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Pinning and de-pinning phenomena in front propagation in heterogeneous media

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This paper investigates the pinning and de-pinning phenomena of some evolutionary partial differential equations which arise in the modelling of the propagation of phase boundaries in materials under the combined effects of an external driving force F and an underlying heterogeneous environment. The phenomenology is the existence of pinning states—stationary solutions—for small values of F, and the appearance of genuine motion when F is above some threshold value. In the case of a periodic medium, we characterise quantitatively, near the transition regime, the scaling behaviour of the interface velocity as a function of F. The results are proved for a class of semilinear and reaction-diffusion equations.

1. Introduction

In this paper we consider the dynamical behaviour of the solutions of two examples of partial differential equations which arise in the modelling of front propagations in a heterogeneous medium. The first example is a semilinear parabolic equation. Let $u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ solve

$$u_t = \Delta u + f(x,u) + F \quad \text{where } u \text{ is } \mathbb{Z}^N\text{-periodic in } x \text{ for all } t \geqslant 0. \tag{1}$$

Here $f(\cdot,\cdot)$ is 1-periodic in both variables, and sufficiently regular, but not constant in the variable u. The constant $F \geqslant 0$ is called the *external driving force*.

The second is the Allen–Cahn equation with spatially periodic forcing. Let $v(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ solve

$$v_t = v_{xx} - W'(v) + \delta(g(x) + F), \quad x \in \mathbb{R}. \tag{2}$$

Here $W(v)$ is a double-well potential, i.e. a function with $W(1) = W(-1) = 0$ and $W(v) > 0$ for $|v| \neq 1$. It usually takes the form of $W(v) = (1 - v^2)^2$. The function $g(\cdot)$ is 1-periodic with mean zero. The constant δ is taken to be positive and sufficiently small. We will concentrate on solutions $v(x,t)$ which have a *front* structure in the sense that they connect two "different end states" of (2)

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as $x \to \pm\infty$. The end states are stationary solutions close to $+1$ and -1. These conditions will be specified later.

The solutions of (1) and (2) we are interested in are stationary solutions on one hand and pulsating waves on the other. The latter are solutions which, up to spatial translations, are *space-time periodic*—the analogue of travelling waves in the case of constant coefficients. We are particularly interested in the *transition* between these two types of solutions as F varies (increases). More precisely, we will show that for both of the above equations, generically, there exists an F_*—*critical forcing*—such that for $0 \leqslant F \leqslant F_*$, the equations have stationary solutions, while for $F > F_*$, they have solutions presenting fronts propagating in space with an average velocity V_F. We also show that in the case of periodic spatial heterogeneity, the velocity satisfies the following scaling law:

$$V_F \sim (F - F_*)^{1/2} \quad \text{for } 0 < F - F_* \ll 1. \tag{3}$$

The above described dynamical behaviour should be contrasted with the situation $f \equiv 0$ or $g \equiv 0$ in which it can be shown (in fact it is obvious for (1)) that there exists for any $F \neq 0$ a solution which moves with constant speed: the linear solution $u(x,t) = U_0 + Ft$ in the case of (1) and the travelling wave $u(x,t) = U(x - ct)$ in the case of (2). So the pinning and de-pinning transition is caused by the *oscillatory* nature of the spatial heterogeneity.

1.1 *Motivations and applications*

Both equations (1) and (2) are simplified models for the motion of an interface in a material under the combined effect of the spatially heterogeneous environment, given by $f(x,u)$ or $g(x)$, the surface tension and the external driving field F. The interface could for example be a domain boundary in a magnetic material with impurities [5, Chapter 13], the interface between immiscible fluids in a porous medium [23], or a dislocation line in a solid [16, 17]. The impurities can in general impede or accelerate the motion of the interface, but they are soft obstacles in the sense that the interface can pass through them. The motions of the interface are characterised by the interaction between an interfacial energy, which keeps the interface flat or straight, and the influence of the impurities, which force the interface to deviate from a flat shape, and the driving field, which makes the interface move one way or the other. For an introduction from the physical point of view, we refer to [11] or [17]. Note that in many of the above applications, the driving forces derived from the interfacial energy, the spatial heterogeneity and the external field can all take the form of nonlocal operators.

Another important ingredient in the modelling is the consideration of random impurities. For an overview of models which are characterised by the competition between a surface energy and a random field, we refer to [21]. For rigorous mathematical results on a one-dimensional model, see [14].

In the dynamic case, the motion of an interface in such a model decomposes into periods of slow motion and sudden jerks, called avalanches [11]. In the specific case of (1) with random spatial forcing $f(x,u)$, the model is known in the physics literature as the quenched Edwards–Wilkinson model which is studied extensively (see for example [4]).

While we see the case of periodic impurities treated in this paper mainly as a first step towards a rigorous mathematical analysis of such models for random media, we would like to point out that the periodicity is fully justified from a modelling point of view if the impurity is related to a periodic crystal structure of an underlying substrate.

Already in the periodic case, the relationship between the mean velocity and the driving force is expected to be a power law *near the transition regime*, which is due to the interplay between some

slow motion *near* the stationary solutions at $F = F_*$ and *fast* motion with speed of order one *away* from the stationary states.

In order to keep the mathematical treatment simple, we consider spatially localised driving forces. As the mean curvature is the first variation of the length or area of the interface which is a simple form of the interfacial energy, one way to model the above phenomenology is by means of the mean curvature flow with a driving force: the normal velocity V_n at any point on the interface $\Gamma(t)$ is given by the following motion law:

$$V_n = \kappa - f(x) + F \quad \text{for } x \in \Gamma(t). \tag{4}$$

Here κ is the mean curvature of the interface, $f(x)$ is some spatial function and F is a constant.

Even in this case, the geometry of the interface can become quite complicated because self-intersection can occur. In addition, the graph representation of the interface might not be preserved under the dynamics. One reason equations (1) and (2) are used is to avoid this technical difficulty. For (1), a linearised version of the curvature operator κ is used and the interface is given by the graph of the function $u(x, t)$. In this way, the graph representation is preserved for all time and estimates of the solutions are easier to obtain.

The second equation (2) (when $\delta = 0$) is a gradient flow of a Ginzburg–Landau type free energy for a physical system with a nonconserved order parameter and two stable phases corresponding to the two "wells" ± 1 of W [2]. It is well known that under large space-time diffusive rescaling, i.e. $t \to \epsilon^2 t$ and $x \to \epsilon x$, the zero level set of suitable solutions converges to an interface moving by its mean curvature [22]. Thus equation (2) is an approximation of (4) on some large spatial and temporal scale, while it is a model in its own right on a smaller scale, usually called the mesoscopic scale.

Of course, in the one-dimensional case studied here the limiting motion is the motion of a point in a spatial potential. Let g be a function of period one. Then under the above mentioned space-time scaling, the zero $R(t)$ of $v(x, t)$ which represents the front location, evolves according to the following ODE:

$$\mu \dot{R} = -\Phi'(R) \tag{5}$$

where $\Phi(R) = -RF - G(R)$, and $G(R)$ is a 1-periodic potential such that $G'(R) = g(R)$. The constant μ is usually called the mobility. In essence, equation (5) describes the negative gradient flow of a particle in a *tilted* periodic potential. This simple ODE already shows some of the qualitative features of the more complex PDEs such as (1) and (2). Thus it is advantageous to briefly describe its behaviour here.

For $F = 0$ the function $G(R)$ is assumed to have global minima periodically spaced leading to stable stationary solutions. When $F = F_* = -g(R_*)$, where $g(R_*) = \inf_{[0,1]} g(x)$, a stationary solution $R(t) \equiv R_*$ still exists. Note that at this point, $g'(R_*) = 0$. Now let $F = F_* + \gamma$, and $y = R - R_*$. If we further assume the nondegeneracy condition $\alpha = g''(R_*) > 0$, then

$$\dot{y} \sim \alpha y^2 + \gamma \quad \text{for } y \text{ near 0 and } 0 < \gamma \ll 1. \tag{6}$$

The explicit solution is given by $y(t) = (\gamma/\alpha)^{1/2} \tan((\alpha\gamma)^{1/2} t)$. Thus it takes the solution a time of order $\gamma^{-1/2}$ to leave the region of slow motion and to reach a region of fast motion where the speed becomes independent of γ.

The above type of argument is also used in [8] in the study of the pinning and de-pinning phenomena for some discrete reaction-diffusion equations. Equation (6) in fact describes the

dynamics along the centre manifold at the critical point $y = 0$. The work [1] also uses an equation similar to (5) in the investigation of the dynamics of some martensitic phase transformations.

1.2 *Statement of theorems*

Here we state precisely the definitions and theorems to be shown in this paper.

For the semilinear equation (1), we first introduce the linearised operator $L_f(u)$ at a function $u(x)$:

$$L_f(u)\varphi := \Delta\varphi + \partial_u f(x, u(x))\varphi \quad \text{for any } \mathbb{Z}^n\text{-periodic } \varphi. \tag{7}$$

The function $f(x, u) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

A1 • f is $\mathbb{Z}^n$-periodic in x and and 1-periodic in u, i.e.

$$f(x, u + 1) = f(x, u) \text{ and } f(x + \hat{e}, u) = f(x, u) \quad \text{for all } (x, u) \in \mathbb{R}^n \times \mathbb{R},$$

where $\hat{e}$ is the unit vector along any coordinate axis.

• $f(x, u)$ is twice continuously differentiable in both variables and $\|f(\cdot, \cdot)\|_{C^2(\mathbb{R}^n \times \mathbb{R})} < \infty$.

A2 At $F = 0$, equation (1) has a stationary periodic solution $u_0(x)$ which is *stable* in the sense that the principal eigenvalue of the linearised operator $L_f(u_0)$ is negative.

Note that the mere existence of a stationary solution at $F = 0$ is nothing more than a normalisation and follows e.g. if the average of $f(x, u)$ is zero (see Lemma 4). The stability condition is indeed an assumption and excludes for example any $f(x, u)$ which is constant in the variable u. The regularity assumption on the nonlinearity serves to keep the technical effort at a minimum. We expect that it can be relaxed.

THEOREM 1 Assume **A1** and **A2**. Then the following statements hold.

1. There exists an $F_* > 0$ such that for $0 \leqslant F \leqslant F_*$, (1) has a stationary solution $U_F(x)$.
2. Assume in addition the condition **A3** (given right afterward), or that there exist only finitely many stationary solutions (up to translation) at $F = F_*$. Then, for $F > F_*$, there exists a unique $0 < T_F < \infty$ and a solution $U_F(x, t)$ of (1) satisfying

$$U_F(x + \hat{e}, \cdot) = U_F(x, \cdot) \quad \text{and} \quad U_F(\cdot, t + T_F) = U_F(\cdot, t) + 1. \tag{8}$$

The condition **A3** is as follows:

A3 For any stationary solution u_* at F_* we have

$$\int_Q \varphi_p^3 \partial_u^2 f(x, u_*(x))\, \mathrm{d}x \neq 0,$$

where $Q = [0, 1]^n$ and φ_p is the principal eigenfunction of $L_f(u_*)$.

