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Error analysis of a finite element method for the Willmore flow of graphs

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The evolution of two-dimensional graphs under Willmore flow is approximated by a continuous-in-time finite element method. The highly nonlinear fourth order problem is split into two coupled second order problems using height and a weighted mean curvature as variables. We prove a-priori error estimates for the resulting time-continuous scheme and present results of test calculations.

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1. Introduction

The purpose of this paper is to analyze a finite element scheme for the approximation of two-dimensional surfaces in $\mathbb{R}^3$ which evolve according to Willmore flow. This flow can be interpreted as the L^2-gradient flow for the Willmore functional

$$W(f) := \frac{1}{2}\int_{\Gamma} H^2 \, \mathrm{d}A, \qquad \Gamma = f(M),$$

where $f : M \to \mathbb{R}^3$ is a smooth immersion, $H = \kappa_1 + \kappa_2$ denotes the mean curvature of Γ and $\mathrm{d}A$ is the area element. Furthermore, M denotes a fixed two-dimensional surface with or without boundary. Considering normal variations $f_\epsilon(x) := f(x) + \epsilon\phi(x)\nu(x)$, $x \in M$, where ν is a unit normal field to Γ and $\phi : M \to \mathbb{R}$ is smooth and vanishes near ∂M, one obtains the formula

$$\langle W'(f), \phi\rangle := \frac{\mathrm{d}}{\mathrm{d}\epsilon} W(f_\epsilon)_{|\epsilon=0} = -\int_{\Gamma} \phi\left(\Delta_\Gamma H + \frac{1}{2}H^3 - 2HK\right)\mathrm{d}A.$$

Here, $K = \kappa_1\kappa_2$ is the Gauss curvature of Γ and Δ_Γ denotes the Laplace–Beltrami operator. The sign of H is chosen in such a way that $H > 0$ for a sphere with outward pointing normal.

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For details on this calculation and more information on the Willmore functional see [18]. Given a smooth, oriented surface $\Gamma_0 \subset \mathbb{R}^3$, the Willmore flow problem now consists in finding a family $(\Gamma(t))_{t\in[0,T]}$ of smooth, oriented surfaces which satisfy

$$V = \Delta_\Gamma H + \frac{1}{2}H^3 - 2HK \quad \text{on } \Gamma(t), \tag{1.1}$$

$$\Gamma(0) = \Gamma_0, \tag{1.2}$$

where V denotes the normal velocity.

In this paper we shall assume that the surfaces $\Gamma(t)$ are graphs over some bounded domain $\Omega \subset \mathbb{R}^2$, i.e.

$$\Gamma(t) = \{(x, u(x,t)) \mid x \in \Omega\}$$

oriented by the unit normal field

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}, \tag{1.3}$$

where $\nabla u = (u_{x_1}, u_{x_2})$. In what follows we shall assume that Ω has a smooth boundary.

It will be convenient to use the abbreviation

$$Q := \sqrt{1+|\nabla u|^2} \tag{1.4}$$

for the area element. The evolution law (1.1) gives rise to a partial differential equation for the height function u. In order to write down this equation we note that the quantities V, H, K and $\Delta_\Gamma H$ appearing in (1.1) are expressed in terms of u as follows:

$$V = -\frac{u_t}{Q}, \quad H = \nabla\cdot\left(\frac{\nabla u}{Q}\right), \quad K = \frac{\det D^2 u}{Q^4}, \tag{1.5}$$

$$\Delta_\Gamma H = \frac{1}{Q}\nabla\cdot\left(\left(QI - \frac{\nabla u\otimes\nabla u}{Q}\right)\nabla H\right), \tag{1.6}$$

where D^2u contains the second space derivatives. The last relation can be rewritten as

$$\Delta_\Gamma H = \nabla\cdot\left(\frac{1}{Q}\left(I - \frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla(QH)\right) - H\nabla\cdot\left(\frac{1}{Q}\left(I - \frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla Q\right). \tag{1.7}$$

Using the expression for H we conclude that

$$\frac{1}{Q}\left(I - \frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla Q = \frac{1}{Q}\left(\nabla Q - \frac{\Delta u}{Q}\nabla u\right) + H\frac{\nabla u}{Q} \tag{1.8}$$

and a calculation shows that

$$\nabla\cdot\left(\frac{1}{Q}\left(\nabla Q - \frac{\Delta u}{Q}\nabla u\right)\right) = -2K. \tag{1.9}$$

Inserting (1.8) and (1.9) into (1.7) we obtain

$$\begin{aligned}\Delta_\Gamma H &= \nabla\cdot\left(\frac{1}{Q}\left(I - \frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla(QH)\right) + 2HK - H\nabla\cdot\left(H\frac{\nabla u}{Q}\right)\\ &= \nabla\cdot\left(\frac{1}{Q}\left(I - \frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla(QH)\right) + 2HK - \frac{1}{2}\nabla\cdot\left(\frac{H^2}{Q}\nabla u\right) - \frac{1}{2}H^3.\end{aligned}$$

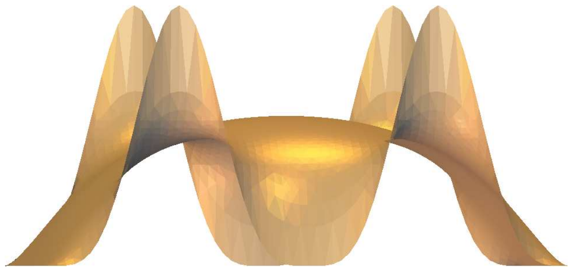

FIG. 1. A stationary solution of (1.10)–(1.11): Willmore surface with prescribed boundary and mean curvature equal to zero on the boundary. The solution was computed with the time dependent algorithm (4.1)–(4.2).

Comparing this expression with (1.1) and recalling that $V = -u_t/Q$ we obtain the following fourth order parabolic PDE for u:

$$u_t + Q\nabla \cdot \left(\frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right)\nabla(QH)\right) - \frac{1}{2}Q\nabla \cdot \left(\frac{H^2}{Q}\nabla u\right) = 0 \quad \text{in } \Omega \times (0,T). \tag{1.10}$$

Note that the above equation has (after division by Q) a nice divergence structure in which the Gauss curvature K no longer appears. This structure was exploited by Droske and Rumpf in [8] for a level set approach to Willmore flow. We shall use the boundary conditions

$$u = g, \qquad H = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.11}$$

for a given smooth function $g : \bar{\Omega} \to \mathbb{R}$. The condition (1.11) is motivated by calculations in [14], where it is shown that if $\Gamma = f(M)$ is stationary for W with respect to variations which keep the boundary of Γ fixed, then necessarily $H = 0$ on $\partial\Gamma$. Finally, we prescribe the initial condition

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega. \tag{1.12}$$

In what follows we shall assume that the initial-boundary value problem (1.10)–(1.12) has a unique solution u, which satisfies

$$u \in L^\infty((0,T); H^{4,\infty}(\Omega)) \cap L^2((0,T); H^5(\Omega)), \tag{1.13}$$

$$u_t \in L^\infty((0,T); H^{2,\infty}(\Omega)) \cap L^2((0,T); H^3(\Omega)), \tag{1.14}$$

$$u_{tt} \in L^\infty((0,T); L^\infty(\Omega)) \cap L^2((0,T); H^1(\Omega)). \tag{1.15}$$

Since the focus of our work is on the analysis of a finite element scheme approximating solutions of (1.10)–(1.12), we do not address the question of whether a function u satisfying the regularity assumptions (1.13)–(1.15) exists. We expect a positive answer at least for small T, but at the moment little seems to be known about boundary value problems for Willmore flow. The situation is different for the evolution of closed surfaces. In [17] it is shown that a unique local solution of (1.1), (1.2) exists provided that Γ_0 is a compact closed immersed and orientable $C^{2,\alpha}$-surface in $\mathbb{R}^3$. The solution exists globally in time if Γ_0 is sufficiently close to a sphere in $C^{2,\alpha}$. Using different methods, Kuwert & Schätzle ([11]) obtain global existence of solutions provided that $\int_{\Gamma_0} |A^\circ|^2$ is sufficiently small, where A° denotes the trace-free part of the second fundamental form. They were subsequently able to remove the smallness assumption and to prove the existence of a global smooth solution provided that $W(f_0) \leqslant 16\pi$, where $\Gamma_0 = f_0(S^2)$ (see [12] and note that our definition differs from the one used in [12] by a factor of 2). The numerical evidence of [13] indicates that the above condition is optimal in the sense that the flow develops a singularity if the initial surface has energy greater than 16π.

2. Variational formulation and discretization

In order to approximate solutions of (1.10)–(1.12) by a finite element scheme, we need to derive a variational formulation of (1.10). To begin, let us introduce

$$E(p)_{ij} := \frac{1}{\sqrt{1+|p|^2}}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right), \quad i, j = 1, 2,\ p \in \mathbb{R}^2. \tag{2.1}$$

It is not difficult to verify that

$$|E(q) - E(p)| \leqslant c|q - p| \qquad \forall p, q \in \mathbb{R}^2, \tag{2.2}$$

$$E(p)q \cdot q \geqslant \frac{|q|^2}{\sqrt{1+|p|^2}^3} \quad \forall p, q \in \mathbb{R}^2. \tag{2.3}$$

Now, (1.10) suggests using $w = -QH$ rather than H as the second variable in a splitting method (cf. [8]). If we divide (1.10) by Q, multiply by a test function $\varphi \in H_0^1(\Omega)$ and integrate by parts we are led to

$$\int_\Omega \frac{u_t \varphi}{Q} + \int_\Omega E(\nabla u)\nabla w \cdot \nabla\varphi + \frac{1}{2}\int_\Omega \frac{w^2}{Q^3}\nabla u \cdot \nabla\varphi = 0 \quad \forall\varphi \in H_0^1(\Omega), \tag{2.4}$$

$$\int_\Omega \frac{w\zeta}{Q} - \int_\Omega \frac{\nabla u \cdot \nabla\zeta}{Q} = 0 \quad \forall\zeta \in H_0^1(\Omega), \tag{2.5}$$

where the second relation stems from the definition of w and (1.5).

Before we proceed and base the discretization of our problem on (2.4), (2.5), we want to deduce the decrease in time of the Willmore energy $\frac{1}{2}\int_{\Gamma(t)} H^2\,\mathrm{d}A = \frac{1}{2}\int_\Omega H^2 Q$ from these relations.

Using $\varphi = u_t$ in (2.4) and observing that $\nabla u \cdot \nabla u_t = Q_t Q$ we deduce

$$\int_\Omega \frac{u_t^2}{Q} + \int_\Omega E(\nabla u)\nabla w \cdot \nabla u_t + \frac{1}{2}\int_\Omega \frac{w^2 Q_t}{Q^2} = 0. \tag{2.6}$$

Next, differentiating (2.5) with respect to time gives

$$\int_\Omega \frac{w_t\zeta}{Q} - \int_\Omega \frac{w\zeta Q_t}{Q^2} - \int_\Omega E(\nabla u)\nabla u_t \cdot \nabla\zeta = 0 \quad \forall\zeta \in H_0^1(\Omega). \tag{2.7}$$

If we insert $\zeta = w$ into (2.7) and combine the resulting identity with (2.6) we obtain

$$0 = \int_\Omega \frac{u_t^2}{Q} - \frac{1}{2}\int_\Omega \frac{w^2 Q_t}{Q^2} + \int_\Omega \frac{w w_t}{Q} = \int_\Omega \frac{u_t^2}{Q} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{w^2}{Q}. \tag{2.8}$$

Observing that $\int_\Omega w^2/Q = \int_\Omega H^2 Q$ we see that the energy decreases in time.

