

A system of degenerate parabolic nonlinear PDE's: a new free boundary problem

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We prove existence of solutions of a new free boundary problem described by a system of degenerate parabolic equations. The problem arises in petroleum engineering and concerns fluid flows in diatomite rocks. The unknown functions represent the pressure of the fluid and a damage parameter of the porous rock. These quantities are not necessarily continuous on the free boundary, which considerably complicates the mathematical analysis.

Keywords: Nonlinear parabolic system; degenerate parabolic PDE's; nonlocal damage mechanics; oil engineering.

1. Introduction

In this paper we study the existence of compactly supported solutions of the system

$$\begin{cases} \omega_t = (\omega^\mu (p - I)_+^\beta \omega_x)_x + A(1 - \omega)(p - I)_+^\gamma & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ p_t = (\omega^\alpha p_x)_x & \text{in } \mathbb{R} \times \mathbb{R}^+, \end{cases} \quad (1)$$

with initial data

$$\omega(x, 0) = \omega_0(x), \quad p(x, 0) = p_0(x) \quad \text{in } \mathbb{R}. \quad (2)$$

Here ω_0 and p_0 are given nonnegative and bounded functions which vanish outside an interval $[a, b]$, and $\alpha, \beta, \gamma, \mu, A$ and I are constants satisfying

$$\alpha, \beta, \gamma > 0, \quad \mu, A, I \geq 0. \quad (3)$$

Physically the problem is motivated by a mathematical model for fluid flow in porous rocks with very low permeability in the pristine state, such as diatomite rocks, introduced by Barenblatt,

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Patzek, Prostokishin and Silin [3] and further developed in [2] and [12]. In [3] the authors stress the importance of the presence of microfractures caused by the fluid flow itself, and try to capture in their model the presence of interfaces, sharp boundaries between regions where ω vanishes (undamaged rocks) and regions where $\omega > 0$ (damaged rocks). So ω is a damage parameter and p is the total pressure of water and oil.

The constant I is related to the strength of the rocks: if p is below this value then no damage can accumulate or diffuse in the rocks. The exact value of I can only be determined by experiments, and in principle we can only say that it is nonnegative, otherwise during the geologic time microcracks would have appeared in the rock. From the mathematical point of view $I = 0$ and $I > 0$ give different results (see also [5]).

In system (1) we have not taken into account the physical constraint of no damage healing, which is mathematically expressed by the condition $\omega_t \geq 0$. In the case of similar but simpler models concerning damage accumulation ([4, 11]) it has been shown that this constraint can be treated mathematically, but since the analysis of our system is already quite complicated we have preferred not to consider it in the present paper.

Roughly speaking, we can describe our main result as follows (for its precise statement we refer to Section 2). If $\omega_0 > 0$ and $p_0 > 0$ in (a, b) and if, in the case $I > 0$, $p > 0$ intersects the level I at most twice, then Problem (1)–(2) has a solution, (ω, p) , which satisfies

$$\begin{cases} \omega(x, t) > 0 \text{ and } p(x, t) > 0 & \text{if } a(t) < x < b(t) \\ \omega(x, t) = p(x, t) = 0 & \text{if } x < a(t), x > b(t). \end{cases}$$

Here $a(t)$ and $-b(t)$ are continuous nonincreasing functions, $a(0) = a$ and $b(0) = b$. In addition

$$\omega(x, t)(p(x, t) - I)_+ \rightarrow 0 \quad \text{as } x \searrow a(t) \text{ and } x \nearrow b(t) \quad \text{for a.e. } t > 0, \quad (4)$$

and

$$p > I \text{ in } \{(x, t) : a(t) < x < b(t)\} \quad \text{if } p_0 > I \text{ in } (a, b). \quad (5)$$

The latter result implies that, at least if $I > 0$, p may be discontinuous across the interfaces. In view of property (4) it is natural to ask whether both ω and $(p - I)_+$ are continuous across the interfaces. An extensive analysis of traveling wave solutions of system (1) in [5] suggests that the answer heavily depends on the various parameters. In particular there exist traveling waves for which ω or $(p - I)_+$ have jumps across the interfaces. We observe that if $I > 0$ and ω is discontinuous across the interface, then the term $(p - I)_+^\beta \omega_x$ is not well defined across the interface, being the product of a discontinuous function and a measure. This indicates that the definition of a solution is a delicate issue. For general solutions of system (1)–(2) the problem how to analyze possible discontinuities at the free boundaries is completely open.

We list some other challenging open problems. We are not able to extend our existence result to the case of higher space dimensions:

$$\begin{cases} \omega_t = \operatorname{div}(\omega^\mu (p - I)_+^\beta \nabla \omega) + A(1 - \omega)(p - I)_+^\gamma, \\ p_t = \operatorname{div}(\omega^\alpha \nabla p); \end{cases} \quad (6)$$

in particular the BV-estimates, which play an essential role in our analysis (see Section 3), are no longer available. Also the problem of the uniqueness of solutions is completely open, even in the one-dimensional case; this is hardly surprising, since even for much simpler systems of degenerate parabolic equations there is no systematic theory for the uniqueness of solutions.

Both the list of open problems and the behavior of solutions near the free boundaries suggest that the degeneracies of the equations of system (1) make the problem much more difficult to analyze than “standard” degenerate parabolic equations such as the porous medium equation and the Stefan problem. For example, formally one could easily write the Rankine–Hugoniot conditions which determine the velocity of the free boundaries, but one has to distinguish several cases according to the occurrence of discontinuities. For the time being, there is no hope to give a rigorous proof of the free boundary conditions.

In [2], [3] and [12], it has been explained that the occurrence of a free boundary has been observed in huge oil fields in California (the observation is based on the phenomenon of subsidence). We emphasize that at this stage, validation of the model is impossible due to the lack of experimental data.

In Section 2 we collect the hypotheses on the data, the definition of a solution, and the main theorem. In Section 3 we approximate system (1) with a uniformly parabolic system and we use some a priori estimates to show the existence of a pair of limit functions (ω, p) . In Section 4 we prove the basic support properties of ω and p , in particular the continuity of the functions $a(t)$ and $b(t)$. In Section 5 we prove that (ω, p) is a solution of our problem.

2. The main results

We shall often use concepts such as support, but since ω and p are not always continuous it is useful to introduce the following notation.

NOTATION.

- (i) If $f : \mathbb{R} \rightarrow [0, \infty)$ and $G = \{x \in \mathbb{R} : f(x) > 0\}$, then

$$\text{supp } f := \overline{\left\{x \in G : \lim_{\delta \rightarrow 0} \frac{1}{2\delta} \mu(G \cap (x - \delta, x + \delta)) > 0\right\}}. \tag{7}$$

Equivalently, the support can be defined as the complement of the largest open set where $f = 0$ a.e.

- (ii) If $g : D \subseteq \mathbb{R}^n \rightarrow [0, \infty)$, with $n \geq 1$, then

$$\{g > 0\} := \text{Int} \left\{x \in D : \liminf_{\rho \rightarrow 0} \int_{B_\rho(x) \cap D} g > 0\right\}, \tag{8}$$

where we have used the notation $f_A f := |A|^{-1} \int_A f$, $B_\rho(x)$ being the ball of radius ρ centered at x .

- (iii) We set $Q_T := \mathbb{R} \times (0, T]$ for all $T > 0$.

Throughout the paper we assume that ω_0 and p_0 satisfy the hypotheses

$$\begin{cases} p_0, \omega_0 \in \text{BV}(\mathbb{R}), \\ p_0 \geq 0, \quad 0 \leq \omega_0 \leq 1 \quad \text{in } \mathbb{R}, \end{cases} \tag{9}$$

$$\begin{cases} \omega_0(x) > 0 \text{ and } p_0(x) > 0 & \text{if } a < x < b, \\ \omega_0(x) = p_0(x) = 0 & \text{if } x < a \text{ or } x > b, \end{cases} \tag{10}$$

and

$$\left\{ \begin{array}{l} \text{there exists } \bar{\rho} > 0 \text{ such that for any } \rho \in [0, \bar{\rho}) \\ \text{the sets } \{x : \omega_0(x) > \rho\} \text{ and } \{x : p_0(x) > I \pm \rho\} \\ \text{are intervals (if they are nonempty).} \end{array} \right. \quad (11)$$

Before stating the main result we have to specify what we mean by solution. As we explained in the introduction, it is not obvious how to define in $\mathbb{R} \times \mathbb{R}^+$ the products involving ω_x and p_x . Therefore we restrict the domains of integration in some of the integrals appearing in the definition of solution.

DEFINITION 2.1 A pair (ω, p) that belongs to $X_T^2 := [L^\infty((0, T); \text{BV}(\mathbb{R})) \cap L^\infty([0, T] \times \mathbb{R}) \cap H^1((0, T); H^{-1}(\mathbb{R}))]^2$ for all $T > 0$ is a *solution* of system (1)–(2) if

- (i) $\omega_x \in L^2_{\text{loc}}(\{p > I\})$, $p_x \in L^2_{\text{loc}}(\{\omega > 0\})$,
- (ii) $\omega^\alpha p_x^2 \in L^1(\{\omega > 0\})$ and $\omega^\mu (p - I)^\beta \omega_x^2 \in L^1(\{p > I\})$,
- (iii) for all $T > 0$ and $\phi, \psi \in H^1(\mathbb{R} \times [0, T])$ with compact support (i.e. for some $M > 0$, $\phi = \psi = 0$ for a.e. $|x| > M$) and for all $t \in [0, T]$,

$$\int_{\mathbb{R}} \phi(t)p(t) - \int_{\mathbb{R}} \phi(0)p_0 = \int_0^t \int_{\mathbb{R}} \phi_t p - \int \int_{\{\omega > 0\} \cap Q_t} \phi_x \omega^\alpha p_x$$

and

$$\begin{aligned} & \int_{\mathbb{R}} \psi(t)\omega(t) - \int_{\mathbb{R}} \psi(0)\omega_0 \\ &= \int_0^t \int_{\mathbb{R}} \psi_t \omega - \int \int_{\{p > I\} \cap Q_t} \psi_x \omega^\mu (p - I)^\beta \omega_x + A \int_0^t \int_{\mathbb{R}} \psi (p - I)_+^\gamma (1 - \omega). \end{aligned}$$

REMARK 2.2 Since X_T^2 is embedded in $[C([0, T]; L^2_{\text{loc}}(\mathbb{R}))]^2$ (see proof of Proposition 3.1 for more details), the integral equalities in (iii) are well defined for all $t \in [0, T]$.