The above is reminiscent of some nondegeneracy condition. It corresponds to the fact that $g''(R_*) \neq 0$ for the ODE example (5). (In fact, this assumption implies that the above quantity is automatically positive. See Claim I on page 90.) In addition, it is also used to give the precise asymptotics of the dynamics for $F > F_*$. Otherwise, some higher (for example, fourth) order information is needed.

Now we are ready to present the asymptotics of the propagating velocity:

THEOREM 2 Assume **A1**, **A2** and **A3**. Then there is a constant A (see (39) for its formula) such that the average velocity V_F defined as T_F^{-1} has the following asymptotics:

$$V_F = A(F - F_*)^{1/2} + o(|F - F_*|^{1/2}) \quad \text{for } 0 < F - F_* \ll 1. \tag{9}$$

REMARK. Note that even though we only consider solutions which are periodic in the spatial variable, this is not as restrictive as it seems because the comparison principle implies that the periodic solutions can already determine the pinning and de-pinning behaviour of arbitrary solutions which are bounded in the L^∞ norm.

In addition, in Section 2.4, we can actually show that there are lots of functions $f(x, u)$ satisfying **A1**–**A3**. See also Section 4.1 for an explicit example. □

For the inhomogeneous Allen–Cahn equation (2), we consider the regime of *weak* spatial heterogeneity. In this case, it turns out that we can obtain quite explicit quantitative information on the speed of the pulsating wave. First we introduce the standing wave solution $m(x)$ (for $\delta = 0$):

$$m_{xx} - W'(m) = 0$$

such that for some constant C it satisfies the estimates

$$||m(x)| - 1| \leqslant Ce^{-a|x|} \quad \text{and} \quad |m_x(x)| \leqslant Ce^{-a|x|} \quad \text{for all } -\infty < x < \infty$$

where a is any positive number less than $W''(\pm 1)$. (The existence of $m(x)$ and its properties are well known [10].) We next introduce the 1-periodic function

$$h(a) = -\int_{-\infty}^{\infty} g(z + a) m_z(z)\, \mathrm{d}z.$$

Without loss of generality, assume that the minimum value of h, denoted by h_*, is attained at $a = 0, \pm 1, \pm 2, \ldots$. Assume further that $h''(0) > 0$. We first define the following quantities:

$$F_* = -\frac{h_*}{2}, \quad \alpha = \frac{2}{\|m_x\|^2}, \quad \beta = \frac{h''(0)}{4\|m_x\|^2} \tag{10}$$

where $\|m_x\|^2 = \int_{-\infty}^{\infty} m_x(x)^2\, \mathrm{d}x$. Then we have:

THEOREM 3 For any $\sigma > 0$, there exist $\delta_*, C_1, C_2 > 0$ such that for any $\delta < \delta_*$ and any F in the range

$$0 < C_1\delta \leqslant F - F_* \leqslant C_2, \tag{11}$$

there exists a constant T_F and a solution $U_F(x, t)$ of (2) satisfying

$$U_F(x, t + T_F) = U_F(x - 1, t). \tag{12}$$

In addition, the velocity of the front, defined as $V_F = T_F^{-1}$, satisfies

$$(1 - \sigma)\frac{\alpha\delta\sqrt{\beta(F - F_*)}}{\pi} \leqslant V_F \leqslant (1 + \sigma)\frac{\alpha\delta\sqrt{\beta(F - F_*)}}{\pi}. \tag{13}$$

1.3 *Structure of the paper and outline of proofs*

The dynamics of (1) and (2) is analysed in Sections 2 and 3 respectively. In Section 4 , we provide some explicit examples of the pinning phenomena. These examples are illustrative in the sense that they clearly demonstrate the existence of a nontrivial pinning state. Furthermore, the constants in the abstract formulation can be computed analytically. In Appendix B, for the reader's convenience, the spectral estimates for the linearised Allen–Cahn operator are provided.

Here we briefly describe the method of the proof. For the semilinear case, we first show the existence of a threshold force and some properties of the stationary solutions. In particular, we prove that the threshold value F_* is characterised by the fact that the principal eigenvalue of the linearised operator at any stationary solution must be zero. The principal eigenfunction and the nondegeneracy condition **A3** are then used to construct sub- and supersolutions near the critical solution. In this way, we can deduce the time a solution spends near a pinned state for F slightly larger than F_*. This is reminiscent of the analysis of the dynamics on the centre-manifold of a stationary solution. Sharp asymptotic estimates for the period of the pulsating wave are also available. The overall phenomenology is very similar to the behaviour already revealed in the simple ODE model (5) and its analysis (6). The existence of pulsating wave solutions is then obtained by the Schauder fixed point theorem. The contracting property of the heat semigroup is crucial in setting up the nonlinear map used in the fixed point theorem.

In Section 3, we obtain the existence of pulsating wave solutions for (2) and quantitative information about their speed of propagation. We work in the weakly spatial heterogeneity regime ($0 < \delta \ll 1$) so that we can employ the techniques of perturbation theory. We heavily use the spectral properties of a one-parameter manifold of stationary solutions of the unperturbed equation ($\delta = 0$). Even though it is conceivable that a proof similar to the one in the semilinear case is also available for the reaction-diffusion case so that δ might not be constrained to be small, we believe that the strategy employed here provides a different perspective and most importantly, it gives explicit quantitative information about the threshold value F_* and the speed of the propagating front.

Here we also mention some related results. The work [20] studies similar equations to (1) but their spatial forcing is required to be positive everywhere. In the sequence of works [26, 27, 28], the pinning phenomena is investigated for the spatially forced Allen–Cahn equation (2). This is commonly referred to as the *bi-stable case*. The existence of pulsating waves is also proved. The inhomogeneity in [26, 27, 28] is different from the case studied in this paper. Recently, [6, 7], and, within a more general framework, [18], analyse similar front propagating phenomena and establish some homogenisation results in the setting of Hamilton–Jacobi and degenerate parabolic differential equations. However, in all of these works, the pinning and de-pinning transition is not studied which is the main emphasis here.

2. The semilinear equation: Proofs

In this section we prove Theorems 1 and 2 for the semilinear equation (1). They are consequences of several propositions.

As the analysis of the dynamics relies very much on the properties of the stationary solutions, we state here the static version of equation (1):

$$0 = \Delta u + f(x, u) + F, \qquad \text{where } u \text{ is } \mathbb{Z}^n\text{-periodic in } x \in \mathbb{R}^n. \tag{14}$$

Recall that F is a constant and that $f(\cdot,\cdot)$ fulfills **A1**. An important role will be played by the linearised operator (7) at a function $u(x)$, which is stated here again for convenience:

$$L_f(u)\varphi = \Delta\varphi + \partial_u f(x,u)\varphi, \qquad \text{where } \varphi \in H^1(\mathbb{R}^n) \text{ is } \mathbb{Z}^n\text{-periodic.} \tag{7}$$

The eigenvalue problem for $L_f(u)$ is given by

$$L_f(u)\varphi = \lambda\varphi \qquad \text{for } \varphi \in H^1(\mathbb{R}^n) \text{ and } \mathbb{Z}^n\text{-periodic.} \tag{15}$$

First we show a proposition which makes assumption **A2** more natural.

PROPOSITION 4 If $f(x,u)$ satisfies **A1** and

$$\int_0^1 \int_Q f(u,x)\,\mathrm{d}x\,\mathrm{d}u = 0,$$

then equation (14) has a solution for $F = 0$.

Proof. Note that the energy functional

$$E(u) := \frac{1}{2}\int_Q |\nabla u|^2\,\mathrm{d}x - \int_Q H(u,x)\,\mathrm{d}x, \quad \text{where} \quad H(u,x) = \int_0^u f(u,x)\,\mathrm{d}u, \tag{16}$$

is lower semicontinuous if $u_n \to u$ strongly and $\nabla u_n \to \nabla u$ weakly in L^2. In addition, because of

$$\left|\int H(x,u(x))\,\mathrm{d}x\right| \leqslant C\|u\|_{L^1},$$

the energy is bounded from below. Thus $E(u)$ has a minimiser in the class H^1_s of all periodic H^1-functions with $\int_Q u(x)\,\mathrm{d}x = s$.

Let $G(s) := \min_{H^1_s} E(u)$. Note that as f has zero average, we have $\int_Q H(u+1,x)\,\mathrm{d}x = \int_Q H(u,x)\,\mathrm{d}x$. Hence $G(s+1) = G(s)$. Comparing $E(u)$ and $E(u+\epsilon)$, we see that G is Lipschitz, so G assumes its minimum for some s_0. This minimum is attained at some function u_0. Therefore a minimiser of E in the class of periodic H^1 functions exists. This function u_0 solves the Euler–Lagrange equation associated with E, which is (14) with $F = 0$. □

REMARK. The u_0 from the above proof is stable in the sense of minimisation. It can further be made stable in the sense of **A2** upon changing f to

$$f(x,u) + \epsilon R(u - u_0(x)) - \epsilon \int_0^1 \int_Q R(u - u_0(x))\,\mathrm{d}x\,\mathrm{d}u$$

(and extended periodically in x and u), where $0 < \epsilon \ll 1$ and R is a smooth function such that $R(s) = -s$ for $|s| \leqslant 1/8$ and $R(s) = 0$ for $|s| \geqslant 1/4$. □

The proof of Theorem 1 is divided into several propositions. For convenience, we use the notation $(1)_F$ and $(14)_F$ to denote the dependence of the equations on F.

PROPOSITION 5 (Existence of threshold force) Assume **A1** and **A2**.

1. There exists an $F_* > 0$ such that $(14)_F$ has a classical solution for $0 \leqslant F \leqslant F_*$ while it has no solution for $F > F_*$.
2. At $F = F_*$, the collection of the stationary solutions is well ordered in the sense that if u_1 and u_2 are two solutions of $(14)_{F_*}$, then either $u_1 < u_2$ or $u_2 < u_1$.
3. Let u be a solution of $(14)_{F_*}$. Then the principal eigenvalue of the linearised operator $L_f(u)$ (7) is zero and simple and the corresponding eigenvector $\varphi_p(\cdot)$ is strictly positive.