Let us now turn to the discretization in space. The variational formulation (2.4)–(2.5) suggests using a second order splitting scheme with linear finite elements in order to approximate the pair (u, w). Let $\mathcal{T}_h$ be a family of triangulations of Ω with maximum mesh size $h := \max_{S\in\mathcal{T}_h} \operatorname{diam}(S)$. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are allowed to be curved. Furthermore we suppose that the triangulation is quasiuniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $S \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. The discrete space is defined by

$$X_h := \{v_h \in C^0(\bar{\Omega}) \mid v_h \text{ is a linear polynomial on each } S \in \mathcal{T}_h\},$$

with suitable modifications for boundary elements. There exists an interpolation operator $I_h : H^2(\Omega) \to X_h$ such that

$$\|v - I_h v\| + h\|\nabla(v - I_h v)\| \leqslant ch^2\|v\|_{H^2(\Omega)} \quad \text{for all } v \in H^2(\Omega). \tag{2.9}$$

Furthermore, let $X_{h0} := X_h \cap H_0^1(\Omega)$ and suppose that $I_h v \in X_{h0}$ for $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Here and throughout the paper we will denote the $L^2(\Omega)$-norm by $\|\cdot\|$.

Our discrete problem now reads: find $(u_h(t), w_h(t))$, $0 \leqslant t \leqslant T$, such that $u_h(t) - I_h g \in X_{h0}$, $w_h(t) \in X_{h0}$, $u_h(0) = u_{0h} \in X_{h0}$ and

$$\int_\Omega \frac{u_{ht}\varphi_h}{Q_h} + \int_\Omega E(\nabla u_h)\nabla w_h \cdot \nabla\varphi_h + \frac{1}{2}\int_\Omega \frac{w_h^2}{Q_h^3}\nabla u_h \cdot \nabla\varphi_h = 0 \quad \forall\varphi_h \in X_{h0}, \tag{2.10}$$

$$\int_\Omega \frac{w_h\zeta_h}{Q_h} - \int_\Omega \frac{\nabla u_h \cdot \nabla\zeta_h}{Q_h} = 0 \quad \forall\zeta_h \in X_{h0}. \tag{2.11}$$

Here, $Q_h = \sqrt{1 + |\nabla u_h|^2}$.

LEMMA 2.1 The system (2.10), (2.11) has a unique solution (u_h, w_h) on $[0, T]$ for all $T < \infty$.

Proof. Local existence on an interval $[0, t_h)$ follows from the theory of ordinary differential equations. Since $u_h(t), w_h(t)$ have values in a finite-dimensional space, it is sufficient to bound some norm of (u_h, w_h) in order to obtain existence on $[0, T]$. If we repeat the argument which led to (2.8) in the discrete setting, the result is

$$\int_0^t \int_\Omega \frac{u_{ht}^2}{Q_h} + \frac{1}{2}\int_\Omega \frac{w_h^2}{Q_h} \leqslant C(h, u_{0h}), \quad 0 \leqslant t < t_h. \tag{2.12}$$

Using $\zeta_h = u_{ht} \in X_{h0}$ in (2.11) we obtain $(\mathrm{d}/\mathrm{d}t)\int_\Omega Q_h = \int_\Omega w_h u_{ht}/Q_h$, which combined with (2.12) yields

$$\int_\Omega Q_h(t) \leqslant c + \left(\int_0^t \int_\Omega \frac{w_h^2}{Q_h}\right)^{1/2}\left(\int_0^t \int_\Omega \frac{u_{ht}^2}{Q_h}\right)^{1/2} \leqslant c(h, T, u_{0h}), \quad 0 \leqslant t < t_h.$$

This implies that $Q_h(x, t) \leqslant c(h, T)$ uniformly in $(x, t) \in \Omega \times [0, t_h)$, and (2.12) then shows that $\|u_h(t)\|$, $\|w_h(t)\|$ remain bounded on $[0, t_h)$. We leave the proof of uniqueness to the reader. □

Our main result is the following error estimate:

THEOREM 2.2 Assume that (1.10)–(1.12) has a unique solution u on the interval $[0, T]$, which satisfies (1.13)–(1.15). Also suppose that $u_{0h} = \widehat{u}_{0h}$, where $\widehat{u}_{0h}$ is defined as the projection of u_0 (see (2.17)). Then

$$\sup_{0\leqslant t\leqslant T} \|(u-u_h)(t)\| + \sup_{0\leqslant t\leqslant T} \|(w-w_h)(t)\| \leqslant ch^2|\log h|^2, \tag{2.13}$$

$$\sup_{0\leqslant t\leqslant T} \|\nabla(u-u_h)(t)\| \leqslant ch, \tag{2.14}$$

$$\int_0^T \|u_t - u_{ht}\|^2 \,\mathrm{d}t \leqslant ch^4|\log h|^4, \tag{2.15}$$

$$\int_0^T \|\nabla(w-w_h)\|^2 \,\mathrm{d}t \leqslant ch^2. \tag{2.16}$$

REMARK 2.3 We expect a similar result if the function g in (1.11) is allowed to depend on time or if we consider a Neumann boundary condition for u. Prescribing an inhomogeneous Dirichlet condition for H or replacing it by a Neumann condition is more difficult since our analysis uses the variable $w = -QH$, so that information on Q on $\partial\Omega \times (0, T)$ is required in order to determine the boundary condition for w.

A detailed proof of Theorem 2.2 will be given in Section 3. One of the crucial points of our error analysis is the use of suitable nonlinear Ritz projections of u and w which we introduce next. To begin, let $\widehat{u}_h$ be defined by: $\widehat{u}_h - I_h g \in X_{h0}$ and

$$\int_\Omega \frac{\nabla\widehat{u}_h \cdot \nabla\zeta_h}{\widehat{Q}_h} = \int_\Omega \frac{\nabla u \cdot \nabla\zeta_h}{Q} \quad \forall\zeta_h \in X_{h0}, \tag{2.17}$$

where $\widehat{Q}_h = \sqrt{1+|\nabla\widehat{u}_h|^2}$. Note that t is just a parameter. Estimates for the error

$$\rho_u := u - \widehat{u}_h$$

in the time-independent case were first carried out in [10], L^∞-estimates are due to [15], [9]. For functions u which depend on time it was proved in [4], [5] that

$$\sup_{0\leqslant t\leqslant T} \|\rho_u(t)\| + h \sup_{0\leqslant t\leqslant T} \|\nabla\rho_u(t)\| \leqslant ch^2, \tag{2.18}$$

$$\sup_{0\leqslant t\leqslant T} \|\rho_u(t)\|_{L^\infty} + h \sup_{0\leqslant t\leqslant T} \|\nabla\rho_u(t)\|_{L^\infty} \leqslant ch^2|\log h|, \tag{2.19}$$

$$\sup_{0\leqslant t\leqslant T} \|\rho_{ut}(t)\| \leqslant ch^2|\log h|^2, \tag{2.20}$$

$$\sup_{0\leqslant t\leqslant T} \|\nabla\rho_{ut}(t)\| \leqslant ch. \tag{2.21}$$

With the help of $\widehat{u}_h$, we next define a projection $\widehat{w}_h \in X_{h0}$ of w as follows:

$$\int_\Omega E(\nabla\widehat{u}_h)\nabla\widehat{w}_h\cdot\nabla\varphi_h = \int_\Omega E(\nabla u)\nabla w\cdot\nabla\varphi_h + \frac{1}{2}\int_\Omega w^2\left(\frac{\nabla u}{Q^3} - \frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)\cdot\nabla\varphi_h \quad \forall\varphi_h \in X_{h0}. \tag{2.22}$$

The proof of the following bounds on the error

$$\rho_w := w - \widehat{w}_h$$

will be given in Lemma A.1 of the Appendix:

$$\sup_{0\leqslant t\leqslant T} \|\nabla\rho_w(t)\| \leqslant ch, \tag{2.23}$$

$$\sup_{0\leqslant t\leqslant T} \|\rho_w(t)\| \leqslant ch^2|\log h|, \tag{2.24}$$

$$\sup_{0\leqslant t\leqslant T} \|\nabla\rho_{wt}(t)\| \leqslant ch, \tag{2.25}$$

$$\sup_{0\leqslant t\leqslant T} \|\rho_{wt}(t)\| \leqslant ch^2|\log h|^2. \tag{2.26}$$

Note that (1.13), (1.14), (2.19)–(2.21), (2.23)–(2.26) together with interpolation and inverse estimates imply that

$$\|\widehat{u}_h\|_{W^{1,\infty}}, \|\widehat{u}_{ht}\|_{W^{1,\infty}}, \|\widehat{w}_h\|_{W^{1,\infty}}, \|\widehat{w}_{ht}\|_{W^{1,\infty}} \leqslant c \tag{2.27}$$

uniformly in h.

We end this section with a few remarks on numerical approaches to Willmore flow and related problems. In [13], a finite difference scheme is derived to approximate axisymmetric solutions of evolution laws $V = f(\kappa_1, \kappa_2)$. The Willmore flow is studied in detail and numerical evidence is provided that the flow may develop singularities in finite time. [8] derives a level set formulation using the level set function and a weighted mean curvature as variables. The formal similarity to the graph approach motivated our choice of variables. For the parametric approach to Willmore flow, [16] derives a variational form which employs position and mean curvature vector as variables and allows the use of linear finite elements to discretize in space. This approach is subsequently extended in [3] to surfaces with boundaries and applied to problems in surface restoration. We also refer to [2], where numerical simulations of anisotropic surface diffusion are carried out for a surface energy which consists of a strongly anisotropic nonconvex part and the Willmore functional (weighted by a small factor).

Let us finally mention evolution by surface diffusion,

$$V = \Delta_\Gamma H \quad \text{on } \Gamma(t),$$

which coincides with Willmore flow in the highest order term, but which is simpler in that the nonlinear terms in the principal curvatures are absent. In the graph case, it is possible to write down a splitting method using height and mean curvature as variables. Finite element error bounds both for time-continuous and fully discrete schemes have been derived in [6], [1], [7].

3. Proof of Theorem 2.2

Before going into details let us sketch the main ideas in the proof of Theorem 2.2. Combining the variational identities (2.4), (2.5) and (2.10), (2.11) with (2.17) and (2.22) we shall derive corresponding relations for $e_u := \widehat{u}_h - u_h$ and $e_w := \widehat{w}_h - w_h$. We then try to mimick the derivation of the a-priori estimate (2.8) in order to gain control on

$$\int_0^T \|e_{ut}\|^2 \,\mathrm{d}t + \sup_{t\in(0,T)} \|e_w(t)\|^2. \tag{3.1}$$

This program is started in Lemmas 3.4 and 3.5. Unlike the case of (2.6)–(2.8) the integrals involving Q_t and the matrix E will no longer cancel and require a subtle analysis. Another difficulty stems from the fact that control of the quantities in (3.1) requires a uniform bound on the discrete area element Q_h. The proof of such a bound heavily relies on the choice of the nonlinear Ritz projections introduced above and a superconvergence property.

