Banach space when endowed with the norm $\int_Q |u| \, \mathrm{d}x + |Du|(Q)$. We recall that $\mathrm{BV}(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N)$, and in particular $\mathrm{BV}(\mathbb{R}^2) \subseteq L^2(\mathbb{R}^2)$. The total variation of u on a Borel set $B \subseteq Q$ is defined as $\inf\{|Du|(A) : A \text{ open},\ B \subseteq A \subseteq Q\}$. For more information about functions of bounded variation we refer to [1], [13], [21].

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of *finite perimeter in Q* if (2.1) is finite when u is the characteristic function χ_E of E. The perimeter of E in Q is defined as $P(E, Q) := |D\chi_E|(Q)$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is countably $(N-1)$-rectifiable with finite $\mathcal{H}^{N-1}$ measure, and compute the outer unit normal $\nu^E(x)$ at $\mathcal{H}^{N-1}$-almost all points x of $\partial^* E$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Moreover, $|D\chi_E|$ coincides with the restriction of $\mathcal{H}^{N-1}$ to $\partial^* E$.

2.2 *A generalized Green formula*

Let $\Omega \subset \mathbb{R}^N$ be an open set. Following [3], let

$$X_2(\Omega) := \{\xi \in L^\infty(\Omega;\mathbb{R}^N) : \operatorname{div}\xi \in L^2(\Omega)\}. \tag{2.2}$$

If $\xi \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap \mathrm{BV}(\Omega)$ we define the functional $(\xi, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (\xi, Dw), \varphi \rangle := -\int_\Omega w\varphi \operatorname{div} \xi \, \mathrm{d}x - \int_\Omega w\xi \cdot \nabla\varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.3}$$

Then (ξ, Dw) is a Radon measure in Ω (see [3]), and

$$\int_\Omega (\xi, Dw) = \int_\Omega \xi \cdot \nabla w \, \mathrm{d}x \quad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega). \tag{2.4}$$

If Ω is a bounded open set with Lipschitz boundary, and ν^Ω denotes the outer unit normal on $\partial\Omega$, we have the following integration by parts formula [3]: given $\xi \in X_2(\Omega)$ there exists a function $[\xi \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[\xi \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leqslant \|\xi\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and such that for any $w \in L^2(\Omega) \cap \mathrm{BV}(\Omega)$ we have

$$\int_\Omega w \operatorname{div} \xi \, \mathrm{d}x = -\int_\Omega (\xi, Dw) + \int_{\partial\Omega} [\xi, \nu^\Omega] w \, \mathrm{d}H^{N-1}. \tag{2.5}$$

If $\Omega = \mathbb{R}^N$, $\xi \in X_2(\mathbb{R}^N)$ and $w \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N)$ we have the following integration by parts formula:

$$\int_{\mathbb{R}^N} w \operatorname{div} \xi \, dx + \int_{\mathbb{R}^N} (\xi, Dw) = 0. \tag{2.6}$$

For convenience, we shall apply the usual notation $\xi \cdot Dw$ instead of (ξ, Dw).

2.3 *Existence and uniqueness of solutions of (1.1)*

Consider the energy functional $\Phi : L^2(\mathbb{R}^N) \to]-\infty, +\infty]$ defined by

$$\Phi(u) = \begin{cases} \int_{\mathbb{R}^N} |Du| & \text{if } u \in \mathrm{BV}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus \mathrm{BV}(\mathbb{R}^N). \end{cases} \tag{2.7}$$

Since the functional Φ is convex, lower semicontinuous and proper, $\partial\Phi$ is a maximal monotone operator with dense domain, generating a contraction semigroup in $L^2(\mathbb{R}^N)$ (see [8]). Let us recall the following characterization of the subdifferential of Φ ([2, 5]).

LEMMA 1 Let $u, v \in L^2(\mathbb{R}^N)$. The following assertions are equivalent:

(a) $(u, v) \in \partial\Phi$;
(b) $u \in L^2(\mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}^N)$, $v \in L^2(\mathbb{R}^N)$, and there exists $z \in X_2(\mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$ such that $v = -\operatorname{div} z$ in $\mathcal{D}'(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} |Du|. \tag{2.8}$$

Thanks to this characterization the semigroup solution of (1.1) can be expressed in more classical terms [2, 5].

THEOREM 3 Let $u_0 \in L^2(\mathbb{R}^N)$. There is a strong solution $u \in C([0,T]; L^2(\mathbb{R}^N))$ of (1.1), (1.2), i.e., $u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{R}^N)) \cap L^1_w(]0,T[; \mathrm{BV}(\mathbb{R}^N))$, $u(0,x) = u_0(x)$, and there exists $z \in L^\infty(]0,T[\times \mathbb{R}^N; \mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$ such that

$$u_t = \operatorname{div} z \quad \text{in } \mathcal{D}'(]0,T[\times \mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} (z(t), Du(t)) = \int_{\mathbb{R}^N} |Du(t)| \quad \text{for a.e. } t > 0. \tag{2.9}$$

The strong solution is unique and it coincides with the semigroup solution.

3. Evolution of the characteristic function of a convex set

The main purpose of this section is to prove Theorem 2. Some preparatory results will be proved in Subsections 3.1 and 3.2.

3.1 *A convex set inside C which decreases without distortion*

The main purpose of this subsection is to prove the following result.

PROPOSITION 1 Let $C \subseteq \mathbb{R}^2$ be a nonempty compact convex set. Then there exists $R > 0$ such that C^R decreases at speed $1/R$. The value of R is characterized by the equation $1/R = \lambda_{C^R}$.

By $\mathrm{int}(X)$ we denote the interior of the set $X \subseteq \mathbb{R}^2$. In the next lemma we recall several facts on convex sets [17, 20]. The proof of (v) can be found, for instance, in [5].

LEMMA 2 Let $C \subseteq \mathbb{R}^2$ be a nonempty compact convex set, and $r > 0$. Then:

(i) C_r and C^r are compact convex sets.
(ii) $(C^r)^r = C^r$ and $(C^r)_r = C_r$.
(iii) $C_r = \bigcap_{s<r} C_s$, and if $\bigcup_{s>r} C_s \neq \emptyset$, then $C_r = \overline{\bigcup_{s>r} C_s}$.
(iv) $C^r = \bigcap_{s<r} C^s$, and if $\bigcup_{s>r} C^s \neq \emptyset$, then $C^r = \overline{\bigcup_{s>r} C^s}$.
(v) If $C = C^r$, then C is of class $C^{1,1}$. In particular, any set C^r is of class $C^{1,1}$.

PROPOSITION 2 Let $C \subset \mathbb{R}^2$ be a bounded set of finite perimeter, and assume that C is connected. Let $\lambda > 0$. The equivalent conditions of Theorem 1 can be complemented with the following ones:

(iv) C is convex, $\lambda = \lambda_C$, and $C = C^{1/\lambda}$.
(v) C is convex, $C = C^{1/\lambda}$, and $|C_{1/\lambda}| = \pi/\lambda^2$.

The equivalence of (iii) and (iv) was proved in [5]. The equivalence (iv)⇔(v) follows from the following lemma.

LEMMA 3 Let $C \subseteq \mathbb{R}^2$ be a nonempty bounded convex set and $r > 0$.

(a) $P(C \oplus B(0,r))/|C \oplus B(0,r)| = 1/r$ if and only if $|C| = \pi r^2$.
(b) Assume that $C = C^r$. Then $P(C)/|C| = 1/r$ if and only if $|C_r| = \pi r^2$.

Proof. (a) By Steiner's formulas in the plane [20], we have (if X is reduced to a segment, then $P(X)$ denotes its length)

$$P(C \oplus B(0,r)) = P(C) + 2\pi r, \quad |C \oplus B(0,r)| = |C| + \pi r^2 + P(C) r,$$

hence

$$|C| - \pi r^2 = |C \oplus B(0,r)| - P(C \oplus B(0,r)) r.$$

We conclude that $C \oplus B(0,r)$ decreases at speed $1/r$ if and only if $|C| = \pi r^2$.

(b) Since $C^r = C_r \oplus B(0,r)$, we see that C_r is a nonempty bounded convex set and the result follows from (a). □

PROPOSITION 3 Let $C \subseteq \mathbb{R}^2$ be a compact convex set. Then $r \in [0, \infty) \mapsto |C_r|$ is a continuous decreasing function. Moreover, there is some $r_0 > 0$ such that $|C_r| = 0$ for all $r > r_0$ and $|C_r| > 0$ for $r < r_0$. As a consequence, there is a unique value of $R \geqslant 0$ such that $|C_R| = \pi R^2$.

Proof. Obviously, the function $Q(r) = |C_r|$ is decreasing, and $|C_r| = 0$ for r large enough. Let $r_0 = \inf\{r > 0 : |C_r| = 0\}$. Then $|C_r| > 0$ for all $0 \leqslant r < r_0$, and $|C_r| = 0$ for all $r > r_0$. Thus it suffices to check that $Q(r)$ is continuous for $r \in [0, r_0]$. Note that $C_{r_0} = \bigcap_{r<r_0} C_r \neq \emptyset$, being the intersection of a decreasing sequence of nonempty compact sets. If $|C_{r_0}| > 0$, then $\mathrm{int}(C_{r_0}) \neq \emptyset$. In that case, there are $\delta > 0$ and $p \in C_{r_0}$ such that $B(p, \delta) \subseteq C_{r_0}$. This implies that $B(p, \delta/2) \subseteq C_s$ for any $s \in (r_0, r_0 + \delta/2)$. This contradicts our definition of r_0. Hence $|C_{r_0}| = 0$ and $Q(r)$ is continuous at $r = r_0$. The continuity of Q when $r < r_0$ is a consequence of Lemma 2(iii) and the fact that the boundary of a convex set has null measure.