Now we are ready to state the main result.

THEOREM 2.3 Let hypotheses (3), (9), (10) and (11) be satisfied. Then Problem (1)–(2) has a solution (ω, p) which has the following properties:

- (i) there exist $a, b \in C([0, \infty))$ such that $a(0) = a$, $b(0) = b$, $a(t)$ is nonincreasing, $b(t)$ is nondecreasing, and

$$\begin{aligned} \omega(x, t) > 0 \text{ and } p(x, t) > 0 & \quad \text{if } a(t) < x < b(t), \\ \omega(x, t) = p(x, t) = 0 & \quad \text{if } x < a(t) \text{ or } x > b(t); \end{aligned}$$

- (ii) for a.e. $t > 0$, $\omega(p - I)_+ \rightarrow 0$ as $x \searrow a(t)$ and $x \nearrow b(t)$;
- (iii) if $p_0 \geq I$ in (a, b) then $p(x, t) \geq I$ if $a(t) < x < b(t)$.

3. A nondegenerate system

We introduce, for $\varepsilon > 0$, the problem

$$\left\{ \begin{array}{ll} p_t = ((\omega^\alpha + \varepsilon)p_x)_x & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ \omega_t = ((\omega^\mu (p - I)_+^\beta + \varepsilon)\omega_x)_x + A(1 - \omega)(p - I)_+^\gamma & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ p(x, 0) = p_0(x), \quad \omega(x, 0) = \omega_0(x) & \text{for } x \in \mathbb{R}. \end{array} \right. \quad (12)$$

A weak solution of (12) is defined in a standard way (requiring that $0 \leq \omega_0 \leq 1$, $p_0 \geq 0$ and $\omega_x, p_x \in L^2_{loc}(\mathbb{R} \times [0, \infty))$).

PROPOSITION 3.1 Let ω_0 and p_0 satisfy

$$\begin{cases} 0 \leq \omega_0 \leq 1, p_0 \geq 0 & \text{in } \mathbb{R}, \\ \omega_0, p_0 \in \text{BV}(\mathbb{R}) \cap L^q(\mathbb{R}) & (1 \leq q \leq \infty). \end{cases} \tag{13}$$

Then, for all $\varepsilon > 0$, problem (12) has a weak solution $(\omega_\varepsilon, p_\varepsilon)$ satisfying

- (i) $0 \leq \omega_\varepsilon \leq 1$ and $0 \leq p_\varepsilon \leq C$ in $\mathbb{R} \times [0, \infty)$;
- (ii) for all $T > 0$ there exists $C_T > 0$ such that

$$\int_0^T \int_{\mathbb{R}} (\omega_\varepsilon^\mu (p_\varepsilon - I)^\beta + \varepsilon) \omega_{\varepsilon x}^2 dx dt + \int_0^T \int_{\mathbb{R}} (\omega_\varepsilon^\alpha + \varepsilon) p_{\varepsilon x}^2 dx dt \leq C_T; \tag{14}$$

- (iii) ω_ε and p_ε are uniformly bounded in

$$L^\infty((0, T); L^q(\mathbb{R})) \cap H^1((0, T); H^{-1}(\mathbb{R})) \cap L^\infty((0, T); \text{BV}(\mathbb{R}))$$

for all $1 \leq q \leq \infty$ and $T > 0$;

- (iv) for all $T > 0$,

$$(\omega_\varepsilon (p_\varepsilon - I)_+)^v$$

is uniformly bounded in $L^2((0, T); H^1(\mathbb{R}))$ if $v \geq \frac{1}{2} \min \{\alpha + 1, \beta + 1, \mu + 2\}$.

Proof. The proof consists of two steps.

Step 1. We assume that problem (12) has a smooth solution $(\omega_\varepsilon, p_\varepsilon)$. Then (i) follows from the comparison principle. Multiplying the equation for p_ε by p_ε^{q-1} ($1 \leq q < \infty$) we obtain a uniform bound of p_ε in $L^\infty((0, \infty); L^q(\mathbb{R}))$. Multiplying the equation for ω_ε by ω_ε^{q-1} and using the bound for p_ε , we find that ω_ε is uniformly bounded in $L^\infty((0, T); L^q(\mathbb{R}))$. In particular, in the case $q = 2$, this procedure yields (14).

Since $((p - I)_+)_x^2 \leq p_{\varepsilon x}^2$, it follows from (14) that $\omega_\varepsilon^\alpha ((p - I)_+)_x^2$ and $\omega_\varepsilon^\mu (p - I)^\beta \omega_{\varepsilon x}^2$ are uniformly bounded in $L^1(\mathbb{R} \times [0, T])$. By Hölder's and Young's inequalities this implies at once that

$$((\omega_\varepsilon (p_\varepsilon - I)_+)^v)_x$$

is uniformly bounded in $L^2(\mathbb{R} \times [0, T])$ if $2v \geq \alpha + 1$, $2v \geq \beta + 1$ and $2v \geq \mu + 2$, and we have proved (iv). It follows from (14) and the equations for ω_ε and p_ε that $\omega_{\varepsilon t}$ and $p_{\varepsilon t}$ are uniformly bounded in $L^2((0, T); H^{-1}(\mathbb{R}))$.

Finally, we prove the uniform boundedness in $L^\infty((0, T); \text{BV}(\mathbb{R}))$. In order to estimate $p_{\varepsilon x}$ and $\omega_{\varepsilon x}$ we adopt a standard technique and introduce a smooth function η_δ whose derivative is an approximation of the sign function:

$$\eta'_\delta(s) = \begin{cases} -1 & \text{if } s < -\delta, \\ 1 & \text{if } s > \delta. \end{cases}$$

We assume that η_δ is even, $\eta''_\delta \geq 0$ and $\eta_\delta \in C^\infty$. Multiplying the equation for $p_{\varepsilon x}$ by $\eta'_\delta(p_{\varepsilon x})$ and integrating yields

$$\frac{d}{dt} \int_{\mathbb{R}} \eta_\delta(p_{\varepsilon x}) dx = \int_{\mathbb{R}} \eta'_\delta(p_{\varepsilon x}) ((\omega_\varepsilon^\alpha + \varepsilon) p_{\varepsilon x})_{xx} dx \leq - \int_{\mathbb{R}} \eta''_\delta(p_{\varepsilon x}) p_{\varepsilon x x} (\omega_\varepsilon^\alpha)_x p_{\varepsilon x} dx.$$

For fixed $\varepsilon > 0$, the latter term vanishes as $\delta \rightarrow 0$ and we see that $\int_{\mathbb{R}} |p_{\varepsilon x}|$ is nonincreasing in time and hence equibounded. The proof for ω_{ε} is similar, and it is enough to observe that the term $A\eta'_{\delta}(\omega_{\varepsilon x})(1 - \omega_{\varepsilon})(1 - p_{\varepsilon})^{\gamma}_{+}$ is uniformly bounded in $L^{\infty}((0, T); L^1(\mathbb{R}))$.

Step 2. For fixed $\varepsilon > 0$ we approximate the nonlinearities ω^{α} , $\omega^{\mu}(p - I)^{\beta}_{+}$ and $(p - I)^{\gamma}_{+}$ by smooth functions of ω and p , and the initial functions ω_0 and p_0 by smooth functions of x . This leads to a family of problems, say problems $P_{\varepsilon, k}$, and following the proof in Step 1 we obtain a priori estimates for problem $P_{\varepsilon, k}$ which are similar to (i)–(ii)–(iii).

From Theorem 1 of [1], it follows that problem $P_{\varepsilon, k}$ admits a unique maximal classical solution in some interval $(0, \tau_{\varepsilon, k})$, which can be continued as long as it remains bounded ([1, Theorem 3]). Hence it exists for all $t > 0$. Since for every compact subset $U \subset \mathbb{R}$,

$$\text{BV}(\mathbb{R}) \cap L^2(\mathbb{R}) \hookrightarrow L^2(U) \hookrightarrow (W^{1,2}(U))'$$

it follows from (ii) and Corollary 4 in [13] that the corresponding solutions $\{(\omega_{\varepsilon, k}, p_{\varepsilon, k})\}$ are relatively compact in $[C([0, T]; L^2_{\text{loc}}(\mathbb{R}))]^2$. Since the spatial derivatives are uniformly bounded in $L^2(\mathbb{R} \times [0, T])$, convergence along a subsequence yields a weak solution, $\{(\omega_{\varepsilon}, p_{\varepsilon})\}$, of (12). This completes the proof of Proposition 3.1. \square

The compactness argument in $[C([0, T]; L^2_{\text{loc}}(\mathbb{R}))]^2$ which we have used before yields a limit function (ω, p) which is a candidate for a solution of (1)–(2):

THEOREM 3.2 Let ω_0 and p_0 satisfy hypothesis (9)–(10) and let $(\omega_{\varepsilon}, p_{\varepsilon})$ be defined by Proposition 3.1. Then there exist a sequence $\varepsilon_n \rightarrow 0$ and functions ω and p such that for all $T > 0$,

$$(\omega_{\varepsilon_n}, p_{\varepsilon_n}) \rightarrow (\omega, p) \quad \text{in } [C([0, T]; L^2_{\text{loc}}(\mathbb{R}))]^2, \tag{15}$$

and

- (i) $0 \leq \omega \leq 1$ and $0 \leq p \leq C$ in $\mathbb{R} \times [0, T]$;
- (ii) $\omega, p \in C([0, T]; L^q(\mathbb{R})) \cap H^1((0, T); H^{-1}(\mathbb{R})) \cap L^{\infty}((0, T); \text{BV}(\mathbb{R}))$, $1 \leq q < \infty$.

