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The N-membranes problem for quasilinear degenerate systems

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We study the regularity of the solution of the variational inequality for the problem of N-membranes in equilibrium with a degenerate operator of p-Laplacian type, $1 < p < \infty$, for which we obtain the corresponding Lewy–Stampacchia inequalities. By considering the problem as a system coupled through the characteristic functions of the sets where at least two membranes are in contact, we analyze the stability of the coincidence sets.

1. Introduction

In an open bounded subset Ω of $\mathbb{R}^d$, $d \geqslant 1$, we consider the quasi-linear operator

$$Av = -\nabla \cdot a(x, \nabla v) \quad \text{in } \mathscr{D}'(\Omega),$$

where $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function, and the *N-membranes problem* that consists in finding $(u_1, \dots, u_N) \in \mathbb{K}_N$ satisfying

$$\sum_{i=1}^{N} \int_\Omega a(x, \nabla u_i) \cdot \nabla(v_i - u_i) \geqslant \sum_{i=1}^{N} \int_\Omega f_i (v_i - u_i), \qquad \forall (v_1, \dots, v_N) \in \mathbb{K}_N. \tag{1}$$

Here $\mathbb{K}_N$ is the convex subset of the Sobolev space $[W^{1,p}(\Omega)]^N$, $1 < p < \infty$, defined by

$$\mathbb{K}_N = \{(v_1, \dots, v_N) \in [W^{1,p}(\Omega)]^N : v_1 \geqslant \cdots \geqslant v_N \text{ a.e. in } \Omega, \\ v_i - \varphi_i \in W_0^{1,p}(\Omega), i = 1, \dots, N\}, \tag{2}$$

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[§] With partial support from the FCT (Fundação para a Ciência e Tecnologia). E-mail: lisa@math.uminho.pt where $\varphi_1, \dots, \varphi_N \in W^{1,p}(\Omega)$ are given and such that $\mathbb{K}_N \neq \emptyset$. For instance, if $\partial\Omega \in C^{0,1}$ is a Lipschitz boundary, it suffices to assume, in the trace sense, that

$$\varphi_1 \geqslant \cdots \geqslant \varphi_N \quad \text{on } \partial\Omega.$$

In (1) we shall assume that

$$f_1, \dots, f_N \in L^q(\Omega) \subset W^{-1,p'}(\Omega) \tag{3}$$

where $W^{-1,p'}(\Omega)$ denotes the dual space of $W_0^{1,p}(\Omega)$, so that $p' = p/(p-1)$ is the conjugate exponent of p and, by Sobolev imbeddings, $q = 1$ if $p > d$, $q > 1$ if $p = d$, and $q = dp/(dp + p - d)$ if $1 < p < d$.

Under the following assumptions for a.e. $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^d$:

$$a(x,\xi) \cdot \xi \geqslant \alpha|\xi|^p, \qquad 1 < p < \infty, \tag{4}$$

$$|a(x,\xi)| \leqslant \beta|\xi|^{p-1}, \tag{5}$$

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) > 0 \quad \text{if } \xi \neq \eta, \tag{6}$$

for given constants $\alpha, \beta > 0$, the general theory of variational inequalities for strictly monotone operators (see [17], [13]) immediately yields the existence and uniqueness of solution to the N-membranes problem (1).

If we choose the minimization functional

$$E(u_1, \dots, u_N) = \sum_{i=1}^{N} \int_\Omega \left[\frac{1}{p}|\nabla u_i|^p - f_i u_i \right]$$

in the convex set of admissible displacements given by (2) as a model for the N-membranes in equilibrium, each one under the action of the forces f_i and attached to rigid supports at height φ_i, we obtain the variational inequality (1) associated with the p-Laplacian

$$Av = -\Delta_p v = -\nabla \cdot (|\nabla v|^{p-2}\nabla v), \qquad 1 < p < \infty.$$

The N-membranes problem was considered in [6] for linear elliptic operators, where for differentiable coefficients the regularity of the solution in Sobolev spaces $W^{2,p}(\Omega)$ was shown for $p \geqslant 2$ (hence also in $C^{1,\lambda}(\Omega)$ for $0 < \lambda = 1 - d/p < 1$) extending earlier results of [26] for the two-membranes problem. Noting the analogy (and relation) with the one-obstacle problem, it was observed in those problems that the C^2-regularity of the solution cannot be expected in general, even for very smooth data.

Considering the analogy of the two- and three-membranes problem with the one- and two-obstacles problems respectively, in [1] we have shown the Lewy–Stampacchia type inequalities

$$\bigwedge_{j=1}^{i} f_j \leqslant Au_i \leqslant \bigvee_{j=i}^{N} f_j \quad \text{a.e. in } \Omega,\ i = 1, \dots, N, \tag{7}$$

for general second order linear elliptic operators with measurable coefficients, and in the cases $N = 2$ and $N = 3$ we have established sufficient conditions on the external forces for the stability of the coincidence sets

$$\{x \in \Omega : u_j(x) = u_{j+1}(x)\}, \qquad j = 1, \dots, N-1, \tag{8}$$

where two consecutive membranes touch each other.

In (7) we use the notation

$$\bigvee_{i=1}^{k} \xi_i = \xi_1 \vee \cdots \vee \xi_k = \sup\{\xi_1, \ldots, \xi_k\} \quad \text{and} \quad \bigwedge_{i=1}^{k} \xi_i = \xi_1 \wedge \cdots \wedge \xi_k = \inf\{\xi_1, \ldots, \xi_k\}$$

and we also write $\xi^+ = \xi \vee 0$ and $\xi^- = -(\xi \wedge 0)$.

In order to prove (7) we shall approximate, in Section 2, the solution $(u_1, \ldots, u_N)$ of (1) by solutions $(u_1^\varepsilon, \ldots, u_N^\varepsilon)$ of a suitable system of Dirichlet problems for the operator A associated to a particular new monotone perturbation that extends the bounded penalization, as $\varepsilon \to 0$, of obstacle problems (see [13] or [22] and their references). Under the further assumptions of strong monotonicity of the vector field $a(x, \xi)$ with respect to ξ, i.e., for some $\alpha > 0$,

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) \geqslant \begin{cases} \alpha|\xi - \eta|^p & \text{if } p \geqslant 2, \\ \\ \alpha\,(|\xi| + |\eta|)^{p-2}\,|\xi - \eta|^2 & \text{if } 1 < p < 2, \end{cases} \tag{9}$$

we are able to establish that the error of the approximating solutions in the $W^{1,p}(\Omega)$-norm is of order $\varepsilon^{1/p}$ if $p > 2$, and of order $\varepsilon^{1/2}$ if $1 < p \leqslant 2$, with a constant that depends only on $\alpha > 0$ and on the L^q-norms of $f_1, \ldots, f_N$. This type of estimate that appears in [23] for the obstacle problem in case $p \geqslant 2$ seems new for $1 < p < 2$.

The inequalities (7) are a consequence of the fact that each Au_i is an L^q function and we can regard u_1 and u_N as solutions of one-obstacle problems and all the other u_i, $1 < i < N$, as solutions of two-obstacles problems, to which we can apply the well-known Lewy–Stampacchia inequalities (see, for instance [22], [25], [23] or [20] and their references). Another important consequence of these properties is the reduction of the regularity of the solution of the N-membranes problem to the regularity of each equation

$$Au_i = h_i \qquad \text{a.e. in } \Omega,\ i = 1, \ldots, N. \tag{10}$$

Therefore, in Section 3, we conclude from the well-known properties of weak solutions of quasilinear elliptic equations (see [14] and [18]) that the solutions u_i are in fact Hölder continuous, provided $q > d/p$ in (3), or have Hölder continuous gradient (see [8]) if $q > dp/(p-1)$ and the operator A has the stronger structural properties, for a.e. $x \in \Omega$,

$$\sum_{i,j=1}^{d} \frac{\partial a_i}{\partial \eta_j}(x,\eta)\xi_i\xi_j \geqslant \alpha_0|\eta|^{p-2}|\xi|^2, \tag{11}$$

$$\left|\frac{\partial a_i}{\partial \eta_j}(x,\eta)\right| \leqslant \alpha_1|\eta|^{p-2} \quad \text{and} \quad \left|\frac{\partial a_i}{\partial x_j}(x,\eta)\right| \leqslant \alpha_1|\eta|^{p-1} \tag{12}$$

for some positive constants α_0, α_1 and all $\eta \in \mathbb{R}^d \setminus \{0\}$, $\xi \in \mathbb{R}^d$ and all $i, j = 1, \ldots, d$. We even conclude that for each $i = 1, \ldots, N$,

$$u_i \in C^{0,\lambda}(\overline{\Omega}) \quad \text{or} \quad u_i \in C^{1,\lambda}(\overline{\Omega}),$$

provided the Dirichlet data φ_i and $\partial\Omega$ have the required regularity (see Section 3).