Proof of 5(1). Consider $\mathcal{F} = \{F \geqslant 0 : \text{equation } (14)_F \text{ has a solution}\}$. By assumption, $0 \in \mathcal{F} \neq \emptyset$. On the other hand, if $F > \|f\|_\infty$, then $(14)_F$ has no solution. Hence we can define $F_* = \sup\{F \in \mathcal{F}\} < \infty$. We claim that $F_* \in \mathcal{F}$. For if not, there exists a sequence of F_n converging to F_* and also a sequence of functions $u_n(x)$ satisfying $(14)_{F_n}$: $\Delta u_n + f(x, u_n) + F_n = 0$. Since f satisfies **A1** and the F_n's are uniformly bounded, by elliptic regularity, the collection of $\{u_n\}$'s is compact and hence has a subsequence converging to some u_* satisfying $\Delta u_* + f(x, u_*) + F_* = 0$. Thus $F_* \in \mathcal{F}$.

According to **A2**, the linearised operator $L_f(u_0)$ at a solution u_0 of $(14)_0$ has negative principal eigenvalue. Thus $L_f(u_0)$ is invertible. A straightforward application of the implicit function theorem (see for example [3]) implies that $(14)_F$ also has a solution for any F which is slightly larger than 0. Hence $F_* > 0$.

Next we show that $(14)_F$ has a solution for any $0 < F < F_*$. Consider the solution u_* of $(14)_{F_*}$ and u_0 of $(14)_0$. Then they are also super- and subsolutions of $(14)_F$, respectively. By the periodicity of $f(x, u)$ in the u variable, we can assume that $u_0 < u_* + N$ for some large enough integer N. Hence $(14)_F$ must have a solution for any F in $(0, F_*)$.

Proof of 5(2). Let u_1 and u_2 be two solutions of $(14)_{F_*}$. Suppose the graphs corresponding to u_1 and u_2 cross each other. Consider

$$U(x) = \min\{u_1(x), u_2(x)\} \quad \text{and} \quad V(x) = \max\{u_1(x), u_2(x)\}.$$

Now $U(x)$ and $V(x)$ are sub- and supersolutions of $(14)_{F_*}$, but not solutions. Again, by translating U or V in the u-direction, we can assume without loss of generality that $V < U$. The idea next is to show that they are still super- and subsolutions for $(1)_{F_*+\gamma}$ for some small enough and positive γ. Thus a stationary solution for $(1)_{F_*+\gamma}$ exists, contradicting the definition of F_*.

This is made precise as follows. Consider the solution $u(x, t)$ of the evolution equation $(1)_{F_*}$ with initial data $U(x)$. Then $u(x, h) < u(x, 0)$ for any $h > 0$. The regularity assumptions satisfied by $f(x, u)$ imply that for $0 < h \ll 1$ the function $v_h(x, t) := u(x, t) - u(x, t + h)$ solves the linear equation

$$(v_h)_t = \Delta v_h + c(x, t) v_h, \quad v(x, 0) \geqslant 0,$$

with a uniformly bounded potential $c(x, t)$. As a consequence, $v_h(x, t)$ cannot attain a negative minimum, which leads to $u(x, t + h) \leqslant u(x, t)$ and hence $u_t(x, t) \leqslant 0$ but $\not\equiv 0$ for $t > 0$. Differentiating $(1)_{F_*}$ with respect to t shows that the function u_t solves a linear parabolic differential equation with bounded coefficients. By the strong maximum principle, there exists a $t_1 > 0$ and c such that $u_t(\cdot, t_1) < -c < 0$. Now consider $(1)_{F_*+\gamma}$ for $0 < \gamma < c$ and the initial data $u(x, t_1)$. Its solution $u_2(x, t)$ will *decrease monotonically* in time. On the other hand, using $V(x)$ we can construct in a similar way a solution $u_3(x, t)$ of $(1)_{F_*+\gamma}$ which will *increase monotonically* in time. Hence there must be a stationary solution of $(14)_{F_*+\gamma}$ leading to the desired contradiction.

Proof of 5(3). Let u_* be a solution of $(14)_{F_*}$. By the spectral theorem for compact self-adjoint operators, the spectrum of $L_f(u_*)$ consists of discrete eigenvalues: $\lambda_p \geqslant \lambda_1 \geqslant \cdots \to -\infty$. The variational characterisation of λ_p, using the Rayleigh quotient, implies that λ_p is simple, i.e. of multiplicity one, and its eigenfunction φ_p is nonnegative. The fact that φ_p is strictly positive follows from the strong maximum principle. Now suppose $\lambda_p \neq 0$. Again, the idea is to construct sub- and supersolutions of $(14)_{F_*+\gamma}$ for some small but positive γ. Let $0 < \epsilon \ll 1$. Consider the function

$$v_\epsilon^{\pm}(x,t) = u(x,t) \pm \epsilon\varphi_p(x).$$

By linearising the nonlinearity $f(u,x)$ up to second order, we obtain

$$\Delta(v_\epsilon^{\pm}) + f(\cdot, v_\epsilon^{\pm}) + F_* + \gamma = \pm\epsilon(\lambda_p\varphi_p + O(\epsilon)) + \gamma.$$

Assume $\lambda_p > 0$. As $\varphi_p(x)$ is strictly positive, we can first choose an ϵ and then a $\gamma > 0$ such that v_ϵ^+ is a supersolution of $(14)_{F_*+\gamma}$, whereas v_ϵ^- is a subsolution. Hence, by reasoning as before, we see that there must be a stationary solution for $F^* + \gamma$ between v_ϵ^+ and $v_\epsilon^- + N$ for some large integer N, which again contradicts the definition of F^*. The case of $\lambda_p < 0$ follows similarly. □

PROPOSITION 6 (Uniqueness of space-time periodic solution) Let F_* be as in Proposition 5. Then for any $F > F_*$, if a space-time periodic solution satisfying (8) exists, then the time period T_F is unique. In addition, the solution is unique up to a time shift, i.e. if U_F^1, U_F^2 are two such solutions, then for some t_0,

$$U_F^1(x,t) = U_F^2(x, t - t_0).$$

Proof. Let $U_1(x,t)$ and $U_2(x,t)$ be two space-time periodic solutions of $(1)_F$ with time-periods $\infty > T_1 > T_2 > 0$. Now consider solutions $u_1(x,t)$ and $u_2(x,t)$ of $(1)_F$ with initial data $U_1(x,0) + N$ and $U_2(x,0)$ where N is a large enough positive integer such that $U_1(x,0) + N > U_2(x,0)$. By (8), there must be a t_1 and x_1 such that $u_2(x_1,t_1) = u_1(x_1,t_1)$, contradicting the comparison principle. Thus $T_1 = T_2 = T_F$.

Now consider again the solution $U_2(x,t)$. There must be a time t_2 and x_2 such that $U_2(x_2,t_2) = U_1(x_2,0) + N$ and $U_2(\cdot,t_2) \leqslant U_1(\cdot,0) + N$ for some large positive integer N. Now solve $(1)_F$ with initial data $U_1(x,0) + N$ and $U_2(x,t_2)$. By time periodicity, after a time T, the two solutions will touch again—contradicting the comparison principle, unless $U_1 \equiv U_2$ (up to an additive integer). Thus the space-time periodic solution must be unique in the sense stated in the proposition. □

PROPOSITION 7 (Existence of space-time periodic solution) For each $F > F_*$, there exists a number $0 < T_F < \infty$ and a function $U_F(x,t)$ satisfying $(1)_F$ and (8).

Proof. It is easy to show that under assumption **A1**, the initial value problem $(1)_F$ is well-posed in L^2 and a solution exists globally in time. We will employ the Schauder fixed point theorem to prove the existence of a solution satisfying (8).

Let $u(x,t)$ be a solution of $(1)_F$. It can be decomposed as $u(x,t) = p(t) + \xi(x,t)$ with $\langle\xi(\cdot,t)\rangle = 0$, where $\langle g(\cdot)\rangle = \int_Q g(x)\,\mathrm{d}x$. Then $p(t)$ and $\xi(\cdot,t)$ satisfy

$$\begin{cases} \dot{p}(t) = \langle f(\cdot, p(t) + \xi(\cdot)) + F\rangle, \\ \xi_t(x,t) - \Delta\xi(x,t) = f(x, p(t) + \xi(x,t)) - \langle f(\cdot, p(t) + \xi(\cdot,t))\rangle, \\ \langle\xi(\cdot,t)\rangle = 0. \end{cases} \tag{17}$$

Let $\xi_0(\cdot) = \xi(\cdot, 0)$ and assume as a normalisation condition that $p(0) = \langle u_*(x)\rangle$. Furthermore, let $\|\xi_0\|_{L^\infty} < A$ for some constant A. We claim that for any $F > F_*$, there exists a unique time $T = T(\xi_0)$ such that $p(T(\xi_0)) = p(0) + 1$ and constants $0 < \tau_1 < \tau_2$ *independent of* ξ_0 such that

$$0 < \tau_1 < T(\xi_0) < \tau_2 < \infty. \tag{18}$$

An obvious lower bound for $T(\xi_0)$ can of course be obtained from $|\dot{p}| \leqslant \|f\|_{L^\infty}$ so that $\|f\|_{L^\infty}^{-1} < T(\xi_0)$. But an upper bound follows from the centre manifold type analysis as described in Theorem 11 (which also gives much more precise estimates).

Now let $S(t)$ be the solution operator for the heat equation $v_t = \Delta v$ on $Q = [0, 1]^n$ with periodic boundary condition. Let also $\psi(x, t) = f(x, p(t) + \xi(x, t)) - \langle f(\cdot, p(t) + \xi(\cdot, t))\rangle$. We introduce the nonlinear map

$$\mathcal{T}(\xi_0) = \xi(x, T(\xi_0)) \quad \text{which equals} \quad S(T(\xi_0))\xi_0 + \int_0^{T(\xi_0)} S(T(\xi_0) - s)\psi(\cdot, s)\,\mathrm{d}s. \tag{19}$$

The existence of a space-time periodic solution is equivalent to finding a $\xi_0(\cdot)$ such that $\xi_0 = \mathcal{T}(\xi_0)$.