To begin, let (u_h, w_h) be the discrete solution and denote by $Q_h := \sqrt{1 + |\nabla u_h|^2}$ the discrete area element. Furthermore, define

$$C_0 := \sup_{x \in \Omega,\, 0 \leqslant t \leqslant T} Q(x,t), \qquad C_1 := \sup_{x \in \Omega,\, 0 \leqslant t \leqslant T} |w(x,t)|.$$

The initial condition together with a continuity argument then yields, for small h,

$$\sup_{x \in \Omega} Q_h(x,t) \leqslant 2C_0, \qquad \sup_{x \in \Omega} |w_h(x,t)| \leqslant 2C_1 \tag{3.2}$$

for small t. Define

$$T_h := \sup\{t \in [0,T] \mid (3.2) \text{ holds on } [0,t]\}. \tag{3.3}$$

Our strategy is to first prove the error estimates on the interval $[0, T_h)$ and subsequently use these bounds to show that $T_h = T$. Thus, in what follows we shall assume (3.2). Let us decompose the errors $u - u_h$ and $w - w_h$ according to

$$u - u_h = \rho_u + e_u, \quad \text{where} \quad e_u = \widehat{u}_h - u_h,$$
$$w - w_h = \rho_w + e_w, \quad \text{where} \quad e_w = \widehat{w}_h - w_h.$$

Furthermore, it will be convenient to work with the following discrete normals:

$$\widehat{\nu}_h := \frac{(\nabla \widehat{u}_h, -1)}{\widehat{Q}_h}, \qquad \nu_h := \frac{(\nabla u_h, -1)}{Q_h}.$$

We infer from Proposition 2 in [5] that

$$|\widehat{\nu}_h - \nu_h| \leqslant |\nabla(\widehat{u}_h - u_h)| \leqslant (1 + \sup_{\Omega} |\nabla \widehat{u}_h|) Q_h |\widehat{\nu}_h - \nu_h|,$$

which, combined with (3.2) and (2.27), yields

$$|\widehat{\nu}_h - \nu_h| \leqslant |\nabla e_u| \leqslant (1 + c) 2C_0 |\widehat{\nu}_h - \nu_h|. \tag{3.4}$$

In order to make the error analysis more transparent we split it up into a series of auxiliary results.

LEMMA 3.1 Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and that $f \in H_0^1(\Omega)$. Then

$$\int_\Omega (F(\nabla u) - F(\nabla \widehat{u}_h)) f = -\int_\Omega \nabla \cdot (f F'(\nabla u)) \rho_u + R,$$

where R satisfies

$$|R| \leqslant C h^2 |\log h| \|f\|.$$

Proof. Clearly,

$$\int_\Omega (F(\nabla u) - F(\nabla \widehat{u}_h))f = \int_\Omega F'(\nabla u)\cdot\nabla\rho_u f + \int_0^1\int_\Omega (F'(\nabla u - s\nabla\rho_u) - F'(\nabla u))\cdot\nabla\rho_u f$$
$$= -\int_\Omega \nabla\cdot(fF'(\nabla u))\rho_u + R,$$

where

$$|R| \leqslant c\|\nabla\rho_u\|_{L^\infty}\|\nabla\rho_u\|\|f\| \leqslant Ch^2|\log h|\|f\|$$

by (1.13), (2.18), (2.19), (2.27) and since F is twice continuously differentiable. □

LEMMA 3.2 For every $\epsilon > 0$ there exists c_ϵ such that

$$\|\nabla e_u(t)\|^2 \leqslant \epsilon\|e_w(t)\|^2 + c_\epsilon\|e_u(t)\|^2 + ch^4|\log h|^2, \quad 0 \leqslant t < T_h.$$

Proof. In view of (2.5), (2.11) and (2.17) we have

$$\int_\Omega\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h}\right)\cdot\nabla\varphi_h = \int_\Omega\left(\frac{w}{Q} - \frac{w_h}{Q_h}\right)\varphi_h \quad \forall\varphi_h \in X_{h0}.$$

Using $\varphi_h = \widehat{u}_h - u_h = e_u \in X_{h0}$ and applying Lemma 3.1 with $F(p) = 1/\sqrt{1+|p|^2}$ and $f = we_u$ we derive

$$\int_\Omega\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h}\right)\cdot\nabla e_u = \int_\Omega\left(\frac{1}{Q} - \frac{1}{\widehat{Q}_h}\right)we_u + \int_\Omega\left(\frac{1}{\widehat{Q}_h} - \frac{1}{Q_h}\right)we_u + \int_\Omega\frac{(w-w_h)e_u}{Q_h}$$
$$\leqslant \int_\Omega\nabla\cdot\left(we_u\frac{\nabla u}{Q^3}\right)\rho_u + ch^2|\log h|\|we_u\| + c\|\nabla e_u\|\,\|e_u\| + (\|e_w\| + \|\rho_w\|)\|e_u\|$$
$$\leqslant ch^2\|\nabla e_u\| + c(\|\nabla e_u\| + \|e_w\| + h^2|\log h|)\|e_u\|, \tag{3.5}$$

in view of (1.13), (2.18) and (2.24). Observing that $(\nabla e_u, 0)^t = \widehat{Q}_h\widehat{\nu}_h - Q_h\nu_h$ we may write

$$\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h}\right)\cdot\nabla e_u = (\widehat{\nu}_h - \nu_h)\cdot(\widehat{Q}_h\widehat{\nu}_h - Q_h\nu_h) = \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2(\widehat{Q}_h + Q_h) \geqslant c_0\|\nabla e_u\|^2$$

by (2.27) and (3.4). Combining this inequality with (3.5) implies the result. □

LEMMA 3.3

$$\|\nabla e_w(t)\|^2 \leqslant c(\|\nabla e_u(t)\|^2 + \|e_{ut}(t)\|^2 + \|e_w(t)\|^2 + h^4|\log h|^4), \quad 0 \leqslant t < T_h.$$

Proof. First note that in view of the definition of $\widehat{w}_h$ we have, for all $\varphi_h \in X_{h0}$,

$$\int_\Omega\frac{\widehat{u}_{ht}\varphi_h}{Q_h} + \int_\Omega E(\nabla\widehat{u}_h)\nabla\widehat{w}_h\cdot\nabla\varphi_h + \frac{1}{2}\int_\Omega\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h\cdot\nabla\varphi_h$$
$$= \int_\Omega\frac{(\widehat{u}_{ht} - u_t)\varphi_h}{Q_h} + \int_\Omega u_t\left(\frac{1}{Q_h} - \frac{1}{Q}\right)\varphi_h + \frac{1}{2}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla\varphi_h,$$

from which we infer

$$\int_\Omega \frac{e_{ut}\varphi_h}{Q_h} + \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla\varphi_h + \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla\varphi_h$$
$$= -\int_\Omega \frac{\rho_{ut}\varphi_h}{Q_h} + \int_\Omega u_t\left(\frac{1}{Q_h} - \frac{1}{Q}\right)\varphi_h + \frac{1}{2}\int_\Omega (\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla\varphi_h. \qquad (3.6)$$

Inserting $\varphi_h = e_w$ into (3.6) we derive

$$\int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_w = -\int_\Omega \frac{e_{ut}e_w}{Q_h} - \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_w$$
$$- \int_\Omega \frac{\rho_{ut}e_w}{Q_h} + \int_\Omega u_t\left(\frac{1}{Q_h} - \frac{1}{Q}\right)e_w + \frac{1}{2}\int_\Omega (\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_w.$$

Using (2.3), (2.2), (2.27) and the fact that $Q_h \leqslant 2C_0$ we obtain

$$\int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_w$$
$$= \int_\Omega E(\nabla u_h)\nabla e_w\cdot\nabla e_w + \int_\Omega (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla\widehat{w}_h\cdot\nabla e_w$$
$$\geqslant \frac{1}{\sqrt{1+4C_0^2}^{\,3}}\|\nabla e_w\|^2 - c\|\nabla e_u\|\|\nabla e_w\| \geqslant \frac{1}{2\sqrt{1+4C_0^2}^{\,3}}\|\nabla e_w\|^2 - c\|\nabla e_u\|^2.$$

Furthermore, we infer from Lemma 3.1 that

$$\int_\Omega u_t\left(\frac{1}{Q_h} - \frac{1}{Q}\right)e_w = \int_\Omega\left(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h}\right)u_te_w - \int_\Omega\left(\frac{1}{Q} - \frac{1}{\widehat{Q}_h}\right)u_te_w$$
$$\leqslant c\|\nabla e_u\|\|e_w\| - \int_\Omega \nabla\cdot\left(u_te_w\frac{\nabla u}{Q^3}\right)\rho_u + ch^2|\log h|\|u_te_w\|$$
$$\leqslant c\|\nabla e_u\|\|e_w\| + ch^2\|\nabla e_w\| + ch^2|\log h|\|e_w\|,$$

by (2.18). The remaining terms are estimated in a straightforward manner so that we finally obtain

$$\frac{1}{2\sqrt{1+4C_0^2}^{\,3}}\|\nabla e_w\|^2 \leqslant c\|\nabla e_u\|^2 + \|e_{ut}\|\|e_w\| + c(\|e_w\| + \|\nabla e_u\|)\|\nabla e_w\| + \|\rho_{ut}\|\|e_w\|$$
$$+ c\|\nabla e_u\|\|e_w\| + ch^2\|\nabla e_w\| + ch^2|\log h|\|e_w\| + c\|\rho_w\|\|\nabla e_w\|.$$

The result now follows from Young's inequality, (2.20) and (2.24). □

The following two lemmas constitute the first steps in proving Theorem 2.2. As mentioned earlier, the strategy is to mimick the derivation of (2.8).

LEMMA 3.4 For $0 \leqslant t < T_h$ we have

$$\frac{1}{2C_0}\|e_{ut}\|^2 + \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_{ut} + \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut}$$
$$\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\rho_u + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u + c\|\nabla e_u\|^2 + ch^4|\log h|^4.$$

Proof. Inserting $\varphi_h = e_{ut}$ into (3.6) and recalling (3.2) we obtain

$$\frac{1}{2C_0}\|e_{ut}\|^2 + \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_{ut} + \frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut}$$
$$\leqslant -\int_\Omega\frac{\rho_{ut}e_{ut}}{Q_h} + \int_\Omega u_t\left(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h}\right)e_{ut} + \int_\Omega u_t\left(\frac{1}{\widehat{Q}_h} - \frac{1}{Q}\right)e_{ut} + \frac{1}{2}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_{ut}$$
$$\equiv I + II + III + IV.$$

We infer from (2.20) that

$$|I| + |II| \leqslant \|\rho_{ut}\|\|e_{ut}\| + c\|\nabla e_u\|\|e_{ut}\| \leqslant \epsilon\|e_{ut}\|^2 + c_\epsilon(h^4|\log h|^4 + \|\nabla e_u\|^2), \tag{3.7}$$

while Lemma 3.1 with $F(p) = 1/\sqrt{1+|p|^2}$ and $f = u_t e_{ut}$ yields

$$III = -\int_\Omega\nabla\cdot\left(u_t e_{ut}\frac{\nabla u}{Q^3}\right)\rho_u + R = -\int_\Omega\nabla\cdot\left(u_t\frac{\nabla u}{Q^3}\right)e_{ut}\rho_u - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\rho_u$$
$$+ \int_\Omega\left(u_t\frac{\nabla u}{Q^3}\right)_t\cdot\nabla e_u\rho_u + \int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\rho_{ut} + R,$$

where $|R| \leqslant Ch^2|\log h|\|u_t e_{ut}\|$. Thus,

$$III \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\rho_u + c\|e_{ut}\|(\|\rho_u\| + h^2|\log h|) + c\|\nabla e_u\|(\|\rho_u\| + \|\rho_{ut}\|)$$
$$\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t\frac{\nabla u}{Q^3}\cdot\nabla e_u\rho_u + \epsilon\|e_{ut}\|^2 + c\|\nabla e_u\|^2 + c_\epsilon h^4|\log h|^4 \tag{3.8}$$

by (2.18), (2.20). Finally,

$$IV = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u - \int_\Omega(\widehat{w}_h\widehat{w}_{ht} - ww_t)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u$$
$$-\frac{1}{2}\int_\Omega(\widehat{w}_h^2 - w^2)\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)_t\cdot\nabla e_u.$$