4. Support properties

In this section we prove that the functions ω and p defined by Theorem 3.2 satisfy

$$\text{supp } \omega(t) = \text{supp } p(t) = [a(t), b(t)] \quad \text{if } t \geq 0, \tag{16}$$

where

$$a, b \in C([0, \infty)), \quad a(0) = a, \quad b(0) = b, \tag{17}$$

and

$$a \text{ is nonincreasing and } b \text{ nondecreasing in } [0, \infty). \tag{18}$$

In addition

$$\text{if } b(t_2) > b(t_1) \text{ for some } 0 \leq t_1 \leq t_2, \text{ then } \{p > I\} \cap ((b(t_1), b(t_2)) \times (t_1, t_2)) \neq \emptyset. \tag{19}$$

We remind the reader the notation (7) for the support.

The proof of these results consists of several lemmas.

LEMMA 4.1 If $\bar{x} \notin \text{supp } \omega(\bar{t})$ for some $(\bar{x}, \bar{t}) \in \mathbb{R} \times \mathbb{R}^+$, then there exists a neighborhood $U(\bar{x})$ of \bar{x} such that $\omega = 0$ a.e. in $U(\bar{x}) \times [0, \bar{t}]$. The same result holds if we replace ω by p .

Proof. Since $\text{supp } \omega(\bar{t})$ is closed, there exists a neighborhood $U(\bar{x})$ of \bar{x} such that $U(\bar{x}) \cap \text{supp } \omega(\bar{t}) = \emptyset$. Let $(\omega_\varepsilon, p_\varepsilon)$ be the solution of the nondegenerate problem (12). We multiply the equation for ω_ε by $-\varphi^2(x)/(\omega_\varepsilon + \delta)$, where $\delta > 0$ and φ is a Lipschitz continuous function such that $\text{supp } \varphi \subseteq U(\bar{x})$. Integrating by parts shows that for $0 \leq t \leq \bar{t}$,

$$\begin{aligned} & - \int_{\mathbb{R}} \varphi^2 \log(\omega_\varepsilon + \delta)|_{\bar{t}} + A \int_t^{\bar{t}} \int_{\mathbb{R}} \varphi^2 \frac{(1 - \omega_\varepsilon)(p_\varepsilon - I)_+^\gamma}{\omega_\varepsilon + \delta} \\ & = -2 \int_t^{\bar{t}} \int_{\mathbb{R}} \varphi \varphi_x (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) (-\log(\omega_\varepsilon + \delta))_x \\ & \quad - \int_t^{\bar{t}} \int_{\mathbb{R}} \varphi^2 [(-\log(\omega_\varepsilon + \delta))_x]^2 (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \\ & \leq \int_t^{\bar{t}} \int_{\mathbb{R}} \varphi_x^2 (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \leq C(\varphi)(t - \bar{t}), \end{aligned}$$

where we have used Hölder's and Young's inequalities. By (15) we have, for fixed $\delta > 0$,

$$\begin{aligned} & - \int_{\mathbb{R}} \varphi^2 \log(\omega + \delta)(\bar{t}) + A \int_t^{\bar{t}} \int_{\mathbb{R}} \varphi^2 \frac{(1 - \omega)(p - I)_+^\gamma}{\omega + \delta} \\ & \leq - \int_{\mathbb{R}} \varphi^2 \log(\omega + \delta)(t) + C(\varphi)(\bar{t} - t), \end{aligned} \tag{20}$$

and since δ is arbitrary ω and p satisfy

$$- \int_{\mathbb{R}} \varphi^2 \log \omega(\bar{t}) + A \int_t^{\bar{t}} \int_{\mathbb{R}} \varphi^2 \frac{(1 - \omega)(p - I)_+^\gamma}{\omega} \leq - \int_{\mathbb{R}} \varphi^2 \log \omega(t) + C(\varphi)(\bar{t} - t). \tag{21}$$

Since $\omega(\bar{t}) = 0$ in $\text{supp } \varphi$ this implies that

$$- \int_{\mathbb{R}} \varphi(x)^2 \log \omega(x, t) = +\infty \quad \text{for } 0 \leq t < \bar{t}.$$

Hence the set $\{x \in U(\bar{x}) : \omega(x, t) = 0\}$ is dense in $U(\bar{x})$, and since $\omega \in L^\infty((0, \bar{t}); \text{BV}(\mathbb{R}))$ this means that $\omega(x, t) = 0$ for all $0 \leq t < \bar{t}$ and for a.e. $x \in U(\bar{x})$. To complete the proof of Lemma 4.1 it is enough to repeat the same argument for p . \square

LEMMA 4.2 For all $t > 0$,

$$\text{supp } \omega(t) = \text{supp } p(t). \tag{22}$$

In addition

$$\text{supp } \omega(t_1) \subseteq \text{supp } \omega(t_2) \quad \text{if } 0 \leq t_1 < t_2. \tag{23}$$

Proof. Property (23) follows at once from Lemma 4.1. We claim that

$$\text{supp } p(t) \subseteq \text{supp } \omega(t) \quad \text{for } t \geq 0. \tag{24}$$

Taking (\bar{x}, \bar{t}) and $U(\bar{x})$ as in Lemma 4.1, it is enough to prove that

$$p(x, t) = 0 \quad \text{for a.e. } x \in U(\bar{x}), 0 \leq t \leq \bar{t}.$$

Since, by Lemma 4.1, $\omega(x, t) = 0$ for a.e. $x \in U(\bar{x})$ if $0 \leq t \leq \bar{t}$, it follows easily from the equation for p that

$$p(x, t) = p_0(x) \quad \text{for a.e. } x \in U(\bar{x}), 0 \leq t \leq \bar{t}. \tag{25}$$

Indeed, for any smooth and nonnegative function ψ in \mathbb{R} such that $\text{supp } \psi \subseteq U(\bar{x})$, we see, by (14) and Hölder's inequality, that for all $0 < t \leq \bar{t}$,

$$\begin{aligned} \left| \int_{\mathbb{R}} p_\varepsilon(t) \psi - \int_{\mathbb{R}} p_{0\varepsilon} \psi \right| &= \left| \int_0^t \int_{\mathbb{R}} \psi_x (\omega_\varepsilon^\alpha + \varepsilon) p_{\varepsilon x} \right| \\ &\leq C \left(\int_0^t \int_{\mathbb{R}} \psi_x^2 (\omega_\varepsilon^\alpha + \varepsilon) \right)^{1/2} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

and (25) follows. Since $U(\bar{x}) \cap \text{supp } p_0 = U(\bar{x}) \cap \text{supp } \omega_0 = \emptyset$, it follows from (24) that $p(x, t) = 0$ for a.e. $x \in U(\bar{x})$ if $0 \leq t \leq \bar{t}$. This proves (24).

Exchanging the roles of ω and p we find that $\text{supp } \omega(t) \subseteq \text{supp } p(t)$, and the proof is complete. \square

LEMMA 4.3 For all $t > 0$ the set $\text{supp } \omega(t)$ is connected.

Proof. Arguing by contradiction we assume that there exist $x_0 \in \mathbb{R}$ and $t_0 > 0$ such that

$$x_0 \notin \text{supp } \omega(t_0), \quad (-\infty, x_0) \cap \text{supp } \omega(t_0) \neq \emptyset, \quad (x_0, \infty) \cap \text{supp } \omega(t_0) \neq \emptyset.$$

By (24), $x_0 \notin \text{supp } \omega_0 = [a, b]$. Without loss of generality we may assume that $x_0 > b$. Since the supports of ω and p coincide, we obtain a contradiction if we prove that

$$p(x, t_0) = 0 \quad \text{for a.e. } x > x_0. \tag{26}$$

In view of Lemmas 4.1 and 4.2 there exists a neighborhood $(x_0 - \delta, x_0 + \delta)$ of x_0 such that

$$\omega = p = 0 \quad \text{a.e. in } (x_0 - \delta, x_0 + \delta) \times (0, t_0). \tag{27}$$

Let $\phi \in C^\infty(\mathbb{R})$ be a nonnegative function such that

$$\phi = 0 \quad \text{if } x < x_0 - \delta \quad \text{and} \quad \phi(x) = 1 \quad \text{if } x \geq x_0.$$

Then

$$\begin{aligned} \int_{x_0}^\infty p_\varepsilon(x, t_0) &\leq \int_{x_0-\delta}^\infty \phi(x) p_\varepsilon(x, t_0) \\ &= \int_{x_0-\delta}^\infty \phi(x) p_{0\varepsilon}(x) - \int_0^{t_0} \int_{x_0-\delta}^{x_0} \phi'(\omega_\varepsilon^\alpha + \varepsilon) p_{\varepsilon x}. \end{aligned} \tag{28}$$

Since $p_0 = 0$ in $(x_0 - \delta, \infty)$, the first term on the right hand side vanishes as $\delta \rightarrow 0$. The second term is bounded by

$$\left(\int_0^t \int_{x_0-\delta}^{x_0} (\omega_\varepsilon^\alpha + \varepsilon) p_{\varepsilon x}^2 \right)^{1/2} \left(\int_0^t \int_{x_0-\delta}^{x_0} (\omega_\varepsilon^\alpha + \varepsilon) (\phi'(x))^2 \right)^{1/2},$$

which, in view of (14) and (27), vanishes as $\varepsilon \rightarrow 0$. Hence (26) follows from (28). \square

Combined with the following theorem, Lemmas 4.1–4.3 imply (17)–(18).