To analyse the map $\mathcal{T}$, we introduce the closed subset of $L^2(Q)$,

$$\mathcal{B}_A = \{v : \|v\|_{L^\infty} \leqslant A,\ \langle v\rangle = 0\}$$

where A is to be determined. First note that $\mathcal{T}$ is a continuous function on $\mathcal{B}$. This follows from the continuous dependence of $\xi(\cdot, t)$ on t and $T(\xi_0)$ on ξ_0. Note that we only need to consider those times t satisfying (18). Next, we claim the following property for $S(t)$ (which is proved immediately afterwards):

For each $\tau > 0$, there exists a $0 < C_1(\tau) < 1$ such that for all $v(\cdot)$, 1-periodic with $\langle v\rangle = 0$,

$$\|S(\tau)v\|_{L^\infty} \leqslant C_1(\tau)\|v\|_{L^\infty} \tag{20}$$

and there are constants M and N such that

$$\|S(t)v\|_{L^\infty} \leqslant Me^{-Nt}\|v\|_{L^\infty} \quad \textit{for } t > 0. \tag{21}$$

With the above estimates and the lower bound for $T = T(\xi_0)$, we have

$$\begin{aligned}
\|\mathcal{T}(\xi_0)\|_{L^\infty} &\leqslant \|S(T)\xi_0\|_{L^\infty} + \int_0^T \|S(T-s)\psi(x,s)\|_{L^\infty}\,\mathrm{d}s \\
&\leqslant C_1(\tau_1)\|\xi_0\|_{L^\infty} + \int_0^{T-1} \|S(T-s)\psi(x,s)\|_{L^\infty}\,\mathrm{d}s + \int_{T-1}^T \|S(T-s)\psi(x,s)\|_{L^\infty}\,\mathrm{d}s \\
&\leqslant C_1(\tau_1)\|\xi_0\|_{L^\infty} + \int_0^{T-1} Me^{-N(T-s)}\|\psi(\cdot,s)\|_{L^\infty}\,\mathrm{d}s + C_2 \\
&\leqslant C_1(\tau_1)\|\xi_0\|_{L^\infty} + C_3
\end{aligned}$$

for some $0 < C_3 < \infty$. In the above, we have used the fact that $\|\psi(\cdot, s)\|_{L^\infty} \leqslant 2\|f\|_{L^\infty} < \infty$.

Now let $A = C_3/(1 - C_1(\tau_1))$. Then $\|\xi_0\| \leqslant A$ implies that $\|\mathcal{T}(\xi)\|_{L^\infty} \leqslant A$. Thus $\mathcal{T}(\cdot)$ maps $\mathcal{B}_A$ into itself. The lower bound for T together with parabolic regularity leads to the compactness of

the map $\mathcal{T}$ in L^2. Hence the Schauder fixed point theorem (see [13, Corollary 11.2]) implies that $\mathcal{T}$ has a fixed point. □

Proof of (20). First, by maximum principle, $\|S(t)v\|_{L^\infty} \leqslant \|v\|_{L^\infty}$ for all $t > 0$. Without loss of generality, assume $\|v\|_{L^\infty} = 1$. Using the heat kernel, it is easily seen that there is a constant $C(\tau)$ such that

$$\|D^2 S(\tau/2)v\|_{L^\infty} \leqslant C(\tau) < \infty. \tag{22}$$

Now introduce the periodically extended version of the function $U(x) := \min(C(\tau)\|x\|^2, 1)$. Let x_* be a point such that

$$(S(\tau/2)v))(x_*) = I_* = \inf_{\mathbb{R}^n}(S(\tau/2)v)(\cdot).$$

Since $\langle S(\tau/2)v\rangle = 0$, we have $I_* \leqslant 0$. Consider $Q(x) = U(x - x_*)$. Then (22) and the definition of U imply that $(S(\tau/2)v)) \leqslant Q(x)$. Hence by the strong maximum principle, for all $t > 0$ we have

$$\sup_{x\in\mathbb{R}^n} (S(t + \tau/2)v)(x) \leqslant \sup_{x\in\mathbb{R}^n} (S(t)Q)(x) = C_1(\tau) < 1,$$

which leads to (20). □

Proof of (21). This follows immediately from (20) once we observe that for any $t \in [k\tau, (k+1)\tau]$,

$$\begin{aligned}\|S(t)v\|_{L^\infty} &\leqslant \|S(k\tau)v\|_{L^\infty} \leqslant C_1(\tau)^k \|v\|_{L^\infty} = e^{-k|\ln(C_1(\tau))|}\|v\|_{L^\infty}\\ &\leqslant e^{|\ln(C_1(\tau))|}e^{-t|\ln(C_1(\tau))|/\tau}\|v\|_{L^\infty}\end{aligned}$$

so that we can set $N = |\ln(C_1(\tau))|/\tau$ and $M = e^{|\ln(C_1(\tau))|}$. □

What is left is proving the asymptotic statement (9) in Theorem 2 about the propagating velocity. The strategy is to construct sub- and supersolutions of $(1)_F$ to provide sharp bounds for T_F. For this, we need to make use of the nondegeneracy assumption **A3**. In particular, this implies that there are only a *finite number* of stationary solutions $\{u^{(i)}\}_{i=1}^N$ of $(14)_{F_*}$. Now, the key observation is that *near* each of the $u^{(i)}$'s, the dynamical behaviour of the solution of $(1)_{F_*}$ is well-approximated by the dynamics on the centre-manifold while *away from* the stationary point, the velocity of the evolving graph is uniformly positive and *independent of* the additional forcing above the critical value. These are made precise by the following definitions and constructions.

2.1 *Construction of sub- and supersolutions near a stationary point*

Let u_* be one of the solutions of $(14)_{F_*}$. By Proposition 5(3), the principal eigenvalue λ_p of the linearised operator (7) is zero and simple with positive eigenfunction $\varphi_p(x)$. The following construction is motivated by the centre-manifold theory for $(1)_{F_*+\gamma}$ near u_* (see [15, pp. 173]). We make the following ansatz:

$$v(x, t) = u_*(x) + (R_0(x) + s(t)\varphi_p(x) + R(x, t)) \tag{23}$$

and compute:

$$\begin{aligned}v_t - \Delta v - f(x, v) - F_* - \gamma &= \dot{s}(t)\varphi_p + R_t\\ &\quad - \{L_f(u_*)R_0 + \gamma + \tfrac{1}{2} f_{uu}(x, u_*)(R_0 + s(t)\varphi_p + R)^2 + L_f(u_*)R + E_3(R_0, s(t), R)\}\end{aligned}$$

where

$$\|E_3\|_\infty \leqslant (|s| + \|R_0\|_\infty + \|R\|_\infty)^2 \omega(|s| + \|R_0\|_\infty + \|R\|_\infty) \tag{24}$$

for some continuous and increasing function $\omega(\cdot)$ such that $\omega(0) = 0$. For simplicity, φ_p is normalised to have norm one: $\|\varphi_p\| = 1$. To construct the *supersolution* $v^+(x,t)$, we define R_0, s, and R_1 by

$$L_f(u_*)R_0 = -\gamma + \langle \gamma, \varphi_p\rangle \varphi_p, \tag{25}$$

$$\dot{s}(t) = (1+\sigma)\{\langle \gamma, \varphi_p\rangle + \langle \tfrac{1}{2} f_{uu}(x, u_*)(s\varphi_p + R_0 + R)^2, \varphi_p\rangle\}, \tag{26}$$

$$R_t = L_f(u^*)R + \Psi, \qquad R(0) = 0, \tag{27}$$

$$\Psi = \tfrac{1}{2} f_{uu}(x, u_*)(s\varphi_p + R_0 + R)^2 - \langle \tfrac{1}{2} f_{uu}(x, u^*)(s\varphi_p + R_0 + R)^2, \varphi_p\rangle \varphi_p, \tag{28}$$

where $0 < \sigma \ll 1$. The initial condition for s is to be determined. With the above choices, v^+ will be a supersolution if for all x, t,

$$|E_3(x,t)| \leqslant \tfrac{1}{2}\sigma |\langle \gamma, \varphi_p\rangle \varphi_p(x) + \langle \tfrac{1}{2} f_{uu}(x, u_*)(R_0(x) + s(t)\varphi_p(x) + R(x,t))^2, \varphi_p(x)\rangle|. \tag{29}$$

Note that φ_p is uniformly positive in the sense that $\varphi_p(x) \geqslant C_p > 0$. To proceed further, we state the following two claims which are proved in Section 2.3:

CLAIM I Assume **A3**. Then

$$\int_Q f_{uu}(x, u_*(x))\varphi_p(x)^3 \, \mathrm{d}x > 0. \tag{30}$$

CLAIM II Let

$$\mathcal{B}_{M,0,T} := \{(s, R) : [0, T] \to \mathbb{R} \times L^\infty(Q) : \sup_{t\in[0,T]} (|s(t)| + \|R(t)\|_{L^\infty}) \leqslant M\}.$$

1. There exists $C_0 > 0$ such that
$$\|R_0\|_\infty \leqslant C_0\gamma.$$
2. There exist $0 < \gamma_0$, $C_1 \ll 1$ and C, $T_* > 0$ such that for all $0 < \gamma < \gamma_0$, there exists a solution $(s(t), R(t)) \in \mathcal{B}_{C_1,0,T_*}$ of (26)–(28) and
$$\|R(t)\|_\infty \leqslant C(\sup_{r\in[0,t]} s(r)^2 + \gamma^2) \quad \text{for } 0 < t < T. \tag{31}$$

Furthermore, T_* can be chosen to be

$$T_* = \inf\{t \geqslant 0 : |s(t)| \geqslant C_1\}$$

The above two claims imply that for γ and s small enough, we have

$$|\langle \tfrac{1}{2} f_{uu}(x, u_*)[(R_0 + s\varphi_p + R)^2 - (s\varphi_p)^2], \varphi_p\rangle| \leqslant o(1)\{\langle \gamma, \varphi_p\rangle + \langle \tfrac{1}{2} f_{uu}(x, u_*)\varphi_p^2, \varphi_p\rangle s^2\}.$$

This leads to the validity of (29) for the same range of γ and s. Furthermore, we have

$$\begin{aligned}(1+\sigma/2)(\langle \gamma, \varphi_p\rangle + \langle \tfrac{1}{2} f_{uu}(x, u_*)\varphi_p^2, \varphi_p\rangle s(t)^2)& \\ \leqslant \dot{s}(t) \leqslant (1+3\sigma/2)(\langle \gamma, \varphi_p\rangle + \langle \tfrac{1}{2} f_{uu}(x, u_*)\varphi_p^2, \varphi_p\rangle s^2(t)),&\end{aligned} \tag{32}$$

which in particular implies that $\dot{s}(t) > 0$ for $0 < t < T_*$. Finally, by Claim II, there exists $0 < K < C_1$ independent of γ such that for $0 < \gamma < \gamma_0$ we have

$$\tfrac{1}{2}K\varphi_p(x) \geqslant |R_0(x) + R(x,t)| \tag{33}$$

for all t as long as (31) holds.

DEFINITION 8 Given $0 < \gamma, \sigma \ll 1$, the supersolution $v_{\text{in}}^{+}(x,t)$ near a stationary point u_* is defined by (23), together with (25)–(28):

$$s(t) : 0 < t < T_{\text{in}}, \quad \text{where } s(0) = -K \text{ and } s(T_{\text{in}}) = K. \tag{34}$$

The subsolution $v_{\text{in}}^{-}(x,t)$ is defined in the same way except with the factor $1+\sigma$ in (26) replaced by $1-\sigma$. (By abuse of notation, we use the same T_{in} for both the sub- and supersolutions, even though they can be two different values.)