We have proved that $Q(r)$ is a continuous function. Finally, note that, since the curves $r \mapsto \pi r^2$ and $r \mapsto |C_r|$ intersect, they do it in a single point $R \geqslant 0$. □

Proof of Proposition 1. Observe that $(C^R)^R = C^R$. Thus C^R decreases at speed $1/R$ if and only if $|(C^R)_R| = \pi R^2$. But $(C^R)_R = C_R$. Since $|C_R| = \pi R^2$, we conclude that C^R indeed decreases at speed $1/R$. □

3.2 *Some preparatory results*

From now on, we assume that $C \subseteq \mathbb{R}^2$ is a fixed nonempty compact convex set and $R > 0$ is the radius given by Proposition 3. Let $r(p)$ be the radius function defined in (1.4), and $p \in C$. Recall that $r(p) = R$ if $p \in C^R$. By Proposition 1 we know that C^R decreases at speed $1/R$. Our first purpose is to prove that the set $\mathrm{int}(C) \setminus C^R$ is foliated by circles whose radius r goes from 0 to R; this will permit us to define the field of unit normals $\nu(p)$ to this family of circles for $p \in \mathrm{int}(C) \setminus C^R$. Then we shall prove that $\nu \in W^{1,1}(\mathrm{int}(C) \setminus C^R)$, and $\mathrm{div}\, \nu(p) = 1/r(p)$ for $p \in \mathrm{int}(C) \setminus C^R$. This will imply that the solution of (1.1) with $u(0, p) = \chi_C(p)$ decreases at speed $1/r$ on $\partial C^r \setminus \partial C$, for $0 < r < R$, until it reaches the value 0 (see Subsection 3.3).

Using Lemma 2, it is not difficult to prove that $p \in \partial C^{r(p)}$ for any $p \in C \setminus C^R$.

LEMMA 4 Let $p \in \partial C^r \setminus \partial C$. Then p is contained in a circular arc of radius r that is part of ∂C^r, and tangent to ∂C. Hence ∂C^r is contained in the union of ∂C and a family of circular arcs of radius r which are tangent to ∂C.

Proof. Since C^r is closed, we have $p \in C^r$. Let $q \in C$ be such that $p \in B = B(q, r) \subseteq C$. Observe that $p \in \partial B$. If $\partial B \cap \partial C$ consists at most of one point, then one can find a direction (either $\overrightarrow{qp}$ if $\partial B \cap \partial C = \emptyset$, or $\overrightarrow{p'p}$ if $\{p'\} = \partial B \cap \partial C$) such that moving slightly B in that direction we find a new ball $B(q', r) \subset C$ with p in its interior, a contradiction. Hence $\partial B \cap \partial C$ has at least two points. Denote by p_- and p_+ the two (different) end points of the connected component of $\partial B \setminus \partial C$ that contains p. Note that, since $B \subseteq C$ and C is convex, ∂B and ∂C are tangent at p_- and p_+. Let γ_1 be the arc of ∂B between p_- and p_+ that contains p, and let γ_2 be the complementary arc in ∂B (see Figure 1). We have to show that $\gamma_1 \subset \partial C^r$. Let α be the angular span of γ_1.

It must be that $\alpha \leqslant \pi$, otherwise, again, one can find a direction in which B can be moved inside C to a new position where it would contain p (away from γ_2). Indeed, one can show that there exists a direction $\vec{n}$ such that

$$\vec{n} \cdot \overrightarrow{qp} > 0 \quad \text{and} \quad \vec{n} \cdot \overrightarrow{qp'} < 0 \quad \text{for all } p' \in \gamma_2, \tag{3.1}$$

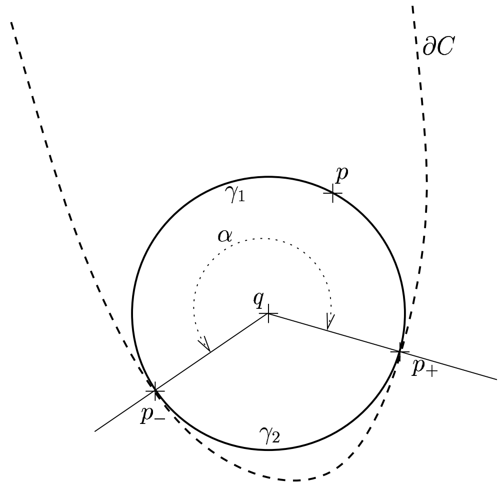

FIG. 1. The geometric situation described in Lemma 4 (the figure corresponds to the case $\alpha > \pi$).

so that if ε is small enough then the ball $B(q + \varepsilon\vec{n}, r)$ is in C and contains p in its interior. By rescaling we may assume that $q = (0, 0)$, $r = 1$, and $p_\pm = (\pm\cos\beta, \sin\beta)$ with $\beta = (\pi - \alpha)/2 \in (-\pi/2, 0)$. Assume that $p = (\cos\theta, \sin\theta)$ ($\beta < \theta < \pi - \beta$). By symmetry, we may assume that $|\theta| \leqslant \pi/2$. If $\theta > 0$, then the vector $\vec{n} = (0, 1)$ does the job. If $\beta < \theta \leqslant 0$, we can take $\vec{n} = \overrightarrow{p_+ p}$. Then it is easy to check that (3.1) holds.

Let L_- and L_+ be the tangent lines to ∂B at the points p_- and p_+, respectively. In case $\alpha < \pi$, those tangent lines intersect at a point P and determine an angular sector Q of vertex P containing C (see Figure 2). In case $\alpha = \pi$, L_- and L_+ are parallel lines, bounding a strip which we call again Q. In this case, we assume that the vertex of Q is a point P at infinity. Assume that there is a point $p' \in C_r \setminus B$ inside Q, between B and P. Note that the convex set C is contained in Q. Hence, if $\alpha < \pi$, there cannot exist a ball B' of radius r contained in Q and containing p'; if $\alpha = \pi$, the convexity would imply that there is a ball B' of radius r such that $p \in \mathrm{int}(B')$, a contradiction. We deduce that the arc γ_1 of ∂B that contains p in its interior is contained in ∂C^r. $\square$

For each $p \in \mathrm{int}(C) \setminus C^R$, we define $\nu(p)$ to be the outer unit normal to $C^{r(p)}$ at p. Our purpose is to prove the following result.

PROPOSITION 4 We have $r \in W^{1,1}(C)$ and $\nu \in W^{1,1}(\mathrm{int}(C) \setminus C^R)$. Moreover,

$$\langle \nabla r(p), \nu(p) \rangle = -|\nabla r(p)| \quad \text{for any } p \in C,$$

and

$$\mathrm{div}\, \nu(p) = \frac{1}{r(p)} \quad \text{for any } p \in \mathrm{int}(C) \setminus C^R.$$

The rest of this section is devoted to the proof of Proposition 4, which is the main technical ingredient of the proof of Theorem 2. Readers not interested in this technical part may go directly to Section 3.3.

The proof of Proposition 4 requires two auxiliary results stated in Proposition 5 and Lemma 7. To prove the former, we recall the following result which combines Theorems 3.1.8 and 3.1.9 (Stepanoff's Theorem) of [14].

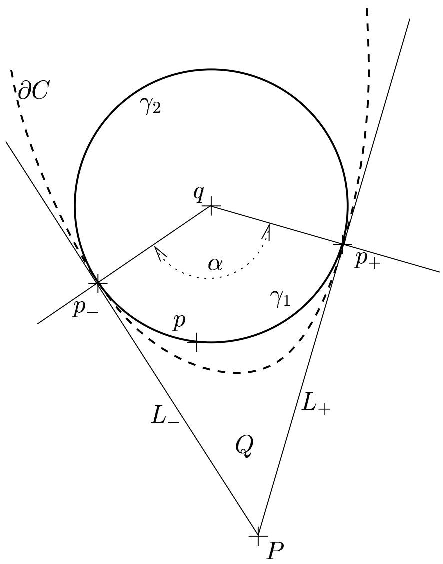

FIG. 2. Geometric situation when $\alpha < \pi$; observe that we cannot move the ball towards P without going out of Q and C.

THEOREM 4 If $A \subseteq B \subseteq \mathbb{R}^N$, A is measurable, B is open, $f : B \to \mathbb{R}$ and

$$\limsup_{x \to a} \frac{|f(x) - f(a)|}{|x - a|} < \infty \quad \text{for all } a \in A, \tag{3.2}$$

then f is differentiable almost everywhere on A. Moreover, A is the union of a countable family of (Lebesgue) measurable sets such that the restriction of f to each member of the family is Lipschitzian.

Strictly speaking we only need the above result for $N = 1$ (see Lemma 7). But it will be clear from the proof of Lemma 7 that we need to prove that both r and ν satisfy condition (3.2) in a domain of $\mathbb{R}^2$, and therefore we recall the result in $\mathbb{R}^N$.

PROPOSITION 5 The function $r(p)$ satisfies

$$\limsup_{q \to p} \frac{|r(q) - r(p)|}{|q - p|} < \infty \quad \text{for any } p \in \text{int}(C) \setminus \text{int}(C^R). \tag{3.3}$$

The function $\nu(p)$ has a similar property in $\text{int}(C) \setminus C^R$.

To prove Proposition 5 we need the following lemma.

LEMMA 5 Let Γ be the circle of center $(0, R_0)$ and radius $R_0 > 0$. Let Γ_h be the circle of center (x_h, y_h) and radius $R_h < R_0$ such that $x_h^2 + y_h^2 < R_h^2$. Assume that

$$(x_h, y_h) \to (0, R_0) \quad \text{and} \quad R_h \to R_0 \quad \text{as } h \to 0.$$

Let $v_h = (R_0 - R_h)/(R_0 - y_h)$ and $w_h = x_h/(R_0 - y_h)$. Let (X_h, Y_h) denote the coordinates of the intersection points of Γ and Γ_h. Then:

(i) If $w_h \to 0$ and $v_h \to v$ along a sequence, then $X_h \to \pm R_0\sqrt{1-v^2}$.
(ii) If $w_h \to \lambda \in \mathbb{R}$ and $v_h \to v$, then X_h tends to some finite value, say, $X_{\lambda,v}$.
(iii) If $|w_h| \to \infty$, then $X_h \to 0$.