We infer from (2.21), (2.24) and (2.27) that

$$IV \leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u + c\|\nabla e_u\|(\|\rho_w\| + \|\rho_{wt}\|)$$
$$\leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\widehat{w}_h^2 - w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u + c\|\nabla e_u\|^2 + ch^4|\log h|^4 \tag{3.9}$$

by (2.24), (2.26). Summing (3.7)–(3.9) and choosing ϵ sufficiently small yields the result. □

LEMMA 3.5 For $0 \leqslant t < T_h$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{e_w^2}{Q_h} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w \leqslant \epsilon\|\nabla e_w\|^2 + c_\epsilon(\|\nabla e_u\|^2 + \|e_w\|^2) + c_\epsilon h^4|\log h|^4.$$

Proof. Starting from (2.5) and recalling the definition of $\widehat{u}_h$ we infer

$$\int_\Omega \frac{w\xi_h}{Q} = \int_\Omega \frac{\nabla u\cdot\nabla\xi_h}{Q} = \int_\Omega \frac{\nabla\widehat{u}_h\cdot\nabla\xi_h}{\widehat{Q}_h} \quad \forall\xi_h \in X_{h0},$$

from which we obtain after differentiation with respect to time

$$\int_\Omega \frac{w_t\xi_h}{Q} - \int_\Omega \frac{w\xi_h}{Q^2}Q_t - \int_\Omega E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht}\cdot\nabla\xi_h = 0 \quad \forall\xi_h \in X_{h0}.$$

Similarly

$$\int_\Omega \frac{w_{ht}\xi_h}{Q_h} - \int_\Omega \frac{w_h\xi_h}{Q_h^2}Q_{ht} - \int_\Omega E(\nabla u_h)\nabla u_{ht}\cdot\nabla\xi_h = 0 \quad \forall\xi_h \in X_{h0}.$$

Taking the difference of the above relations we obtain

$$\int_\Omega \frac{e_{wt}\xi_h}{Q_h} - \int_\Omega \frac{e_w\xi_h}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)\xi_h - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla\xi_h$$
$$= \int_\Omega \widehat{w}_{ht}\left(\frac{1}{Q_h} - \frac{1}{Q}\right)\xi_h - \int_\Omega \rho_{wt}\frac{1}{Q}\xi_h + \int_\Omega \rho_w\frac{Q_t}{Q^2}\xi_h - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_t}{Q^2}\right)\xi_h.$$

If we use $\xi_h = e_w$ the result is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{e_w^2}{Q_h} - \frac{1}{2}\int_\Omega \frac{e_w^2 Q_{ht}}{Q_h^2} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w - \int_\Omega (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w$$
$$= \int_\Omega \widehat{w}_{ht}\left(\frac{1}{Q_h} - \frac{1}{Q}\right)e_w - \int_\Omega \rho_{wt}\frac{1}{Q}e_w + \int_\Omega \rho_w\frac{Q_t}{Q^2}e_w - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_t}{Q^2}\right)e_w$$
$$\equiv I + \cdots + IV.$$

In order to deal with the first term we apply again Lemma 3.1 with $F(p) := 1/\sqrt{1+|p|^2}$ and $f = \widehat{w}_{ht}e_w$:

$$I = \int_\Omega \left(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h}\right)\widehat{w}_{ht}e_w + \int_\Omega \left(\frac{1}{\widehat{Q}_h} - \frac{1}{Q}\right)\widehat{w}_{ht}e_w$$
$$= \int_\Omega \left(\frac{1}{Q_h} - \frac{1}{\widehat{Q}_h}\right)\widehat{w}_{ht}e_w - \int_\Omega \nabla\cdot\left(\widehat{w}_{ht}e_w\frac{\nabla u}{Q^3}\right)\rho_u + R,$$

where $|R| \leqslant ch^2|\log h| \|\widehat{w}_{ht} e_w\|$. As a consequence,

$$|I| \leqslant c\|\widehat{w}_{ht}\|_{L^\infty}(\|\nabla e_u\|\|e_w\| + \|\nabla e_w\|\|\rho_u\| + h^2|\log h|\|e_w\|) + c\|\nabla \widehat{w}_{ht}\|_{L^\infty}\|e_w\|\|\rho_u\| \\ \leqslant \epsilon\|\nabla e_w\|^2 + c_\epsilon h^4|\log h|^2 + c\|\nabla e_u\|^2 + c\|e_w\|^2 \tag{3.10}$$

by (2.27). Next, (2.24) and (2.26) imply

$$|II + III| \leqslant c(\|\rho_{wt}\| + \|\rho_w\|)\|e_w\| \leqslant c\|e_w\|^2 + ch^4|\log h|^4. \tag{3.11}$$

Abbreviating $G_i(p) := p_i/\sqrt{1+|p|^2}^3$ and $i = 1, 2$, we obtain

$$\begin{aligned} IV &= \int_\Omega \left(\frac{\nabla u \cdot \nabla u_t}{Q^3} - \frac{\nabla \widehat{u}_h \cdot \nabla \widehat{u}_{ht}}{\widehat{Q}_h^3} \right) \widehat{w}_h e_w \\ &= \int_\Omega (G_i(\nabla u) - G_i(\nabla \widehat{u}_h)) u_{tx_i} \widehat{w}_h e_w - \int_\Omega (G_i(\nabla u) - G_i(\nabla \widehat{u}_h)) \rho_{utx_i} \widehat{w}_h e_w \\ &\quad + \int_\Omega G_i(\nabla u) \rho_{utx_i} \widehat{w}_h e_w \\ &= -\int_\Omega \nabla \cdot (G_i'(\nabla u) \widehat{w}_h e_w u_{tx_i}) \rho_u + R - \int_\Omega (G_i(\nabla u) - G_i(\nabla \widehat{u}_h)) \rho_{utx_i} \widehat{w}_h e_w \\ &\quad - \int_\Omega \frac{\partial}{\partial x_i} G_i(\nabla u) \rho_{ut} \widehat{w}_h e_w - \int_\Omega G_i(\nabla u) \widehat{w}_{hx_i} \rho_{ut} e_w - \int_\Omega G_i(\nabla u) e_{wx_i} \widehat{w}_h \rho_{ut}, \end{aligned}$$

where $|R| \leqslant ch^2|\log h| \|\widehat{w}_h e_w \nabla u_t\|$. Using (2.27) we can estimate

$$\begin{aligned} |IV| &\leqslant c\|\widehat{w}_h\|_{W^{1,\infty}}\|e_w\|_{H^1}\|\rho_u\| + ch^2|\log h|\|e_w\| + c\|\nabla \rho_u\|_{L^\infty}\|\nabla \rho_{ut}\|\|e_w\| \\ &\quad + c\|\rho_{ut}\|\|e_w\| + c\|\nabla \widehat{w}_h\|_{L^\infty}\|\rho_{ut}\|\|e_w\| + c\|\nabla e_w\|\|\widehat{w}_h\|_{L^\infty}\|\rho_{ut}\| \\ &\leqslant \epsilon\|\nabla e_w\|^2 + c_\epsilon h^4|\log h|^4 + c\|e_w\|^2. \end{aligned} \tag{3.12}$$

Combining (3.10)–(3.12) we finally obtain the result. □

We are now in a position to complete the proof of Theorem 2.2. It follows from Lemmas 3.4 and 3.5 that

$$\begin{aligned} &\frac{1}{2C_0}\|e_{ut}\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{e_w^2}{Q_h} \\ &\quad + \frac{1}{2}\int_\Omega \left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla \widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h \right) \cdot \nabla e_{ut} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2} Q_{ht} - \int_\Omega \widehat{w}_h \left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2} \right) e_w \\ &\quad + \int_\Omega (E(\nabla \widehat{u}_h)\nabla \widehat{w}_h - E(\nabla u_h)\nabla w_h) \cdot \nabla e_{ut} - \int_\Omega (E(\nabla \widehat{u}_h)\nabla \widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht}) \cdot \nabla e_w \\ &\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t \frac{\nabla u}{Q^3} \cdot \nabla e_u \rho_u + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\widehat{w}_h^2 - w^2)\frac{\nabla \widehat{u}_h}{\widehat{Q}_h^3} \cdot \nabla e_u \\ &\quad + c_\epsilon h^4|\log h|^4 + \epsilon\|\nabla e_w\|^2 + c_\epsilon(\|e_u\|_{H^1}^2 + \|e_w\|^2). \end{aligned} \tag{3.13}$$

The main problem now is to deal with the terms appearing in the second and third line of the above inequality. A short calculation shows that

$$\frac{1}{2}\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} - \frac{1}{2}\frac{e_w^2}{Q_h^2}Q_{ht} - \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w$$
$$= -\frac{1}{2}\widehat{w}_h^2\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{1}{2}\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h\cdot\nabla u_{ht} - \frac{1}{2}\frac{w_h^2}{Q_h^3}\nabla u_h\cdot\nabla\widehat{u}_{ht} + \frac{1}{2}\widehat{w}_h^2\frac{Q_{ht}}{Q_h^2} + \widehat{w}_h w_h\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2}$$
$$\equiv S_1 + \cdots + S_5.$$

Clearly,

$$S_1 = \frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{1}{\widehat{Q}_h}\right), \quad S_4 = -\frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{1}{Q_h}\right).$$

Observing that $|\widehat{\nu}_h - \nu_h|^2 = 2 - 2\frac{1+\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h Q_h}$ we obtain

$$S_2 = -\frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^3}\right) + \frac{1}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_{ht}\cdot\nabla u_h}{\widehat{Q}_h^3} - \frac{3}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^4}\widehat{Q}_{ht}$$
$$= -\frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^3}\right) + \frac{1}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_{ht}\cdot\nabla u_h}{\widehat{Q}_h^3} + \frac{3}{4}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3}|\widehat{\nu}_h - \nu_h|^2$$
$$-\frac{3}{2}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3} - \frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{1}{\widehat{Q}_h^3}\right),$$

while

$$S_3 = -\frac{1}{2}(\widehat{w}_h - w_h)^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} - \widehat{w}_h w_h\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} + \frac{1}{2}\widehat{w}_h^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3}.$$

Summation of $S_1, \ldots, S_5$ yields

$$S_1 + \cdots + S_5 = \frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{1}{\widehat{Q}_h} - \frac{1}{Q_h} - \frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^3} - \frac{1}{\widehat{Q}_h^3}\right)$$
$$+ \frac{3}{4}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3}|\widehat{\nu}_h - \nu_h|^2 - \frac{1}{2}e_w^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3}$$
$$+ \frac{1}{2}\widehat{w}_h^2\nabla(u_h - \widehat{u}_h)\cdot\nabla\widehat{u}_{ht}\left(\frac{1}{\widehat{Q}_h^3} - \frac{1}{Q_h^3}\right) + \widehat{w}_h^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} + \frac{1}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_h\cdot\nabla\widehat{u}_{ht}}{\widehat{Q}_h^3}$$
$$- \frac{1}{2}\widehat{w}_h^2\frac{\nabla\widehat{u}_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} + w_h\widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3}\right) - \frac{3}{2}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3}$$
$$= \frac{1}{2}\widehat{w}_h^2\frac{\partial}{\partial t}\left(\frac{2}{\widehat{Q}_h} - \frac{1}{Q_h} - \frac{\nabla\widehat{u}_h\cdot\nabla u_h}{\widehat{Q}_h^3} - \frac{1}{\widehat{Q}_h^3}\right) + \frac{3}{4}\widehat{w}_h^2\frac{Q_h\widehat{Q}_{ht}}{\widehat{Q}_h^3}|\widehat{\nu}_h - \nu_h|^2$$
$$- \frac{1}{2}e_w^2\frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3} + \frac{1}{2}\widehat{w}_h^2\nabla(u_h - \widehat{u}_h)\cdot\nabla\widehat{u}_{ht}\left(\frac{1}{\widehat{Q}_h^3} - \frac{1}{Q_h^3}\right)$$
$$+ \widehat{w}_h(w_h - \widehat{w}_h)\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3}\right) - \frac{1}{2}\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^3}\left(-4\widehat{Q}_h + 3Q_h + \frac{\widehat{Q}_h^4}{Q_h^3}\right).$$

