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Critical size of crystals in the plane

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We study a modified Stefan problem (and its quasi-steady approximation) for crystalline motion in the plane. We are interested in the behaviour of solution for a symmetric problem, in particular we assume that the Wulff shape W is a regular polygon with N sides. We describe two situations. In the first one we show that ice will be melting. In the second one we examine the properties of $V(t)$ for small t assuming that $V(0) = 0$, where V is the velocity of the interfacial curve.

Keywords: Stefan problem; free boundary; Gibbs–Thompson law; ice ball melting.

1. Introduction

In this paper we study a modified Stefan problem in the plane. This model describes evolution of crystals. We assume that the interfacial curve is a polygon, in particular it is nonsmooth (this assumption is natural from the thermodynamical point of view). We examine the behavior of the velocity V of the interfacial curve. We are mostly interested in the sign of V.

The process of melting and growing is described by thermodynamical laws (see Gurtin's book [1]). The problem we study here comes from the paper by Matias and Gurtin (see [2]). Our crystal $\Omega_1(t)$ is contained in a bounded Ω, and $\Omega_2(t) = \Omega \setminus \Omega_1(t)$ is a fluid. The temperature u is a continuous function across the interface $s = \partial\Omega_1 \cap \partial\Omega_2$. In our case s is a polygon with facets s_i, $s = \bigcup_{i=1}^N s_i$, and we assume that the number of facets is constant. The condition which must be satisfied by the temperature on the polygon is described by the so-called Gibbs–Thompson law, which says that the temperature on the interface is proportional to the curvature of the interface. This yields the equation

$$\int_{s_i(t)} u \,\mathrm{d}l = \Gamma_i - \beta_i L_i(t) V_i(t), \qquad i = 1, \ldots, N.$$

Here Γ_i, β_i are constants and $L_i(t)$ is the length of the i-th facet s_i. The meaning of Γ_i and β_i will be explained in the next section. V_i is the velocity of s_i in the direction of the outer normal ν_i. We will work with the following system:

$$\begin{cases} \Delta u = \varepsilon u_t & \text{in } \bigcup_{0<t<T} (\Omega_1(t) \cup \Omega_2(t)), \\ \int_{s_i(t)} u \,\mathrm{d}l = \Gamma_i - \beta_i L_i(t) V_i(t), & i = 1, \ldots, N, \\ V_i = [\![\nabla u]\!] \nu_i, & i = 1, \ldots, N. \end{cases} \tag{1}$$

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We must add initial and boundary conditions to the above system. We consider only Dirichlet boundary data. In the case $\varepsilon = 0$ we require that

$$s(0) = s_0, \quad u_{|\partial\Omega} = 0 \quad \text{for } t \geqslant 0,$$

and for $\varepsilon > 0$,

$$u(0, x) = u_0(x), \quad s(0) = s_0, \quad u_{|\partial\Omega} = 0 \quad \text{for } t \geqslant 0.$$

We wish to find the time evolution of the temperature u as well as the interface s.

In this paper we will be particularly interested in solutions to the so-called *symmetric problem*. Let us define what it means.

DEFINITION We say that problem (1) is *symmetric* if the following conditions are satisfied:

(i) Ω_1 is a regular polygon with N sides, Ω_2 is a ball, and Ω_1 and Ω_2 have common center,
(ii) $\Gamma_i = \Gamma$, $\beta_i = \beta$, $V_i = V$ for all $i = 1, \dots, N$.

A similar problem was examined by Herrero and Velazquez (see [4]). They considered the melting of an ice ball surrounded by water, and obtained an asymptotic expansion for the radius of the melting ball. Here we have a new situation: we examine a critical configuration of our system; we can say something about the dependence of the velocity (for small t) on the initial temperature, but we do not examine the asymptotic behaviour of V.