Note that in both cases,

$$v_{\text{in}}^{\pm}(x,0) < u_*(x) - \tfrac{1}{2}KC_p \quad \text{and} \quad v_{\text{in}}^{\pm}(x,T_{\text{in}}) > u_*(x) + \tfrac{1}{2}KC_p. \tag{35}$$

2.2 *Construction of sub- and supersolution away from a stationary point*

Now, by assumption, we have only finitely many (up to vertical translations) periodic solutions of $(14)_{F_*}$: $u_*^{(1)}, \ldots, u_*^{(N)}$. By Proposition 5(2), they can be well-ordered: $u_*^{(1)} < \cdots < u_*^{(N)}$.

We now describe the construction of subsolutions between any two stationary solutions $u_*^{(i)}$ and $u_*^{(i+1)}$. (Supersolution can be trivially constructed upon the introduction of discontinuous jumps.) Let $v_{\text{in},i}^{-}$ be the subsolutions near the $u_*^{(i)}$'s constructed in the previous section. Let also $[0, T_{\text{in},i}]$ be the time interval defined in (34) for $u_*^{(i)}$. Denote by $S_{0,t}^{*}(v)$ the solution at time t of $(1)_{F_*}$ with initial datum v at time 0. It has the following property which will be proved in Section 2.3:

CLAIM III There exists a time $T > 0$ *bounded independently of* γ *and* σ such that

$$u_*^{(i+1)} - \tfrac{1}{2}KC_p < S_{0,T}^{*}(v_{\text{in},i}^{-}(T_{\text{in},i})) < u_*^{(i+1)}$$

where K, C_p are the constants for the $u_*^{(i)}$ as in (35).

We now give the following definition:

DEFINITION 9 The subsolution $v_{\text{out},i,i+1}^{-}$ between $u_*^{(i)}$ and $u_*^{(i+1)}$ is given by

$$v_{\text{out},i,i+1}^{-}(x,t) := S_{0,t}^{*}(v_{\text{in},i}^{-}(\cdot, T_{\text{in},i})) \quad \text{for } 0 < t \leqslant T_{\text{out},i,i+1} \tag{36}$$

where

$$T_{\text{out},i,i+1} := \inf\{t : v_{\text{out},i,i+1}^{-}(\cdot,t) - v_{\text{in},i+1}^{-}(\cdot,0) \geqslant 0\}. \tag{37}$$

Note that the $v_{\text{out},i,i+1}^{\pm}$ and $T_{\text{out},i,i+1}$ *do not depend* on γ.

With the above construction, we can define the following sub- and supersolutions from $u_*^{(1)}(x)$ to $u_*^{(1)}(x)+1$, i.e. they evolve vertically through one spatial period. They are used to provide upper and lower bounds for the propagating speed of the front.

DEFINITION 10 The subsolution $V^-(t)$ is given by

$$V^-(t) := \begin{cases} v^-_{\mathrm{in},i}(P_{i-1}+t) & \text{for } 0 < t \leqslant T_{\mathrm{in},i}, \\ v^-_{\mathrm{out},i,i+1}(P_{i-1}+T_{\mathrm{in},i}+t) & \text{for } 0 < t \leqslant T_{\mathrm{out},i,i+1}, \end{cases} \qquad i = 1, \dots N,$$

where $P_0 = 0$ and $P_i = \sum_{j=1}^{i}(T_{\mathrm{in},j} + T_{\mathrm{out},j,j+1})$.

The supersolution $V^+(t)$ is given by

$$V^+(t) := v^+_{\mathrm{in},i}(Q_{i-1}+t) \quad \text{for } 0 < t \leqslant T_{\mathrm{in},i},\ i = 1, \dots N,$$

where $Q_0 = 0$ and $Q_i = \sum_{j=1}^{i} T_{\mathrm{in},j}$.

REMARK. Note that we do not need the construction of the supersolution in between two stationary solutions $v^+_{\mathrm{out},i,i+1}$ as we introduce the upward discontinuous jumps at the Q_i's. The above sub- and supersolutions are piecewise continuous in time, the subsolution is upper semicontinuous and the supersolution lower semicontinuous. In such a case a comparison theorem between solutions and, respectively, sub- and supersolutions holds. For such a comparison theorem see e.g. [25].

The following statement gives the sharp asymptotic value for the propagation speed.

THEOREM 11 Let T_{tot} be the time it takes for the solution to travel one period in space. Then for each $\sigma > 0$, there exists a $\gamma_0(\sigma)$ such that for $0 < \gamma < \gamma_0(\sigma)$, we have

$$(1-\sigma)A\gamma^{-1/2} \leqslant T_{\mathrm{tot}} \leqslant (1+\sigma)A\gamma^{-1/2} \tag{38}$$

where

$$A = \pi \sum_{i=1}^{N} \left\{ \left(\int_Q \varphi_p^{(i)}(x)\,\mathrm{d}x \right) \langle \tfrac{1}{2} f_{uu}(x, u_*^{(i)})(\varphi_p^{(i)})^2, \varphi_p^{(i)} \rangle \right\}^{-1/2}. \tag{39}$$

Proof. This is a simple consequence of (32) and the uniform bound for the $T_{\mathrm{out},i,i+1}$'s which is independent of γ. Consider the supersolution, $V^+(t)$. Let

$$A_1 := \int_Q \varphi_p^{(i)}(x)\,\mathrm{d}x \quad \text{and} \quad A_2 := \langle \tfrac{1}{2} f_{uu}(x, u^{(i)})(\varphi_p^{(i)})^2, \varphi_p^{(i)} \rangle.$$

Then upon integrating (32), we have

$$\frac{1}{(1+3\sigma/2)\sqrt{A_1A_2\gamma}} \left\{ \tan^{-1}\left(\sqrt{\frac{A_2}{A_1\gamma}}\, s(t) \right) - \tan^{-1}\left(\sqrt{\frac{A_2}{A_1\gamma}}\, s(0) \right) \right\}$$
$$\leqslant t \leqslant \frac{1}{(1+\sigma/2)\sqrt{A_1A_2\gamma}} \left\{ \tan^{-1}\left(\sqrt{\frac{A_2}{A_1\gamma}}\, s(t) \right) - \tan^{-1}\left(\sqrt{\frac{A_2}{A_1\gamma}}\, s(0) \right) \right\}.$$

Choosing $s(0) = -K$, $s(T_{\mathrm{in},i}) = K$ as in Definition 8, for $0 < \gamma \ll 1$ we have

$$\frac{\pi}{(1+3\sigma/2)\sqrt{A_1A_2\gamma}} \leqslant T_{\mathrm{in},i} \leqslant \frac{\pi}{(1+\sigma/2)\sqrt{A_1A_2\gamma}}.$$

A similar estimate applies to $T_{\mathrm{in},i}$ for the case of subsolution.

Estimate (38) then follows immediately upon summing over the i's:

$$\sum_{i=1}^{N} T_{\mathrm{in},i} \leqslant T_{\mathrm{tot}} \leqslant \sum_{i=1}^{N} T_{\mathrm{in},i} + T_{\mathrm{out},i,i+1}. \qquad \square$$

REMARK. Assumption **A3** is needed for the precise time asymptotics. If we are only interested in the existence of a piecewise subsolution without further information, then we can simply use

$$v^{-}(x,t) := u_*(x) + \frac{\gamma t}{2\|\varphi_p\|_\infty}\varphi_p(x)$$

for $|t| < T_1(\gamma)$, and continue this solution away from the stationary points by the evolution at F_* as in (36). $\square$

2.3 *Proof of Claims I, II, and III*

Proof of Claim I. We argue by contradiction. Consider for $0 < s \ll 1$ the functions

$$\begin{aligned} w &:= u_* + s\varphi_p - s^2 R, \\ R &= (L_f(u_*))^{-1}(\tfrac{1}{2} f_{uu}(x,u_*)\varphi_p^2 - \langle \tfrac{1}{2} f_{uu}(x,u_*)\varphi_p^2, \varphi_p\rangle \varphi_p). \end{aligned} \tag{40}$$

Clearly u_* is a supersolution of $(14)_{F_*+\gamma}$ for any $\gamma > 0$, and $u_* < w$ for s small enough. If we can show that w is a subsolution, then there exists a stationary solution for $F_* + \gamma$, which contradicts the assumption that F_* is critical.

By expanding the nonlinearity up to second order we obtain

$$\Delta w + f(x,w) + F_* + \gamma = s^2\langle \tfrac{1}{2} f_{uu}(x,u_*)\varphi_p^2, \varphi_p\rangle + o(s^2) + \gamma,$$

which can be made negative by first choosing s so small that the $o(s^2)$-terms are dominated by the second order terms, and then choosing γ to be sufficiently small. $\square$

Proof of Claim II. 1. The right hand side of (25) is orthogonal to φ_p. Hence there exists $c > 0$ such that $\|R_0\|_2 \leqslant c\gamma$. The regularity assumption **A1** for the nonlinearity implies further that $\|R_0\|_{W^{2,2}} \leqslant c''\gamma$. The claim follows directly (for $n < 4$) or from $\|R_0\|_p \leqslant c'''\gamma$ for some $p > 0$. We can then conclude by a standard bootstrapping procedure.

2. The local existence for $R(\cdot,t) \in L^2(0,1)$ follows from a Banach fixed point type argument. To prove (31), let Ψ be as in (28). Then by Step 1, we can find a T and $(s,R) \in \mathcal{B}_{C_1,0,T}$ for any C_1, provided the initial datum satisfies $\|R(0)\| + |s| \leqslant C_1/2$. Given a small constant ρ, we can choose C_1 sufficiently small in order to ensure that there exists a large constant $C_3(\rho)$ such that

$$|\Psi(x,t)| \leqslant \rho R(x,t) + C(\gamma^2 + s(t)^2).$$

Moreover, there exist $A > 0$ and $b > 0$ such that if $\langle R, \varphi_p\rangle = 0$, then

$$\|e^{tL_f(u_*)}R\|_\infty \leqslant Ae^{-bt}\|R\|_\infty. \tag{41}$$

The above is proved by first establishing that $\|e^{L_f(u_*)}R\|_\infty \leqslant M\|R\|_{L^2}$ for some $M > 0$. As standard regularity theory for the periodic Laplacian implies that u_* is at least C^2, this follows

from short time regularity estimates for parabolic equations (see for example [12, Chapter 1.6, Thm. 11], adapted to the periodic case, for Hölder continuous coefficients, or for an alternative approach [15, Chapter 3.5, Exercise 4] which requires less regularity assumption for the potential). Then (41) follows from the L^2-exponential decay in time due to the spectral gap for $L_f(u_*)$.