A short calculation shows that

$$D := \frac{2}{\widehat{Q}_h} - \frac{1}{Q_h} - \frac{\nabla\widehat{u}_h \cdot \nabla u_h}{\widehat{Q}_h^3} - \frac{1}{\widehat{Q}_h^3}$$

can be written as

$$D = \frac{1}{2}\frac{Q_h}{\widehat{Q}_h^2}|\widehat{\nu}_h - \nu_h|^2 - \frac{1}{Q_h\widehat{Q}_h^2}(\widehat{Q}_h - Q_h)^2. \tag{3.14}$$

Integration over Ω together with (3.4) gives

$$\begin{aligned}\frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \\ \geqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \widehat{w}_h^2 D - \int_\Omega \widehat{w}_h\widehat{w}_{ht}D - c\|\nabla e_u\|^2 - c\|e_w\|^2 \\ - c\int_\Omega |e_w|\left|\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla u_h\cdot\nabla\widehat{u}_{ht}}{Q_h^3}\right| - c\int_\Omega\left|-4\widehat{Q}_h + 3Q_h + \frac{\widehat{Q}_h^4}{Q_h^3}\right|.\end{aligned}$$

Since

$$\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla\widehat{u}_{ht}\cdot\nabla u_h}{Q_h^3} = \frac{\nabla\widehat{u}_{ht}}{\widehat{Q}_h^2}\left(\frac{\nabla\widehat{u}_h}{\widehat{Q}_h} - \frac{\nabla u_h}{Q_h}\right) + \frac{\nabla u_h}{Q_h}\cdot\nabla\widehat{u}_{ht}\frac{Q_h + \widehat{Q}_h}{\widehat{Q}_h^2 Q_h}\frac{Q_h - \widehat{Q}_h}{Q_h}$$

we obtain

$$\left|\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{\nabla\widehat{u}_{ht}\cdot\nabla u_h}{Q_h^3}\right| \leqslant c|\widehat{\nu}_h - \nu_h| \leqslant c\|\nabla e_u\|.$$

It is easily shown that

$$\left|-4\widehat{Q}_h + 3Q_h + \frac{\widehat{Q}_h^4}{Q_h^3}\right| \leqslant c(Q_h - \widehat{Q}_h)^2 \leqslant c|\nabla e_u|^2.$$

Recalling (3.14) we have in conclusion

$$\begin{aligned}\frac{1}{2}\int_\Omega\left(\frac{\widehat{w}_h^2}{\widehat{Q}_h^3}\nabla\widehat{u}_h - \frac{w_h^2}{Q_h^3}\nabla u_h\right)\cdot\nabla e_{ut} - \frac{1}{2}\int_\Omega \frac{e_w^2}{Q_h^2}Q_{ht} - \int_\Omega \widehat{w}_h\left(\frac{\widehat{Q}_{ht}}{\widehat{Q}_h^2} - \frac{Q_{ht}}{Q_h^2}\right)e_w \\ \geqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \widehat{w}_h^2\left\{\frac{1}{2}\frac{Q_h}{\widehat{Q}_h^2}|\widehat{\nu}_h - \nu_h|^2 - \frac{1}{Q_h\widehat{Q}_h^2}(\widehat{Q}_h - Q_h)^2\right\} - c(\|e_w\|^2 + \|\nabla e_u\|^2).\end{aligned} \tag{3.15}$$

Next,

$$\begin{aligned}(E(\nabla\widehat{u}_h)\nabla\widehat{w}_h - E(\nabla u_h)\nabla w_h)\cdot\nabla e_{ut} - (E(\nabla\widehat{u}_h)\nabla\widehat{u}_{ht} - E(\nabla u_h)\nabla u_{ht})\cdot\nabla e_w \\ = (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla(\widehat{u}_{ht} - u_{ht})\cdot\nabla\widehat{w}_h - (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla\widehat{u}_{ht}\cdot\nabla e_w.\end{aligned}$$

In order to deal with the first term we introduce $\widehat{P}_h, P_h \in \mathbb{R}^{3\times3}$ by

$$\widehat{P}_{h,ij} = \delta_{ij} - \widehat{\nu}_{hi}\widehat{\nu}_{hj}, \quad P_{h,ij} = \delta_{ij} - \nu_{hi}\nu_{hj}, \quad i, j = 1, 2, 3.$$

Since $(\nabla(\widehat{u}_h - u_h), 0)^t = \widehat{Q}_h \widehat{\nu}_h - Q_h \nu_h$ we may calculate

$$\begin{aligned}\left(\frac{1}{\widehat{Q}_h}\widehat{P}_h - \frac{1}{Q_h}P_h\right)(\nabla(\widehat{u}_{ht} - u_{ht}), 0)^t &= \left(\frac{1}{\widehat{Q}_h}\widehat{P}_h - \frac{1}{Q_h}P_h\right)(\widehat{Q}_{ht}\widehat{\nu}_h + \widehat{Q}_h\widehat{\nu}_{ht} - Q_{ht}\nu_h - Q_h\nu_{ht})\\ &= \widehat{\nu}_{ht} - \frac{Q_{ht}}{\widehat{Q}_h}\widehat{P}_h\nu_h - \frac{Q_h}{\widehat{Q}_h}\widehat{P}_h\nu_{ht} - \frac{\widehat{Q}_{ht}}{Q_h}P_h\widehat{\nu}_h - \frac{\widehat{Q}_h}{Q_h}P_h\widehat{\nu}_{ht} + \nu_{ht}\\ &= \frac{\partial}{\partial t}\left\{\nu_h - \widehat{\nu}_h - \frac{Q_h}{\widehat{Q}_h}\widehat{P}_h\nu_h\right\} + S, \end{aligned} \tag{3.16}$$

where

$$S = -\frac{\widehat{Q}_{ht}Q_h}{\widehat{Q}_h^2}\widehat{P}_h\nu_h + \frac{Q_h}{\widehat{Q}_h}\widehat{P}_{ht}\nu_h + 2\widehat{\nu}_{ht} - \frac{\widehat{Q}_{ht}}{Q_h}P_h\widehat{\nu}_h - \frac{\widehat{Q}_h}{Q_h}P_h\widehat{\nu}_{ht}.$$

Observe that

$$\widehat{P}_h\nu_h = \nu_h - (\nu_h\cdot\widehat{\nu}_h)\widehat{\nu}_h = \nu_h - \widehat{\nu}_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h$$

and similarly

$$P_h\widehat{\nu}_h = \widehat{\nu}_h - \nu_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\nu_h.$$

This gives

$$\begin{aligned} S &= -\frac{\widehat{Q}_{ht}Q_h}{\widehat{Q}_h^2}\left(\widehat{\nu}_h - \nu_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) + \frac{Q_h}{\widehat{Q}_h}(-(\nu_h\cdot\widehat{\nu}_h)\widehat{\nu}_{ht} - (\widehat{\nu}_{ht}\cdot\nu_h)\widehat{\nu}_h)\\ &\quad + 2\widehat{\nu}_{ht} - \frac{\widehat{Q}_{ht}}{Q_h}\left(\nu_h - \widehat{\nu}_h + \frac{1}{2}|\widehat{\nu}_h - \nu_h|^2\nu_h\right) - \frac{\widehat{Q}_h}{Q_h}(\widehat{\nu}_{ht} - (\widehat{\nu}_{ht}\cdot\nu_h)\nu_h)\\ &= -\frac{1}{2}\widehat{Q}_{ht}|\widehat{\nu}_h - \nu_h|^2\left(\frac{Q_h}{\widehat{Q}_h^2}\widehat{\nu}_h + \frac{1}{Q_h}\nu_h\right) + \frac{\widehat{Q}_{ht}}{\widehat{Q}_h}\left(\frac{\widehat{Q}_h}{Q_h} - \frac{Q_h}{\widehat{Q}_h}\right)(\widehat{\nu}_h - \nu_h)\\ &\quad + \left(2 - \frac{Q_h}{\widehat{Q}_h}(\widehat{\nu}_h\cdot\nu_h) - \frac{\widehat{Q}_h}{Q_h}\right)\widehat{\nu}_{ht} + (\widehat{\nu}_{ht}\cdot(\nu_h - \widehat{\nu}_h))\left(\frac{\widehat{Q}_h}{Q_h}\nu_h - \frac{Q_h}{\widehat{Q}_h}\widehat{\nu}_h\right) \end{aligned}$$

since $\widehat{\nu}_h\cdot\widehat{\nu}_{ht} = 0$. We infer from (3.2) and (3.4) that

$$\left|\frac{\widehat{Q}_h}{Q_h} - \frac{Q_h}{\widehat{Q}_h}\right|, \left|\frac{\widehat{Q}_h}{Q_h}\nu_h - \frac{Q_h}{\widehat{Q}_h}\widehat{\nu}_h\right| \leqslant c|\nabla e_u|,$$

as well as

$$\left|2 - \frac{Q_h}{\widehat{Q}_h}(\widehat{\nu}_h\cdot\nu_h) - \frac{\widehat{Q}_h}{Q_h}\right| = \left|\frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2 - \frac{(\widehat{Q}_h - Q_h)^2}{\widehat{Q}_h Q_h}\right| \leqslant c|\nabla e_u|^2,$$

which implies

$$|S| \leqslant c|\nabla e_u|^2. \tag{3.17}$$

In conclusion

$$\begin{aligned} &(E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla(\widehat{u}_{ht} - u_{ht})\cdot\nabla\widehat{w}_h\\ &\quad = \left(\frac{1}{\widehat{Q}_h}\widehat{P}_h - \frac{1}{Q_h}P_h\right)(\nabla(\widehat{u}_{ht} - u_{ht}), 0)^t\cdot(\nabla\widehat{w}_h, 0)^t\\ &\quad = \left\{\frac{\partial}{\partial t}\left(\left(\frac{Q_h}{\widehat{Q}_h} - 1\right)(\widehat{\nu}_h - \nu_h) - \frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h - \nu_h|^2\widehat{\nu}_h\right) + S\right\}\cdot(\nabla\widehat{w}_h, 0)^t. \end{aligned}$$

If we integrate this relation over Ω and recall (3.17) and (2.27) we obtain

$$\begin{aligned}&\int_\Omega (E(\nabla\widehat{u}_h) - E(\nabla u_h))\nabla(\widehat{u}_{ht} - u_{ht})\cdot\nabla\widehat{w}_h\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigg(\bigg(\frac{Q_h}{\widehat{Q}_h}-1\bigg)(\widehat{\nu}_h-\nu_h)-\frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h-\nu_h|^2\widehat{\nu}_h\bigg)\cdot(\nabla\widehat{w}_h,0)^t\\ &\quad-\int_\Omega\bigg(\bigg(\frac{Q_h}{\widehat{Q}_h}-1\bigg)(\widehat{\nu}_h-\nu_h)-\frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h-\nu_h|^2\widehat{\nu}_h\bigg)\cdot(\nabla\widehat{w}_{ht},0)^t+\int_\Omega S\cdot(\nabla\widehat{w}_h,0)^t\\ &\geqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigg(\bigg(\frac{Q_h}{\widehat{Q}_h}-1\bigg)(\widehat{\nu}_h-\nu_h)-\frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h-\nu_h|^2\widehat{\nu}_h\bigg)\cdot(\nabla\widehat{w}_h,0)^t-c\|\nabla e_u\|^2. \qquad (3.18)\end{aligned}$$