THEOREM 4.4 Let (ω, p) be a weak solution of Problem (1)–(2) constructed in Theorem 2.3. There exists a nondecreasing function $r \in C([0, \infty); [0, \infty))$ with $r(0) = 0$ such that if $\text{supp } \omega_0 \subseteq B(x_0, r_0)$ then $\text{supp } \omega(t) \subseteq B(x_0, r_0 + r(t))$.

In other words, Theorem 4.4 states the property of *finite speed of propagation* for solutions of Problem (1)–(2). The proof of this theorem is postponed to the Appendix.

Finally, we prove property (19). Arguing by contradiction, we assume

$$\{p > I\} \cap ((b(t_1), b(t_2)) \times (t_1, t_2)) = \emptyset.$$

Testing the equation for ω_ε with a smooth, nonnegative function $\varphi : (b(t_1), b(t_2)) \rightarrow \mathbb{R}$ with compact support yields

$$\begin{aligned} \int_{\mathbb{R}} \varphi \omega_\varepsilon(t) - \int_{\mathbb{R}} \varphi \omega_\varepsilon(t_1) &= - \int_{t_1}^t \int_{\mathbb{R}} \varphi_x (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \omega_{\varepsilon x} + A \int_{t_1}^t \int_{\mathbb{R}} \varphi (p_\varepsilon - I)_+^\gamma (1 - \omega_\varepsilon) \\ &\leq C \left(\int_{t_1}^{t_2} \int_{b(t_1)}^{b(t_2)} (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon)^{1/2} + \int_{t_1}^{t_2} \int_{b(t_1)}^{b(t_2)} (p_\varepsilon - I)_+^\gamma \right) \end{aligned}$$

for all $t_1 < t \leq t_2$. Here we have used (14). If we let $\varepsilon \rightarrow 0$, the right hand side vanishes, and we obtain

$$\int_{\mathbb{R}} \varphi \omega_\varepsilon(t) = \int_{\mathbb{R}} \varphi \omega_\varepsilon(t_1) = 0 \quad \text{for all } t \in (t_1, t_2].$$

Since φ is arbitrary, we get

$$\int_{b(t_1)}^{b(t_2)} \omega_\varepsilon(t) = 0 \quad \text{for all } t \in (t_1, t_2],$$

which implies $b(t_1) = b(t_2)$, and we have found a contradiction.

5. Proof of the main result

In this section we show that the pair (ω, p) , defined by Theorem 3.2, is a solution of problem (1)–(2) and has all properties listed in Theorem 2.3. We shall need condition (11) on the initial functions, and without loss of generality we may assume that for all $\varepsilon > 0$,

$$\omega_{0\varepsilon} \text{ and } p_{0\varepsilon} \text{ satisfy condition (11).} \tag{29}$$

It follows from standard lap-number theory ([10]) that for all $t > 0$ and $0 \leq \rho < \bar{\rho}$,

$$\{x : p_\varepsilon(x, t) > I \pm \rho\} \text{ is an interval,} \tag{30}$$

and hence, by (15), we find that $\{x : p(t) > I \pm \rho\}$ is an interval.

This section is organized as follows. First we prove Theorem 2.3, except property (iii), assuming that the following result holds:

PROPOSITION 5.1 Let (ω, p) and the function $a(t)$ and $b(t)$ be defined by, respectively, Theorem 3.2 and (16)–(18). Then for any compact set

$$K \subset \{(x, t) : a(t) < x < b(t), t \geq 0\}$$

there exists $\delta_K > 0$ such that

$$\omega_\varepsilon \geq \delta_K \quad \text{in } K \text{ for all } 0 \leq \varepsilon \leq \varepsilon_K. \tag{31}$$

In particular it follows that $\{\omega > 0\} = \{(x, t) : a(t) < x < b(t), t \geq 0\}$.

Proof of Theorem 2.3. In view of Theorem 3.2, (ω, p) is a solution of problem (1)–(2) if we show that it has properties (ii) and (iii) of Definition 2.1. By (31) and standard results on bounded weak solutions of uniformly parabolic equations ([9]), the functions $\{p_\varepsilon, \varepsilon > 0\}$ are uniformly Hölder continuous in any compact set $K \subseteq \{\omega > 0\} \cap (\mathbb{R} \times (0, T))$. Hence, the functions $\{\omega_\varepsilon, \varepsilon > 0\}$ are uniformly Hölder continuous in compact subsets of $\{p > I\} \cap (\mathbb{R} \times (0, T))$ (here we have used the fact that the supports of ω and p coincide). In particular

$$\omega_{\varepsilon x} \rightarrow \omega_x \quad \text{weakly in } L^2_{\text{loc}}(\{p > I\}), \quad p_{\varepsilon x} \rightarrow p_x \quad \text{weakly in } L^2_{\text{loc}}(\{\omega > 0\}). \quad (32)$$

Combining (14), (15) and (32) we obtain, for all $T > 0$,

$$\iint_{\{p > I\} \cap Q_T} \omega^\mu (p - I)^\beta \omega_x^2 + \iint_{\{\omega > 0\} \cap Q_T} \omega^\alpha p_x^2 < C_T,$$

where the constant C_T depends only on T , which implies (ii) of Definition 2.1.

To prove (iii), we observe that for every φ smooth enough, with compact support in $\mathbb{R} \times [0, T]$,

$$\int_{\mathbb{R}} \varphi(t) p_\varepsilon(t) - \int_{\mathbb{R}} \varphi(0) p_0 - \int_0^t \int_{\mathbb{R}} \varphi_t p_\varepsilon = - \int_0^t \int_{\mathbb{R}} \varphi_x (\omega_\varepsilon^\alpha + \varepsilon) p_{\varepsilon x}.$$

By (15) the terms on the left hand side converge to the corresponding ones with p_ε replaced by p . On the right hand side the term $\int_0^t \int_{\mathbb{R}} \varepsilon p_{\varepsilon x} \varphi_x$ vanishes as $\varepsilon \rightarrow 0$. The term $\int_0^t \int_{\mathbb{R}} \varphi_x \omega_\varepsilon^\alpha p_{\varepsilon x}$ can be rewritten as

$$\int_0^t \int_{\mathbb{R}} ((1 - \psi_n) + \psi_n) \varphi_x \omega_\varepsilon^\alpha p_{\varepsilon x}$$

where ψ_n is a smooth function such that $\text{supp } \psi_n \subseteq \{\omega > 0\}$ and $\psi_n \rightarrow \chi_{\{\omega > 0\}}$ a.e. as $n \rightarrow \infty$. Therefore

$$\int_0^t \int_{\mathbb{R}} \psi_n \varphi_x \omega_\varepsilon^\alpha p_{\varepsilon x} \xrightarrow{(\varepsilon \rightarrow 0)} \int_0^t \int_{\mathbb{R}} \varphi_x \psi_n \omega^\alpha p_x \xrightarrow{(n \rightarrow \infty)} \iint_{\{\omega > 0\} \cap Q_t} \varphi_x \omega^\alpha p_x,$$

and, by (14),

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} (1 - \psi_n) \varphi_x \omega_\varepsilon^\alpha p_{\varepsilon x} \right| &\leq C \left(\int_0^t \int_{\mathbb{R}} \omega_\varepsilon^\alpha (1 - \psi_n)^2 \right)^{1/2} \\ &\xrightarrow{(\varepsilon \rightarrow 0)} C \left(\int_0^t \int_{\mathbb{R}} \omega^\alpha (1 - \psi_n)^2 \right)^{1/2} \xrightarrow{(n \rightarrow \infty)} 0. \end{aligned} \quad (33)$$

This completes the proof of the first integral identity of (iii).