We consider only the symmetric problems. We shall look at the velocity of the interface. In Subsection 3.1 we work with system (1) when $\varepsilon = 0$. We recall the definition of weak solution and theorems about existence and uniqueness. The main result tells us that if $\varepsilon = 0$, then the velocity is negative for all time. In Subsection 3.2 we examine the problem in the case $\varepsilon > 0$. Again, we recall the definition of weak solution and theorems about existence and uniqueness. We examine the sign of the velocity for small time assuming that $V(0) = 0$ and that V solves a symmetric problem. This is interesting because the condition $V(0) = 0$ describes the critical configuration of our model. In order to obtain this estimate we have to divide the expression for a weak solution into two pieces: regular and singular, and control each of them.

In the next section we explain some notation and complete the description of our problem.

2. Preliminaries

We assume that sets Ω, $\Omega_1(t)$, $\Omega_2(t)$ are bounded regions in $\mathbb{R}^2$, where $\Omega = \Omega_1(t) \cup s(t) \cup \Omega_2(t)$. We also assume that $\Omega_1(t) \subset\subset \Omega$ and the boundary $\partial\Omega$ is smooth. The interface $s(t) = \partial\Omega_1(t) \cap \partial\Omega_2(t)$ is a polygon with facets s_i, and $s(t) = \bigcup_{i=1}^{N} s_i(t)$. The length of s_i is $L_i = |w_i - w_{i+1}|$, where w_i and w_{i+1} are the vertices of s_i. The length of $s(t)$ is $L = \sum_{i=1}^{N} L_i(t)$. We shall consider only admissible polygonal interfaces, which means that the outer normals ν_i to the facets s_i belong to the set of normals of a given Wulff shape W. Moreover, we require that the normals to successive facets must be neighbouring normals to W (for our purposes we can think of W as a given convex polygon; see [1, Sections 7 and 12]).

Let $V_i(t)$ denote the velocity of $s_i(t)$ in the direction of the outer normal ν_i. We can write

$$V_i(t) = \frac{\mathrm{d}}{\mathrm{d}t} z_i(t),$$

where

$$z_i(t) = \begin{cases} \mathrm{dist}(l_i(t), l_i(0)) & \text{if } (w_i(t) - w_i(0)) \cdot \nu_i > 0, \\ -\,\mathrm{dist}(l_i(t), l_i(0)) & \text{if } (w_i(t) - w_i(0)) \cdot \nu_i < 0, \end{cases} \tag{2}$$

and $l_i(t)$ is the line containing $s_i(t)$. In fact z_i is a signed distance between the lines containing $s_i(0)$ and $s_i(t)$.

The symbol $[\![*]\!]$ used in our equations denotes the jump across $s(t)$. Its definition is as follows:

$$[\![\phi]\!](x) = \lim_{\Omega_2(t)\ni y\to x} \phi(y) - \lim_{\Omega_1(t)\ni y\to x} \phi(x), \quad \text{where } y \in s(t).$$

The Γ_i, $i = 1, \ldots, N$, are constants defined by

$$\Gamma_i = -l_i$$

(in this paper $\Gamma_i = -l$), where l_i denotes the length of the facet of ∂W that has normal ν_i (see [1, Section 12.5]). We note that it is possible to interpret Γ_i / L_i as the crystalline curvature of s_i. The kinetic coefficients β_i are constant and positive.

3. Main results

3.1 *Quasi-steady approximation*

In this case our equation becomes the Laplace equation with a free boundary. First of all we show that if the initial data are symmetric, then this property is preserved at later times. The main result is that for any symmetric problem the normal velocity of the facets V_i for $i = 1, \ldots, N$ is always negative.