Finally, in Section 4 we study the stability of the coincidence sets (8) in terms of the convergence of their characteristic functions. For this purpose, we define, for a.e. $x \in \Omega$ and for $1 \leqslant j < k \leqslant N$, the following $N(N-1)/2$ coincidence sets:

$$I_{j,k} = \{x \in \Omega : u_j(x) = \cdots = u_k(x)\} \tag{13}$$

and notice that the sets defined in (8) are simply $I_{j,j+1}$. Moreover, $I_{j,k} = I_{j,j+1} \cap \cdots \cap I_{k-1,k}$. Set

$$\chi_{j,k}(x) = \chi_{I_{j,k}}(x) = \begin{cases} 1 & \text{if } u_j(x) = \cdots = u_k(x), \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In [1] we have shown that the solution (u_1, u_2, u_3) of (1) for $N = 3$ with a linear operator in fact satisfies, a.e. in Ω,

$$\begin{cases} Au_1 = f_1 + \frac{1}{2}(f_2 - f_1)\chi_{1,2} & & + \frac{1}{6}(2f_3 - f_2 - f_1)\chi_{1,3}, \\ Au_2 = f_2 - \frac{1}{2}(f_2 - f_1)\chi_{1,2} & + \frac{1}{2}(f_3 - f_2)\chi_{2,3} & + \frac{1}{6}(2f_2 - f_1 - f_3)\chi_{1,3}, \\ Au_3 = f_3 & - \frac{1}{2}(f_3 - f_2)\chi_{2,3} & + \frac{1}{6}(2f_1 - f_2 - f_3)\chi_{1,3}, \end{cases} \tag{15}$$

which extends the remark of [27] for the case $N = 2$ that corresponds to the first two equations of (15) with $\chi_{2,3} \equiv 0$ (and consequently also $\chi_{1,3} \equiv 0$). As

$$f_1 \neq f_2 \quad \text{a.e. in } \Omega$$

is a sufficient condition for the convergence of the unique coincidence set $I_{1,2}$ in case $N = 2$, additionally

$$f_2 \neq f_3, \quad f_1 \neq \frac{f_2 + f_3}{2}, \quad f_3 \neq \frac{f_1 + f_2}{2} \quad \text{a.e. in } \Omega$$

in case $N = 3$ are sufficient conditions for the convergence of the three coincidence sets $I_{1,2}$, $I_{2,3}$ and $I_{1,3}$, with respect to the perturbation of the forces f_1, f_2, f_3 (see [1] for a direct proof).

In Section 4 we extend the system (15) to arbitrary N by showing that, for given forces $(f_1, \ldots, f_N)$ the solution $(u_1, \ldots, u_N)$ of (1) solves a system of the form

$$Au_i = f_i + \sum_{1 \leqslant j < k \leqslant N,\ j \leqslant i \leqslant k} b_i^{j,k}[f]\chi_{j,k} \quad \text{a.e. in } \Omega,\ i = 1, \ldots, N, \tag{16}$$

where each $b_i^{j,k}[f]$ represents a certain linear combination of the forces.

We denote the average of $f_j, \ldots, f_k$ by

$$\langle f \rangle_{j,k} = \frac{f_j + \cdots + f_k}{k - j + 1}, \quad 1 \leqslant j \leqslant k \leqslant N, \tag{17}$$

and we shall establish that

$$\langle f \rangle_{i,j} \neq \langle f \rangle_{j+1,k} \quad \text{a.e. in } \Omega, \quad \text{for all } i, j, k \in \{1, \ldots, N\} \text{ with } i \leqslant j < k, \tag{18}$$

is a sufficient condition for the stability of the coincidence sets $I_{j,k}$ in the N-membranes problem.

2. Approximation by bounded penalization

In this section we approximate the variational inequality using bounded penalization. Defining

$$\begin{aligned} \xi_0 &= \max\left\{\frac{f_1+\cdots+f_i}{i} : i=1,\dots,N\right\},\\ \xi_i &= i\xi_0-(f_1+\cdots+f_i) \quad \text{for } i=1,\dots,N, \end{aligned} \tag{19}$$

we observe that

$$\begin{cases} \xi_i \geqslant 0 & \text{if } i\geqslant 1,\\ (\xi_{i-1}-\xi_{i-2})-(\xi_i-\xi_{i-1}) = f_i-f_{i-1} & \text{if } i\geqslant 2. \end{cases} \tag{20}$$

For $\varepsilon > 0$, let θ_ε be defined as follows:

$$\theta_\varepsilon : \overline{\mathbb{R}} \to \overline{\mathbb{R}}, \qquad s \mapsto \begin{cases} 0 & \text{if } s\geqslant 0,\\ s/\varepsilon & \text{if } -\varepsilon < s < 0,\\ -1 & \text{if } s \leqslant -\varepsilon. \end{cases} \tag{21}$$

The approximate problem is given by the system

$$\begin{cases} Au_i^\varepsilon + \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) - \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon) \ = \ f_i & \text{in } \Omega,\\ u^\varepsilon_{i|\partial\Omega} = \varphi_i, & i=1,\dots,N, \end{cases} \tag{22}$$

with the convention $u_0^\varepsilon = +\infty$, $u_{N+1}^\varepsilon = -\infty$.

PROPOSITION 2.1 If the operator A satisfies the assumptions (4)–(6), then problem (22) has a unique solution $(u_1^\varepsilon,\dots,u_N^\varepsilon) \in [W^{1,p}(\Omega)]^N$. This solution satisfies

$$u_i^\varepsilon \leqslant u_{i-1}^\varepsilon + \varepsilon \quad \text{for } i=2,\dots,N. \tag{23}$$

Proof. Existence and uniqueness of solution of problem (22) is an immediate consequence of the theory of strictly monotone and coercive operators (see [17]). In fact, if we sum the N equations of the system, each one multiplied by a test function w_i, then problem (22) implies that

$$\sum_{i=1}^N \int_\Omega \langle Au_i^\varepsilon, w_i\rangle + \langle Bu^\varepsilon, w\rangle = \sum_{i=1}^N \int_\Omega f_i w_i, \qquad \forall w = (w_1,\dots,w_N) \in [W^{1,p}(\Omega)]^N,$$

where

$$\langle Bv, w\rangle = \sum_{i=1}^N \int_\Omega (\xi_i\theta_\varepsilon(v_i - v_{i+1}) - \xi_{i-1}\theta_\varepsilon(v_{i-1}-v_i))w_i$$

with the same convention $v_0 = +\infty$, $v_{N+1} = -\infty$, satisfies

$$\begin{aligned} &\langle Bv - Bw, v-w\rangle \\ &\quad = \sum_{i=1}^{N-1} \int_\Omega \xi_i(\theta_\varepsilon(v_i - v_{i+1}) - \theta_\varepsilon(w_i - w_{i+1}))((v_i - v_{i+1}) - (w_i - w_{i+1})) \geqslant 0, \end{aligned} \tag{24}$$

since $\xi_i \geqslant 0$ for $i=1,\dots,N$ and θ_ε is nondecreasing.