Hence we can estimate

$$\begin{aligned}\|R(t)\|_\infty &\leqslant \int_0^t Ae^{-b(t-t')}(\rho\|R(t')\|_\infty + C(\gamma^2 + s(t')^2)\,\mathrm{d}t' \\ &\leqslant \frac{A}{b}(\rho \sup_{t'\in[0,t]} \|R(\cdot,t')\|_\infty + \sup_{t'\in[0,t]} |s(t')| + \gamma^2).\end{aligned}$$

The claim follows if ρ is chosen so small that $Ab^{-1}\rho \leqslant 1/2$. □

Proof of Claim III. Let $v(x) := v^-_{\mathrm{in},i}(x, T_{\mathrm{in}})$. Consider the sequence of functions

$$v_n := (S^*_{0,n})(v).$$

As $u_*^{(i)} < v_n \leqslant u_*^{(i+1)}$, the sequence is bounded in L^∞; then parabolic regularity (recall the regularity assumption **A1**) implies that it is bounded in $C^{0,\alpha}$ for some $\alpha > 0$. Hence it has a convergent subsequence in C^0. This sequence converges to either $u_*^{(i)}$ or $u_*^{(i+1)}$.

It remains to show that v_n cannot converge to $u_*^{(i)}$. Let w be as in (40). By expanding the nonlinearity as in the proof of Claim I, we conclude that w is a supersolution of $(14)_{F_*}$ for s sufficiently small. Moreover we can choose w in such a way that $u_*^{(i)} < w < v$. Therefore, convergence of v_n to $u_*^{(i)}$ contradicts the comparison principle.

By the same reasoning, $(S^*_{0,t})(w)$ will converge to $u_*^{(i+1)}$. This behaviour does not depend on γ and σ and hence the time it takes for w to evolve from $u_*^{(i)} + \frac{1}{2}KC_p$ to $u_*^{(i+1)} - \frac{1}{2}KC_p$ is independent of them. The claim follows as $S^*_{0,t}(w) \leqslant S^*_{0,t}(v)$. □

2.4 *Nondegeneracy condition* **A3**

This section demonstrates that there are "abundant" examples of nonlinear functions $f(x,u)$ satisfying the nondegeneracy condition **A3**. See also Section 4.1 for an explicit example.

PROPOSITION 12 Given any $f(x,u)$ satisfying **A1** and **A2** and any $\epsilon > 0$, there exists a function $g(x,u)$ satisfying **A1**–**A3** with $\|g - f\|_\infty < \epsilon$. Moreover, g can be chosen such that:

1. The critical forces for f and g are the same.
2. The set $\{u : \Delta u + g(x,u) + F_* = 0\}$ can be taken to be any finite subset of $\{u : \Delta u + f(x,u) + F_* = 0\}$.
3. Instead of property 1 above, we can require that

$$\int_0^1 \int_{[0,1]^n} g(x,u)\,\mathrm{d}x\,\mathrm{d}u = \int_0^1 \int_{[0,1]^n} f(x,u)\,\mathrm{d}x\,\mathrm{d}u$$

(so that Lemma 4 can be applied to g). In this case, the critical forces for g and f can differ by at most order ϵ.

Proof. Given any finite subset of $\{u : \Delta u + f(x,u) + F_* = 0\} = \{u^{(1)} < \cdots < u^{(N)}\}$ (and setting $u^{(j+N)}(x) = u^{(j)}(x) + 1$ for $j \in \mathbb{Z}$) we introduce

$$g(x,u) = f(x,u) + \epsilon \prod_{j=-\infty}^{\infty} Q\left(\frac{u - u^{(j)}(x)}{\frac{1}{8}\min_j \|u^{(j+1)} - u^{(j)}\|_\infty}\right)$$

where Q is a smooth even function having the form

$$Q(r) = \begin{cases} \frac{1}{2}r^2 & \text{for } |r| \leqslant 1, \\ 1 & \text{for } |r| \geqslant 2, \end{cases} \quad \text{and} \quad Q(|r|) \text{ is an increasing function,} \tag{42}$$

so that $Q(0) = Q'(0) = 0$ and $Q''(0) > 0$.

Let F_* be the critical force for f. As $g \geqslant f$, the critical force for g is no greater than F_*. On the other hand, it is immediate that

$$\{u^{(j)}\}_{j=-\infty}^{\infty} \subset \{u : \Delta u + g(x,u) + F_* = 0\}.$$

Thus g and f have the same critical force value.

Analogous to (7), consider the linearised operator

$$L_g(u^{(i)})\varphi = \Delta\varphi + \partial_u g(x, u^{(i)})\varphi.$$

Let $\varphi_p^{(i)}(x)$ (> 0) be the principal eigenfunction of $L_f(u^{(i)})$. Note that

$$L_g(u^{(i)})\varphi_p^{(i)} = L_f(u^{(i)})\varphi_p^{(i)} + \epsilon\left\{\prod_{j=-\infty}^{\infty} Q\left(\frac{u - u^{(j)}(x)}{\frac{1}{8}\min_j \|u^{(j+1)} - u^{(j)}\|_\infty}\right)\right\}'\Bigg|_{u=u^{(i)}} \varphi_p^{(i)} = 0.$$

Hence it is also the principal eigenfunction of $L_g(u^{(i)})$ with eigenvalue 0. Moreover,

$$\int_Q \partial_u^2 g(x, u^{(i)}(x))\varphi_p(x)^3\,\mathrm{d}x = \int_Q \partial_u^2 f(x, u^{(i)}(x))\varphi_p(x)^3\,\mathrm{d}x + \epsilon\int_Q \varphi_p(x)^3\,\mathrm{d}x > 0.$$

Thus **A3** is satisfied for g at any $u^{(i)}$.

To continue, suppose $\Delta v + g(x,v) + F_* = 0$ and v is not equal to any of the $u^{(i)}$'s. Then we must have $\Delta v + f(x,v) + F_* \leqslant 0$ but *not identically zero*. Now, similar to the proof of Proposition 5(2), consider

$$u_t = \Delta u + f(x,u) + F_*, \qquad u(x,0) = v(x).$$

There exist $\gamma, t_* > 0$ such that $\Delta u(x,t_*) + f(x,u(x,t_*)) + F_* + \gamma < 0$, i.e. $u(x,t_*)$ is a subsolution of $(14)_{F_*+\gamma}$. Combining this with the fact that each $u^{(i)}$ is a supersolution of $(14)_{F_*+\gamma}$, we get a contradiction with the definition of F_*. Thus $\{u : \Delta u + g(x,u) + F_* = 0\} \subset \{u^{(j)}\}_{j=-\infty}^{\infty}$.

Finally, by assumption, for $f(x,u)$, there is a stable solution of $(14)_0$ in the sense of **A2**. By the implicit function theorem, for any small enough $\epsilon > 0$, the same condition also holds for $g(x,u)$.

Statement 3 follows simply by changing g to $g(x,u) - \int_0^1 \int_{[0,1]^n} g(x,u)\,\mathrm{d}x\,\mathrm{d}u$. □

PROPOSITION 13 Given any collection $\{u^{(1)} < \cdots < u^{(N)}\}$ of $\mathbb{Z}^n$-periodic functions (and setting $u^{(j+N)}(x) = u^{(j)}(x) + 1$ for $j \in \mathbb{Z}$), there exists an $f(x,u)$ satisfying $\int_0^1 \int_{[0,1]^n} f(x,u)\,\mathrm{d}x\,\mathrm{d}u = 0$ and conditions **A1**–**A3** such that at its critical forcing F_*, we have

$$\{u : \Delta u + f(x,u) + F_* = 0\} = \{u^{(1)} < \cdots < u^{(N)}\}.$$

Proof. The proof is similar to that of the previous proposition once we take f to be $\tilde{f}(x,u) - \int_0^1 \int_{[0,1]^2} \tilde{f}\,\mathrm{d}x\,\mathrm{d}u$ where

$$\tilde{f}(x,u) = -\prod_{j=-\infty}^{\infty} \Delta u^{(j)}(x) P\left(\frac{u - u^{(j)}(x)}{\frac{1}{8}\min_j \|u^{(j+1)} - u^{(j)}\|_\infty}\right) + A \prod_{j=-\infty}^{\infty} Q\left(\frac{u - u^{(j)}(x)}{\frac{1}{8}\min_j \|u^{(j+1)} - u^{(j)}\|_\infty}\right)$$

and

- $A \gg \{\max_i \|\Delta u^{(i)}\|_\infty\}$;
- P is a smooth positive even function such that $P(r) = 1$ for $|r| < 1$ and $P(r) = 0$ for $|r| > 2$;
- Q is as in (42).

With the above f, for large enough A, in fact, we have $F_* = \iint \tilde{f}\,\mathrm{d}x\,\mathrm{d}u$. First of all, it is simple to check that

$$\{u^{(j)}\}_{j=-\infty}^{\infty} \subset \{u : \Delta u + f(x,u) + F_* = 0\}.$$

In addition, at F_* and for each $u^{(i)}$, the linearised operator $L_f(u^{(i)})$ has the principal eigenvalue $\lambda_p = 0$ and eigenfunction $\varphi_p \equiv 1$. Furthermore, as $\partial_u^2 f(x, u^{(i)}) \equiv 1/2 > 0$, **A3** is satisfied. Also, the facts that $\{u : \Delta u + f(x,u) + F_* = 0\} \subset \{u^{(j)}\}_{j=-\infty}^{\infty}$ and F_* is indeed the critical force follow from the following one-parameter family of supersolutions of $(14)_{F_*}$ "foliating" the region between any $u^{(j)}$ and $u^{(j+1)}$:

$$v_\lambda(x) = \begin{cases} u^{(j)}(x) + d\lambda & \text{for } 0 < \lambda \leqslant 1, \\ (2-\lambda)(u^{(j)}(x) + d) + (\lambda - 1)(u^{(j+1)}(x) - d) & \text{for } 1 < \lambda \leqslant 2, \\ u^{(j+1)}(x) - d(3-\lambda) & \text{for } 2 < \lambda \leqslant 3, \end{cases}$$

where $d = \frac{1}{8}\min_j \|u^{(j+1)} - u^{(j)}\|_\infty$. (Recall that we have chosen A to be sufficiently large.)

Since $\iint f\,\mathrm{d}x\,\mathrm{d}u = 0$, we can apply Proposition 4 and the remark afterward to ensure that **A2** is also satisfied. □

3. Forced Allen–Cahn equation in the perturbative regime

The equation studied in this section is the spatially inhomogeneous Allen–Cahn equation (2) in the regime $0 < \delta \ll 1$. We assume that $g(\cdot)$ is a C^3 spatial 1-periodic function: $g(x+1) = g(x)$.