A weak solution of (1) in the case $\varepsilon = 0$ was defined in [5]. Namely, we say that $u \in C([0, T], H_0^1(\Omega))$ is a *weak solution* to problem (1) if the following conditions are satisfied:

$$\begin{cases} \displaystyle\int_\Omega \nabla\varphi\nabla u \,\mathrm{d}\Omega - \sum_{i=1}^{N} \int_{s_i} V_i \varphi \,\mathrm{d}l = 0 \quad \forall \varphi \in H_0^1(\Omega), \\ \displaystyle\int_{s_i(t)} u = \Gamma_i - \beta_i L_i(t) V_i(t), \quad i = 1, \ldots, N. \end{cases} \tag{3}$$

Let us recall some existence and uniqueness results for our problem (see [5, Corollary 5]). There exists a unique weak solution to problem (1) for $\varepsilon = 0$, which satisfies

$$u \in C^{1/2}([0, T_{\max}), H_0^1(\Omega)), \qquad V_i \in C^{0,1}([0, T_{\max})).$$

Now we have to show the symmetry for solution of our problem.

PROPOSITION 1 Assume that $\varepsilon = 0$ and the system is symmetric at $t = 0$. Then:

(a) the system is symmetric for all $t \in [0, T_{\max})$,
(b) $V_i(t) < 0$ for all $i = 1, \ldots, N$.

Proof. (a) Recall that the Laplace equation is invariant under rotations, that is, if $\Delta u(x) = 0$ then $\Delta u(Lx) = 0$, where $L \in \mathrm{SO}(n, \mathbb{R})$.

Now suppose that our claim is false, so there exist i, j and t such that $V_i(t) \neq V_j(t)$. We can assume that $j = i + 1$. Now we rotate our polygon by $2\pi/N$. We get the same Laplace equation with the same initial data. Uniqueness of solution (see [5, Corollary 5]) implies that $V_j(t) = V_i(t)$.

(b) Taking u as a test function in the definition of a weak solution, we obtain

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}\Omega = \sum_{i=1}^{N} \int_{s_i} V_i u \,\mathrm{d}l.$$

Here $V_i = V$, so

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}\Omega = V \int_s u \,\mathrm{d}l,$$

where $s = \bigcup_{i=1}^{N} s_i$. But

$$\int_s u \,\mathrm{d}l = N\Gamma - \beta L V,$$

where $L = \sum_{i=1}^{N} L_i$, and we get

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}\Omega = VN\Gamma - V^2 \beta L.$$

Viewing this as a quadratic equation in V, it is easy to see that $V < 0$; for example, $V_1 V_2 > 0$ and $V_1 + V_2 < 0$. □

3.2 *Full system*

In this subsection we examine the solution to a symmetric problem for small t assuming that $V(0) = 0$. This situation is interesting because if $V(0) < 0$ then $V(t) < 0$ for small t and similarly if $V(0) > 0$ then $V(t) > 0$ for small t (V is continuous).

We first recall some facts about the weak formulation for (1). A weak solution of (1) on $[0, T)$ was defined in [6] as a pair (z, u), where $z \in C^1([0, T), \mathbb{R}^N)$, $z(0) = 0$, $u \in C^\alpha([0, T), H_0^1(\Omega))$, $\alpha \in (0, 1/2)$, with $u(0) = u_0$, $u_t \in L^\infty_{\mathrm{loc}}([0, T), H^{-1}(\Omega))$, and the following conditions hold:

$$\varepsilon\langle u_t, \varphi\rangle = -\int_\Omega \nabla u \nabla\varphi \,\mathrm{d}\Omega + \sum_{i=1}^{N} \int_{s_i} V_i \varphi \,\mathrm{d}l, \qquad \forall \varphi \in H_0^1(\Omega), \tag{4}$$

$$\int_{s_i(t)} u \,\mathrm{d}l = \Gamma_i - \beta_i L_i(t) V_i(t), \qquad i = 1, \ldots, N. \tag{5}$$

Here $\langle\cdot,\cdot\rangle$ is the pairing between H^{-1} and H_0^1. Define elements $f_i \in H_0^1(\Omega)$, $i = 1, \ldots, N$, as follows:

$$f_i = -\Delta^{-1}\delta_{s_i}.$$

It was shown in [6, Corollary 3.2] that if $\varepsilon > 0$, $u_0 \in H_0^1(\Omega)$ and

$$u_0 - \sum_{j=1}^{N} V_j(0) f_j(0) \in H^2(\Omega) \cap H_0^1(\Omega),$$

then there exists exactly one weak solution to (1) on the interval $[0, T_{\max})$ and

$$u \in C^\alpha([0, T_{\max}), H_0^1), \qquad z_i \in C^{1,\alpha}([0, T_{\max}))$$

for any $\alpha < 1/2$.