Suppose that the coordinate X_h of the intersection points remains bounded away from zero. Then w_h is bounded and v_h is bounded away from 1.

Proof. Our assumptions imply that $0 \leqslant v_h \leqslant 1$ and Γ_h is below Γ near $(0,0)$ and, thus, it intersects Γ. The equations of Γ and Γ_h are, respectively, $x^2+(y-R_0)^2 = R_0^2$ and $(x-x_h)^2+(y-y_h)^2 = R_h^2$. The equations of the lower semicircles of Γ and Γ_h are $y = R_0 - \sqrt{R_0^2 - x^2}$ and $y = y_h - \sqrt{R_h^2 - x_h^2 - x^2 + 2xx_h}$, respectively. Equating both equations we compute the x coordinate of the intersection points (X, Y) of Γ and Γ_h. After some computations we find that X satisfies the second order equation

$$\begin{aligned}4(x_h^2+(R_0-y_h)^2)X^2 + 4(R_h^2-R_0^2-x_h^2-(R_0-y_h)^2)x_hX \\ = -[(R_h^2-R_0^2)-x_h^2]^2 + (R_0-y_h)^2[2R_h^2+2R_0^2-2x_h^2-(R_0-y_h)^2],\end{aligned}$$

which can be written in terms of v_h and w_h as

$$\begin{aligned}4(1+w_h^2)X^2 - 4(v_h(R_0+R_h)+x_hw_h+(R_0-y_h))w_hX \\ = -(v_h(R_h+R_0)+w_hx_h)^2 + 2R_h^2+2R_0^2-2x_h^2-(R_0-y_h)^2. \qquad (3.4)\end{aligned}$$

If $w_h \to 0$ and $v_h \to v$ along a sequence of h, then the solution X_h of equation (3.4) tends to the solution of

$$X^2 = R_0^2(1-v^2).$$

If $w_h \to \lambda$ and $v_h \to v$, then X_h tends to the solution of

$$(1+\lambda^2)X^2 - 2\lambda R_0 vX = R_0^2(1-v^2).$$

If $|w_h| \to \infty$ along a sequence h, then, dividing equation (3.4) by w_h^2 and letting $h \to 0$, we see that the solution of (3.4) converges to the solution of $X^2 = 0$, i.e., to $X = 0$.

Assume now that the coordinates X_h are bounded away from 0. Then by (iii) we know that w_h must be bounded. Let us prove that, under the same assumption, also $v_h \to 1$ implies that $w_h \to 0$. On the contrary, suppose that $v_h \to 1$ and there is a sequence $h_j \to 0$ such that w_{h_j} is bounded away from 0. Without loss of generality, we may assume that $w_{h_j} \to \eta \neq 0$. Letting $j \to \infty$ we observe that both coordinates X_{h_j} tend to the solutions of

$$(1+\eta^2)X^2 - 2\eta R_0 X = 0$$

whose solutions are $X = 0$ and $X = 2R_0\eta/(1+\eta^2)$, and therefore X_{h_j} cannot be bounded away from 0. This contradiction proves that $w_h \to 0$. Summarizing, if v_h or a subsequence converges to 1, then $w_h \to 0$ along the same subsequence and, using (i), we deduce that for the same subsequence $X_h \to \pm R_0\sqrt{1-v^2} = 0$, contradicting the fact that X_h is bounded away from 0. We conclude that v_h is bounded away from 1. □

REMARK Observe that $b_h := (R_h - y_h)/(R_0 - y_h)$ is bounded. Indeed,

$$1 = v_h + b_h$$

and the result follows from the bound $0 \leqslant v_h \leqslant 1$. We also note that if $0 \leqslant v_h \leqslant \eta < 1$ then the above identity proves that

$$b_h \geqslant 1 - \eta. \tag{3.5}$$

To be able to use Lemma 5 we prove the following lemma.

LEMMA 6 (i) The function $r(p)$ is continuous in $\mathrm{int}(C)$.
(ii) The function $\nu(p)$ is continuous for $p \in \mathrm{int}(C) \setminus \mathrm{int}(C^R)$.
(iii) For each $p \in \mathrm{int}(C) \setminus C^R$, let $\tilde{r}(t, p) = \tilde{r}(p + \nu(p)t)$ be defined for $|t| < \delta(p)$ for some $\delta(p) > 0$. Then for each $p \in \mathrm{int}(C) \setminus C^R$, there is $\epsilon(p) > 0$ such that if $s, t \in (-\epsilon(p), \epsilon(p))$ with $s < t$ then $\tilde{r}(t, p) < \tilde{r}(s, p)$.

Proof. Let us first prove (ii). Let $p \in \mathrm{int}(C) \setminus \mathrm{int}(C^R)$ and let $\delta > 0$ be its distance from ∂C. Let $p_n \in \mathrm{int}(C) \setminus C_R$, $p_n \to p$, $p_n \in B(p, \delta/2)$. Notice that this implies that $r(p_n) \geqslant \delta/2$ and $r(p) \geqslant \delta$. Suppose that $\nu(p_n)$ converges to a unit vector $\nu' \neq \nu(p)$. Since $\nu(p_n)$ is the unit vector orthogonal to $\partial C^{r(p_n)}$ at p_n, the fact that $\nu' \neq \nu(p)$ implies that the circular arc containing p_n which forms part of the boundary of $C^{r(p_n)}$ intersects $C^{r(p)}$ and its complement. This contradicts the fact that both sets are nested.

(i) It suffices to prove the continuity in $\mathrm{int}(C) \setminus \mathrm{int}(C^R)$. Let $p \in \mathrm{int}(C) \setminus \mathrm{int}(C^R)$ and $p_n \in \mathrm{int}(C) \setminus C^R$, $p_n \to p$. Since $p_n \in C^{r(p_n)}$, there exist $q_n \in C$ such that

$$p_n \in \partial B(q_n, r(p_n)) \subseteq B(q_n, r(p_n)) \subseteq C. \tag{3.6}$$

Since $r(p_n)$ is bounded and bounded away from zero, we may assume that $r(p_n)$ converges to some value $\mu > 0$, and also that $q_n \to q$ for some $q \in C$. Passing to the limit in (3.6) we obtain $p \in \partial B(q, \mu) \subseteq B(q, \mu) \subseteq C$. Hence $\mu \leqslant r(p)$. If there is a subsequence of p_n such that $r(p_n) \geqslant r(p)$, then $r(p) = \mu$. Thus, we may assume that $r(p_n) < r(p)$ for all n. Hence $p_n \notin C^{r(p)}$. If $\mu < r(p)$, using (ii), we deduce that $\partial B(q_n, r(p_n)) \cap \partial C^{r(p_n)}$ would intersect $\partial C^{r(p)}$ near p, a contradiction. This implies that $r(p) = \mu$.

(iii) Observe that $p + t\nu(p) \notin C^{r(p)}$ for any $t \in (0, \delta(p))$. This implies that $r(p) > r(p + t\nu(p))$, i.e., $\tilde{r}(0, p) > \tilde{r}(t, p)$. Observe that, by (ii), this argument can be extended to $0 < s < t < \epsilon(p)$ for some small $\epsilon(p)$. Indeed, since $\nu(p)$ is a continuous function of p, we may assume that there is some $\eta > 0$ such that, if $q \in B(p, \eta)$, then the angle between $\nu(p)$ and $\nu(q)$ is less than $\pi/4$. We choose $\epsilon(p)$ such that $p + t\nu(p) \in B(p, \eta)$ for all $t \in (0, \epsilon(p))$. Let $0 < s < t < \epsilon(p)$. Since the angle between $\nu(p)$ and $\nu(p + s\nu(p))$ is $< \pi/4$, it follows that $\nu(p)$ is transversal and points outwards on $\partial C^{\tilde{r}(s,p)}$. Hence $p + t\nu(p) \notin C^{\tilde{r}(s,p)}$. We conclude that $\tilde{r}(s, p) > r(p + t\nu(p)) = \tilde{r}(t, p)$. In a similar way we consider the case $s < t \leqslant 0$. □

Proof of Proposition 5. Let us prove (3.3). Let $p \in \mathrm{int}(C) \setminus \mathrm{int}(C^R)$. Use Lemma 3.3 of [6] with $\partial E = \partial C^{r(p)}$ and $\eta(x) = \nu(x)$ for $x \in \partial C^{r(p)}$ to find a neighborhood V of p where the map $F_\eta : C^{r(p)} \times (-\epsilon, \epsilon) \to V$ defined by $F_\nu(x, t) = x + t\nu(x)$ is bilipschitz. Let $q \in \mathrm{int}(C) \setminus C^R$ be a point in V. Then either $q \in C^{r(p)}$ or $q \notin C^{r(p)}$. Since we can proceed in both cases in a similar way, we shall only consider the case where $q \notin C^{r(p)}$ in detail. Let $p' \in \partial C^{r(p)}$ and $t > 0$, be such that $q = p' + t\nu(p')$ (see Figure 3). Then

$$\frac{|r(q) - r(p)|}{|q - p|} \leqslant C' \frac{|r(q) - r(p')|}{\sqrt{|p' - p|^2 + |q - p'|^2}} \leqslant C' \frac{|r(q) - r(p')|}{|q - p'|}$$

for some constant $C' > 0$. Take p' as the origin of coordinates and the radius of $C^{r(p)}$ through p' as the y-axis, with the positive y-axis directed towards the interior of $C^{r(p)}$, and the x-axis tangent