To prove (23), multiplying the i-th equation of (22) by $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$ and integrating on Ω, noticing that $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+_{|\partial\Omega} = 0$ we obtain

$$\begin{aligned}\int_\Omega Au_i^\varepsilon (u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ &= \int_\Omega [f_i - \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) + \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon)](u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \\ &= \int_\Omega [f_i - \xi_i\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) - \xi_{i-1}](u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+\end{aligned}$$

since $\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon)(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ = -(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$. In particular, because $\theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) \geqslant -1$, we have

$$\int_\Omega Au_i^\varepsilon (u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \leqslant \int_\Omega [f_i + \xi_i - \xi_{i-1}](u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+. \tag{25}$$

With similar arguments, if we multiply, for $i \geqslant 2$, the $(i-1)$-th equation of (22) by $(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+$ and integrate on Ω we obtain

$$\int_\Omega Au_{i-1}^\varepsilon (u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \geqslant \int_\Omega [f_{i-1} + \xi_{i-1} - \xi_{i-2}](u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+. \tag{26}$$

From inequalities (25) and (26) we have, using (20),

$$\begin{aligned}\int_\Omega (a(x, \nabla u_i^\varepsilon) - a(x, \nabla u_{i-1}^\varepsilon)) \cdot \nabla(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ &= \int_\Omega \left(Au_i^\varepsilon - Au_{i-1}^\varepsilon\right)(u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ \\ &\leqslant \int_\Omega [f_i - f_{i-1} + (\xi_i - \xi_{i-1}) - (\xi_{i-1} - \xi_{i-2})](u_i^\varepsilon - u_{i-1}^\varepsilon - \varepsilon)^+ = 0.\end{aligned}$$

From the strict monotonicity (6) of a, it follows that $u_i^\varepsilon \leqslant u_{i-1}^\varepsilon + \varepsilon$ a.e. in Ω. □

PROPOSITION 2.2 If $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ and $(u_1, \dots, u_N)$ are the solutions of problems (22) and (1) respectively then

$$(u_1^\varepsilon, \dots, u_N^\varepsilon) \rightharpoonup (u_1, \dots, u_N) \quad \text{in } [W^{1,p}(\Omega)]^N\text{-weak as } \varepsilon \to 0.$$

Proof. Multiplying the i-th equation of (22) by $v_i - u_i^\varepsilon$, where $(v_1, \dots, v_N) \in \mathbb{K}_N$ and $u^\varepsilon = (u_1^\varepsilon, \dots, u_N^\varepsilon)$, integrating over Ω and summing, we obtain

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i^\varepsilon) \cdot \nabla(v_i - u_i^\varepsilon) + \langle Bu^\varepsilon, v - u^\varepsilon \rangle = \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^\varepsilon).$$

Noticing that $\langle Bv, v - u^\varepsilon \rangle = 0$ and due to the monotonicity of the operator B proved in (24),

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i^\varepsilon) \cdot \nabla(v_i - u_i^\varepsilon) \geqslant \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^\varepsilon) \tag{27}$$

and using (6) we conclude that

$$\sum_{i=1}^N \int_\Omega a(x, \nabla v_i) \cdot \nabla(v_i - u_i^\varepsilon) \geqslant \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^\varepsilon). \tag{28}$$

From (4) and (5) we easily deduce the uniform boundedness of $\{(u_1^\varepsilon, \dots, u_N^\varepsilon)\}_\varepsilon$ in $[W^{1,p}(\Omega)]^N$. So, there exists $(u_1^*, \dots, u_N^*) \in [W^{1,p}(\Omega)]^N$ such that

$$(u_1^\varepsilon, \dots, u_N^\varepsilon) \rightharpoonup (u_1^*, \dots, u_N^*) \quad \text{in } [W^{1,p}(\Omega)]^N\text{-weak as } \varepsilon \to 0,$$

and letting $\varepsilon \to 0$ in (28) we obtain

$$\sum_{i=1}^N \int_\Omega a(x, \nabla v_i) \cdot \nabla(v_i - u_i^*) \geqslant \sum_{i=1}^N \int_\Omega f_i(v_i - u_i^*), \qquad \forall (v_1, \dots, v_N) \in \mathbb{K}.$$

Furthermore, by (23), $u_1^* \geqslant \cdots \geqslant u_n^*$. Since we also have $u^*_{i\,|\partial\Omega} = \varphi_i$ for $i = 1, \dots, N$, it follows that $(u_1^*, \dots, u_N^*) \in \mathbb{K}_N$. The hemicontinuity of the operator A allows us to conclude that $(u_1^*, \dots, u_N^*)$ actually solves the variational inequality (1) and the uniqueness of solution of the variational inequality implies that $u_i^* = u_i$, $i = 1, \dots, N$. ☐

We now present two lemmas that will be used to prove the next theorem. The first lemma states that, under certain circumstances, weak convergence implies strong convergence. The second lemma is a reverse Hölder inequality.

LEMMA 2.3 ([5, p. 190]) Under the assumptions (4)–(6), when $\varepsilon \to 0$, if

$$u^\varepsilon - u \rightharpoonup 0 \quad \text{in } W_0^{1,p}(\Omega) \tag{29}$$

and

$$\int_\Omega [a(x, \nabla u^\varepsilon) - a(x, \nabla u)] \cdot \nabla(u^\varepsilon - u) \to 0 \tag{30}$$

then

$$u^\varepsilon - u \to 0 \quad \text{in } W_0^{1,p}(\Omega)\text{-strong.}$$ ☐

LEMMA 2.4 ([24, p. 8]) Let $0 < r < 1$ and $r' = r/(r-1)$. If $F \in L^r(\Omega)$, $FG \in L^1(\Omega)$ and $\int_\Omega |G(x)|^{r'}\,\mathrm{d}x < \infty$ in a bounded domain Ω of $\mathbb{R}^d$, then

$$\left(\int_\Omega |F(x)|^r\,\mathrm{d}x\right)^{1/r} \leqslant \left(\int_\Omega |F(x)G(x)|\,\mathrm{d}x\right)\left(\int_\Omega |G(x)|^{r'}\,\mathrm{d}x\right)^{-1/r'}. \tag{31}$$

THEOREM 2.5 Let $(u_1^\varepsilon, \dots, u_N^\varepsilon)$ and $(u_1, \dots, u_N)$ denote, respectively, the solutions of problems (22) and (1). Under the assumptions (4)–(6):

(i) $(u_1^\varepsilon, \dots, u_N^\varepsilon) \to (u_1, \dots, u_N)$ in $[W^{1,p}(\Omega)]^N$ as $\varepsilon \to 0$.

(ii) If, in addition, a is strongly monotone, i.e., satisfies (9), then there exists a positive constant C, independent of ε, such that, for all $i = 1, \dots, N$,

$$\|\nabla(u_i^\varepsilon - u_i)\|_{L^p(\Omega)} \leqslant \begin{cases} C\varepsilon^{1/p} & \text{if } p \geqslant 2, \\ C\varepsilon^{1/2} & \text{if } 1 < p \leqslant 2. \end{cases}$$

Proof. (i) Choose, for $i = 1, \dots, N$, $v_i = \bigvee_{k=i}^N u_k^\varepsilon$ in (1). Indeed, since $v_{i-1} \geqslant v_i$ a.e. in Ω and $v_i - \varphi_i \in W_0^{1,p}(\Omega)$, we have $(v_1, \dots, v_N) \in \mathbb{K}_N$ and

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i) \cdot \nabla\Big(\bigvee_{k=i}^N u_k^\varepsilon - u_i\Big) \geqslant \sum_{i=1}^N \int_\Omega f_i\Big(\bigvee_{k=i}^N u_k^\varepsilon - u_i\Big).$$

So,

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i) \cdot \nabla(u_i^\varepsilon - u_i) \geqslant \sum_{i=1}^N \int_\Omega f_i(u_i^\varepsilon - u_i) \\ + \sum_{i=1}^N \int_\Omega a(x, \nabla u_i) \cdot \nabla\Big(u_i^\varepsilon - \bigvee_{k=i}^N u_k^\varepsilon\Big) + \sum_{i=1}^N \int_\Omega f_i\Big(\bigvee_{k=i}^N u_k^\varepsilon - u_i^\varepsilon\Big).$$