We shall use the following expression for the weak solution (see [7]):

$$u(t) = e^{(t/\varepsilon)\Delta}u_0 - \frac{1}{\varepsilon}\int_0^t \Delta e^{((t-\tau)/\varepsilon)\Delta}\sum_{j=1}^{N} V_j(\tau) f_j(\tau)\,\mathrm{d}\tau. \tag{6}$$

Using the same argument as in Proposition 1(a), based on invariance of Δ under rotations, we can show that if (1) is a symmetric problem, $\varepsilon \geqslant 0$ and u_0 is invariant under rotation by $2\pi/N$, then $V_i(t) = V(t)$ for all $i = 1, \dots, N$.

We now state a fact which will be useful in the proof of our main theorem.

LEMMA 1 Let $\gamma : [0, 1] \times [\alpha, \beta] \to \mathbb{R}^2$ be a parameterization of a union of curves $s(t)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)} F(x, t)\,\mathrm{d}l(x) = \int_{s(t)} \frac{\partial F}{\partial t}\,\mathrm{d}l(x) + F(\gamma(t, \tau), t)\dot{\gamma}(t, \tau)|_{\alpha}^{\beta},$$

where the dot denotes the derivative with respect to t.

Proof. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{s(t)} F(x, t)\,\mathrm{d}l(x)\right) &= \int_{\alpha}^{\beta} \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t, \tau), t)\frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(t, \tau)\,\mathrm{d}\tau + \int_{\alpha}^{\beta} F(\gamma(t, \tau), t)\frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t, \tau)\,\mathrm{d}\tau \\
&= \int_{s(t)} \frac{\partial F}{\partial t}\,\mathrm{d}l(x) + \int_{s(t)} v\frac{\partial F}{\partial x}(x, t)\,\mathrm{d}l(x) + F(\gamma(t, \tau), t)\dot{\gamma}(t, \tau)|_{\alpha}^{\beta} \\
&\quad - \int_{\alpha}^{\beta} \frac{\partial}{\partial x} F(\gamma(t, \tau), t)\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\dot{\gamma}(t, \tau)\,\mathrm{d}\tau \\
&= \int_{s(t)} \frac{\partial F}{\partial t}\,\mathrm{d}l(x) + F(\gamma(t, \tau), t)\dot{\gamma}(t, \tau)|_{\alpha}^{\beta}. \qquad \square
\end{aligned}$$

Now we can formulate our theorem.

THEOREM 1 Assume that $V_0 = 0$, u_0 is invariant under rotation by $2\pi/N$ and $\int_{s(0)} \Delta u_0\,\mathrm{d}l > 0$ (respectively $\int_{s(0)} \Delta u_0\,\mathrm{d}l < 0$). Then there exists $\eta > 0$ such that $V(t) < 0$ for all $t \in (0, \eta)$ (respectively $V(t) > 0$ for all $t \in (0, \eta)$).

REMARK We assume high regularity on initial data, $u_0 \in H^3$, in order to compute the trace of Δu_0 on $s(0)$.