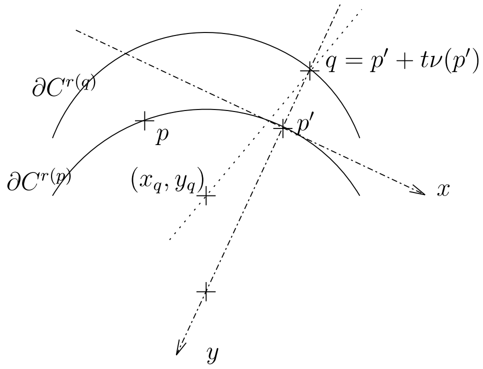

FIG. 3. The construction in the proof of Proposition 5.

to $C^{r(p)}$. Then the center of the arc contained in $\partial C^{r(p)}$ going through p' has coordinates $(0, r(p))$. The point q is the point of intersection of $\partial C^{r(q)}$ with the y-axis at distance $t = |q - p'|$ from the origin. Let (x_q, y_q) be the center of the arc contained in $\partial C^{r(q)}$ and containing q (see Figure 3). Then

$$|q - p'| = \sqrt{r(q)^2 - x_q^2} - y_q.$$

We shall denote by C' a positive constant which may be different from line to line. First observe that, by Lemma 6, as $q \to p$, the x-coordinates of the intersection points of $\partial C^{r(p)}$ and $\partial C^{r(q)}$ are bounded away from 0. Thus, using Lemma 5 and (3.5), we obtain

$$\frac{|x_q|}{r(q) - y_q} = \frac{|x_q|}{r(p) - y_q}\frac{r(p) - y_q}{r(q) - y_q} \leqslant C' \tag{3.7}$$

and

$$\begin{aligned}\frac{|r(q) - r(p')|}{|q - p'|} &= \frac{|r(q) - r(p')|}{\sqrt{r(q)^2 - x_q^2} - y_q} \leqslant \frac{|r(q) - r(p')|}{r(q) - x_q^2/r(q) - y_q}\\ &= \frac{r(p') - r(q)}{r(p') - y_q}\frac{r(p') - y_q}{r(q) - y_q}\frac{r(q) - y_q}{r(q) - x_q^2/r(q) - y_q}\\ &\quad \text{and using (3.5), (3.7), and } x_q \to 0\\ &\leqslant C'\frac{r(p') - r(q)}{r(p') - y_q} \leqslant C'.\end{aligned}$$

Let us prove the analogous statement for the function ν. First observe that

$$\begin{aligned}\frac{|\nu(q) - \nu(p)|}{|q - p|} &\leqslant C'\frac{|\nu(q) - \nu(p')|}{\sqrt{|p' - p|^2 + |q - p'|^2}} + C'\frac{|\nu(p') - \nu(p)|}{\sqrt{|p' - p|^2 + |q - p'|^2}}\\ &\leqslant C'\frac{|\nu(q) - \nu(p')|}{|q - p'|} + C'\frac{|\nu(p') - \nu(p)|}{|p' - p|}.\end{aligned}$$

Since $|\nu(p') - \nu(p)|/|p' - p| \leqslant C'$ for some constant C' depending on $r(p)$, it suffices to prove that $|\nu(q) - \nu(p')|/|q - p'|$ remains bounded as $q \to p$. The vector $\nu(q)$ is the unit vector in the direction of the vector joining (x_q, y_q) and $q = (0, y_q - \sqrt{r(q)^2 - x_q^2})$, i.e.,

$$\nu(q) = -\frac{(x_q, \sqrt{r(q)^2 - x_q^2})}{r(q)}.$$

Thus

$$\nu(q) - \nu(p') = -\frac{(x_q, \sqrt{r(q)^2 - x_q^2})}{r(q)} - (0, -1) = \frac{(-x_q, r(q) - \sqrt{r(q)^2 - x_q^2})}{r(q)}.$$

Since $x_q \to 0$ and $x_q/(r(q) - y_q)$ is bounded as $q \to p$, we have

$$\begin{aligned}\frac{|x_q|}{|q-p'|} &= \frac{|x_q|}{\sqrt{r(q)^2 - x_q^2} - y_q} \leqslant \frac{|x_q|}{r(q) - x_q^2/r(q) - y_q} \\ &\leqslant \frac{|x_q|}{(r(q) - y_q)(1 - \frac{x_q^2}{r(q)(r(q)-y_q)})} \leqslant C' \frac{|x_q|}{r(q) - y_q} \leqslant C'.\end{aligned}$$

In a similar way,

$$\begin{aligned}\frac{r(q) - \sqrt{r(q)^2 - x_q^2}}{r(q)|q-p'|} &= \frac{r(q)(1 - \sqrt{1 - x_q^2/r(q)^2})}{r(q)|q-p'|} \leqslant C' \frac{x_q^2/r(q)^2}{r(q) - x_q^2/r(q) - y_q} \\ &\to 0 \qquad \text{as } q \to p.\end{aligned}$$

We have shown that

$$\limsup_{q \to p} \frac{|\nu(q) - \nu(p)|}{|q-p|} < \infty \qquad \text{for all } p \in \operatorname{int}(C) \setminus C^R. \qquad \square$$

We prepare the proof of Proposition 6 with the following lemma. Since the proof in $\mathbb{R}^2$ and $\mathbb{R}^N$, $N \neq 2$, is the same, we state it in the general case. By $\mathcal{L}^1$ we denote the Lebesgue measure in $\mathbb{R}$.

LEMMA 7 Let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $u \in \mathrm{BV}(\Omega)$ and satisfies condition (3.2) for any $x \in \Omega$, then $u \in W^{1,1}(\Omega)$.

Proof. Notice that condition (3.2) implies that u is continuous. First we shall prove the result for $N = 1$. By Theorem 4 applied to u (with $A = B = \Omega$), we may write $\Omega = \bigcup_n E_n$, where E_n are measurable sets such that $E_n \subseteq E_{n+1}$, and $u|_{E_n}$ is Lipschitz with a constant L_n. Let μ be the measure $\mu(A) = |Du|(A)$ for any Borel set $A \subseteq \mathbb{R}$. Let us prove that $\mu \ll \mathcal{L}^1$. Let A be a subset of $\mathbb{R}$ such that $\mathcal{L}^1(A) = 0$. Since $\mu(A) = \lim_n \mu(A \cap E_n)$, it will follow that $\mu(A) = 0$ if we show that $\mu(A \cap E_n) = 0$ for all n. Thus, we may assume that $A \subseteq E_n$. Let $\epsilon > 0$. Since μ is Borel regular, there is an open set $U \supset A$ such that $\mathcal{L}^1(U) < \epsilon$ and $\mu(U) \leqslant \mu(A) + \epsilon$. Let $U = \bigcup_{i=1}^{\infty} I_i$, where $I_i =]x_i, y_i[$. Let $I_{ij} =]x_{ij}, y_{ij}[$, $j = 1, \ldots, N$, be a partition of I_i such that

$$\mu(I_i) - \sum_j |u(x_{ij}) - u(y_{ij})| \leqslant \epsilon/2^i.$$

Since u is continuous the measure μ does not charge points and we have $\mu(I_i) = \sum_j \mu(I_{ij})$. Hence

$$\sum_j (\mu(I_{ij}) - |u(x_{ij}) - u(y_{ij})|) \leqslant \epsilon/2^i.$$

Let $I'_{ij} =]x'_{ij}, y'_{ij}[$ be the largest subinterval of I_{ij} with end points in $\overline{A \cap I_i}$. We have

$$\mu(A \cap I_i) = \sum_j \mu(A \cap I_{ij}) \leqslant \sum_j \mu(I'_{ij}).$$

Now,

$$\sum_j (\mu(I'_{ij}) - |u(x'_{ij}) - u(y'_{ij})|) \leqslant \sum_j (\mu(I_{ij}) - |u(x_{ij}) - u(y_{ij})|) \leqslant \epsilon/2^i.$$

Thus

$$\mu(A \cap I_i) \leqslant \sum_j |u(x'_{ij}) - u(y'_{ij})| + \epsilon/2^i \leqslant L_n \sum_j |x'_{ij} - y'_{ij}| + \epsilon/2^i$$
$$\leqslant L_n|x_i - y_i| + \epsilon/2^i.$$

Hence

$$\mu(A) = \sum_i \mu(A \cap I_i) \leqslant L_n \sum_i |x_i - y_i| + \epsilon \leqslant L_n \mathcal{L}^1(U) + \epsilon \leqslant (L_n + 1)\epsilon.$$

Since the above inequality holds for all $\epsilon > 0$ we conclude that $\mu(A) = 0$.