In order to study the front propagation in heterogeneous medium, we only consider solutions resembling a front structure. For $0 < \delta \ll 1$, the equation (2) has two end states close to 1 and -1. These are described by means of the functions $m^{\pm}(x)$ which satisfy

$$\delta m_{xx}^{-}(x) - W'(1 + \delta m^{-}(x)) + \delta(g(x) + F) = 0,$$
$$\delta m_{xx}^{+}(x) - W'(-1 + \delta m^{+}(x)) + \delta(g(x) + F) = 0.$$

We now take the following ansatz for the solution $v(x,t)$ of (2):

$$v(x,t) = m(z) + \delta E(z,t) + \delta\varphi(z,t)$$

where $z = x - \delta c(t)$, $E(z,t) = \alpha(z)m^-(x) + \beta(z)m^+(x)$ and $\alpha(z)$ and $\beta(z)$ are two smooth positive functions such that for some large but fixed positive constant K,

$$\alpha(z) = \begin{cases} 1 & \text{for } z < -K, \\ 0 & \text{for } z > -K+1, \end{cases} \qquad \beta(z) = \begin{cases} 0 & \text{for } z < K-1, \\ 1 & \text{for } z > K. \end{cases}$$

Note that the above ansatz for $v(x,t)$ connects $-1 + \delta m^+(x)$ as $x \to +\infty$ and $1 + \delta m^-(x)$ as $x \to -\infty$. Now $\varphi(z,t)$ satisfies an equation of the form

$$\begin{aligned} \varphi_t &= \varphi_{zz} + \dot{c}(m_z + \delta E_z + \delta\varphi_z) - E_t \\ &\quad + \frac{1}{\delta}(m_{zz} + \delta E_{zz} - W'(m + \delta E + \delta\varphi) + \delta(g(z + \delta c(t)) + F)) \end{aligned} \tag{43}$$

so that

$$\varphi_t = \varphi_{zz} - W''(m)\varphi + H(t,z,\varphi) \tag{44}$$

where

$$H(t,z,\varphi) = \dot{c}(m_z + \delta E_z + \delta\varphi_z) - E_t \tag{45}$$

$$+ \frac{1}{\delta}\{m_{zz} + \delta E_{zz} - W'(m + \delta E) + \delta(g(z + \delta c(t)) + F)\} \tag{46}$$

$$+ O_1(\delta\varphi) + O_2(\delta\varphi^2).$$

We will work with the following weighted function spaces:

$$\mathcal{B}_{0,b} = \{f : \|f\|_{0,b} = \sup\{|f(z)e^{b|z|}| : z \in \mathbb{R}\} < \infty\}, \tag{47}$$

$$\mathcal{B}_{1,b} = \{f : \|f\|_{1,b} = \|f\|_{0,b} + \|f'\|_{0,b}\} \tag{48}$$

where $0 < b < W''(\pm 1)$. The value of b will be chosen later according to the spectral estimates (83).

Expressions (45) and (46) are estimated as follows. For (45),

$$(45) = \dot{c}(m_z + \delta E_z + \delta\varphi_z) - E_t = \dot{c}\{m_z + \delta(\alpha'(z)m^-(x) + \beta'(z)m^+(x)) + \delta\varphi_z\}$$

so that for some constant A,

$$\|(45)\|_{0,b} < A|\dot{c}(t)|(1 + \delta\|\varphi(\cdot,t)\|_{1,b}). \tag{49}$$

For (46), it can be rewritten as

$$\begin{aligned} (46) = \frac{1}{\delta}\{&m_{zz} - W'(m + \delta E) + W'(\alpha(z) + \delta E - \beta(z)) \\ &+ \delta E_{zz} - W'(\alpha(z) + \delta E - \beta(z)) + \delta(g(x) + F)\}. \end{aligned}$$

After some routine computation, it is easily verified that for some constant A again,

$$\|(46)\|_{0,b} < A.$$

Hence

$$\|H(\cdot,t)\|_{0,b} \leqslant A\{1 + |\dot{c}(t)|(1 + \delta\|\varphi(\cdot,t)\|_{1,b}) + \delta\|\varphi\|_{1,b} + \delta\|\varphi\|_{1,b}^2\}. \tag{50}$$

We now choose $c(t)$ to make $\int \varphi(z,t)m_z(z)\,\mathrm{d}z = 0$ for all $t > 0$. Such a choice can always be made if $\int \varphi(z,0)m_z(z)\,\mathrm{d}z = 0$ and $c(t)$ satisfies

$$\dot{c}(t) = -\frac{1}{\delta}\frac{\langle m_{zz} + \delta E_{zz} - W'(m+\delta E) + \delta(g(z+\delta c(t)) + F), m_z\rangle}{\|m_z\|^2 + \delta\langle \alpha'(z)m^-(x) + \beta'(z)m^+(x) + \varphi_z, m_z\rangle}. \tag{51}$$

The above expression is well defined provided that the denominator of the right hand side does not vanish. In particular, there exist some positive constants B, C such that

$$\text{if} \quad C > \delta\|\varphi\|_{1,b}, \quad \text{then} \quad |\dot{c}(t)| \leqslant \frac{B}{C - \delta\|\varphi(\cdot,t)\|_{1,b}}. \tag{52}$$

Such a choice of $c(t)$ also implies that

$$\int H(z,t)m_z(z)\,\mathrm{d}z = 0 \quad \text{for all } t \geqslant 0. \tag{53}$$

Next we will make use of the decay properties of the linear operator

$$L_W\psi = \psi_{zz} - W''(m(z))\psi \tag{54}$$

to deduce some long time estimates for $\|\varphi(\cdot,t)\|_{1,b}$. First we provide the following preliminary result.

LEMMA 14 (Gronwall type estimate) Let $f(t)$ ($t \geqslant 0$) be a continuous positive function such that for some constants $P \geqslant 1$, Q, R, ω, and $\delta > 0$,

$$f(t) \leqslant Pe^{-\omega t}f(0) + \int_0^t e^{-\omega(t-s)}(Q + R\delta f(s))\,\mathrm{d}s.$$

If further $R\delta \leqslant \omega/2$, then $f(t) \leqslant Pf(0) + 2Q/\omega$ for all $t \geqslant 0$.

Proof. Let $h(t) = e^{\omega t}f(t)$. Then

$$h(t) \leqslant Ph(0) + \frac{Q}{\omega}(e^{\omega t} - 1) + R\delta\int_0^t h(s)\,\mathrm{d}s$$

Gronwall's inequality implies that

$$\begin{aligned} h(t) &\leqslant Ph(0) + \frac{Q}{\omega}(e^{\omega t} - 1) + R\delta\int_0^t e^{R\delta(t-s)}\left(Ph(0) + \frac{Q}{\omega}(e^{\omega s} - 1)\right)\mathrm{d}s \\ &\leqslant Ph(0) + \frac{Q}{\omega}(e^{\omega t} - 1) + Ph(0)(e^{R\delta t} - 1) + \frac{QR\delta}{\omega(\omega - R\delta)}(e^{\omega t} - e^{R\delta t}) + \frac{Q}{\omega}(1 - e^{R\delta t}). \end{aligned}$$

Hence switching back to $f(t)$ and upon simplification, we get

$$f(t) \leqslant Pf(0)e^{-(\omega - R\delta)t} + \frac{Q}{\omega - R\delta}(1 - e^{-(\omega - R\delta)t}),$$

leading to the desired statement. □

Now the solution $\varphi(\cdot, t)$ of (44) can be written as

$$\varphi(t) = S(t)\varphi(0) + \int_0^t S(t-s)H(s)\,\mathrm{d}s \tag{55}$$

where $S(t)$ is the solution operator for the linear equation $\varphi_t = L_W\varphi$. We have the following estimate for $\varphi(t)$:

PROPOSITION 15 There exist $D > 1$ and $\omega > 0$ such that if $\sup_{s\in[0,t]} \delta\|\varphi(s)\|_{1,b} < C/2$, then

$$\|\varphi(t)\|_{1,b} \leqslant De^{-\omega t}\|\varphi(0)\|_{1,b} + \int_0^t De^{-\omega(t-s)}A\left\{1 + \frac{2B}{C}(1+\delta\|\varphi(s)\|_{1,b}) + \delta\|\varphi(s)\|_{1,b} + \delta\|\varphi(s)\|_{1,b}^2\right\}\mathrm{d}s.$$

Proof. From (55), we can write

$$\|\varphi(t)\|_{1,b} \leqslant \|S(t)\varphi_0\|_{1,b} + \int_0^t \|S(t-s)H(s)\|_{1,b}\,\mathrm{d}s.$$

The estimate for $S(t)$ (from Theorem 25) gives $\|S(t)\varphi(0)\|_{1,b} \leqslant De^{-\omega t}\|\varphi_0\|_{1,b}$. To estimate the remaining term, consider, for $0 < t < 2$,

$$\int_0^t \|S(t-s)H(s)\|_{1,b}\,\mathrm{d}s \leqslant \int_0^t \frac{De^{-\omega(t-s)}}{\sqrt{t-s}}\|H(s)\|_{0,b}\,\mathrm{d}s \leqslant C\|\varphi(0)\|_{1,b}$$

where we have made use of (80). For $t > 2$, consider

$$\begin{aligned}\int_0^t \|S(t-s)H(s)\|_{1,b}\,\mathrm{d}s &\leqslant \int_0^{t-1} \|S(t-s)H(s)\|_{1,b}\,\mathrm{d}s + \int_{t-1}^t \|S(t-s)H(s)\|_{1,b}\,\mathrm{d}s \\ &\leqslant \int_0^{t-1} \frac{De^{-\omega(t-s)}}{\sqrt{t-s}}\|H(s)\|_{0,b}\,\mathrm{d}s + \int_{t-1}^t De^{-\omega(t-s)}\|H(s)\|_{1,b}\,\mathrm{d}s.\end{aligned}$$

By (50) and (52), the first integral can be bounded by the term stated in the proposition. For the second integral, an extension of (50) and parabolic regularity as in (67) give

$$\begin{aligned}&\int_{t-1}^t De^{-\omega(t-s)}\|H(s)\|_{1,b}\,\mathrm{d}s \\ &\quad\leqslant \int_{t-1}^t De^{-\omega(t-s)}A\{1 + |\dot{c}(s)|(1+\delta\|\varphi(s)\|_{2,b}) + \delta\|\varphi(s)\|_{2,b} + \delta\|\varphi(s)\|_{2,b}^2\}\,\mathrm{d}s \\ &\quad\leqslant \int_{t-1}^t D'e^{-\omega(t-s)}A\left\{1 + \frac{2B}{C}(1+\delta\|\varphi(s-1)\|_{1,b}) + \delta\|\varphi(s-1)\|_{1,b} + \delta\|\varphi(s-1)\|_{1,b}^2\right\}\mathrm{d}s \\ &\quad\leqslant \int_{t-2}^{t-1} D'e^{-\omega(t-s)}A\left\{1 + \frac{2B}{C}(1+\delta\|\varphi(s)\|_{1,b}) + \delta\|\varphi(s)\|_{1,b} + \delta\|\varphi(s)\|_{1,b}^2\right\}\mathrm{d}s,\end{aligned}$$

which can then be absorbed into the first integral. The stated conclusion is thus proved. □

From the above result, we see that *if* for some $T > 0$, φ satisfies $\sup_{t\in[0,T]} \|\varphi(s)\|_{1,b} \leqslant M$ (with M to be chosen later) then for $\delta \leqslant C/2M$, there are positive constants E (independent of M) and F_M such that

$$\|\varphi(t)\|_{1,b} \leqslant De^{-\omega t}\|\varphi(0)\|_{1,b} + \int_0^t e^{-\omega(t-s)}(E + F_M\delta\|\varphi(s)\|_{1,b})\,\mathrm{d}s \quad \text{for } 0 \leqslant t \leqslant T. \tag{56}$$

The next two results provide some long time estimates for the solution $\varphi(t)$. In all of the following, the constants $A, B, C, D, E, F_M, \omega$ are as in (50), (52) and (56).