The equation for ω_ε yields

$$\begin{aligned} \int_{\mathbb{R}} \varphi(t) \omega_\varepsilon(t) - \int_{\mathbb{R}} \varphi(0) \omega_\varepsilon(0) - \int_0^t \int_{\mathbb{R}} \varphi_t \omega_\varepsilon - A \int_0^t \int_{\mathbb{R}} \varphi (p_\varepsilon - I)_+^\gamma (1 - \omega_\varepsilon) \\ = - \int_0^t \int_{\mathbb{R}} \varphi_x (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \omega_{\varepsilon x}. \end{aligned}$$

Arguing as before, letting $\varepsilon \rightarrow 0$, we can replace ω_ε and p_ε with ω and p on the left hand side. The term on the right hand side can be rewritten as

$$\int_0^t \int_{\mathbb{R}} ((1 - \psi_n) + \psi_n) \varphi_x (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \omega_{\varepsilon x},$$

where ψ_n is chosen as before. Again the terms containing the factor $1 - \psi_n$ or ε vanish as $\varepsilon \rightarrow 0$ and as $n \rightarrow \infty$. We consider the remaining term

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \psi_n \varphi_x \omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta \omega_{\varepsilon x} &= I_1 + I_2 \\ &:= \iint_{\{p \geq I + \delta\} \cap Q_t} \psi_n \varphi_x \omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta \omega_{\varepsilon x} + \iint_{\{p < I + \delta\} \cap Q_t} \psi_n \varphi_x \omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta \omega_{\varepsilon x}. \end{aligned} \quad (34)$$

We observe that

$$\begin{aligned} |I_2| &\leq \iint_{\{p < I + \delta\} \cap Q_t} |\psi_n \varphi_x \omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta \omega_{\varepsilon x}| \leq C \left(\iint_{\{p < I + \delta\} \cap Q_t} \psi_n^2 \varphi_x^2 \omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta \right)^{1/2} \\ &\leq K \delta^{\beta/2} \xrightarrow{(\delta \rightarrow 0)} 0, \end{aligned} \quad (35)$$

and

$$\begin{aligned} I_1 &= \iint_{\{p \geq I + \delta\} \cap Q_t} \psi_n \varphi_x \omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta \omega_{\varepsilon x} \xrightarrow{(\varepsilon \rightarrow 0)} \iint_{\{p \geq I + \delta\} \cap Q_t} \psi_n \varphi_x \omega^\mu (p - I)_+^\beta \omega_x \\ &\xrightarrow{(n \rightarrow \infty)} \iint_{\{p \geq I + \delta\} \cap Q_t} \varphi_x \omega^\mu (p - I)_+^\beta \omega_x \xrightarrow{(\delta \rightarrow 0)} \iint_{\{p > I\} \cap Q_t} \varphi_x \omega^\mu (p - I)_+^\beta \omega_x. \end{aligned}$$

Hence (ω, p) is a solution of Problem (1)–(2).

Part (i) of Theorem 2.3 follows at once from (16)–(18). Part (ii) is a consequence of Proposition 3.1(iv):

$$(\omega_\varepsilon (p_\varepsilon - I)_+)^v \in L^2((0, T); H^1(\mathbb{R})) \subseteq L^2((0, T); C(\mathbb{R})),$$

i.e., the function $x \mapsto \omega(x, t)(p(x, t) - I)_+$ is continuous for a.e. $t > 0$. Finally, part (iii) follows from the following result, which we prove at the end of this section.

PROPOSITION 5.2 If $p_0 \geq I$, respectively $p_0 > I$, a.e. in (a, b) , then for all $t > 0$, $p(t) \geq I$, respectively $p(t) > I$, a.e. in $(a(t), b(t))$.

Proof of Proposition 5.1. Let $\underline{\omega}_\varepsilon$ be the solution of

$$\begin{cases} z_t = ((\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) z_x)_x & \text{in } (a, b) \times \mathbb{R}^+, \\ z(x, 0) = \omega_0(x) & \text{for } x \in (a, b), \\ z(a, t) = z(b, t) = 0 & \text{for } t > 0. \end{cases} \quad (36)$$

Then, by the comparison principle,

$$\omega_\varepsilon \geq \underline{\omega}_\varepsilon \quad \text{in } (a, b) \times \mathbb{R}^+,$$

and, up to subsequences, for all $T > 0$,

$$\underline{\omega}_\varepsilon \rightarrow \underline{\omega} \leq \omega \quad \text{in } C([0, T]; L^2(a, b)).$$

Arguing as in the proof for $\omega(x, t)$, we find that

$$[a, b] = \text{supp } \underline{\omega}(t_1) = \text{supp } \underline{\omega}(t_2) \quad \text{if } 0 \leq t_1 \leq t_2. \tag{37}$$

We fix $T > 0$. We divide the proof into several steps.

CLAIM 1 For all sufficiently small $\sigma > 0$ there exists $\lambda_\sigma > 0$ such that

$$\omega_\varepsilon \geq \underline{\omega}_\varepsilon \geq \lambda_\sigma \quad \text{in } [a + \sigma, b - \sigma] \times [0, T]. \tag{38}$$

Proof of Claim 1. By (37) we have, for $0 \leq t \leq T$,

$$A_\sigma(t) := \int_{[a, a+\sigma]} \underline{\omega}(t) \neq 0 \quad \text{and} \quad B_\sigma(t) := \int_{[b-\sigma, b]} \underline{\omega}(t) \neq 0.$$

We define $A_\sigma^\varepsilon(t)$ and $B_\sigma^\varepsilon(t)$ in a similar way. Since A_σ and B_σ are continuous in $[0, T]$ there exists $\lambda_\sigma > 0$ such that

$$A_\sigma(t), B_\sigma(t) \geq 2\lambda_\sigma \quad \text{for } 0 \leq t \leq T.$$

Since $\underline{\omega}_\varepsilon \rightarrow \underline{\omega}$ in $C([0, T]; L^2(a, b))$,

$$A_\sigma^\varepsilon \rightarrow A_\sigma \quad \text{and} \quad B_\sigma^\varepsilon \rightarrow B_\sigma \quad \text{uniformly in } [0, T].$$

Hence, for $\varepsilon > 0$ small enough,

$$A_\sigma^\varepsilon(t), B_\sigma^\varepsilon(t) > \lambda_\sigma \quad \text{for } 0 \leq t \leq T.$$

Now (38) follows from standard lap-number theory ([10]), where σ is chosen so small that $\lambda_\sigma < \bar{\rho}$ ($\bar{\rho}$ is defined by (11)).

REMARK 5.3 If $A = 0$, we can apply the lap-number theory to the equation for ω_ε itself, and arguing as in the proof of Claim 1, with A_σ replaced by $\int_{[a(t), a(t)+\sigma]} \omega$, we obtain Proposition 5.1. So it remains to consider the case

$$A > 0. \tag{39}$$

CLAIM 2 Let $\{p_0 > I\} \neq \emptyset$ and let $0 < t_0 < t_1$ and $x_0 < x_1$ be such that

$$K = [x_0, x_1] \times [t_0, t_1] \subseteq \{p > I\}.$$

Then w_ε and p_ε are uniformly Hölder continuous in K , $\omega, p \in C(K)$ and there exists $\delta_K > 0$ such that for all $\varepsilon > 0$ small enough

$$\omega_\varepsilon \geq \delta_K \quad \text{and} \quad p_\varepsilon \geq I + \delta_K \quad \text{in } K.$$

REMARK 5.4

- (i) If $\{p_0 > I\} = \emptyset$, then $p \leq I$ in $\mathbb{R} \times [0, \infty)$, $\omega(x, t) = \omega_0(x)$ for all x and t , and problem (1)–(2) becomes trivial.
- (ii) As a consequence of Claim 2, ω_ε and p_ε are uniformly Hölder continuous in any finite union of rectangles.

Proof of Claim 2. Since $\{p(t) > I + \rho\}$ is an interval if $0 \leq \rho < \bar{\rho}$ and $p \in C([0, \infty); L^2(\mathbb{R}))$, it follows that

$$t \geq 0 \text{ and } x \in \{p(t) > I\} \Rightarrow (x, t) \in \{p > I\}. \tag{40}$$

Moreover, (40) holds with \Rightarrow replaced by \Leftrightarrow . Indeed, let $(x, t) \in \{p > I\}$. Since the set $\{p > I\}$ is open and $p \in C([0, \infty); L^2_{\text{loc}}(\mathbb{R}))$, without loss of generality, we may assume that there exist $c, d \in \mathbb{R}^+$ such that

$$\begin{aligned} [x - c, x + c] \times [t - d, t + d] &\subseteq \{p > I\}, \\ [x - c, x + c] &\subseteq \{p(s) \geq I\} \quad \text{for all } s \in (t - d, t + d) \end{aligned}$$

and

$$[x - c, x + c] \subseteq \{p(t - d) > I + \rho'\}$$

for some $\rho' > 0$. Arguing by contradiction we assume that $p(y, t) = I$ for all $y \in [x - c, x + c]$. Denote by $(\omega_\varepsilon, p_\varepsilon)$ the sequence defined by Theorem 3.2. By property (15) it follows that for each $\delta > 0$,

$$p_\varepsilon - I > -\delta \quad \text{in } [x - c, x + c] \times [t - d, t + d],$$

for $\varepsilon = \varepsilon(\delta)$ small enough; in particular,

$$p_\varepsilon(y, t - d) > I + \rho'/2 \quad \text{in } [x - c, x + c].$$

Now, we proceed as in the proof of Lemma 4.1. We multiply the equation for p_ε by $-\phi^2/((p_\varepsilon - I) + 2\delta)$, where $\phi = \phi(y)$ is a Lipschitz continuous function with $\text{supp } \phi \subseteq [x - c, x + c]$. Integrating by parts and letting $\varepsilon \rightarrow 0$, we find

$$-\int_{\mathbb{R}} \phi^2 \log(p - I + 2\delta)(t) \leq -\int_{\mathbb{R}} \phi^2 \log(p - I + 2\delta)(t - d) + C(\phi)d \leq \bar{C}.$$

Since δ is arbitrary, we have proved that

$$t \geq 0 \text{ and } x \in \{p(t) > I\} \Leftrightarrow (x, t) \in \{p > I\}. \tag{41}$$

The set $\{p > I\}$ is connected, has a nonempty intersection with $(a, b) \times \{0\}$, and

$$p(x, t) \leq I \quad \text{for all } x \in \overline{\{p(t) > I\}}, t \geq 0.$$

Since $\{p > I\}$ is open there exists $\eta_0 \in (0, t_0)$ such that for all $0 \leq \eta \leq \eta_0$,

$$K_\eta := [x_0 - \eta, x_1 + \eta] \times [t_0 - \eta, t_1] \subseteq \{p > I\}.$$

Arguing as in the proof of Claim 1 it follows that for all $0 \leq \eta < \eta_0$ there exists $\delta(\eta)$ such that

$$p_\varepsilon \geq \delta(\eta) > 0 \quad \text{in } K_\eta,$$

for ε sufficiently small. Hence, by [8], ω_ε , and therefore ω , are equicontinuous in K_η for $0 \leq \eta < \eta_0$.