Now we consider the general case $N \geqslant 2$. Let $\vec{e}$ be any direction in $\mathbb{R}^N$, let $\pi_{\vec{e}}$ be the plane through the origin orthogonal to $\vec{e}$, and let $\Omega_{\vec{e}}$ be the projection of Ω onto $\pi_{\vec{e}}$. For any $y \in \Omega_{\vec{e}}$, let $\Omega_y^{\vec{e}} = \{t \in \mathbb{R} : y + t\vec{e} \in \Omega\}$. Let $u_y^{\vec{e}} : \Omega_y^{\vec{e}} \to \mathbb{R}$ be defined by $u_y^{\vec{e}}(t) = u(y + t\vec{e})$ for $t \in \Omega_y^{\vec{e}}$. Since $u \in \mathrm{BV}(\Omega)$ there exist N independent directions $\vec{e}_i$, $i = 1, \ldots, N$, such that $u_y^{\vec{e}_i} \in \mathrm{BV}(\Omega_y^{\vec{e}_i})$ for $\mathcal{L}^{N-1}$-a.e. $y \in \Omega_{\vec{e}_i}$ (say for $y \in \Omega'_{\vec{e}_i}$ with $\mathcal{L}^{N-1}(\Omega_{\vec{e}_i} \setminus \Omega'_{\vec{e}_i}) = 0$) and

$$\int_{\Omega_{\vec{e}_i}} |Du_y^{\vec{e}_i}|(\Omega_y^{\vec{e}_i})\,\mathrm{d}y < \infty \quad \forall i = 1, \ldots, N. \tag{3.8}$$

Fix $i \in \{1, \ldots, N\}$ and $y \in \Omega'_{\vec{e}_i}$. Since $u_y^{\vec{e}_i}$ also satisfies condition (3.2), by the first part of the proof, we conclude that $u_y^{\vec{e}_i} \in W^{1,1}(\Omega_y^{\vec{e}_i})$ and

$$\int_{\Omega_y^{\vec{e}_i}} |\nabla u_y^{\vec{e}_i}(t)|\,\mathrm{d}t = |Du_y^{\vec{e}_i}|(\Omega_y^{\vec{e}_i}).$$

Now, using (3.8) we conclude that $u \in W^{1,1}(\Omega)$. □

When $N = 1$ and u is increasing the above result is contained in [18, Corollary to Theorem 8.1.11].

Proof of Proposition 4. Observe that the coarea formula applied to r implies that $r \in \mathrm{BV}(C)$. Since also $r \in C(\mathrm{int}(C))$, by Proposition 5 and Lemma 7 we conclude that $r \in W^{1,1}(C)$. Moreover, using Lemma 6(iii) we deduce that $\partial\tilde{r}(t, p)/\partial t > 0$ for almost all $t \in (-\epsilon(p), \epsilon(p))$ and any $p \in \mathrm{int}(C)\backslash C^R$. Therefore $\nabla r(p) \neq 0$ for almost all $p \in \mathrm{int}(C)\backslash C^R$. At each point $p \in \mathrm{int}(C)\backslash C^R$, let $X(p)$ be the unit tangent vector to $C^{r(p)}$ at p oriented so that $\{X(p), \nu(p)\}$ forms an orthonormal frame. Then

$$\nabla r(p) = \langle \nabla r(p), \nu(p)\rangle \nu(p) + \langle \nabla r(p), X(p)\rangle X(p) = \langle \nabla r(p), \nu(p)\rangle \nu(p)$$

on C, and this implies that

$$|\langle \nabla r(p), \nu(p)\rangle| = -\langle \nabla r(p), \nu(p)\rangle = |\nabla r(p)| \quad \text{on } C.$$

Observe that ν is also a (vector) function of bounded variation in $\mathrm{int}(C) \setminus C^R$. For that, it suffices to note that, since $\nu(p)$ is continuous, it suffices to integrate its variation on the arcs of the family of circles $\partial C^{r'}$, $r' \in [0, R]$, contained in $\mathrm{int}(C) \setminus C^R$. Now, since the sets $C^{r'}$, $r' \in [0, R]$, are convex the variation of the function $\nu(p)$ on the boundary of each set $C^{r'}$ is bounded by 2π. Integrating all these variations we find that ν is of bounded variation in $\mathrm{int}(C) \setminus C^R$. Using again Proposition 5 and Lemma 7 we deduce that each coordinate of $\nu(p)$ is in $W^{1,1}(\mathrm{int}(C) \setminus C^R)$, hence is differentiable a.e. Moreover, the computations made for ν in Proposition 5 prove that

$$\lim_{t\to 0} \nu(p) \cdot \frac{\nu(p + t\nu(p)) - \nu(p)}{t} = 0.$$

In other words, at any point of differentiability of ν we have

$$\langle D\nu(p)(\nu(p)), \nu(p)\rangle = 0.$$

Finally, on $\mathrm{int}(C) \setminus C^R$ we have

$$\operatorname{div} \nu(p) = \langle D\nu(p)(X(p)), X(p)\rangle + \langle D\nu(p)(\nu(p)), \nu(p)\rangle = 1/r(p).$$

3.3 *Proof of Theorem 2 and some extensions*

Let C be a nonempty bounded convex set in $\mathbb{R}^2$; we will apply the notation introduced in Sections 1, 3.1, and 3.2. Let $R > 0$ be the radius given in Proposition 1. Since C^R decreases at speed $1/R$, we know ([5]) that there exists a vector field $z^{C^R} \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $|z^{C^R}| \leqslant 1$ such that $z^{C^R}(x) = -\nu(x)\ \mathcal{H}^1$-a.e. in ∂C^R, and

$$\operatorname{div} z^{C^R} = -\frac{P(C^R)}{|C^R|} \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Theorem 2 will be a consequence of the following proposition.

PROPOSITION 6 Let C be a nonempty bounded convex set in $\mathbb{R}^2$. Let

$$z(x) = \begin{cases} -\nu(x) & \text{if } x \in \mathrm{int}(C) \setminus C^R, \\ z^{C^R}(x) & \text{if } x \in C^R, \\ z^{\mathbb{R}^2\setminus C} & \text{if } x \in \mathbb{R}^2 \setminus C. \end{cases}$$

Extend r to $\mathbb{R}^2$ by defining $r(x) = 0$ if $x \in \mathbb{R}^2 \setminus C$, and set $\frac{1}{r(x)}\chi_C(x) = 0$ if $x \in \mathbb{R}^2 \setminus C$. Then

$$z \cdot Dr = |Dr| \qquad \text{in } \mathbb{R}^2, \tag{3.9}$$

$$\operatorname{div} z = -1/r\,\chi_C \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{3.10}$$

Proof. The coarea formula applied to r implies that $r \in \mathrm{BV}(\mathbb{R}^2)$. By Proposition 4, (3.9) holds in C. Since $r(x) = 0$ outside C, and $z(x) = -\nu^C(x)\ \mathcal{H}^1$-a.e. on ∂C, it follows that (3.9) holds.

Since $\operatorname{div} z = -P(C^R)/|C^R| = -1/R$ in C^R, $\operatorname{div} z = 0$ in $\mathbb{R}^2 \setminus C$, $-\nu(x) = \nu^{\mathbb{R}^2\setminus C}(x)$ $\mathcal{H}^{N-1}$-a.e. in ∂C, and $\nu^{C^R}(x) = -\nu(x)$ on ∂C^R, we obtain (3.10). □

Let us finally mention that, though stated in a different form, related results have been obtained in the case of crystalline norms by G. Bellettini, M. Novaga, and M. Paolini [7].

Proof of Theorem 2. We have

$$u_t(t,x) = -\mathrm{sign}^+\left(1 - \frac{t}{r(x)}\right)\frac{1}{r(x)}\chi_C(x).$$

Now, observe that $\mathrm{sign}^+(1 - t/r)^+ = 1$ if and only if $t \leqslant r(x)$, i.e., $x \in C^t$. Otherwise, $\mathrm{sign}^+(1 - t/r(x)) = 0$. In particular, for $t \geqslant R$ we have $u_t = 0$ and also $u(t) = 0$. Thus $u_t(t,x) = -(1/r(x))\chi_{C^t}(x)\chi_{[0,T]}(t)$. Since C^t is a convex set, there is a vector field $z^{\mathbb{R}^2\setminus C^t} \in L^\infty(\mathbb{R}^2 \setminus C^t)$ with $\|z^{\mathbb{R}^2\setminus C^t}\|_\infty \leqslant 1$ such that

$$\operatorname{div} z^{\mathbb{R}^2\setminus C^t} = 0 \quad \text{in } \mathbb{R}^2 \setminus C^t, \qquad z^{\mathbb{R}^2\setminus C^t} \cdot \nu^{\mathbb{R}^2\setminus C^t} = 1.$$

Let $z(x)$ be the vector field defined in Proposition 6. When $t \leqslant R$, let

$$z(t,x) = \begin{cases} z(x) & \text{if } x \in C^t, \\ z^{\mathbb{R}^2\setminus C^t}(x) & \text{if } x \in \mathbb{R}^2 \setminus C^t. \end{cases}$$

When $t \geqslant R$, let $z(t,x) = 0$. Now, using the results in [3], we have

$$\begin{aligned} \int_{\mathbb{R}^2} (z(t), Du(t)) &= \int_0^{\|u(t)\|_\infty} \int_{\mathbb{R}^2} (z(t), D\chi_{[u(t)\geqslant\lambda]})\,\mathrm{d}\lambda \\ &= \int_0^{\|u(t)\|_\infty} \int_{\partial^*[u(t)\geqslant\lambda]} (z(t), \nu^{[u(t)\geqslant\lambda]})\,\mathrm{d}\lambda \end{aligned}$$

for any $t > 0$. Since for any $\lambda \in (0, \|u(t)\|_\infty)$, $\partial^*[u(t) \geqslant \lambda] = \partial[u(t) \geqslant \lambda]$ is of class C^1, by the results in [4], we have $(z(t), \nu^{[u(t)\geqslant\lambda]}) = 1$ $\mathcal{H}^1$-a.e. on $\partial[u(t) \geqslant \lambda]$. Thus we obtain

$$\int_{\mathbb{R}^2} (z(t), Du(t)) = \int_0^{\|u(t)\|_\infty} P(\partial^*[u(t) \geqslant \lambda])\,\mathrm{d}\lambda = \int_{\mathbb{R}^2} |Du(t)|.$$

On the other hand, by construction of $z(t,x)$ we have

$$\operatorname{div} z(t) = -\frac{1}{r(x)}\chi_{C^t}(x)$$

if $t \leqslant R$. If $t > R$, we have $\operatorname{div} z(t) = 0$. Thus $u_t(t) = \operatorname{div} z(t)$ for almost all $t \in (0,T)$ (also in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$ for any $T > 0$). By the characterization of $\partial\Phi$ given in Lemma 1, $u(t)$ is a strong solution in the sense of semigroups of (1.1) which coincides with the solution given by Theorem 3. □