On the other hand, by (27),

$$\sum_{i=1}^N \int_\Omega a(x, \nabla u_i^\varepsilon) \cdot \nabla(u_i^\varepsilon - u_i) \leqslant \sum_{i=1}^N \int_\Omega f_i(u_i^\varepsilon - u_i),$$

and we conclude that

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega [a(x, \nabla u_i^\varepsilon) - a(x, \nabla u_i)] \cdot \nabla(u_i^\varepsilon - u_i) \\ &\qquad \leqslant \sum_{i=1}^N \int_\Omega a(x, \nabla u_i) \cdot \nabla\Big(\bigvee_{k=i}^N u_k^\varepsilon - u_i^\varepsilon\Big) - \sum_{i=1}^N \int_\Omega f_i\Big(\bigvee_{k=i}^N u_k^\varepsilon - u_i^\varepsilon\Big) \\ &\qquad = \sum_{i=1}^N \int_\Omega (Au_i - f_i)\Big(\bigvee_{k=i}^N u_k^\varepsilon - u_i^\varepsilon\Big). \end{aligned} \tag{32}$$

Here we have used the fact that $Au_i \in L^q(\Omega)$ for $i = 1, \dots, N$, since we know that

$$f_i - \xi_{i-1} \leqslant Au_i^\varepsilon = -\xi_i \theta_\varepsilon(u_i^\varepsilon - u_{i+1}^\varepsilon) + \xi_{i-1}\theta_\varepsilon(u_{i-1}^\varepsilon - u_i^\varepsilon) + f_i \leqslant f_i + \xi_i,$$

by (22) and $-1 \leqslant \theta_\varepsilon \leqslant 0$.

Noticing that, from (23),

$$0 \leqslant \bigvee_{k=i}^N u_k^\varepsilon - u_i^\varepsilon \leqslant u_i^\varepsilon + (N - i + 1)\varepsilon - u_i^\varepsilon \leqslant (N - i + 1)\varepsilon \tag{33}$$

it is immediate to conclude that

$$0 \leqslant \sum_{i=1}^N \int_\Omega [a(x, \nabla u_i^\varepsilon) - a(x, \nabla u_i)] \cdot \nabla(u_i^\varepsilon - u_i) \leqslant C\varepsilon, \tag{34}$$

and, since (29) and (30) hold, Lemma 2.3 shows that for each $i = 1, \dots, N$,

$$u_i^\varepsilon \to u_i \quad \text{in } W^{1,p}(\Omega) \text{ as } \varepsilon \to 0.$$

(ii) From (34) and using the strong monotonicity of a, for $p \geqslant 2$ we have

$$\alpha \sum_{i=1}^{N} \int_{\Omega} |\nabla(u_i^{\varepsilon} - u_i)|^p \leqslant \sum_{i=1}^{N} \int_{\Omega} [a(x, \nabla u_i^{\varepsilon}) - a(x, \nabla u_i)] \cdot \nabla(u_i^{\varepsilon} - u_i) \leqslant C\varepsilon.$$

Let now $1 < p < 2$. Using also the strong monotonicity of a and (34), we obtain

$$\alpha \sum_{i=1}^{N} \int_{\Omega} (|\nabla u_i^{\varepsilon}| + |\nabla u_i|)^{p-2} |\nabla(u_i^{\varepsilon} - u_i)|^2 \leqslant \sum_{i=1}^{N} \int_{\Omega} [a(x, \nabla u_i^{\varepsilon}) - a(x, \nabla u_i)] \cdot \nabla(u_i^{\varepsilon} - u_i) \leqslant C\varepsilon. \tag{35}$$

Let $\hat{\Omega}_i = \{x \in \Omega : |\nabla u_i^{\varepsilon}| + |\nabla u_i| \neq 0\}$. We may use the reverse inequality (31) with $r = p/2$, noticing that $0 < r < 1$ and $r' = p/(p-2)$, setting $F = |\nabla(u_i^{\varepsilon} - u_i)|^2$ and $G = (|\nabla u_i^{\varepsilon}| + |\nabla u_i|)^{p-2}$. Then we obtain, for $i = 1, \dots, N$,

$$\left(\int_{\hat{\Omega}_i} |\nabla(u_i^{\varepsilon} - u_i)|^p\right)^{2/p} \mathrm{d}x \leqslant \left(\int_{\hat{\Omega}_i} |\nabla(u_i^{\varepsilon} - u_i)|^2 (|\nabla u_i^{\varepsilon}| + |\nabla u_i|)^{p-2}\, \mathrm{d}x\right)\left(\int_{\hat{\Omega}_i} (|\nabla u_i^{\varepsilon}| + |\nabla u_i|)^p\, \mathrm{d}x\right)^{(2-p)/p}.$$

Since by (35),

$$\int_{\hat{\Omega}_i} |\nabla(u_i^{\varepsilon} - u_i)|^2 (|\nabla u_i^{\varepsilon}| + |\nabla u_i|)^{p-2}\, \mathrm{d}x \leqslant \frac{1}{\alpha} C\varepsilon,$$

and

$$\exists M_p > 0 : \quad \left(\int_{\hat{\Omega}_i} (|\nabla u_i^{\varepsilon}| + |\nabla u_i|)^p\, \mathrm{d}x\right)^{(2-p)/p} \leqslant M_p,$$

the conclusion follows immediately by summing the N inequalities above. □

3. Lewy–Stampacchia inequalities and regularity

As a consequence of the approximation by bounded penalization we already know that $Au_i \in L^q(\Omega)$, $i = 1, \dots, N$, and so we can use the analogy with the obstacle problem to show further regularity of the solution u_i.

In [15] Lewy and Stampacchia have shown that the solution of the obstacle problem for the Laplacian satisfies a dual inequality, which in fact holds in more general cases, as observed in [10] or [4] for nonlinear operators. Summarizing the known results for the one- and two-obstacles problem that we shall apply to the N-membranes problem, the following theorem may be proved as in [22] or [20].

THEOREM 3.1 Given $\varphi, \psi_1, \psi_2 \in W^{1,p}(\Omega)$ $(1 < p < \infty)$, with f, $(A\psi_2 - f)^+$ and $(A\psi_1 - f)^-$ in $L^q(\Omega) \subset W^{-1,p'}(\Omega)$ ($q = 1$ if $p > d$, $q > 1$ if $p = d$, and $q = dp/(dp + p - d)$ if $1 < p < d$) such that

$$\mathbb{K}_{\psi_2}^{\psi_1} = \{v \in W^{1,p}(\Omega) : \psi_1 \geqslant v \geqslant \psi_2 \text{ a.e. in } \Omega,\ v - \varphi \in W_0^{1,p}(\Omega)\} \neq \emptyset, \tag{36}$$

the unique solution $u \in \mathbb{K}_{\psi_2}^{\psi_1}$ to the variational inequality

$$\int_\Omega a(x, \nabla u) \cdot \nabla(v - u) \geqslant \int_\Omega f(v - u), \qquad \forall v \in \mathbb{K}_{\psi_2}^{\psi_1}, \tag{37}$$

under the assumptions (4)–(6) satisfies the Lewy–Stampacchia inequality

$$f \wedge A\psi_1 \leqslant Au \leqslant f \vee A\psi_2 \quad \text{a.e. in } \Omega. \tag{38}$$

REMARK 3.2 Setting $\xi_1 = (A\psi_1 - f)^-$ and $\xi_2 = (A\psi_2 - f)^+$ and using the penalization function θ_ε of the previous section we may approach, as $\varepsilon \to 0$, the solution of (37) by the solutions u^ε of the equation

$$Au^\varepsilon + \xi_2\theta_\varepsilon(u^\varepsilon - \psi_2) - \xi_1\theta_\varepsilon(\psi_1 - u^\varepsilon) = f \quad \text{in } \Omega \tag{39}$$

with the Dirichlet boundary condition $u^\varepsilon = \varphi$ on $\partial\Omega$. Noting that

$$f \wedge A\psi_1 = f - (A\psi_1 - f)^- \quad \text{and} \quad f \vee A\psi_2 = f + (A\psi_2 - f)^+$$

we easily deduce (38) from the analogous inequalities that are satisfied for each u^ε.