Recalling the definitions of $\hat{P}_h$, P_h and observing (3.2) we may estimate

$$\begin{aligned}\bigg|\int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u_h))\nabla\widehat{u}_{ht}\cdot\nabla e_w\bigg| &= \bigg|\int_\Omega\bigg(\frac{1}{\widehat{Q}_h}\hat{P}_h-\frac{1}{Q_h}P_h\bigg)(\nabla\widehat{u}_{ht},0)^t\cdot(\nabla e_w,0)^t\bigg|\\ &\leqslant c\int_\Omega|\nabla e_u||\nabla e_w|\leqslant\epsilon\|\nabla e_w\|^2+c_\epsilon\|\nabla e_u\|^2. \qquad (3.19)\end{aligned}$$

If we insert (3.15), (3.18) and (3.19) into (3.13) we finally obtain

$$\begin{aligned}\frac{1}{2C_0}\|e_{ut}\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{e_w^2}{Q_h} \leqslant& -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t\frac{\nabla u}{Q^3}\nabla e_u\rho_u+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\widehat{w}_h^2-w^2)\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\cdot\nabla e_u\\ &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widehat{w}_h^2\bigg\{\frac{1}{2}\frac{Q_h}{\widehat{Q}_h^2}|\widehat{\nu}_h-\nu_h|^2-\frac{1}{Q_h\widehat{Q}_h^2}(\widehat{Q}_h-Q_h)^2\bigg\}\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\bigg(\bigg(\frac{Q_h}{\widehat{Q}_h}-1\bigg)(\widehat{\nu}_h-\nu_h)-\frac{1}{2}\frac{Q_h}{\widehat{Q}_h}|\widehat{\nu}_h-\nu_h|^2\widehat{\nu}_h\bigg)\cdot(\nabla\widehat{w}_h,0)^t\\ &+\epsilon\|\nabla e_w\|^2+c_\epsilon(\|e_w\|^2+\|\nabla e_u\|^2+h^4|\log h|^4).\end{aligned}$$

Integrating with respect to time between 0 and t $(0\leqslant t<T_h)$ and taking into account Lemmas 3.2 and 3.3 yields

$$\begin{aligned}\int_0^t\|e_{ut}\|^2\,\mathrm{d}s+\|e_w(t)\|^2 &\leqslant c\|\nabla e_u(t)\|(\|\rho_u(t)\|+\|e_w(t)\|+\|\nabla e_u(t)\|)\\ &\quad+\epsilon\int_0^t\|\nabla e_w\|^2\,\mathrm{d}s+c_\epsilon\int_0^t(\|e_w\|^2+\|\nabla e_u\|^2)\,\mathrm{d}s+c_\epsilon h^4|\log h|^4\\ &\leqslant\epsilon\bigg(\|e_w(t)\|^2+\int_0^t\|e_{ut}\|^2\,\mathrm{d}s\bigg)+c_\epsilon\bigg(\|e_u(t)\|^2+h^4|\log h|^4+\int_0^t(\|e_u\|^2+\|e_w\|^2)\,\mathrm{d}s\bigg).\end{aligned}$$

After choosing ϵ sufficiently small we obtain

$$\int_0^t\|e_{ut}\|^2\,\mathrm{d}s+\|e_w(t)\|^2\leqslant c_1\|e_u(t)\|^2+c\int_0^t(\|e_u\|^2+\|e_w\|^2)\,\mathrm{d}s+ch^4|\log h|^4. \qquad (3.20)$$

On the other hand, in view of $e_u(0)=0$ we have

$$\|e_u(t)\|^2\leqslant 2\int_0^t\|e_u\|\|e_{ut}\|\,\mathrm{d}s\leqslant\frac{1}{2}\int_0^t\|e_{ut}\|^2\,\mathrm{d}s+c\int_0^t\|e_u\|^2\,\mathrm{d}s.$$

If we use this inequality in (3.20) we finally obtain

$$\int_0^t \|e_{ut}\|^2 \,\mathrm{d}s + \|e_w(t)\|^2 + \|e_u(t)\|^2 \leqslant c \int_0^t (\|e_u\|^2 + \|e_w\|^2) \,\mathrm{d}s + ch^4|\log h|^4, \quad 0 \leqslant t < T_h,$$

and Gronwall's inequality yields

$$\|e_w(t)\|^2 + \|e_u(t)\|^2 \leqslant ch^4|\log h|^4, \quad 0 \leqslant t \leqslant T_h. \tag{3.21}$$

Lemma 3.2 implies that $\|\nabla e_u(t)\| \leqslant ch^2|\log h|^2$ for $0 \leqslant t \leqslant T_h$, which together with (2.19) and an inverse estimate gives

$$\begin{aligned}\|Q_h(t)\|_{L^\infty} &\leqslant \|Q(t)\|_{L^\infty} + c\|\nabla(u - \widehat{u}_h)(t)\|_{L^\infty} + c\|\nabla e_u(t)\|_{L^\infty} \\ &\leqslant C_0 + ch|\log h| + ch|\log h|^2 \leqslant \tfrac{3}{2}C_0,\end{aligned}$$

and similarly $\|w_h(t)\|_{L^\infty} \leqslant \frac{3}{2}C_1$, $0 \leqslant t \leqslant T_h$, provided that $h \leqslant h_0$. Now we are in a position to prove that $T_h = T$. If not, the above argument would imply that $Q_h(x, T_h) \leqslant \frac{3}{2}C_0$ and $|w_h(x, T_h)| \leqslant \frac{3}{2}C_1$ for $x \in \bar{\Omega}$, and we could establish (3.2) on $[0, T_h + \delta]$ for some $\delta > 0$, contradicting the definition of T_h. The estimates (2.13)–(2.16) now follow from (3.21), Lemma 3.2, Lemma 3.3 and the interpolation results for $\widehat{u}_h$, $\widehat{w}_h$.

4. Numerical results

For our numerical tests we have to use a time discretization. We have chosen a semi-implicit discretization with respect to the time variable in the spatially discrete scheme (2.10), (2.11). For a generic function v we denote its evaluation at the m-th time level $t_m = m\tau$ by $v^m = v(\cdot, t_m)$. The time discretization is then given by:

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega \frac{(u_h^{m+1} - u_h^m)\varphi_h}{Q_h^m} + \int_\Omega E(\nabla u_h^m)\nabla w_h^{m+1} \cdot \nabla\varphi_h \\ &\qquad + \frac{1}{2}\int_\Omega \frac{(w_h^m)^2}{(Q_h^m)^3}\nabla u_h^{m+1} \cdot \nabla\varphi_h = \int_\Omega f^m\varphi_h \quad \forall\varphi_h \in X_{h0},\end{aligned} \tag{4.1}$$

$$\int_\Omega \frac{w_h^{m+1}\zeta_h}{Q_h^m} - \int_\Omega \frac{\nabla u_h^{m+1} \cdot \nabla\zeta_h}{Q_h^m} = 0 \quad \forall\zeta_h \in X_{h0}, \tag{4.2}$$

for $m = 0, 1, \ldots, m(T)$ with $\tau m(T) = T$ and $Q_h^m = \sqrt{1 + |\nabla u_h^m|^2}$. We have introduced an additional given right hand side f, which we will need for our numerical tests. Denote by $\{\varphi_j\}_{j=1,\ldots,N}$ the usual nodal basis of X_{h0}. Then (4.1), (4.2) represents a linear system for the coefficients of u_h^{m+1} and w_h^{m+1} in the expansions

$$u_h^m - I_h g = \sum_{j=1}^N U_j^m \varphi_j, \quad w_h^m = \sum_{j=1}^N W_j^m \varphi_j.$$

Denote by $U^m = (U_1^m, \ldots, U_N^m)$, $W^m = (W_1^m, \ldots, W_N^m)$ the coefficient vectors and set

$$M_{i,j}^m = \int_\Omega \frac{\varphi_i\varphi_j}{Q_h^m}, \quad E_{i,j}^m = \int_\Omega E(\nabla u_h^m)\nabla\varphi_i \cdot \nabla\varphi_j, \quad A_{i,j}^m = \int_\Omega \frac{\nabla\varphi_i \cdot \nabla\varphi_j}{Q_h^m},$$

as well as

$$B_{i,j}^m = \frac{1}{2}\int_\Omega \frac{(w_h^m)^2}{(Q_h^m)^3}\nabla\varphi_i\cdot\nabla\varphi_j, \qquad F_i^m = \int_\Omega f^m\varphi_i$$

for $i, j = 1, \dots, N$. With these settings we can write the linear system (4.1), (4.2) in the form

$$\frac{1}{\tau}M^m U^{m+1} + E^m W^{m+1} + B^m U^{m+1} = \frac{1}{\tau}M^m U^m + F^m, \tag{4.3}$$

$$M^m W^{m+1} - A^m U^{m+1} = 0. \tag{4.4}$$

Eliminating W^{m+1} from the first equation by inverting the (weighted) mass matrix in the second equation leads to the linear system

$$\left(\frac{1}{\tau}M^m + E^m(M^m)^{-1}A^m + B^m\right)U^{m+1} = \frac{1}{\tau}M^m U^m + F^m. \tag{4.5}$$

We solve this nonsymmetric system by the biconjugate gradient method. In the practical computations we use mass lumping, so that M^m becomes a diagonal matrix.

We have tested our algorithm with the help of the following problem: let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $(0, T) = (0, 0.5)$ and

$$u(x,t) = 2.5\cos(2\pi t)(|x|-1)^3|x|^5.$$

The function f is calculated in such a way that u is a solution of the PDE

$$\frac{u_t}{Q} + \nabla\cdot\left(\frac{1}{Q}\left(I - \frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla(QH)\right) - \frac{1}{2}\nabla\cdot\left(\frac{H^2}{Q}\nabla u\right) = f \quad \text{in } \Omega\times(0,T).$$

In order not to destroy the second order convergence properties in some norms we choose the time step as $\tau = 0.1h^2$. This has proved experimentally to be a good choice for our computations for the Willmore flow of graphs. In Tables 1 and 2 we show the absolute errors for u in the norms

$$E_{\infty,2,u} = \max_{m=1,\dots,m(T)}\|u^m - u_h^m\|, \qquad E_{\infty,2,\nabla u} = \max_{m=1,\dots,m(T)}\|\nabla u^m - \nabla u_h^m\|,$$

$$E_{2,2,u_t} = \left(\tau\sum_{m=1}^{m(T)}\left\|u_t^m - \frac{u_h^m - u_h^{m-1}}{\tau}\right\|^2\right)^{1/2}, \qquad E_{\infty,\infty,u} = \max_{m=1,\dots,m(T)}\sup_\Omega|u^m - u_h^m|,$$

as well as for w in the norms

$$E_{\infty,2,w} = \max_{m=1,\dots,m(T)}\|w^m - w_h^m\|, \quad E_{2,2,\nabla w} = \left(\tau\sum_{m=1}^{m(T)}\|\nabla w^m - \nabla w_h^m\|^2\right)^{1/2},$$

$$E_{\infty,\infty,w} = \max_{m=1,\dots,m(T)}\sup_\Omega|w^m - w_h^m|.$$

The computational results confirm the theoretical results of Theorem 2.2. Between two spatial grid levels with maximal grid size h_1 and h_2 and errors $E(h_1)$, $E(h_2)$ we computed the experimental order of convergence according to

$$\operatorname{eoc}(h_1,h_2) = \log\frac{E(h_1)}{E(h_2)}\left(\log\frac{h_1}{h_2}\right)^{-1}.$$