EXAMPLES (i) Let $C = [0,L]^2$. If we compute the value of R such that $P(C^R)/|C^R| = 1/R$, we obtain

$$R = \frac{16 - \sqrt{24\pi}}{32 - 3\pi} L.$$

We can also compute the value of $r(x,y)$, $(x,y) \in \mathbb{R}^2$, in the connected component of $C \setminus C^R$ containing $(0,0)$. In this case $r(x,y) = x + y + \sqrt{2xy}$. The arcs foliating $C \setminus C^R$ are quadrants of circles which are tangent to C. The solution is

$$u(t,x,y) = \left(1 - \frac{t}{r(x,y)}\right)^+ \chi_C(x,y), \qquad (x,y) \in \mathbb{R}^2.$$

(ii) We can also compute the solution of (1.1) corresponding to the characteristic function of an angular sector in $\mathbb{R}^2$. Let

$$S_{\theta,\vec{v}} = \{(x, y) \in \mathbb{R}^2 : \text{the angle formed by } (x, y) \text{ and } \vec{v} \text{ is less than } \theta\}, \quad \vec{v} \in \mathbb{R}^2.$$

Even if angular sectors are not bounded, the ideas above can be applied to compute the explicit solution $u_{\theta,\vec{v}}$ of (1.1) with $u_{\theta,\vec{v}}(0, x, y) = \chi_{S_{\theta,\vec{v}}}(x, y)$ (whose existence and uniqueness is guaranteed by the results in [5]). Moreover, these solutions exhibit the behavior of solutions $u(t, x, y)$ of (1.1) with $u(0, x, y) = \chi_C(x, y)$ at corners of ∂C for any bounded convex set $C \subseteq \mathbb{R}^2$. Indeed, if C has a corner at $p \in \partial C$ of angular aperture 2θ, then, by a blow-up of $u(t, x, y)$ around p, we obtain a solution $u_{\theta,\vec{v}}$ for some $\vec{v} \in \mathbb{R}^2$ (the bisector of the corner at p). Notice that $u_{\theta,\vec{v}}$ is self-similar, i.e., $u_{\theta,\vec{v}}(\lambda t, \lambda x, \lambda y) = u_{\theta,\vec{v}}(t, x, y)$ for any $\lambda > 0$, $t \geqslant 0$, $(x, y) \in \mathbb{R}^2$. Let us compute it.

To fix ideas, let $\vec{v} = (1, 1)$ and $S_\theta := S_{\theta,(1,1)}$. Observe that for each $(x, y) \in S_\theta$ there is a unique circle centered at some point $(a(x, y), a(x, y))$ and tangent to the boundary of S_θ. Let $r(x, y)$ be the radius of this circle. Then $r(x, y) = \sqrt{2} \sin\theta\, a(x, y)$ and $a(x, y)$ solves the equation $2a^2 \cos^2\theta - 2a(x + y) + x^2 + y^2 = 0$. Then the solution is $u(t, x, y) = (1 - t/r(x, y))^+ \chi_{S_\theta}$. Note that this solution lives in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Let us recall the following result which was proved in [5], though stated in a different context.

THEOREM 5 Let $\Omega \subset \mathbb{R}^2$ be a set which is the union of a finite number of connected components $C_1, \ldots, C_m$ which are nonempty bounded convex sets. Then there is a vector field $z \in L^\infty(\mathbb{R}^2 \setminus \Omega)$ with $\|z\|_\infty \leqslant 1$ such that

$$\operatorname{div} z = 0 \quad \text{in } \mathbb{R}^2 \setminus \Omega \tag{3.11}$$

and

$$z(x) \cdot \nu^{\mathbb{R}^2\setminus\Omega}(x) = 1 \quad \text{in } \partial\Omega \tag{3.12}$$

if and only if the following condition (c) holds: let $0 \leqslant k \leqslant m$ and let $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ be any k-tuple of indices; if we denote by $E_{i_1,\ldots,i_k}$ a solution of the variational problem

$$\min\left\{P(E) : E \text{ of finite perimeter}, \bigcup_{j=1}^{k} C_{i_j} \subseteq E \subseteq \mathbb{R}^2 \setminus \bigcup_{j=k+1}^{m} C_{i_j}\right\}, \tag{3.13}$$

then

$$P(E_{i_1,\ldots,i_k}) \geqslant \sum_{j=1}^{k} P(C_{i_j}). \tag{3.14}$$

In particular, if $\Omega \subseteq \mathbb{R}^2$ is convex, a vector field $z^{\mathbb{R}^2\setminus\Omega}$ satisfying (3.11), (3.12) always exists.

Condition (c) amounts to saying that the sets C_i, $i = 1, \ldots, m$, are sufficiently separated from each other. Indeed, the perimeter of the convex envelope of any k-tuple of them is larger than the sum of the perimeters of the sets in the k-tuple. As an example (see [5]), if $\Omega \subseteq \mathbb{R}^2$ is the union of two disjoint balls of radius r whose centers are at distance $L > 0$, then condition (c) is equivalent to $L \geqslant \pi r$. If Ω is the disjoint union of three balls of radius $r > 0$ whose centers are the vertices of an equilateral triangle with edges of length L, then condition (c) amounts to the inequality $r \leqslant 3L/4\pi$ (see [5]).

By using Theorem 5, Proposition 6 can be extended to the more general situation formulated in the next result.

PROPOSITION 7 Let $\Omega = C_1 \cup \cdots \cup C_m \subseteq \mathbb{R}^2$, where the C_i are nonempty bounded convex sets satisfying condition (c) in Theorem 5. Let $r_i(p)$ and $\nu_i(p)$ be the functions constructed above associated to C_i, $i = 1, \ldots, m$. Let $C_i^{R_i}$ be the opening of C_i such that $|(C_i)_{R_i}| = \pi R_i^2$. Let

$$z(x) = \begin{cases} -\nu_i(x) & \text{if } x \in \operatorname{int}(C_i) \setminus C_i^{R_i}, \\ z^{C_i^{R_i}}(x) & \text{if } x \in C_i^{R_i}, \\ z^{\mathbb{R}^2 \setminus \Omega} & \text{if } x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Let

$$r(x) = \begin{cases} r_i(x) & \text{if } x \in C_i, \\ 0 & \text{if } x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Set $(1/r(x))\chi_\Omega(x) = 0$ if $x \notin \Omega$. Then

$$z \cdot Dr = |Dr| \quad \text{in } \mathbb{R}^2, \tag{3.15}$$

$$\operatorname{div} z = -\frac{1}{r}\chi_\Omega \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{3.16}$$

As a consequence, the function $u(t,x) = \sum_{i=1}^m \operatorname{sign}(b_i)(|b_i| - t/r_i(x))^+ \chi_{C_i}$ is the solution of (1.1) corresponding to the initial condition $u(0,x) = \sum_{i=1}^m b_i \chi_{C_i}$.

4. Explicit solutions of the denoising problem

The previous results allow us to explicitly compute the minimum of the denoising problem

$$\min_{u \in \mathrm{BV}(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |Du| + \frac{1}{2\lambda} \int_{\mathbb{R}^2} (u - f)^2 \, \mathrm{d}x \right\}, \tag{4.1}$$

where $\lambda > 0$, for some data $f \in L^2(\mathbb{R}^2)$. We shall only compute the explicit solutions of (4.1) which can be derived from Propositions 6 and 7; other explicit solutions have been given in [5].

PROPOSITION 8 Let $\Omega = C_1 \cup \cdots \cup C_m \subseteq \mathbb{R}^2$, where C_i are bounded convex sets satisfying condition (c) in Theorem 5, and such that $C_i = C_i^{r_i}$ for some $r_i > 0$, $i = 1, \ldots, m$. Let $r_i(x)$ be the corresponding functions defined in Proposition 7. Let $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$, and $f := \sum_{i=1}^m (b_i / r_i(x))\chi_{C_i}$. Let $\lambda > 0$. The solution u of the variational problem (4.1) is

$$u := \sum_{i=1}^m \operatorname{sign}(b_i) \frac{(|b_i| - \lambda)^+}{r_i(x)} \chi_{C_i}.$$

Proof. Observe that, under the assumptions of the proposition, $f \in L^2(\mathbb{R}^2)$. Recall that a function $u \in \mathrm{BV}(\mathbb{R}^2)$ is the solution of (4.1) if and only if u is the solution of

$$u - \lambda \operatorname{div}\left(\frac{Du}{|Du|}\right) = f. \tag{4.2}$$

We have to prove that $u = \sum_{i=1}^m \operatorname{sign}(b_i) \frac{(|b_i| - \lambda)^+}{r_i(x)} \chi_{C_i}$ is the solution of (4.2). Let us define

$I_\lambda := \{i \in \{1, \ldots, m\} : |b_i| \geqslant \lambda\}$ and $J_\lambda := \{i \in \{1, \ldots, m\} : |b_i| < \lambda\}$. Since, in this case,

$$f - u = \lambda \sum_{i \in I_\lambda} \operatorname{sign}(b_i) \frac{\chi_{C_i}}{r_i(x)} + \sum_{i \in J_\lambda} b_i \frac{\chi_{C_i}}{r_i(x)},$$

to prove that u is a solution of (4.2) we have to construct a vector field $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $\|\xi\|_\infty \leqslant 1$ such that

$$-\operatorname{div} \xi = \sum_{i \in I_\lambda} \operatorname{sign}(b_i) \frac{\chi_{C_i}}{r_i(x)} + \sum_{i \in J_\lambda} \frac{b_i}{\lambda} \frac{\chi_{C_i}}{r_i(x)} \tag{4.3}$$

and $(\xi, Du) = |Du|$. Let $F \in L^2(\mathbb{R}^2)$ denote the right-hand side of (4.3). As proved in [5], a solution $\xi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ with $\|\xi\|_\infty \leqslant 1$ exists if and only if $\|F\|_* \leqslant 1$, where

$$\|F\|_* := \sup \left\{ \left| \int_{\mathbb{R}^2} F(x) v(x)\, dx \right| : v \in L^2(\mathbb{R}^2) \cap \mathrm{BV}(\mathbb{R}^2),\ \int_{\mathbb{R}^2} |Dv| \leqslant 1 \right\}.$$

As in [5, Proposition 8], $\|F\|_* \leqslant 1$ follows from the inequality

$$|F(x)| \leqslant \sum_{i=1}^m \frac{\chi_{C_i}}{r_i(x)}$$

and the fact that, by Proposition 7, we have $\| \sum_{i=1}^m \chi_{C_i} / r_i(x) \|_* \leqslant 1$.