REMARK 3.3 Theorem 3.1, although stated for the two-obstacles problem, also contains the case of only one obstacle. Indeed, by taking $\psi_1 \equiv +\infty$, (37) is a lower obstacle problem and (38) reads

$$f \leqslant Au \leqslant f \vee A\psi_2 \quad \text{for } u \geqslant \psi_2, \text{ a.e. in } \Omega, \tag{40}$$

and by taking $\psi_2 \equiv -\infty$, (37) is an upper obstacle problem for which (38) reads

$$f \wedge A\psi_1 \leqslant Au \leqslant f \quad \text{for } u \leqslant \psi_1, \text{ a.e. in } \Omega. \tag{41}$$

REMARK 3.4 In [20], for more general operators and under a strong monotonicity assumption of the type (9), which however is not necessary in our Theorem 3.1, it was shown that the inequalities of (38) still hold independently of one another in the duality sense, provided $A\psi_1 - f$ and/or $A\psi_2 - f$ are in $V_{p'}^* = [W^{-1,p'}(\Omega)]^+ - [W^{-1,p'}(\Omega)]^+$, i.e., in the ordered dual space of $W_0^{1,p}(\Omega)$.

THEOREM 3.5 The solution $(u_1, \ldots, u_N)$ of the N-membranes problem, under the assumptions (4)–(6), satisfies the following Lewy–Stampacchia type inequalities:

$$\left.\begin{array}{rl} f_1 \leqslant Au_1 & \leqslant f_1 \vee \cdots \vee f_N \\ f_1 \wedge f_2 \leqslant Au_2 & \leqslant f_2 \vee \cdots \vee f_N \\ \vdots & \\ f_1 \wedge \cdots \wedge f_{N-1} \leqslant Au_{N-1} & \leqslant f_{N-1} \vee f_N \\ f_1 \wedge \cdots \wedge f_N \leqslant Au_N & \leqslant f_N \end{array}\right\} \text{a.e. in } \Omega. \tag{42}$$

Proof. Observe that choosing $(v, u_2, \ldots, u_N) \in \mathbb{K}_N$, with $v \in \mathbb{K}_{u_2}$, we see that $u_1 \in \mathbb{K}_{u_2}$ (as in (36) with $\psi_1 = +\infty$) solves the variational inequality (37) with $f = f_1$, and so by (40) we have

$$f_1 \leqslant Au_1 \leqslant f_1 \vee Au_2 \quad \text{a.e. in } \Omega.$$

Analogously, we see that $u_j \in \mathbb{K}_{u_{j+1}}^{u_{j-1}}$ solves the two-obstacles problem (37) with $f = f_j$, $j = 2, \dots, N-1$, and satisfies, by (38),

$$f_j \wedge Au_{j-1} \leqslant Au_j \leqslant f_j \vee Au_{j+1} \quad \text{a.e. in } \Omega.$$

Since $u_N \in \mathbb{K}^{u_{N-1}}$, by (41), also satisfies

$$f_N \wedge Au_{N-1} \leqslant Au_N \leqslant f_N \quad \text{a.e. in } \Omega,$$

(42) is easily obtained by simple iteration. □

For $p > d$, the Sobolev inclusion $W^{1,p}(\Omega) \subset C^{0,\lambda}(\Omega)$ for $0 < \lambda = 1 - d/p < 1$ immediately implies the Hölder continuity of the solutions u_i of the N-membranes problem; however, this property still holds for $1 < p \leqslant d$ by using the fact that each Au_i is in the same $L^q(\Omega)$ as the forces f_i, $i = 1, \dots, N$. So under the classical assumptions of [14] (see also [18]) we may state for completeness the following regularity result.

COROLLARY 3.6 Under the assumptions (3)–(6) for $1 < p \leqslant d$ with $q > d/p$ in (3), the solution $(u_1, \dots, u_N)$ of (1) is such that

$$u_i \in C^{0,\lambda}(\Omega) \quad \text{for some } 0 < \lambda < 1,\ i = 1, \dots, N,$$

and is also in $C^{0,\lambda}(\overline{\Omega})$ if, in addition, each $\varphi_i \in C^{0,\lambda}(\partial\Omega)$ and $\partial\Omega$ is smooth, for instance, of class $C^{0,1}$. □

REMARK 3.7 The above classical result for equations was also shown to hold for the one-obstacle problem, for instance, in [7] and [19], and for the two-obstacles problems in [12], under more general assumptions on the data. It would be interesting to obtain the Hölder continuity of the solution of (1) directly under the classical and more general assumptions that each f_i is in $W^{-1,s}(\Omega)$ for $s > d/(p-1)$.

A more interesting regularity is the Hölder continuity of the gradient of the solution, by analogy with the results for solutions of degenerate elliptic equations. For instance, as a consequence of the inequalities (42) and the results of [8] on the $C^{1,\lambda}$ local regularity of weak solutions, as well as on the regularity up to the boundary in [16], we may also state the following results.

COROLLARY 3.8 Under the stronger differentiability properties (11), (12), if (3) holds with $q > dp/(p-1)$, then the solution $(u_1, \dots, u_N)$ of (1) is such that

$$u_i \in C^{1,\lambda}(\Omega) \quad \text{for some } 0 < \lambda < 1,\ i = 1, \dots, N,$$

and is also in $C^{1,\lambda}(\overline{\Omega})$ if, in addition, each $\varphi_i \in C^{1,\gamma}(\partial\Omega)$ for some γ ($\lambda \leqslant \gamma < 1$), and $f_i \in L^\infty(\Omega)$ for all $i = 1, \dots, N$. □

REMARK 3.9 Additional regularity can be obtained for p-Laplacian type operators. For instance, as a consequence of recent results of [9], for $p > 2$, in a convex polyhedral domain with $\varphi_i = 0$ and $f_i \in W^{(p-2)/p,p}(\Omega)$, we could obtain solutions in the fractional order Sobolev spaces $W^{1+2/p-\varepsilon,p}(\Omega)$ for all $\varepsilon > 0$.

Another example for the p-Laplacian is provided by the results of [2], for 2-dimensional domains ($d = 2$), with $\partial\Omega$ of class C^2, in the case $1 < p < 2$: the solutions are in $H^2(\Omega) =$

$W^{2,2}(\Omega)$ if $f_i \in L^q(\Omega)$, $q > 2$, and $\varphi_i \in H^2(\Omega)$. These regularity results may be important in finite element approximations of the N-membranes problem for degenerate systems (see, for instance, [3]). To our knowledge that extension has not yet been considered in the literature for the N-membranes problem.

For differentiable strongly coercive vector fields satisfying the assumptions (11), (12), with $p = 2$, there is no degeneration of the operator A and stronger regularity in $W^{2,s}(\Omega)$ may be obtained also from the fact that (42) holds for the solution of the N-membranes problem. For instance, as in Theorem 3.3 of [13, p. 114] (see also [22, Remark 4.5, p. 244]), we can prove the following result.