TABLE 1
u-errors for the test problem, $\tau = 0.1h^2$

h	$E_{\infty,2,u}$	eoc	$E_{\infty,2,\nabla u}$	eoc	$E_{2,2,u_t}$	eoc	$E_{\infty,\infty,u}$	eoc
1.0	0.59486	-	1.41194	-	1.90956	-	1.09159	-
0.73681	0.30771	2.158	0.90635	1.451	0.97860	2.188	0.30057	4.222
0.42033	0.11159	1.807	0.53259	0.947	0.50503	1.178	0.11449	1.719
0.22192	0.047902	1.324	0.34495	0.680	0.23983	1.165	0.046528	1.409
0.11373	0.013300	1.916	0.18205	0.956	0.070064	1.840	0.012886	1.920
0.057535	0.0034522	1.979	0.094278	0.965	0.018898	1.922	0.0033490	1.977

TABLE 2
w-errors for the test problem, $\tau = 0.1h^2$

h	$E_{\infty,2,w}$	eoc	$E_{2,2,\nabla w}$	eoc	$E_{\infty,\infty,w}$	eoc
1.0	1.58839	-	4.58182	-	3.83664	-
0.73681	1.44454	0.310	2.59686	1.859	1.79998	2.477
0.42033	0.74736	1.174	2.99178	-0.252	0.86798	1.299
0.22192	0.40324	0.965	1.76851	0.823	0.41747	1.146
0.11373	0.11953	1.818	0.90402	1.003	0.11833	1.885
0.057535	0.031087	1.976	0.44252	1.048	0.028734	2.077

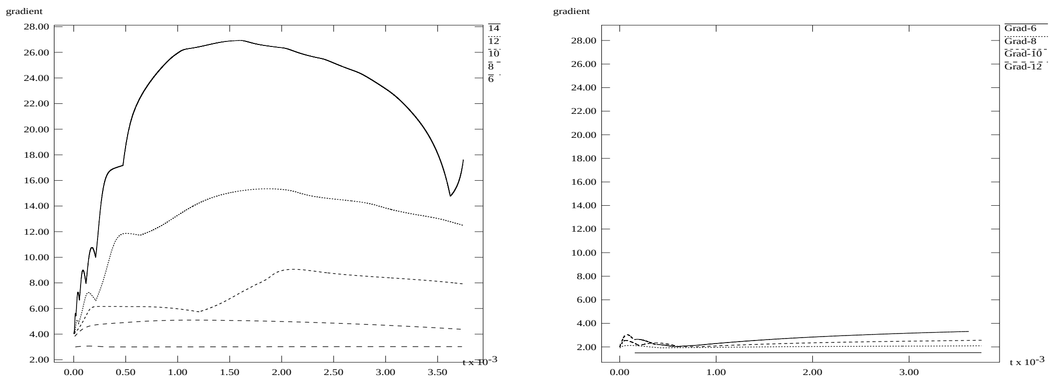

FIG. 2. Maximal gradient of the solution u for initial data (4.6) versus time $t \in (0.0, 0.00375)$ for global refinement levels 3, 4, 5, 6 (left and right), and 7 (left only) of the grid. The plot on the left corresponds to the choice $\delta = 1$ in (4.6), the plot on the right to $\delta = 0.5$.

We demonstrate the possible effect that the gradient of the continuous solution may blow up at some time. Figure 2 shows the norm gradient$(t) = \|\nabla u(\cdot, t)\|_{L^\infty(\Omega)}$. From Lemma 2.1 we know that the discrete solution exists for all times. The continuous one may exist for finite time only. A blow up of the continuous gradient may be deduced practically from the behaviour of the discrete gradient. As initial function for the computations in Figure 2 we have chosen

$$u_0(x_1, x_2) = \delta \sin^2(\pi(1 + x_1)) \sin^2(\pi x_1) \tag{4.6}$$

on the domain $\Omega = (-1, 1) \times (-1, 1)$. Figure 2 shows the maximal gradient of the solution corresponding to $\delta = 1$ and $\delta = 0.5$ respectively on various refinement levels. The results indicate a gradient blow up in the first case while the gradient remains bounded in the latter.

We finally give a computational example for the Willmore flow of a graph. The domain is $\Omega = (-2, 2) \times (-2, 2)$ and the initial value for u is given by

$$u_0(x_1, x_2) = 0.75 \sin^2(\pi(1 + x_1)) \sin^2(\pi x_1) + 0.1 \sin(4\pi x_1) \sin(5\pi x_2).$$

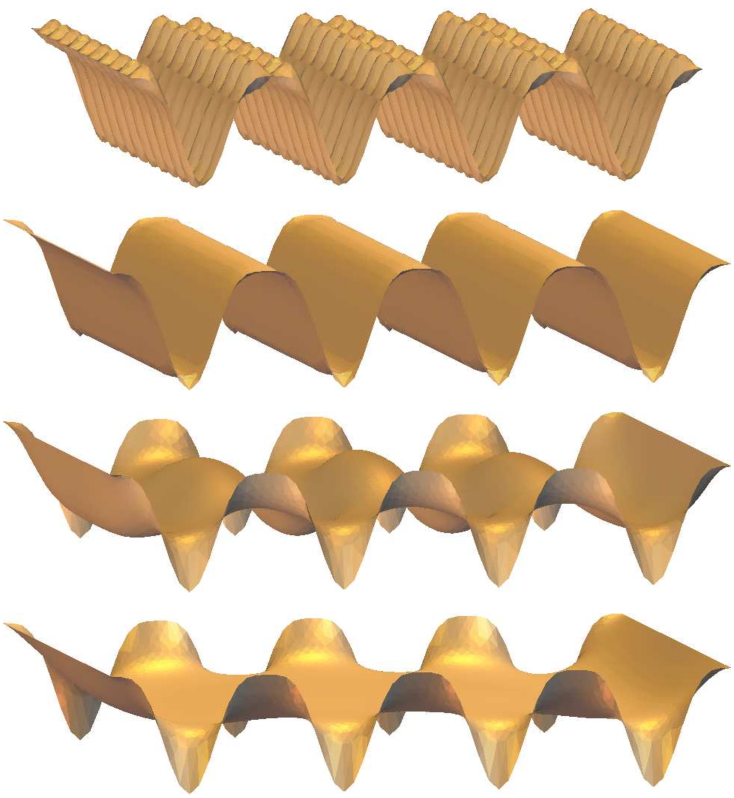

FIG. 3. Graph of u for the time steps $t = 0.0$, 0.000671, 0.02448 and 0.05072.

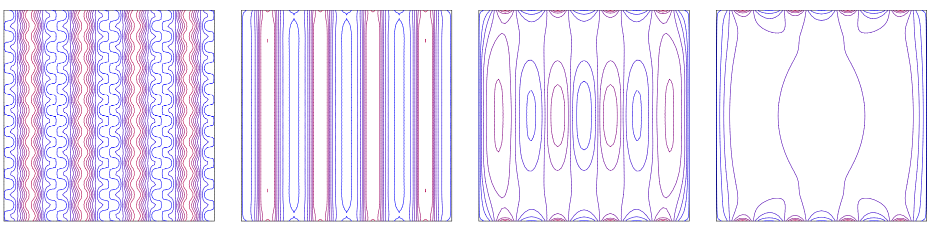

FIG. 4. Level lines of u for the time steps from Figure 3.

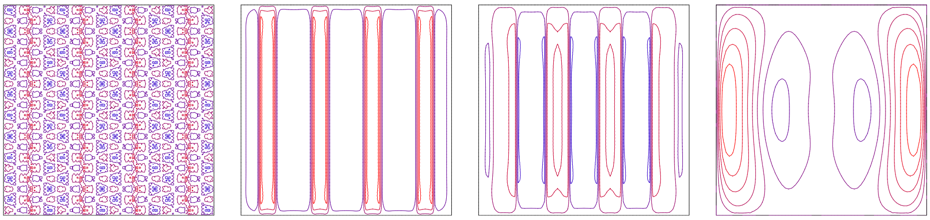

FIG. 5. Level lines of w for the times $t = 0.0$, 0.000671, 0.02448, 0.05072.

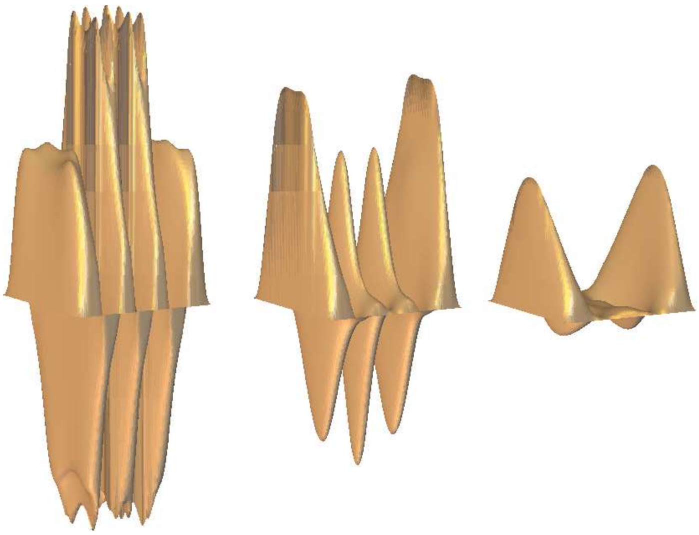

FIG. 6. Graph of w for the time steps $t = 0.01837$, 0.03241 and $t = 0.04279$.

In Figure 3 we show the graph of the solution u at several time steps. Level lines of u are plotted in Figure 4. The levels run from 0.0 to 1.0 with an increment of 0.1. We show the graph of w for some time steps in Figure 6 and level lines in Figure 5. The spatial grid size was $h = 0.08839$ and in order to capture the rapid smoothing of the solution the time step was chosen as $\tau = 6.1035e - 6$.