Let C be any of the sets C_i and $r(x)$ the corresponding $r_i(x)$. Multiplying (3.10) by $\chi_C(x)/r(x)$, integrating by parts the divergence term, and using (3.9), we obtain

$$\int_{\mathbb{R}^2} \frac{\chi_C(x)}{r(x)^2} \, \mathrm{d}x = \int_{\mathbb{R}^2} \left| D \frac{\chi_C(x)}{r(x)} \right|. \tag{4.4}$$

Since $(|b_i| - \lambda)^+ = 0$ for all $i \in J_\lambda$, we have

$$\int_{\mathbb{R}^2} |Du| = \sum_{i \in I_\lambda} (|b_i| - \lambda) \int_{\mathbb{R}^2} \left| D \frac{\chi_{C_i}(x)}{r_i(x)} \right|$$

Since

$$\int_{\mathbb{R}^2} (\xi, Du) = -\int_{\mathbb{R}^2} \operatorname{div} \xi \, u \, \mathrm{d}x = \int_{\mathbb{R}^2} F u \, \mathrm{d}x = \sum_{i \in I_\lambda} (|b_i| - \lambda) \int_{\mathbb{R}^2} \frac{\chi_{C_i}(x)}{r_i(x)^2} \, \mathrm{d}x,$$

and we may apply the equality in (4.4) to C_i, we obtain

$$\int_{\mathbb{R}^2} (\xi, Du) = \sum_{i \in I_\lambda} (|b_i| - \lambda) \int_{\mathbb{R}^2} \left| D \frac{\chi_{C_i}(x)}{r_i(x)} \right| \mathrm{d}x = \int_{\mathbb{R}^2} |Du|,$$

which in turn implies that $(\xi, Du) = |Du|$, since $\|\xi\|_\infty \leqslant 1$. □

5. Experimental results

In this section, we show some numerical examples of evolutions. In order to implement equation (1.1), we first discretize the evolution in time. We choose a time step $\delta t > 0$ and, given the initial value u_0, we define a sequence $(u_n)_{n\geqslant 0}$ by letting, for every $n \geqslant 0$,

$$u_{n+1} = \arg\min_{u\in \mathrm{BV}(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} |Du| + \frac{1}{2\delta t}\int_{\mathbb{R}^2} (u-u_n)^2 \,\mathrm{d}x \right\}. \tag{5.1}$$

It is standard that if we let $u^{\delta t}(t,x) = u_{[t/\delta t]}(x)$ ($[\cdot]$ denotes the integer part) for every $t \geqslant 0$ and $x \in \mathbb{R}^2$, then $u^{\delta t} \to u(t,x)$ as $\delta t \to 0$, where u is the solution of (1.1), (1.2) (see, for instance, [8]).

Then we discretize the problem (5.1) in space. We fix a space step $\delta x > 0$. We will consider the case where $u_0 = \chi_C$, with C a bounded convex subset of $\mathbb{R}^2$, and we discretize (5.1) on a bounded open subset $\mathcal{R} =]a,b[\times]c,d[\subset \mathbb{R}^2$, containing the initial set C, and impose the Dirichlet condition $u = 0$ on the boundary $\partial\mathcal{R}$. This agrees with the fact that the solution of (1.1) in $\mathbb{R}^2$ is shown to remain constant, equal to zero, off C.

We follow the algorithm proposed in [10] based on a dual formulation of the problem (see [16] for a related approach). We consider the fixed grid $\mathcal{R} \cap \delta x\mathbb{Z}^2$, which, up to translation, consists of the points $\{(i\delta x, j\delta x) : i = 1,\dots,N,\ j = 1,\dots,M\}$ for two integers $N, M \geqslant 0$. The discretization of u is a matrix $(u_{i,j})_{1\leqslant i\leqslant N, 1\leqslant j\leqslant M} \in X = \mathbb{R}^{N\times M}$, where $u_{i,j}$ represents the value of u at $(i\delta x, j\delta x)$. The gradient of u is represented by the linear operator $\nabla : X \to Y$, with $Y = \mathbb{R}^{(N+1)\times(M+1)} \times \mathbb{R}^{(N+1)\times(M+1)}$, defined for $i = 0,\dots,N$, $j = 0,\dots,M$ by

$$(\nabla u)_{i,j} = ((\nabla u)^1_{i,j}, (\nabla u)^2_{i,j}),$$

with $(\nabla u)^1_{i,0} = 0$ for all i, and, for $j \geqslant 1$,

$$(\nabla u)^1_{i,j} = \begin{cases} u_{1,j}/\delta x & \text{if } i = 0,\\ (u_{i+1,j} - u_{i,j})/\delta x & \text{if } 0 < i < N,\\ -u_{N,j}/\delta x & \text{if } i = N,\end{cases}$$

and $(\nabla u)^2_{0,j} = 0$ for all j, and, for $i \geqslant 1$,

$$(\nabla u)^2_{i,j} = \begin{cases} u_{i,1}/\delta x & \text{if } j = 0,\\ (u_{i,j+1} - u_{i,j})/\delta x & \text{if } 1 < j < M,\\ -u_{i,M}/\delta x & \text{if } j = M.\end{cases}$$

This definition takes into account the fact that we have imposed the Dirichlet condition $u = 0$ on $\partial\mathcal{R}$. Then the discrete total variation of u is

$$V(u) = \delta x^2 \sum_{i=0}^{N}\sum_{j=0}^{M} |(\nabla u)_{i,j}|.$$

In X and Y we consider the Euclidean scalar products

$$\langle x, x'\rangle = \delta x^2 \sum_{i=1}^{N}\sum_{j=1}^{M} x_{i,j}x'_{i,j}\,, \quad \forall x, x' \in X,$$

$$\langle y, y' \rangle = \delta x^2 \sum_{i=0}^{N} \sum_{j=0}^{M} y_{i,j} \cdot y'_{i,j} = \delta x^2 \sum_{i=0}^{N} \sum_{j=0}^{M} (y^1_{i,j} y'^1_{i,j} + y^2_{i,j} y'^2_{i,j}), \quad \forall y, y' \in Y.$$

In order to find a definition of $V(u)$ similar to (2.1), we introduce the operator div : $Y \to X$, defined by div $= -\nabla^*$, that is, for $\xi = (\xi^1, \xi^2) \in Y$,

$$(\operatorname{div} \xi)_{i,j} = \frac{\xi^1_{i,j} - \xi^1_{i-1,j}}{\delta x} + \frac{\xi^2_{i,j} - \xi^2_{i,j-1}}{\delta x}.$$

It satisfies $\langle \operatorname{div} \xi, u \rangle = -\langle \xi, \nabla u \rangle$ for any $\xi \in Y$ and $u \in X$. Then one easily checks that

$$V(u) = \sup\{\langle u, \operatorname{div} \xi \rangle : \max_{i,j} \|\xi_{i,j}\| \leqslant 1\} = \delta^*_K(u),$$

where the set K is given by

$$K = \{\operatorname{div} \xi : \xi \in Y, \ \max_{i,j} \|\xi_{i,j}\| \leqslant 1\}, \tag{5.2}$$

the function $\delta_K(v)$ is 0 if $v \in K$, $+\infty$ otherwise, and $\delta^*_K(u) = \sup_{v \in X}(\langle u, v \rangle - \delta_K(v))$ denotes the Legendre–Fenchel transform of δ_K. Since K is closed and convex, one also has $V^* = \delta_K$.

The evolution (5.1) is discretized in space by first considering a discretization u_0 of the initial data, and letting, for all $n \geqslant 0$, u_{n+1} be the (unique) solution of

$$u_{n+1} = \arg\min_{u \in X} \left\{ V(u) + \frac{1}{2\delta t} \|u - u_n\|^2_X \right\}. \tag{5.3}$$

It is shown in [10], by classical convex duality techniques, that the solution of this problem is given by

$$u_{n+1} = u_n - \Pi_{\delta t K}(u_n),$$

where $\Pi_{\delta t K}$ denotes the projection onto the convex set $\delta t K$. Hence, the problem amounts to computing the nonlinear projection $\Pi_{\delta t K}(u_n)$, that is, in view of the definition (5.2) of K, to solving

$$\min\{\|\delta t \operatorname{div} \xi - u_n\|^2_X : \max_{i,j} \|\xi_{i,j}\| \leqslant 1\}.$$

This can be done through the following fixed point algorithm:

- Fix an initial $\xi^0 \in Y$ with $\max_{i,j} \|\xi^0_{i,j}\| \leqslant 1$, and choose $\tau > 0$;
- For each $k \geqslant 0$, compute $\xi^{k+1} \in Y$ by the following iteration:

$$\xi^{k+1}_{i,j} = \frac{\xi^k_{i,j} + \tau(\nabla(\operatorname{div} \xi^k - u_n/\delta t))_{i,j}}{1 + \tau |(\nabla(\operatorname{div} \xi^k - u_n/\delta t))_{i,j}|},$$

  $i = 1, \ldots, N$, $j = 1, \ldots, M$.