COROLLARY 3.10 Let (11), (12) hold for $p = 2$, suppose $\partial\Omega \in C^{1,1}$ and $f_i \in L^\infty(\Omega)$, $\varphi_i \in W^{2,\infty}(\Omega)$ for all $i = 1, \dots, N$. Then the solution $(u_1, \dots, u_N)$ of (1) is such that

$$u_i \in W^{2,s}(\Omega) \cap C^{1,\gamma}(\overline{\Omega}), \qquad i = 1, \dots, N, \text{ for all } 1 \leqslant s < \infty \text{ and } 0 \leqslant \gamma < 1. \tag{43}$$

REMARK 3.11 For N linear operators of the form

$$a_i^k(x, \xi) = \sum_{j=1}^{d} a_{ij}^k(x)\xi_j, \qquad k = 1, \dots, N,$$

the regularity (43) was shown in [6] for every $s \geqslant 2$ and, for the same operators with lower order terms in [1] for $s > 1$ if $d = 2$, and for $s \geqslant 2d/(d+2)$ if $d \geqslant 3$. For the case of two membranes with linear operators, earlier results in [26] were shown by using similar regularity results for the one-obstacle problem. In spite of this analogy, the optimal $W^{2,\infty}$ regularity of solutions to obstacle problems is an open problem for the N-membranes system.

REMARK 3.12 In the case of two membranes with constant mean curvature, i.e., when A is the minimal surface operator and f_1 and f_2 are constants in a smooth domain with mean curvature $H_{\partial\Omega}$ of $\partial\Omega$ greater than or equal to $|f_1| \vee |f_2|/(d-1)$, in [27] the existence of a unique solution with the regularity (43) was shown. The N-membranes problem for the minimal surface operator is, in general, an open problem.

4. Convergence of coincidence sets

In this section we prove that, if $(u_1^n, \dots, u_N^n)$ is the solution of the N-membranes problem, under the assumptions (4)–(6) with given data $(f_1^n, \dots, f_N^n)$, $n \in \mathbb{N}$, and if $(f_1^n, \dots, f_N^n)$ converges in $[L^q(\Omega)]^N$ to $(f_1, \dots, f_N)$, we have the stability result in $L^s(\Omega)$, $1 \leqslant s < \infty$, for the corresponding coincidence sets:

$$\chi_{\{u_k^n = \cdots = u_l^n\}} \underset{n}{\to} \chi_{\{u_k = \cdots = u_l\}} \qquad \text{for } 1 \leqslant k < l \leqslant N.$$

We begin by presenting a lemma that will be needed.

LEMMA 4.1 ([23]) Given functions $u, v \in W^{1,p}(\Omega)$, $1 < p < \infty$, such that $Au, Av \in L^1(\Omega)$, we have

$$Au = Av \qquad \text{a.e. in } \{x \in \Omega : u(x) = v(x)\}. \qquad \square$$

In what follows we continue using the convention $u_0 = +\infty$ and $u_{N+1} = -\infty$. Given $1 \leqslant j \leqslant k \leqslant N$, we define the following sets:

$$\Theta_{j,k} = \{x \in \Omega : u_{j-1}(x) > u_j(x) = \cdots = u_k(x) > u_{k+1}(x)\}. \tag{44}$$

The first part of the following proposition identifies the value of Au_i a.e. on each coincidence set $I_{j,k}$ defined in (13). The second part states a necessary condition on the forces in order that there exists contact among consecutive membranes.

PROPOSITION 4.2 If $j, k \in \mathbb{N}$ are such that $1 \leqslant j \leqslant k \leqslant N$, we have

(i) $Au_i = \begin{cases} \langle f\rangle_{j,k} & \text{a.e. in } \Theta_{j,k} \text{ if } i \in \{j, \dots, k\}, \\ f_i & \text{a.e. in } \Theta_{j,k} \text{ if } i \notin \{j, \dots, k\}, \end{cases}$

(ii) if $j < k$ then for all $i \in \{j, \dots, k\}$, $\langle f\rangle_{i+1,k} \geqslant \langle f\rangle_{j,i}$ a.e. in $\Theta_{j,k}$.

Proof. (i) Suppose $i \in \{j, \dots, k\}$ (the other case has a similar and simpler proof). For a.e. $x \in \Theta_{j,k}$ we have $u_{j-1}(x) - u_j(x) = \alpha > 0$ and $u_k(x) - u_{k+1}(x) = \beta > 0$, for some $\alpha = \alpha(x)$ and $\beta = \beta(x)$. Since x belongs to the open set $\{y \in \Omega : u_{j-1}(y) - u_j(y) - \alpha/2 > 0\} \cap \{y \in \Omega : u_k(y) - u_{k+1}(y) - \beta/2 > 0\}$, there exists $\delta > 0$ such that, for all $\varphi \in \mathcal{D}(B(x, \delta))$, there exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, then $u_{j-1} \geqslant u_j \pm \varepsilon\varphi$ and $u_k \geqslant u_{k+1} \pm \varepsilon\varphi$.

Choose for test functions

$$v_r = \begin{cases} u_r & \text{if } r \notin \{j, \dots, k\}, \\ u_r \pm \varepsilon\varphi & \text{if } r \in \{j, \dots, k\}. \end{cases}$$

Then

$$\pm\varepsilon \sum_{r=j}^{k} \int_\Omega a(x, \nabla u_r) \cdot \nabla\varphi \geqslant \pm\varepsilon \sum_{r=j}^{k} \int_\Omega f_r\varphi, \qquad \forall\varphi \in \mathcal{D}(B(x, \delta)),$$

and

$$\sum_{r=j}^{k} \int_\Omega a(x, \nabla u_r) \cdot \nabla\varphi = \sum_{r=j}^{k} \int_\Omega f_r\varphi, \qquad \forall\varphi \in \mathcal{D}(B(x, \delta)).$$

So we conclude that

$$\sum_{r=j}^{k} Au_r = \sum_{r=j}^{k} f_r \quad \text{a.e. in } B(x, \delta).$$

We know that $Au_i \in L^1(\Omega)$, for all $i = 1, \dots, N$. So, using Lemma 4.1, we have

$$Au_j = \cdots = Au_i = \cdots = Au_k \quad \text{in } \Theta_{j,k}$$

and we conclude that

$$(k - j + 1)Au_i = f_j + \cdots + f_k \quad \text{a.e. in } \Theta_{j,k}.$$

(ii) The proof of this item is analogous to the previous one. We choose for test functions

$$v_r = \begin{cases} u_r & \text{if } r \notin \{j, \dots, i\}, \\ u_r + \varepsilon\varphi & \text{if } r \in \{j, \dots, i\}, \end{cases}$$

with $\varphi \in \mathcal{D}(B(x, \delta))$, $\varphi \geqslant 0$, $\varepsilon > 0$ such that $(v_1, \dots, v_N) \in \mathbb{K}_N$. We then conclude that

$$\sum_{r=j}^{i} \int_\Omega a(x, \nabla u_r) \cdot \nabla\varphi \geqslant \sum_{j=r}^{i} \int_\Omega f_r\varphi, \qquad \forall\varphi \in \mathcal{D}(B(x, \delta)),\ \varphi \geqslant 0,$$

and so, we have $Au_i \geqslant \langle f\rangle_{j,i}$ a.e. in $\Theta_{j,k}$. Then using the first part of the proposition we conclude that

$$\langle f\rangle_{j,k} \geqslant \langle f\rangle_{j,i} \quad \text{a.e. in } \Theta_{j,k},$$

or equivalently, that

$$\langle f\rangle_{i+1,k} \geqslant \langle f\rangle_{j,i} \quad \text{a.e. in } \Theta_{j,k}. \qquad \square$$

Our goal is to determine a system of N equations, coupled by the characteristic functions of the $N(N-1)/2$ coincidence sets, which is equivalent to problem (1).

This was done in [26] for the case $N = 2$ and in [1] for the case $N = 3$. The system for $N = 2$ is simply

$$\begin{cases} Au_1 = f_1 + \dfrac{f_2 - f_1}{2}\chi_{\{u_1=u_2\}}, \\ Au_2 = f_2 - \dfrac{f_2 - f_1}{2}\chi_{\{u_1=u_2\}}, \end{cases}$$

and for $N = 3$ it is the system (15). From these two examples we see that the determination of the coefficients of this system is not a very simple problem of combinatorics. We present the result for the case of general N in Theorem 4.5.