Appendix

LEMMA A.1 Let $\widehat{w}_h \in X_{h0}$ be given by (2.22). Then

$$\sup_{0\leqslant t\leqslant T} \|\nabla(w - \widehat{w}_h)(t)\| \leqslant ch \tag{A.1}$$

$$\sup_{0\leqslant t\leqslant T} \|(w - \widehat{w}_h)(t)\| \leqslant ch^2|\log h|, \tag{A.2}$$

$$\sup_{0\leqslant t\leqslant T} \|\nabla(w_t - \widehat{w}_{ht})(t)\| \leqslant ch, \tag{A.3}$$

$$\sup_{0\leqslant t\leqslant T} \|(w_t - \widehat{w}_{ht})(t)\| \leqslant ch^2|\log h|^2. \tag{A.4}$$

Proof. Clearly, (2.22) implies

$$\int_\Omega E(\nabla\widehat{u}_h)\nabla(w-\widehat{w}_h)\cdot\nabla\varphi_h = \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w\cdot\nabla\varphi_h - \frac{1}{2}\int_\Omega w^2\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)\cdot\nabla\varphi_h \tag{A.5}$$

for all $\varphi_h \in X_{h0}$. Inserting $\varphi_h = I_h w - \widehat{w}_h$ we obtain

$$\int_\Omega E(\nabla\widehat{u}_h)\nabla(w-\widehat{w}_h)\cdot\nabla(w-\widehat{w}_h) = \int_\Omega E(\nabla\widehat{u}_h)\nabla(w-\widehat{w}_h)\cdot\nabla(w-I_h w) + \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w\cdot\nabla(I_h w-\widehat{w}_h) - \frac{1}{2}\int_\Omega w^2\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)\cdot\nabla(I_h w-\widehat{w}_h),$$

from which we deduce (A.1) in view of (2.3), (2.2), (2.18), standard interpolation estimates and the uniform boundedness of $\widehat{Q}_h$. The L^2-norm of $w-\widehat{w}_h$ is estimated with the help of the usual duality argument. Solve

$$-\operatorname{div}(E(\nabla u)\nabla z) = w-\widehat{w}_h \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega.$$

The ellipticity of E together with our smoothness assumptions on Ω and u implies that the above boundary value problem has a unique solution $z \in H^2(\Omega)$ (t is a parameter) with $\|z\|_{H^2} \leqslant c\|w-\widehat{w}_h\|$. Using (A.5) with $\varphi_h = I_h z$ we derive

$$\begin{aligned}\|w-\widehat{w}_h\|^2 &= \int_\Omega (-\operatorname{div}(E(\nabla u)\nabla z)(w-\widehat{w}_h)) = \int_\Omega E(\nabla u)\nabla z\cdot\nabla(w-\widehat{w}_h)\\ &= \int_\Omega (E(\nabla u)-E(\nabla\widehat{u}_h))\nabla z\cdot\nabla(w-\widehat{w}_h) + \int_\Omega E(\nabla\widehat{u}_h)\nabla(z-I_h z)\cdot\nabla(w-\widehat{w}_h)\\ &\quad + \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w\cdot\nabla I_h z - \frac{1}{2}\int_\Omega w^2\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)\cdot\nabla I_h z\\ &\equiv I+\cdots+IV.\end{aligned}$$

We infer from (2.2), (2.19), (A.1) and the a-priori estimate for z that

$$|I| \leqslant c\|\nabla(u-\widehat{u}_h)\|_{L^\infty}\|\nabla z\|\|\nabla(w-\widehat{w}_h)\| \leqslant ch^2|\log h|\|w-\widehat{w}_h\| \leqslant \epsilon\|w-\widehat{w}_h\|^2 + c_\epsilon h^4|\log h|^2.$$

An interpolation estimate together with (A.1) implies

$$|II| \leqslant ch\|D^2 z\|\|\nabla(w-\widehat{w}_h)\| \leqslant \epsilon\|w-\widehat{w}_h\|^2 + c_\epsilon h^4.$$

Next, Lemma 3.1 yields

$$\begin{aligned} III &= \int_\Omega (E_{ij}(\nabla\widehat{u}_h)-E_{ij}(\nabla u))w_{x_i}z_{x_j} + \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w\cdot\nabla(I_h z - z)\\ &= \int_\Omega \nabla\cdot(E'_{ij}(\nabla u)w_{x_i}z_{x_j})\rho_u + R + \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w\cdot\nabla(I_h z-z),\end{aligned}$$

where $|R| \leqslant ch^2|\log h|\|\nabla w\|_{L^\infty}\|\nabla z\| \leqslant ch^2|\log h|\|\nabla z\|$. This implies

$$|III| \leqslant c(\|\rho_u\| + h^2|\log h| + h\|\nabla\rho_u\|)\|\nabla z\|_{H^1} \leqslant \epsilon\|w-\widehat{w}_h\|^2 + c_\epsilon h^4|\log h|^2$$

and in a similar way it follows that

$$|IV| \leqslant \epsilon\|w-\widehat{w}_h\|^2 + c_\epsilon h^4|\log h|^2.$$

Combining the estimates for $I, \ldots, IV$ implies (A.2). Let us next turn to the estimates for $w_t - \widehat{w}_{ht}$. Differentiating (A.5) with respect to time yields

$$\begin{aligned}\int_\Omega E(\nabla\widehat{u}_h)\nabla(w_t-\widehat{w}_{ht})\cdot\nabla\varphi_h &= -\int_\Omega E_{p_i}(\nabla\widehat{u}_h)\widehat{u}_{ht,x_i}\nabla(w-\widehat{w}_h)\cdot\nabla\varphi_h \\ &+ \int_\Omega (E_{p_i}(\nabla\widehat{u}_h)\widehat{u}_{htx_i} - E_{p_i}(\nabla u)u_{tx_i})\nabla w\cdot\nabla\varphi_h + \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w_t\cdot\nabla\varphi_h \\ &- \int_\Omega ww_t\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)\cdot\nabla\varphi_h - \frac{1}{2}\int_\Omega w^2\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)_t\cdot\nabla\varphi_h\end{aligned}$$

for all $\varphi_h \in X_{h0}$. From this relation for $\varphi_h = I_h w_t - \widehat{w}_{ht}$ it is not difficult to deduce (A.3), using (2.2), (2.18), (2.21), (A.1) and an interpolation estimate. It remains to bound $\|w_t - \widehat{w}_{ht}\|$. Denoting by z the solution of

$$\begin{aligned} -\operatorname{div}(E(\nabla u)\nabla z) &= w_t - \widehat{w}_{ht} &&\text{in } \Omega,\\ z &= 0 &&\text{on } \partial\Omega,\end{aligned}$$

and proceeding in the same way as above we obtain

$$\begin{aligned}&\|w_t-\widehat{w}_{ht}\|^2\\ &\quad= \int_\Omega (E(\nabla u)-E(\nabla\widehat{u}_h))\nabla z\cdot\nabla(w_t-\widehat{w}_{ht}) + \int_\Omega E(\nabla\widehat{u}_h)\nabla(z-I_hz)\cdot\nabla(w_t-\widehat{w}_{ht})\\ &\qquad- \int_\Omega E_{p_i}(\nabla\widehat{u}_h)\widehat{u}_{htx_i}\nabla(w-\widehat{w}_h)\cdot\nabla I_hz + \int_\Omega (E_{p_i}(\nabla\widehat{u}_h)\widehat{u}_{htx_i} - E_{p_i}(\nabla u)u_{tx_i})\nabla w\cdot\nabla I_hz\\ &\qquad+ \int_\Omega (E(\nabla\widehat{u}_h)-E(\nabla u))\nabla w_t\cdot\nabla I_hz - \int_\Omega ww_t\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)\cdot\nabla I_hz\\ &\qquad- \frac{1}{2}\int_\Omega w^2\left(\frac{\nabla u}{Q^3}-\frac{\nabla\widehat{u}_h}{\widehat{Q}_h^3}\right)_t\cdot\nabla I_hz\\ &\quad\equiv I_1+\cdots+I_7.\end{aligned}$$

Clearly,

$$|I_1|+|I_2| \leqslant c(\|\nabla(u-\widehat{u}_h)\|_{L^\infty}\|\nabla z\| + ch\|z\|_{H^2})\|\nabla(w_t-\widehat{w}_{ht})\| \leqslant \epsilon\|w_t-\widehat{w}_{ht}\|^2 + c_\epsilon h^4|\log h|^2.$$

Next,

$$\begin{aligned}I_3 &= \int_\Omega (E_{p_i}(\nabla u)u_{tx_i} - E_{p_i}(\nabla\widehat{u}_h)\widehat{u}_{htx_i})\nabla(w-\widehat{w}_h)\cdot\nabla z\\ &\quad+ \int_\Omega E_{p_i}(\nabla\widehat{u}_h)\widehat{u}_{htx_i}\nabla(w-\widehat{w}_h)\cdot\nabla(z-I_hz) - \int_\Omega E_{p_i}(\nabla u)u_{tx_i}\nabla(w-\widehat{w}_h)\cdot\nabla z\\ &= I_{31}+I_{32}+I_{33}.\end{aligned}$$

Observing that $\|f\|_{L^p} \leqslant cp\|f\|_{H^1}$ for $p > 2$ and using (2.18), (2.21), (A.1) as well as an inverse estimate we deduce

$$\begin{aligned}|I_{31}| &\leqslant c(\|\nabla(u-\widehat{u}_h)\| + \|\nabla(u_t-\widehat{u}_{ht})\|)(\|\nabla(I_hw-\widehat{w}_h)\|_{L^{2p/(p-2)}}\|\nabla z\|_{L^p} + h\|w\|_{W^{2,\infty}}\|\nabla z\|)\\ &\leqslant c(phh^{-2/p}\|\nabla(I_hw-\widehat{w}_h)\| + ch^2)\|z\|_{H^2} \leqslant cph^2h^{-2/p}\|w_t-\widehat{w}_{ht}\|.\end{aligned}$$

The choice $p = |\log h|$ then implies that

$$|I_{31}| \leqslant ch^2 |\log h| \|w_t - \widehat{w}_{ht}\| \leqslant \epsilon \|w_t - \widehat{w}_{ht}\|^2 + c_\epsilon h^4 |\log h|^2.$$

Clearly,

$$|I_{32}| \leqslant ch \|\nabla(w - \widehat{w}_h)\| \|z\|_{H^2} \leqslant ch^2 \|w_t - \widehat{w}_{ht}\| \leqslant \epsilon \|w_t - \widehat{w}_{ht}\|^2 + c_\epsilon h^4.$$

Integration by parts together with (A.2) yields

$$|I_{33}| = \left| \int_\Omega \nabla \cdot (E_{p_i}(\nabla u) u_{tx_i} \nabla z)(w - \widehat{w}_h) \right| \leqslant ch^2 |\log h| \|z\|_{H^2} \leqslant \epsilon \|w_t - \widehat{w}_{ht}\|^2 + c_\epsilon h^4 |\log h|^2.$$

In order to deal with I_4 we use the splitting

$$\begin{aligned} I_4 &= \int_\Omega (E_{p_i}(\nabla \widehat{u}_h) \widehat{u}_{htx_i} - E_{p_i}(\nabla u) u_{tx_i}) \nabla w \cdot \nabla (I_h z - z) \\ &\quad + \int_\Omega (E_{p_i}(\nabla \widehat{u}_h) - E_{p_i}(\nabla u))(\widehat{u}_{htx_i} - u_{tx_i}) \nabla w \cdot \nabla z \\ &\quad + \int_\Omega (E_{p_i}(\nabla \widehat{u}_h) - E_{p_i}(\nabla u)) u_{tx_i} \nabla w \cdot \nabla z + \int_\Omega E_{p_i}(\nabla u)(\widehat{u}_{htx_i} - u_{tx_i}) \nabla w \cdot \nabla z \\ &\equiv I_{41} + \cdots + I_{44}. \end{aligned}$$

Using (2.18), (2.21) and an interpolation estimate we obtain

$$|I_{41}| \leqslant ch(\|\nabla(u - \widehat{u}_h)\| + \|\nabla(u_t - \widehat{u}_{ht})\|) \|z\|_{H^2} \leqslant \epsilon \|w_t - \widehat{w}ht\|^2 + c_\epsilon h^4.$$

Next, (2.19) and (2.21) imply

$$|I_{42}| \leqslant c \|\nabla(u - \widehat{u}_h)\|_{L^\infty} \|\nabla(u_t - \widehat{u}_{ht})\| \, \|\nabla z\| \leqslant \epsilon \|w_t - \widehat{w}_{ht}\|^2 + c_\epsilon h^4 |\log h|^2.$$

An application of Lemma 3.1 yields

$$|I_{43}| \leqslant ch^2 |\log h| \|z\|_{H^2} \leqslant \epsilon \|w_t - \widehat{w}_{ht}\|^2 + c_\epsilon h^4 |\log h|^2.$$

Finally, integration by parts together with (2.20) implies that

$$|I_{44}| \leqslant ch^2 |\log h|^2 \|z\|_{H^2} \leqslant \epsilon \|w_t - \widehat{w}_{ht}\|^2 + c_\epsilon h^4 |\log h|^4.$$

The remaining terms I_5, I_6 and I_7 can be dealt with in a similar manner so that we obtain (A.4) after collecting the above estimates and choosing ϵ sufficiently small. □

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