It is shown in [10, Thm. 3.1] that if $\tau \leqslant \delta x^2/8$, then $\delta t \operatorname{div} \xi^k$ converges to $\Pi_{\delta t K}(u_n)$ as $k \to \infty$.

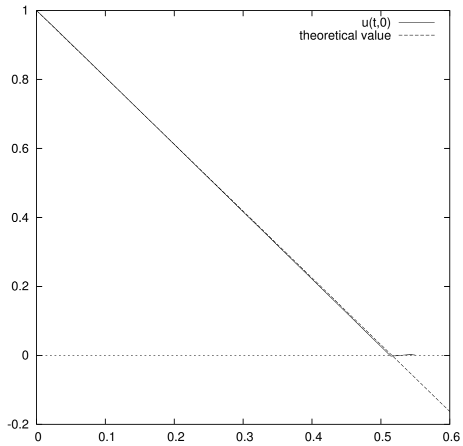

FIG. 4. The values $u(t, 0)$ when $u(0, x) = \chi_C(x)$.

FIG. 5. The value of u at time $t = 0.05$, $t = 0.25$, and $t = 0.5$ when the initial condition is $\chi_C(x)$. The grey scale is different for the last image, since the maximum value of u at time 0.5 is around 0.03.

Although the convergence of this algorithm is experimentally quite fast for an arbitrary u_n, when u_n has large "flat" areas (which is the case we are considering in this paper), it is not as efficient, since the information has to propagate from the boundaries of the flat zones into the middle. For this reason, the first iterations (in particular, $n = 0$) are computed in our example with a very small error, that is, large k. The first iteration (computing u_1 from u_0) is done with an initial $\xi^0 = 0$. Then, when computing u_{n+1} from u_n with $n \geqslant 1$, we choose for ξ^0 the last value of ξ^k found in the iteration for computing u_n. After a few iterations ($n = 1$ to 3), the convergence is obtained in a few steps.

We show the results of two experiments, with two different initial convex sets C. The first C is the union of the rectangle $[0, 1] \times [-1, 1]$ and the half-disc $\{(x, y) : x^2 + y^2 \leqslant 1, x \leqslant 0\}$. For this convex set, it is easy to check that $C_r = (1 - r)C$ if $r \in [0, 1]$ (and $C_r = \emptyset$ if $r > 1$), so that $|C_r| = (1 - r)^2|C|$. Thus, $|C_R| = \pi R^2$ if and only if $R = 1/(1 + \sqrt{\pi/|C|})$. Since $|C| = \pi/2 + 2$, we find the approximate value of R to be

$$R = 0.5160019.$$

In Fig. 4, the value $u(t, 0)$ is plotted as a function of the time t, together with the line $(1 - t/R)^+$. The actual slope corresponds to an effective radius $R \simeq 0.513$, that is, less than 0.6% smaller.

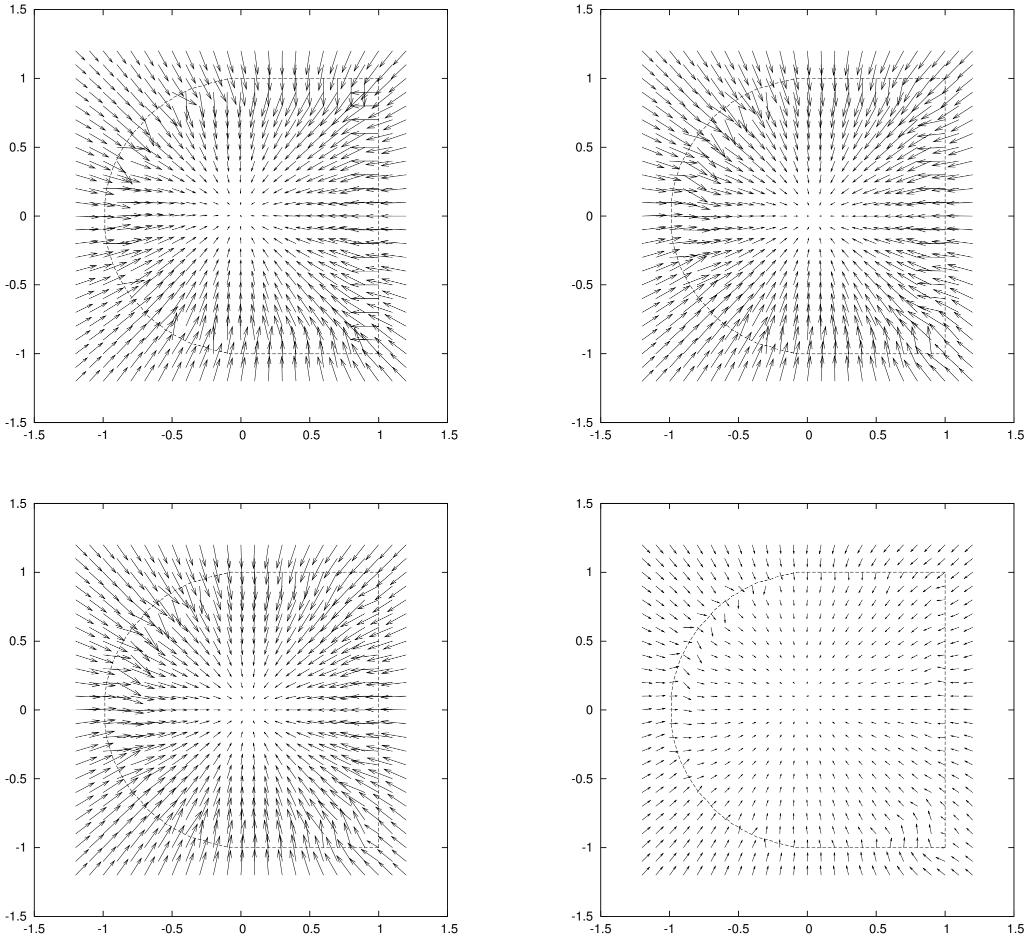

FIG. 6. The field z at time $t = 0.05$, 0.25, 0.5 and 0.515 for the evolution of $\chi_C(x)$.

Figure 5 shows the function u (plotted as the grey-level function of an image) at three successive values of t, while Fig. 6 shows the vector field z at the same times, and after disappearance of the convex C.

A second experiment was performed with the convex set K defined as the union of the disc of center zero and radius 1, and the triangle of vertices $(0, -1)$, $(2/\sqrt{5}, 1/\sqrt{5})$, $((1 + \sqrt{5})/2, -1)$. The area of K is now $1/\tan(\alpha/2) + (\pi + \alpha)/2$, where $\alpha = \arctan 2$, and since again one has $|K_r| = (1 - r)^2|K|$, this gives a value of R of approximately

$$R = 0.5218609.$$

Again, the computed value, of about 0.519, is very close to the theoretical value (see Fig. 7). Figures 8 and 9 show respectively the function u and the field z at times $t = 0.05$ and $t = 0.25$.

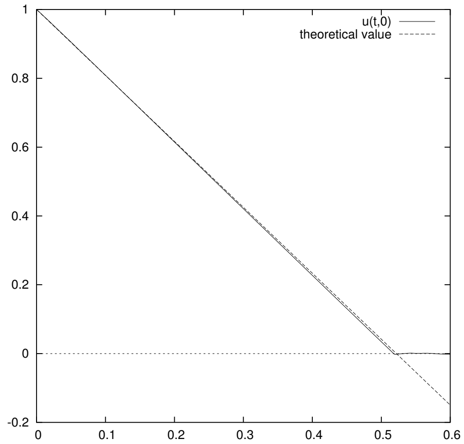

FIG. 7. The values $u(t, 0)$ when $u(0, x) = \chi_K(x)$.

FIG. 8. The value of u at time $t = 0.05$ and $t = 0.25$ for the convex set K which is the union of a disc and a triangle (as in the text.

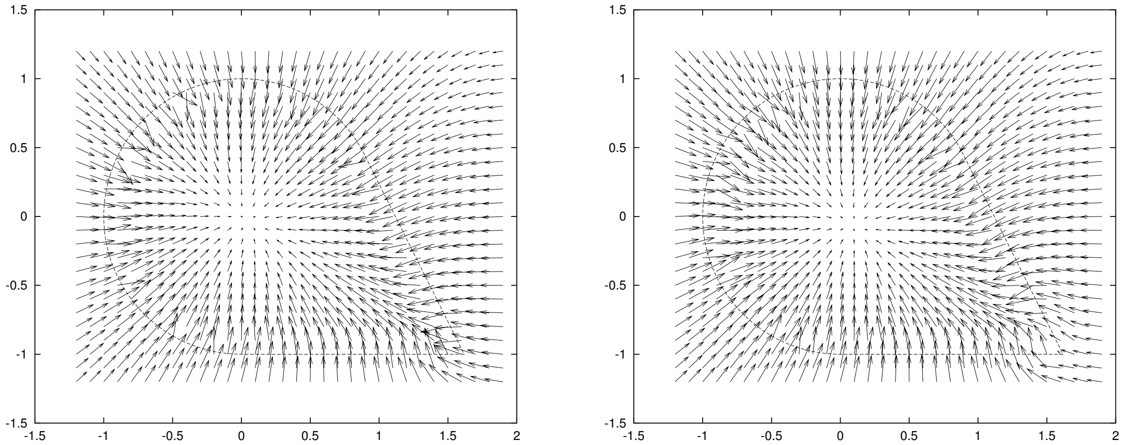

FIG. 9. The field z at time $t = 0.05$ and $t = 0.25$ for the evolution of the convex set K.

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