DEFINITION 4.3 Given $f_1, \dots, f_N \in L^q(\Omega)$ we define, for $j, k, i \in \{1, \dots, N\}$, with $j < k$ and $j \leqslant i \leqslant k$,

$$b_i^{j,k}[f] = \begin{cases} \langle f\rangle_{j,k} - \langle f\rangle_{j,k-1} & \text{if } i = j, \\ \langle f\rangle_{j,k} - \langle f\rangle_{j+1,k} & \text{if } i = k, \\ \dfrac{2}{(k-j)(k-j+1)}\left(\langle f\rangle_{j+1,k-1} - \dfrac{1}{2}(f_j + f_k)\right) & \text{if } j < i < k. \end{cases}$$

Observe that, if $j < i < k$, then $b_i^{j,k}[f]$ does not depend on i. It is also not difficult to see that $\sum_{i=j}^{k} b_i^{j,k}[f] = 0$. We first record some auxiliary results concerning the coefficients $b_i^{j,k}[f]$ that will be needed. From now on we drop the dependence of $b_i^{j,k}[f]$ on f in notation.

LEMMA 4.4

(i) If $j \leqslant l < r$ then

$$\sum_{k=l+1}^{r} b_j^{j,k} = \frac{r-l}{r-j+1}(\langle f\rangle_{l+1,r} - \langle f\rangle_{j,l}).$$

In particular $\sum_{k=l+1}^{r} b_j^{j,k}$ is positive if and only if the average of $f_{l+1}, \dots, f_r$ is greater than or equal to the average of $f_j, \dots, f_l$.

(ii) If $m < i$ then

$$\forall r \in \{i, \dots, N\} \qquad \sum_{k=i}^{r} b_i^{m,k} = b_r^{m,r}.$$

Proof. (i) We have

$$\begin{aligned}\sum_{k=l+1}^{r} b_j^{j,k} &= \sum_{k=l+1}^{r} (\langle f\rangle_{j,k} - \langle f\rangle_{j,k-1}) = \langle f\rangle_{j,r} - \langle f\rangle_{j,l}\\
&= \frac{f_j+\cdots+f_r}{r-j+1} - \frac{f_j+\cdots+f_l}{l-j+1}\\
&= \frac{f_j+\cdots+f_l}{r-j+1} + \frac{f_{l+1}+\cdots+f_r}{r-j+1} - \frac{f_j+\cdots+f_l}{l-j+1}\\
&= \frac{f_{l+1}+\cdots+f_r}{r-j+1} - \frac{(r-l)(f_j+\cdots+f_l)}{(r-j+1)(l-j+1)}\\
&= \frac{r-l}{r-j+1}\left(\frac{f_{l+1}+\cdots+f_r}{r-l} - \frac{f_j+\cdots+f_l}{l-j+1}\right)\\
&= \frac{r-l}{r-j+1}(\langle f\rangle_{l+1,r} - \langle f\rangle_{j,l}).\end{aligned}$$

(ii) We prove the equality by induction on r. If $r=i$, the equality is trivial. For $r>i$ we have

$$\begin{aligned}\sum_{k=i}^{r+1} b_i^{m,k} &= \sum_{k=i}^{r} b_i^{m,k} + b_i^{m,r+1}\\
&= b_r^{m,r} + b_i^{m,r+1} \quad \text{by induction hypothesis}\\
&= \langle f\rangle_{m,r} - \langle f\rangle_{m+1,r} + \frac{2}{(r-m+1)(r-m+2)}\left(\langle f\rangle_{m+1,r} - \frac{1}{2}(f_m+f_{r+1})\right)\\
&= \frac{f_m+\cdots+f_r}{r-m+1} - \frac{f_{m+1}+\cdots+f_r}{r-m}\\
&\quad + \frac{2(f_{m+1}+\cdots+f_r)}{(r-m)(r-m+1)(r-m+2)} - \frac{f_m+f_{r+1}}{(r-m+1)(r-m+2)}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=i}^{r+1} b_i^{m,k} &= \left(\frac{1}{r-m+1} - \frac{1}{(r-m+1)(r-m+2)}\right) f_m - \frac{1}{(r-m+1)(r-m+2)} f_{r+1}\\
&\quad + \left(\frac{1}{r-m+1} - \frac{1}{r-m} + \frac{2}{(r-m)(r-m+1)(r-m+2)}\right)(f_{m+1}+\cdots+f_r)\\
&= \frac{f_m}{r-m+2} - \frac{f_{r+1}}{(r-m+1)(r-m+2)} - \frac{f_{m+1}+\cdots+f_r}{(r-m+1)(r-m+2)}\\
&= \frac{f_m+\cdots+f_{r+1}}{r-m+2} - \frac{f_{m+1}+\cdots+f_{r+1}}{r-m+1} = b_{r+1}^{m,r+1}.\end{aligned}$$ □

We are now able to deduce the system of equations involving the characteristic functions of the coincidence sets which is equivalent to problem (1).

THEOREM 4.5

$$Au_i = f_i + \sum_{1\leqslant j<k\leqslant N,\ j\leqslant i\leqslant k} b_i^{j,k}\,\chi_{j,k} \quad \text{a.e. in } \Omega. \tag{45}$$

Proof. We prove that the equality is valid a.e. in $\Theta_{m,r}$ for m, r such that $1 \leqslant m \leqslant r \leqslant N$. This is enough because $\bigcup_{1\leqslant m\leqslant r\leqslant N} \Theta_{m,r} = \Omega$.

If $i \notin \{m, \dots, r\}$, then (45) results immediately from Proposition 4.2(i).

Suppose that $i \in \{m, \dots, r\}$. In view of Lemma 4.2, the equality (45) for $x \in \Theta_{m,r}$ becomes

$$f_i + \sum_{m\leqslant j<k\leqslant r,\ j\leqslant i\leqslant k} b_i^{j,k} = \langle f\rangle_{m,r}.$$

We now prove this equality by induction on $i - m$. If $i - m = 0$, then

$$\begin{aligned} f_i + \sum_{m\leqslant j<k\leqslant r,\ j\leqslant i\leqslant k} b_i^{j,k} &= f_m + \sum_{m<k\leqslant r} b_m^{m,k} \\ &= f_m + \sum_{m<k\leqslant r} (\langle f\rangle_{m,k} - \langle f\rangle_{m,k-1}) = \langle f\rangle_{m,r}. \end{aligned}$$

For the induction step, if $i - m > 0$, then

$$\begin{aligned} f_i + \sum_{m\leqslant j<k\leqslant r,\ j\leqslant i\leqslant k} b_i^{j,k} &= f_i + \sum_{m+1\leqslant j<k\leqslant r,\ j\leqslant i\leqslant r} b_i^{j,k} + \sum_{i\leqslant k\leqslant r} b_i^{m,k} \\ &= \langle f\rangle_{m+1,r} + \sum_{k=i}^{r} b_i^{m,k} \quad \text{by induction hypothesis} \\ &= \langle f\rangle_{m+1,r} + b_r^{m,r} \quad \text{by Lemma 4.4(ii)} \\ &= \langle f\rangle_{m,r}. \end{aligned}$$

□

We now state the main result of this section.

THEOREM 4.6 Given $n \in \mathbb{N}$, let $(u_1^n, \dots, u_N^n)$ denote the solution of problem (1) with given data $(f_1^n, \dots, f_N^n) \in [L^q(\Omega)]^N$, with q as in (3). Suppose that

$$f_i^n \underset{n}{\to} f_i \quad \text{in } L^q(\Omega),\ i = 1, \dots, N. \tag{46}$$

Then

$$u_i^n \underset{n}{\to} u_i \quad \text{in } W^{1,p}(\Omega),\ i = 1, \dots, N. \tag{47}$$

If, in addition, the limit forces satisfy

$$\langle f\rangle_{i,j} \neq \langle f\rangle_{j+1,k} \qquad \text{for all } i, j, k \in \{1, \dots, N\} \text{ with } i \leqslant j < k, \tag{48}$$

then, for any $1 \leqslant s < \infty$,

$$\forall j, k \in \{1, \dots, N\},\ j < k, \qquad \chi_{\{u_j^n=\cdots=u_k^n\}} \underset{n}{\to} \chi_{\{u_j=\cdots=u_k\}} \quad \text{in } L^s(\Omega). \tag{49}$$

Before proving the theorem we need another auxiliary lemma:

LEMMA 4.7 Let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ be such that $\sum_{r=j}^n a_r > 0$ for all $j = 1, \dots, n$. Then the inequality

$$a_1Y_1 + \cdots + a_nY_n \leqslant 0$$

with the restrictions $0 \leqslant Y_1 \leqslant \cdots \leqslant Y_n$ has only the trivial solution $Y_1 = \cdots = Y_n = 0$.