It remains to prove that $\omega > 0$ in K . Arguing by contradiction, assume that $\omega(\bar{x}, \bar{t}) = 0$ for some $(\bar{x}, \bar{t}) \in K$. We choose $\eta \in (0, \eta_0)$. Since $[x_1 - \eta, x_2 + \eta] \subseteq \text{supp } p(\bar{t}) = \text{supp } \omega(\bar{t})$, there exist $x^- \in (x_1 - \eta, \bar{x})$ and $x^+ \in (\bar{x}, x_2 + \eta)$ such that $\omega(x^\pm, t) > 0$. By the continuity of ω there exists $t^- \in (t_0 - \eta, \bar{t})$ such that

$$\omega(x^\pm, t) > C_0 > 0 \quad \text{for } t^- \leq t \leq \bar{t}.$$

Since $A > 0$ and $p \geq \delta(\eta) > 0$, the comparison principle shows that

$$\omega(x, t) \geq C_1(t - t^-) \quad \text{for } x^- \leq x \leq x^+, t^- \leq t \leq \bar{t},$$

where $C_1 > 0$ is a sufficiently small constant. Hence $\omega(\bar{x}, \bar{t}) \geq C_1(\bar{t} - t^-) > 0$ and we have found a contradiction. \square

Now we are ready to complete the proof of Proposition 5.1. We argue by contradiction and suppose that $\omega(x_0, t_0) = 0$ for some $t_0 > 0$. By Claim 1, we can assume $x_0 \geq b$ (the case $x_0 \leq a$ is completely similar). Since $b(t)$ is continuous and $b(t_0) > x_0 \geq b = b(0)$, we have

$$\tau := \inf \{t > 0 : b(t) > x_0\} > 0 \quad \text{and} \quad x_0 = b(\tau).$$

We claim that

$$\{p > I\} \cap \Lambda(\tau, t_0) \neq \emptyset,$$

where we have set

$$\Lambda(\tau, t_0) := \{(x, t) : x_0 < x < b(t), \tau < t < t_0\}.$$

Indeed, otherwise $\omega = 0$ in $\Lambda(\tau, t_0)$, and since $[a(t), b(t)] = \text{supp } \omega(t)$ we obtain a contradiction.

Let $(x_1, t_1) \in \{p > I\} \cap \Lambda(\tau, t_0)$. From the connectedness of $\{p > I\}$, it follows that there exist $(\eta_1, \tau_1) \in \{p > I\}$ and a smooth function $g(t)$ ($\tau_1 \leq t \leq t_1$) such that

$$\begin{aligned} 0 \leq \tau_1 < t_0, \quad \eta_1 \in (a, b), \quad g(\tau_1) = \eta_1, \quad g(t_1) = x_1, \\ (g(t), t) \in \{p > I\}, \quad g(t) > a \quad \text{if } \tau_1 \leq t \leq t_1. \end{aligned}$$

Let $x^- > a$ be such that $x^- < g(t)$ for all $\tau_1 \leq t \leq t_1$. By Claims 1 and 2 there exists $C > 0$ such that

$$\omega(x, t) \geq 2C > C$$

if (x, t) satisfies one of the following conditions:

- (i) $x = x^-, \tau_1 \leq t \leq t_1$;
- (ii) $t = \tau_1, x^- < x < \eta_1 = g(\tau_1)$;
- (iii) $x = g(t), \tau_1 \leq t \leq t_1$.

Observing that

$$E := \{(g(t), t) : t \in [\tau_1, t_1]\}$$

is compact, we can find a finite number of sets of the form

$$K_j := [\underline{x}_j, \bar{x}_j] \times [\underline{t}_j, \bar{t}_j], \quad j = 1, \dots, n,$$

such that

$$E \subset \tilde{K} := \bigcup_{j=1}^n K_j \subseteq \{p > I\},$$

and $\omega_\varepsilon \geq C$ in \tilde{K} for all ε sufficiently small. In addition, by Claim 1, $\omega_\varepsilon(x, t) \geq C$ for ε small enough, if (x, t) satisfies conditions (i) and (iii). Hence, by the maximum principle,

$$\omega_\varepsilon(x, t), \omega(x, t) \geq C \quad \text{if } x^- \leq x \leq g(t), \tau_1 \leq t \leq t_1.$$

Arguing as in Claim 1, with $(a, b) \times (0, T)$ replaced by $(x^-, x_1) \times (t, t_0)$, we conclude that $\omega_\varepsilon(x_0, t_0), \omega(x_0, t_0) > C'$ and we have found a contradiction. \square

Proof of Proposition 5.2. First we consider the case $p_0 \geq I$ in (a, b) . We assume $\{p_0 > I\} \neq \emptyset$, otherwise the result is trivial. We argue by contradiction and assume that the set $\{p < I\} \cap \{\omega > 0\}$ is not empty. Define

$$\tau = \sup\{t : \{p(t) > I\} \neq \emptyset\} \quad (\text{possibly } \tau = +\infty).$$

We distinguish two different cases:

CASE 1 There exists $(x_1, t_1) \in \{p < I\} \cap \{\omega > 0\}$ such that $t_1 \leq \tau$.

CASE 2 The set $\{p < I\} \cap \{\omega > 0\} \cap ([0, \tau] \times \mathbb{R})$ is empty.

Case 1. Since $\{p(t) \geq I\}$ is an interval and $p \in C([0, T]; L^2_{\text{loc}}(\mathbb{R}))$, without loss of generality, we may suppose that there exist $\delta > 0$ and $x_2 \geq x_1$ such that

$$p < I \quad \text{in } [x_2, \infty) \times [t_1 - \delta, t_1 + \delta],$$

and

$$x_2 < b(t) \quad \text{for } t_1 - \delta \leq t \leq t_1 + \delta.$$

As a consequence, by (19), $b(t) = b(t_1)$ for all $t \in [t_1 - \delta, t_1 + \delta]$. Let

$$t_0 := \inf\{t \leq t_1 : \text{for all } t \leq s \leq t_1 \text{ there exists } x_s \in (a(s), b(s)) \text{ such that } p(x, s) < I \text{ for all } x \geq x_s\}.$$

Then $b(t) = b(t_0)$ for $t_0 \leq t \leq t_1$. By construction, there exists $x_0 \in (a(t_0), b(t_0))$ such that

$$p(x, t_0) \geq I \quad \text{in } (x_0, b(t_0)). \tag{42}$$

Let us still denote by $(x_0, b(t_0))$ the maximal interval for which (42) is satisfied. Since $\tau > t_0$ we can assume that

$$(y_0, t_0) \in \{p > I\} \tag{43}$$

for some $y_0 \in (x_0, b(t_0))$. On the other hand $\{p > I\}$ is open, therefore there exists $\delta_0 > 0$ such that

$$p > I \quad \text{in } [y_0 - \delta_0, y_0 + \delta_0] \times [t_0, t_0 + \delta].$$

Now let $\varphi \in \text{Lip}(\mathbb{R})$ be such that $0 \leq \varphi \leq 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in [y_0 + \delta_0, b(t_0)], \\ 0 & \text{if } x \in (-\infty, y_0 - \delta_0] \cup [b(t_0) + \delta, \infty). \end{cases}$$

Testing the equation for p_ε with $\varphi^2(p_\varepsilon - I)_-$ in $\mathbb{R} \times [t_0, t_0 + \delta]$, we find

$$\begin{aligned} -\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} \varphi^2(p_\varepsilon - I)_-^2 &= 2 \int_{\mathbb{R}} (\omega_\varepsilon^\alpha + \varepsilon)(p_\varepsilon - I)_- [(p_\varepsilon - I)_-]_x \varphi \varphi_x \\ &\quad + \int_{\mathbb{R}} [(p_\varepsilon - I)_-]_x^2 (\omega_\varepsilon^\alpha + \varepsilon) \varphi^2 \\ &\geq -C \int_{\mathbb{R}} [(p_\varepsilon - I)_-]^2 (\omega_\varepsilon^\alpha + \varepsilon) \varphi_x^2, \end{aligned}$$

where the last term vanishes as $\varepsilon \rightarrow 0$. Since $\int_{b(t_0)}^\infty (p - I)_- \varphi^2$ is constant, this yields

$$\int_{y_0+\delta_0}^{b(t_0)} (p - I)_-^2(t) \leq \int_{y_0+\delta_0}^{b(t_0)} \varphi^2 (p - I)_-^2(t_0) = 0 \quad \text{for } t \in (t_0, t_0 + \delta_0].$$

In view of the definition of t_0 , we have found a contradiction.

Case 2. We have

$$p \geq I \quad \text{in } \{\omega > 0\} \cap Q_\tau, \quad p(\tau) = I \quad \text{in } (a(\tau), b(\tau)), \quad p(t) \leq I \quad \text{for } t > \tau.$$

Therefore, for all $t > \tau$ we have

$$\omega(x, t) = \omega(x, \tau) \quad \text{for } x \in \mathbb{R},$$

and

$$a(t) = a(\tau), \quad b(t) = b(\tau).$$

Since

$$\int_{a(\tau)}^{b(\tau)} p(t) = \int_{a(\tau)}^{b(\tau)} p(\tau) = I(b(\tau) - a(\tau)),$$

we conclude that

$$p = I \quad \text{in } (a(\tau), b(\tau)) \times [\tau, \infty),$$

and we have found a contradiction.