Proof. We give a proof only in the case when $\int_{s(0)} \Delta u_0\,\mathrm{d}l > 0$. The other case is similar. Suppose that there exists $\delta > 0$ such that $V(t) \geqslant 0$ for all $t \in (0, \delta)$. We know that

$$V(t) = \frac{1}{\beta L}\left(N\Gamma - \int_{s(t)} u\,\mathrm{d}l\right).$$

Because

$$\int_{s(0)} u_0\,\mathrm{d}l = N\Gamma$$

we can rewrite our expression for V in the following way:

$$V(t) = \frac{1}{L\beta}\left(\int_{s(0)} u_0\,\mathrm{d}l - \int_{s(t)} u\,\mathrm{d}l\right).$$

We shall look at the upper limit $\overline{\lim}_{t\to 0^+} V(t)\beta L(t)/t$. By formula (6),

$$
\begin{aligned}
\overline{\lim_{t\to 0^+}} \frac{V(t)\beta L(t)}{t}
&= \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(\int_{s(0)} u_0\,\mathrm{d}l - \int_{s(t)}\left(e^{(t/\varepsilon)\Delta}u_0 - \frac{1}{\varepsilon}\int_0^t \Delta e^{((t-\tau)/\varepsilon)\Delta}\sum_{j=1}^N V_j(\tau) f_j(\tau)\,\mathrm{d}\tau\right)\right)\\
&\leqslant \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(\int_{s(0)} u_0\,\mathrm{d}l - \int_{s(t)} e^{(t/\varepsilon)\Delta}u_0\,\mathrm{d}l\right)\\
&\quad + \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(\frac{1}{\varepsilon}\int_0^t\int_{s(t)} \Delta e^{((t-\tau)/\varepsilon)\Delta}\sum_{j=1}^N V_j(\tau) f_j(\tau)\,\mathrm{d}\tau\right)\\
&= I_1 + I_2.
\end{aligned}
$$

We begin by estimating I_1:

$$
I_1 \leqslant \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(\int_{s(0)} u_0\,\mathrm{d}l - \int_{s(0)} e^{(t/\varepsilon)\Delta}u_0\,\mathrm{d}l\right) + \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(\int_{s(0)} e^{(t/\varepsilon)\Delta}u_0\,\mathrm{d}l - \int_{s(t)} e^{(t/\varepsilon)\Delta}u_0\,\mathrm{d}l\right).
$$

From the previous lemma we have

$$
I_1 \leqslant \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(-\int_{s(0)} (e^{(t/\varepsilon)\Delta}u_0 - u_0)\,\mathrm{d}l\right).
$$

It is essential to assume that u_0 is in H^3. This guarantes that the trace of Δu_0 is well defined on s_0, and the above expression makes sense. From the Fatou lemma and the fact that $e^{t\Delta}$ is the semigroup generated by the Laplacian (see [3]) we have

$$
I_1 \leqslant \int_{s(0)} -\Delta u_0\,\mathrm{d}l,
$$

which implies that $I_1 < 0$.

Turning to

$$
I_2 = \overline{\lim_{t\to 0^+}} \frac{1}{t}\left(\frac{1}{\varepsilon}\int_0^t\int_{s(t)} \Delta e^{((t-\tau)/\varepsilon)\Delta}\sum_{j=1}^N V_j(\tau) f_j(\tau)\,\mathrm{d}\tau\right),
$$

from the fact that $-\Delta e^{t\Delta} f_i \geqslant 0$ (see [6, Lemma 4.5]) and the assumption that $V(t) \leqslant 0$ we get $I_2 \leqslant 0$. We conclude that

$$
I_1 + I_2 < 0.
$$

This ends the proof, because it implies that $V(t) < 0$ for all $t \in (0, \eta)$ and sufficiently small η. □

The previous theorem does not cover the situation when $V_0 = 0$ and $\int_{s(0)} \Delta u_0\,\mathrm{d}l = 0$. In this case we do not know what will happen with V for small t.

It is not difficult to show that if

$$
\lim_{t\to 0^+} \frac{1}{t}\left(\frac{1}{\varepsilon}\int_0^t\int_{s(t)} \Delta e^{((t-\tau)/\varepsilon)\Delta}\sum_{j=1}^N V_j(\tau) f_j(\tau)\,\mathrm{d}\tau\right) = 0
$$

then $\int_{s(0)} \Delta u_0\,\mathrm{d}l = 0$ is equivalent to the condition that $\frac{dv}{dt}|_{t=0}$ exists and is equal to zero.

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