Proof. If $n = 1$ the conclusion is immediate. Supposing the result proved for n, let us prove it for $n + 1$:

$$0 \geqslant a_1Y_1 + \cdots + a_nY_n + a_{n+1}Y_{n+1} \geqslant a_1Y_1 + \cdots + a_nY_n + a_{n+1}Y_n$$

since $Y_{n+1} \geqslant Y_n \geqslant 0$ and $a_{n+1} > 0$. Then

$$0 \geqslant a_1Y_1 + \cdots + (a_n + a_{n+1})Y_n$$

and, because the result is true for n, we have $Y_1 = \cdots = Y_n = 0$ and, therefore, since $a_{n+1} > 0$, also $Y_{n+1} = 0$. $\square$

Proof of Theorem 4.6. The convergence (47) of the solutions is an immediate consequence of a theorem due to Mosco.

For simplicity, we write $\chi_{\{u_j=\cdots=u_k\}} = \chi_{j,k}$ and we denote $\chi_{\{u_j^n=\cdots=u_k^n\}}$ by $\chi_{j,k}^n$.

Let $j, k \in \{1, \ldots, N\}$ with $j < k$. Since $0 \leqslant \chi_{j,k} \leqslant 1$, there exists $\chi_{j,k}^* \in L^q(\Omega)$ such that $(\chi_{j,k}^n)_{n\in\mathbb{N}}$ converges to $\chi_{j,k}^*$ in $L^q(\Omega)$-weak. Of course we have

$$\begin{cases} 0 \leqslant \chi_{j,k}^* \leqslant 1, & \text{because } 0 \leqslant \chi_{j,k}^n \leqslant 1, \\ \chi_{m,r}^* \leqslant \chi_{j,k}^* \ (\text{if } m \leqslant j < k \leqslant r), & \text{because } \chi_{m,r}^n \leqslant \chi_{j,k}^n. \end{cases} \tag{50}$$

Moreover, letting $n \to \infty$ in the equality $\chi_{j,k}^n(u_j^n - u_k^n)^+ \equiv 0$, we conclude

$$\chi_{j,k}^*(u_j - u_k)^+ = 0 \quad \text{a.e. in } \Omega. \tag{51}$$

Consider now the system (45), with the coefficients b replaced by b_n, for data $f_1^n, \ldots, f_N^n$, with $n \in \mathbb{N}$,

$$Au_i^n = f_i^n + \sum_{j<k\leqslant N,\ j\leqslant i\leqslant k} (b_n)_i^{j,k}\ \chi_{j,k}^n \quad \text{a.e. in } \Omega,\ i = 1, \ldots, N.$$

Passing to the weak limit in $L^q(\Omega)$ as $n \to \infty$, we have

$$Au_i = f_i + \sum_{j<k\leqslant N,\ j\leqslant i\leqslant k} b_i^{j,k}\ \chi_{j,k}^* \quad \text{a.e. in } \Omega,\ i = 1, \ldots, N.$$

Subtracting the equality (45) for the limit solution from this one, we obtain

$$\sum_{j<k\leqslant N,\, j\leqslant i\leqslant k} b_i^{j,k}(\chi_{j,k} - \chi_{j,k}^*) = 0 \quad \text{a.e. in } \Omega,\ i = 1, \ldots, N. \tag{52}$$

For $k > j$, let $Y_{j,k}$ denote $\chi_{j,k} - \chi_{j,k}^*$. To complete the proof we only need to show that, for $j < k$, $Y_{j,k} \equiv 0$, i.e., $(\chi_{j,k}^n)_{n\in\mathbb{N}}$ converges to $\chi_{j,k}$ in $L^q(\Omega)$-weak.

From equation (51) we know that

$$\forall j < k \quad Y_{j,k} \equiv 0 \quad \text{in } \{u_j \neq u_k\} = \{u_j > u_k\}. \tag{53}$$

Fix j_0 and k_0 such that $j_0 < k_0$. Using (53), we only need to see that $Y_{j_0,k_0} \equiv 0$ in $I_{j_0,k_0} = \{u_{j_0} = \cdots = u_{k_0}\}$. It is then enough to prove this in two cases:

(i) in $\Theta_{j_0,r}$ for $r \geqslant j_0$;
(ii) in $\Theta_{m,r}$ for $m < j_0$ and $r \geqslant k_0$.

In the first case, using (53), we have $Y_{j,k} \equiv 0$ in $\Theta_{j_0,r}$ if $j < j_0$ or $k > r$. So, letting $i = j_0$ in equation (52), we have, in $\Theta_{j_0,r}$,

$$0 = \sum_{j<k\leqslant N,\ j\leqslant j_0\leqslant k} b_{j_0}^{j,k} Y_{j,k} = \sum_{j_0\leqslant j<k\leqslant N,\ j\leqslant j_0\leqslant k\leqslant r} b_{j_0}^{j,k} Y_{j,k} = \sum_{k=j_0+1}^{r} b_{j_0}^{j_0,k} Y_{j_0,k}.$$

We can now apply Lemma 4.7 to conclude that $Y_{j_0,k} = 0$ in $\Theta_{j_0,r}$ for $k \in \{j_0+1, \dots, r\}$, since

- for $x \in \Theta_{j_0,r}$, $Y_{j_0,r}(x) = 1 - \chi^*_{j_0,k}(x)$ and, using (50), $Y_{j_0,j_0+1}(x) \leqslant \cdots \leqslant Y_{j_0,r}(x)$;
- for $l \geqslant j_0$, by Lemma 4.4(i),

$$\sum_{k=l+1}^{r} b_{j_0}^{j_0,k} = \frac{r-l}{r-j_0+1}(\langle f\rangle_{l+1,r} - \langle f\rangle_{j_0,l}),$$

which is positive, by Proposition 4.2(ii), as $x \in \Theta_{j_0,r}$, and (48).

In the second case, in $\Theta_{m,r}$ ($m < j_0$ and $r \geqslant k_0$),

$$\begin{aligned} 0 \leqslant Y_{j_0,k_0} &= \chi_{j_0,k_0} - \chi^*_{j_0,k_0} \\ &= 1 - \chi^*_{j_0,k_0} \quad \text{since } m < j_0 < k_0 \leqslant r \\ &\leqslant 1 - \chi^*_{m,k_0} \quad \text{by (50)} \\ &= \chi_{m,k_0} - \chi^*_{m,k_0} \\ &= Y_{m,k_0} \\ &= 0 \quad \text{as in the previous case.} \end{aligned}$$

Notice that, since χ_{j_0,k_0} is a characteristic function, $(\chi^n_{j_0,k_0})_{n\in\mathbb{N}}$ converges in fact to χ_{j_0,k_0} in $L^s(\Omega)$-strong, for all $1 \leqslant s < \infty$. ☐

REMARK 4.8 Arguing as in Theorem *2.5*, under the strong monotonicity assumption (9), it is easy to show the following continuous dependence on the data:

$$\sum_{i=1}^{N} \|u_i^n - u_i\|_{W_0^{1,p}(\Omega)} \leqslant C_q \sum_{i=1}^{N} \|f_i^n - f_i\|_{L^q(\Omega)},$$

for q defined as in (3). However, a corresponding L^1 estimate for the characteristic functions of the coincidence sets, similar to the one in the obstacle problem ([22], [23]), seems more difficult to obtain.

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