Finally, consider the case $p_0 > I$ in (a, b) . Then $p \geq I$ in $\{\omega > 0\}$ and arguing as in the proof of (41) we obtain the result easily. \square

Appendix A. Finite speed of propagation

This section is devoted to the proof of Theorem 4.4. Our approach will use the techniques originally developed in [6, 7] to establish the occurrence of waiting time phenomena for the thin film equation as well as for higher order doubly nonlinear equations with variational structure. Therefore the main tool will be to derive a suitable integral inequality, to which we can apply Stampacchia’s iteration lemma [14, Lemma 4.1]:

LEMMA A.1 Assume that a given nonnegative nonincreasing function $G : [0, \xi_0] \rightarrow \mathbb{R}$ satisfies

$$G(\xi) \leq \frac{c_0}{(\xi - \eta)^\lambda} G(\eta)^{1+\kappa} \tag{44}$$

for $0 \leq \eta \leq \xi \leq \xi_0$ and positive numbers c_0, λ and κ . If

$$\xi_0^\lambda \geq 2^{\lambda(1+\kappa)/\kappa} c_0 G(0)^\kappa, \tag{45}$$

then $G(\xi_0) = 0$.

In the proof we will use the Gagliardo–Nirenberg inequality. In the version stated below, some of the summability powers are allowed to be less than one (see [6] for a more general statement).

THEOREM A.2 (Gagliardo–Nirenberg) Let $0 < s < p$, $1 \leq r \leq \infty$, $n \in \mathbb{N}$, $n > 0$, $0 \leq \theta < 1$. Let $\Omega \subseteq \mathbb{R}^N$ be open and diffeomorphic to a cone. Suppose that $u \in L^s(\Omega)$ and its derivative of order n belongs to $L^r(\Omega)$. Then the following inequality holds (with a constant K depending only on N, n, r, s and Ω):

$$\|u\|_p \leq K \|D^n u\|_r^\theta \|u\|_s^{1-\theta}$$

if

$$\frac{1}{p} = \theta \left(\frac{1}{r} - \frac{n}{N} \right) + (1 - \theta) \frac{1}{s}.$$

Proof of Theorem 4.4. First we will derive the integral inequality. Let $\varepsilon > 0$ and let $(\omega_\varepsilon, p_\varepsilon)$ be the solution of Problem (12) defined by Proposition 3.1. For a compactly supported function $\varphi \in W^{1,\infty}(\mathbb{R})$, we set

$$\Phi := \varphi^m (p_\varepsilon - I)_+^\delta, \quad \Psi := \varphi^m \omega_\varepsilon^\sigma.$$

We keep the possibility to choose later the constants $\delta, \sigma, m \in \mathbb{R}^+, m \geq 2$. Taking Φ (resp. Ψ), as test function for the equation for p_ε (resp. ω_ε), and integrating by parts in $\mathbb{R} \times [0, t]$, $t \leq T$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \varphi^m \frac{(p_\varepsilon - I)_+^{\delta+1}}{\delta + 1}(t) - \int_{\mathbb{R}} \varphi^m \frac{(p_0 - I)_+^{\delta+1}}{\delta + 1} \\ &= -m \int_0^t \int_{\mathbb{R}} \varphi^{m-1} \varphi_x (\omega_\varepsilon^\alpha + \varepsilon) (p_\varepsilon - I)_+^\delta p_{\varepsilon x} - \delta \int_0^t \int_{\mathbb{R}} \varphi^m (\omega_\varepsilon^\alpha + \varepsilon) (p_\varepsilon - I)_+^{\delta-1} (p_{\varepsilon x})^2 \end{aligned} \quad (46)$$

and

$$\begin{aligned} & \int_{\mathbb{R}} \varphi^m \frac{\omega_\varepsilon^{\sigma+1}}{\sigma + 1}(t) - \int_{\mathbb{R}} \varphi^m \frac{\omega_0^{\sigma+1}}{\sigma + 1} = -m \int_0^t \int_{\mathbb{R}} \varphi^{m-1} \varphi_x (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \omega_\varepsilon^\sigma \omega_{\varepsilon x} \\ & - \sigma \int_0^t \int_{\mathbb{R}} \varphi^m (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \omega_\varepsilon^{\sigma-1} (\omega_{\varepsilon x})^2 + A \int_0^t \int_{\mathbb{R}} \varphi^m \omega_\varepsilon^\sigma (p_\varepsilon - I)_+^\gamma (1 - \omega_\varepsilon). \end{aligned} \quad (47)$$

Adding (46) and (47) and applying Hölder's and Young's inequalities, we find

$$\begin{aligned} & \int_{\mathbb{R}} \varphi^m \left(\frac{(p_\varepsilon - I)_+^{\delta+1}}{\delta + 1}(t) + \frac{\omega_\varepsilon^{\sigma+1}}{\sigma + 1}(t) \right) \leq \int_{\mathbb{R}} \varphi^m \left(\frac{(p_0 - I)_+^{\delta+1}}{\delta + 1} + \frac{\omega_0^{\sigma+1}}{\sigma + 1} \right) \\ & + C_1 \int_0^t \int_{\mathbb{R}} \varphi^{m-2} \varphi_x^2 ((\omega_\varepsilon^\alpha + \varepsilon) (p_\varepsilon - I)_+^{\delta+1} + \omega_\varepsilon^\mu ((p_\varepsilon - I)_+^\beta + \varepsilon) \omega_\varepsilon^{\sigma+1}) \\ & - C_2 \int_0^t \int_{\mathbb{R}} \varphi^m ((\omega_\varepsilon^\alpha + \varepsilon) (p_\varepsilon - I)_+^{\delta-1} p_{\varepsilon x}^2 + (\omega_\varepsilon^\mu (p_\varepsilon - I)_+^\beta + \varepsilon) \omega_\varepsilon^{\sigma-1} \omega_{\varepsilon x}^2) \\ & + C_3 \int_0^t \int_{\mathbb{R}} \varphi^m \omega_\varepsilon^\sigma (p_\varepsilon - I)_+^\gamma =: I_0 + C_1 I_1 - C_2 I_2 + C_3 I_3, \end{aligned}$$

where the constants C_1, C_2, C_3 depend only on m, δ, σ and A . Letting $\bar{\beta} := \min\{\beta, \alpha\gamma\}$, we choose δ and σ such that

$$\sigma + 1 > \alpha, \quad \delta + 1 > \max\{\bar{\beta}, \gamma\}. \quad (48)$$

To estimate I_1 , we rewrite

$$I_1 = \int_0^t \int_{\mathbb{R}} \varphi^{m-2} \varphi_x^2 (\omega_\varepsilon^\alpha (p_\varepsilon - I)_+^{\delta+1} + \omega_\varepsilon^{\mu+\sigma+1} (p_\varepsilon - I)_+^\beta) + \varepsilon \int_0^t \int_{\mathbb{R}} \varphi^{m-2} \varphi_x^2 ((p_\varepsilon - I)_+^{\delta+1} + \omega_\varepsilon^{\mu+\sigma+1}) =: I_1^1 + I_1^2$$

and we observe that

$$I_1^2 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Hence

$$I_1^1 = \int_0^t \int_{\mathbb{R}} \varphi^{m-2} \varphi_x^2 \omega_\varepsilon^\alpha (p_\varepsilon - I)_+^{\bar{\beta}} ((p_\varepsilon - I)_+^{\delta+1-\bar{\beta}} + \omega_\varepsilon^{\mu+\sigma+1-\alpha} (p_\varepsilon - I)_+^{\beta-\bar{\beta}}),$$

and from Hölder’s and Young’s inequalities it follows that

$$I_1^1 \leq \frac{q-1}{q} \int_0^t \int_{\mathbb{R}} \varphi^{(m-2)\frac{q}{q-1}} ((p_\varepsilon - I)_+^{(\delta+1-\bar{\beta})\frac{q}{q-1}} + \omega_\varepsilon^{(\mu+\sigma+1-\alpha)\frac{q}{q-1}}) + \frac{\|\varphi_x\|_\infty}{q} \int_0^t \int_{\text{supp } \varphi} \omega_\varepsilon^{\alpha q} (p_\varepsilon - I)_+^{\bar{\beta}q}, \tag{49}$$

where $q > 1$. Choosing δ, σ and m such that

$$(i) \frac{\delta+1}{\bar{\beta}} = q = \frac{\sigma+1}{\alpha}, \quad (ii) m = 2q, \tag{50}$$

we obtain

$$I_1^1 \leq C_4 \int_0^t \int_{\mathbb{R}} \varphi^m ((p_\varepsilon - I)_+^{\delta+1} + \omega_\varepsilon^{\sigma+1}) + C_5 \int_0^t \int_{\text{supp } \varphi} \omega_\varepsilon^{\sigma+1} (p_\varepsilon - I)_+^{\delta+1}. \tag{51}$$

To handle I_2 we have to determine $l \in \mathbb{R}^+$ such that

$$[(\omega_\varepsilon^{\sigma+1} (p_\varepsilon - I)_+^{\delta+1})^l]_x^2 \leq C_6 (\omega_\varepsilon^\alpha (p_\varepsilon - I)_+^{\delta-1} p_{\varepsilon x}^2 + \omega_\varepsilon^{\mu+\sigma+1} (p_\varepsilon - I)_+^\beta \omega_{\varepsilon x}^2). \tag{52}$$

A straightforward computation leads to the condition

$$l \geq \max \left\{ \frac{\mu + \sigma + 1}{2(\sigma + 1)}, \frac{\beta}{2(\delta + 1)} \right\}.$$

As a consequence,

$$I_2 \geq C_7 \int_0^t \int_{\mathbb{R}} \varphi^m [(\omega_\varepsilon^{\sigma+1} (p_\varepsilon - I)_+^{\delta+1})^l]_x^2. \tag{53}$$

Finally, we estimate I_3 . By Hölder’s and Young’s inequalities, we get

$$I_3 \leq C_8 \int_0^t \int_{\mathbb{R}} \varphi^m (\omega_\varepsilon^{\frac{\sigma}{\delta+1-\gamma}} + (p_\varepsilon - I)_+^{\delta+1}).$$

It follows from assumption (50)(i) that $\sigma + 1 \geq (\delta + 1)/\gamma$. Hence $\sigma(\delta + 1)/(\delta + 1 - \gamma) \geq \sigma + 1$, and

$$I_3 \leq C_8 \int_0^t \int_{\mathbb{R}} \varphi^m ((p_\varepsilon - I)_+^{\delta+1} + \omega_\varepsilon^{\sigma+1}). \tag{54}$$

Combining (51), (53) and (54) and letting $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \varphi^m (\omega^{\sigma+1} + (p - I)_+^{\delta+1})(t) + C_9 \int_0^t \int_{\mathbb{R}} \varphi^m [(\omega^{\sigma+1} (p - I)_+^{\delta+1})']_x^2 \\ & \leq C_{10} \int_{\mathbb{R}} \varphi^m (\omega_0^{\sigma+1} + (p_0 - I)_+^{\delta+1}) + C_{11} \int_0^t \int_{\mathbb{R}} \varphi^m (\omega^{\sigma+1} + (p - I)_+^{\delta+1}) \\ & \quad + C_{12} \int_0^t \int_{\text{supp } \varphi} \omega^{\sigma+1} (p - I)_+^{\delta+1} \\ & \leq C_{10} \int_{\mathbb{R}} \varphi^m (\omega_0^{\sigma+1} + (p_0 - I)_+^{\delta+1}) + C_{11} t \sup_{\tau} \int_{\mathbb{R}} \varphi^m (\omega^{\sigma+1} + (p - I)_+^{\delta+1}) \\ & \quad + C_{12} \int_0^t \int_{\text{supp } \varphi} \omega^{\sigma+1} (p - I)_+^{\delta+1} \end{aligned}$$

for all $0 < t < \tau \leq T$. Taking the supremum in $(0, \tau)$ on the left hand side and imposing $C_{11} \tau \leq 1/2$, we find

$$\begin{aligned} & \sup_{t \in (0, \tau)} \int_{\mathbb{R}} \varphi^m (\omega^{\sigma+1} + (p - I)_+^{\delta+1})(t) + C_9 \int_0^\tau \int_{\mathbb{R}} \varphi^m [(\omega^{\sigma+1} (p - I)_+^{\delta+1})']_x^2 \\ & \leq C'_{10} \int_{\mathbb{R}} \varphi^m (\omega_0^{\sigma+1} + (p_0 - I)_+^{\delta+1}) + C'_{12} \int_0^\tau \int_{\text{supp } \varphi} \omega^{\sigma+1} (p - I)_+^{\delta+1}. \end{aligned} \tag{55}$$

Finally, defining

$$w = (\omega^{(\sigma+1)/2} (p - I)_+^{(\delta+1)/2})^{l'}, \quad l' = 2l,$$

it follows from (55) that for some positive constant \bar{C} ,

$$\begin{aligned} & \sup_{t \in (0, \tau)} \int_{\mathbb{R}} \varphi^m w^{1/l'}(t) + \bar{C}^{-1} \int_0^\tau \int_{\mathbb{R}} \varphi^m w_x^2 \\ & \leq \bar{C} \int_{\mathbb{R}} \varphi^m (\omega_0^{\sigma+1} + (p_0 - I)_+^{\delta+1}) + \bar{C} \int_0^\tau \int_{\text{supp } \varphi} w^{2/l'}, \end{aligned} \tag{56}$$

which is the desired integral inequality.

Since $\omega, p \in L^\infty_{\text{loc}}([0, \infty); L^q(\mathbb{R}))$ for all $1 \leq q < \infty$, we observe that (56) is still satisfied for all $\varphi \in W^{1, \infty}(\mathbb{R})$.

Without loss of generality, we may assume that $a = \min \text{supp } \omega_0 = 1$. For a positive number $r < 1$ we take

$$\varphi = \varphi_r(x) := \begin{cases} 1 & \text{if } x < r - 1, \\ r - x & \text{if } x \in [r - 1, r], \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for arbitrary $0 < \rho < r$, (56) can be rewritten in the following way:

$$\sup_{t \in (0, \tau)} \int_{-\infty}^{\rho} w^{1/l'} + \bar{C}^{-1} \int_0^{\tau} \int_{-\infty}^{\rho} w_x^2 \leq \frac{\bar{C}}{(r - \rho)^m} \int_0^{\tau} \int_{-\infty}^r w^{2/l'}. \tag{57}$$

We estimate $\int_0^{\tau} \int_{-\infty}^{\rho} w^{2/l'}$ by Gagliardo–Nirenberg’s inequality (cf. Theorem A.2):

$$\left(\int_{-\infty}^{\rho} w^{2/l'} \right)^{l'/2} \leq K \left(\int_{-\infty}^{\rho} w_x^2 \right)^{\theta/2} \left(\int_{-\infty}^{\rho} w^{1/l'} \right)^{l'(1-\theta)}$$

with

$$\theta = \frac{l'}{1 + 2l'} < 1.$$

As a consequence,

$$\begin{aligned} \int_0^{\tau} \int_{-\infty}^{\rho} w^{2/l'} &\leq K_1 \int_0^{\tau} \left(\int_{-\infty}^{\rho} w_x^2 \right)^{\frac{1}{1+2l'}} \left(\int_{-\infty}^{\rho} w^{1/l'} \right)^{2\frac{1+l'}{1+2l'}} \\ &\leq K_1 \left(\int_0^{\tau} \int_{-\infty}^{\rho} w_x^2 \right)^{\frac{1}{1+2l'}} \left(\int_0^{\tau} \left(\int_{-\infty}^{\rho} w^{1/l'} \right)^{2\frac{1+l'}{1+2l'} \frac{1+2l'}{2l'}} \right)^{\frac{2l'}{1+2l'}} \\ &\leq K_1 \left(\int_0^{\tau} \int_{-\infty}^{\rho} w_x^2 \right)^{\frac{1}{1+2l'}} \tau^{\frac{2l'}{1+2l'}} \left(\sup_{(0, \tau)} \int_{-\infty}^{\rho} w^{1/l'} \right)^{2\frac{1+l'}{1+2l'}}. \end{aligned}$$

Combining this with (57) yields

$$\int_0^{\tau} \int_{-\infty}^{\rho} w^{2/l'} \leq K_2 \tau^{\frac{2l'}{1+2l'}} \left(\frac{1}{(r - \rho)^m} \int_0^{\tau} \int_{-\infty}^r w^{2/l'} \right)^{1 + \frac{2}{1+2l'}}. \tag{58}$$

Let us introduce for $\xi \geq 0$ the decreasing nonnegative function

$$G(\xi) := \int_0^{\tau} \int_{-\infty}^{1-\xi} w^{2/l'}.$$

Hence, for $\xi = 1 - \rho$ and $\eta = 1 - r$, (58) becomes

$$G(\xi) \leq K_2 \frac{\tau^{\frac{2l'}{1+2l'}}}{(\xi - \eta)^{m(1+\kappa)}} G(\eta)^{1+\kappa}$$

for $0 \leq \eta \leq \xi \leq 1$, $\kappa = 2/(1 + 2l')$. Applying Stampacchia’s Lemma A.1, we find $G(\xi_0) = 0$ provided

$$\xi_0^{m(1+\kappa)} \geq 2^{m(1+\kappa)^2/\kappa} K_2 \tau^{(2-\kappa)/2} G(0)^\kappa. \tag{59}$$

Since $G(0) \leq \tau \|p_0\|_1$, (59) is satisfied if

$$\xi_0 \geq K_3 \tau^{\frac{2+\kappa}{2m(1+\kappa)}} =: 1 - \rho_0(\tau). \tag{60}$$

Thus, for sufficiently small $\tau^* = \tau^*(K_3)$, from (60) we conclude that

$$\int_0^t \int_{-\infty}^{\rho_0(t)} \omega^{\sigma+1} (p - I)_+^{\delta+1} = 0$$

for all $0 \leq t \leq \tau^*$. This yields $\omega(x, t) = 0$ for all $x < \rho_0(t)$, $0 \leq t \leq \tau^*$, i.e.

$$r(t) = K_3 t^{\frac{2+\kappa}{2m(1+\kappa)}}, \quad 0 \leq t \leq \tau^*.$$

Since the constants involved in the proof are independent of time the proof of the theorem is complete. \square

Conclusions

The system we consider is motivated by the study of subterranean fluid flows in the framework of nonlocal continuum damage mechanics, according to a model which has recently been introduced by Barenblatt, Patzek, Prostokishin and Silin [3]. In particular they conjectured the existence of a free moving boundary of the damaged region. The system is nonstandard and, even after a slight simplification explained in the introduction, the definition and the existence of solutions turn out to be nontrivial problems. In fact both the pressure p and the damage ω may become discontinuous across the free boundary, as was pointed out in [5]. Therefore a detailed analysis of the support properties is essential in our study.

One of the main open problems is to extend our analysis to higher space dimension, since even the validity of the basic BV estimates, which play an essential role in the compactness results, is not clear. The problem of uniqueness of solutions is completely open, even in the case of one space dimension.

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