PROPOSITION 16 Let $M \geqslant 8E/\omega$ and $\delta < \min\{\omega/2F_M, C/2M\}$. If the initial condition satisfies $\|\varphi(0)\|_{1,b} \leqslant M/4D$, then $\|\varphi(t)\|_{1,b} \leqslant M/2$ for all $t > 0$.

Proof. Suppose $T_M = \inf\{t : \|\varphi(t)\|_{1,b} = M\} < \infty$. Then Lemma 14 says

$$\sup_{t\in[0,T_M]} \|\varphi(t)\|_{1,b} \leqslant D\|\varphi(0)\|_{1,b} + \frac{2E}{\omega} \leqslant \frac{M}{4} + \frac{M}{4} \leqslant \frac{M}{2}.$$

Then standard parabolic regularity implies that $\|\varphi(t)\|_{1,b} \leqslant M$ for $t \in [T_M, T_M+\epsilon]$ where ϵ is some small enough number. This contradicts the definition of T_M. Hence $T_M = \infty$ and so $\|\varphi(t)\|_{1,b} \leqslant M/2$ for all $t \geqslant 0$. □

PROPOSITION 17 Let $M \geqslant 16DE/\omega$ and $\delta \leqslant \min\{\omega/8DF_M, C/2M\}$. If $\|\varphi(0)\|_{1,b} \leqslant M/4D$, then $\|\varphi(t)\|_{1,b} \leqslant M/4D$ for all $t \geqslant \omega^{-1}\ln(2D)$.

Proof. Note that under the current hypothesis, the assumptions of the previous Proposition 16 are also satisfied. Hence we can legitimately write

$$\begin{aligned}\|\varphi(t)\|_{1.b} &\leqslant De^{-\omega t}\|\varphi(0)\|_{1,b} + \int_0^t e^{-\omega(t-s)}\left(E + \frac{F_M\delta M}{2}\right)\mathrm{d}s \leqslant \frac{M}{4}e^{-\omega t} + \frac{1}{\omega}\left(E + \frac{1}{2}F_M\delta M\right)\\ &\leqslant \frac{M}{8D} + \frac{E}{\omega} + \frac{F_M\delta}{2\omega}M \leqslant \frac{M}{8D} + \frac{M}{16D} + \frac{M}{16D} = \frac{M}{4D}.\end{aligned}$$

□

The next quantitative result implies that as long as the external forcing F is near the threshold value, it takes a *very long* time for the front to travel across the heterogeneous spatial environment. Thus, the condition of large time in Proposition 17 is automatically satisfied. Recall the definitions given in (10).

THEOREM 18 (Estimation of velocity) Let $M \geqslant 16DE/\omega$. Then for any $\sigma > 0$, there exist $0 < \delta_* \leqslant \min\{\omega/8DF_M, C/2M\}$ and $C_1, C_2 > 0$ such that for any $0 < \delta < \delta_*$ and $\|\varphi_0\|_{1,b} \leqslant M/4D$ and the forcing F within the range

$$0 < C_1\delta \leqslant F - F_* \leqslant C_2,$$

the velocity V_F of the front satisfies the estimate (13).

Proof. From (51), as $\|\varphi(t)\|_{1,b} \leqslant M/2$ for all t, we see that

$$\begin{aligned}\dot{c}(t) &= \frac{1}{\delta}\frac{\langle m_{zz} + \delta E_{zz} - W'(m) - \delta W''(m)E + O(\delta^2) + \delta(g(z+\delta c(t)) + F), -m_z\rangle}{\|m_z\|^2 + O(\delta)}\\ &= \frac{\langle g(z+\delta c(t)) + F, -m_z\rangle}{\|m_z\|^2} + O(\delta)\end{aligned}$$

where we have used the facts that $m_{zz} - W'(m) = 0$ and $m_{zzz} - W''(m)m_z = 0$.

Using the notations of (10), we can write

$$\begin{aligned}\dot{c}(t) &= \frac{(2F + h_*) + (h(\delta c(t))) - h_*)}{\|m_z\|^2} + O(\delta)\\ &= \alpha(\gamma + O(\delta) + \beta(\delta c(t))^2), \qquad \text{where } \gamma = F + h_*/2.\end{aligned}$$

Now fix a positive constant $0 < P \ll 1$ which is *independent of* γ *and* δ. Then the time t_P it takes for the front location $\delta c(t)$ to travel from 0 to P can be computed as

$$t_P = \frac{1}{\alpha}\int_0^{P/\delta} \frac{\mathrm{d}c}{\gamma + O(\delta) + \beta(\delta c)^2}.$$

Similar to the proof of Theorem 11, by choosing C_1 large enough and δ_* and C_2 small enough, we can estimate t_P as

$$\frac{(1-\sigma)\pi}{2\alpha\delta\sqrt{\beta\gamma}} \leqslant t_P \leqslant \frac{(1+\sigma)\pi}{2\alpha\delta\sqrt{\beta\gamma}}. \tag{57}$$

Again, the time it takes for the front to travel from P to $1 - P$ is some number *independent of* γ *and* δ. We conclude that the total time T_γ for the front to travel one period of the spatial heterogeneity follows the same asymptotics as t_P, leading to the assertion of the theorem. □

The previous results facilitate the use of the Schauder fixed point theorem to prove the existence of a pulsating wave.

THEOREM 19 (Existence of fixed point) For any M, δ and F as in Theorem 18, there exists a φ_0 with $\|\varphi_0\|_{1,b} \leqslant M/4D$ and $0 < T(\varphi_0) < \infty$ such that

$$\varphi(T(\varphi_0)) = \varphi_0 \tag{58}$$

where $\varphi(t)$ is the solution of (44) with initial data φ_0.

Proof. Let $(\varphi(t), c(t))$ be the solution of (44) and (51) with initial condition (φ_0, c_0). In addition, let

$$T(\varphi_0) = \inf\{t : c(t) = c_0 + 1\}.$$

Then (57) implies that $0 < T(\varphi_0) < \infty$ and it is *uniformly bounded* independently of φ_0. Proposition 17 also implies that the set $\{\psi : \|\psi\|_{1,b} \leqslant M/4D\}$ is mapped into itself by the nonlinear map

$$\mathcal{S} : \varphi_0 \mapsto \varphi(T(\varphi_0)).$$

Parabolic regularity (67) implies that it is a continuous map.

Now consider the Banach space $\mathcal{BC}$ of bounded continuous functions of $z \in \mathbb{R}$ with the uniform norm. Since the set $\{\psi : \|\psi\|_{1,b} \leqslant M/4D\}$ is a compact convex subset of $\mathcal{BC}$, [13, Corollary 11.2] gives the desired existence of a fixed point of $\mathcal{S}$. □

4. Analytical examples

This section provides two analytical examples of the pinning and de-pinning behaviour, one in the semilinear graph case and the other in the reaction-diffusion case.

4.1 *Semilinear graph case*

Consider for $\delta > 0$ the equation

$$0 = u_{xx} + \delta \sin(2\pi u(x)) \cos(2\pi x) + F, \quad u(0) = u(1). \tag{59}$$

We have the following lemma:

LEMMA 20 1. Let $0 < \delta < 1/4\pi$ so that $2\pi\delta/(1 - 2\pi\delta) < 1$. Then for $F > \delta^2 2\pi/(1 - 2\pi\delta)$, (59) does not have a solution.
2. Let $0 < \delta < 1/2$. Then for $0 < F < \delta^2(1/4 - 2\delta^3)$, (59) has a stable stationary solution.

REMARK. Statement 2 shows that there is pinning while statement 1 shows that the de-pinning is nontrivial, i.e. it occurs before $\delta \sin(2\pi u(x)) \cos(2\pi x) + F$ has a definite sign. Note that the ranges of the parameters are not expected to be sharp for δ small. □

Proof. For statement 1, the claim is obvious if $F > \delta$. Hence assume $F = f\delta$, $0 < f < 1$. We argue by contradiction and assume that there exists a stationary solution u. By standard regularity this solution is smooth, and because it is periodic, there exists an $x_0 \in [0, 1]$ such that $u'(x_0) = 0$. Hence $|u'(x)| \leqslant \delta(1 + f)$ and $|u(x) - u(0)| \leqslant \delta(1 + f)$.

Integrating (59) over $[0, 1]$ we obtain

$$\begin{aligned} 0 &= \delta f + \delta \int_0^1 \sin(2\pi u(0)) \cos(2\pi x)\, \mathrm{d}x \\ &\quad + \delta \int_0^1 \left(\cos(2\pi x)(u(0) - u(x)) 2\pi \int_0^1 \cos(2\pi\{(1 - s)u(0) + su(x)\})\, \mathrm{d}s \right) \mathrm{d}x \\ &\geqslant \delta f - 2\pi\delta \sup_{0<x<1} |u(0) - u(x)| \geqslant \delta(f - 2\pi\delta(1 + f)) > 0, \end{aligned}$$

which proves the first claim.

For statement 2, we will construct sub- and supersolutions. The comparison principle implies that there exists a stable solution between the sub- and the supersolution. We write $F = \delta^2 f$, $0 < f < 1$. Let $\rho > \delta^3$ and choose $c^\pm \in (1/8, 3/8)$ such that

$$\cos(2\pi c^\pm) \sin(2\pi c^\pm) = -4\pi(f \mp \rho),$$

which is possible for $f \leqslant (8\pi)^{-1} - \delta^3$. Let

$$u^\pm(x) := c^\pm + \delta(2\pi)^{-2} \sin(2\pi c^\pm) \cos(2\pi x) - \frac{1}{2}(2\pi)^{-2}\delta^2(f \mp \rho) \cos^2(2\pi x).$$

Then

$$\begin{aligned} u^\pm_{xx} &= -\delta \sin(2\pi c^\pm) \cos(2\pi x) + \delta^2 2(f \mp \rho) \cos^2(2\pi x) - \delta^2(f \mp \rho), \\ \delta \sin(2\pi u^\pm) \cos(2\pi x) &= \delta \sin(2\pi c^\pm) \cos(2\pi x) + \delta^2(2\pi)^{-1} \cos(2\pi c^\pm) \sin(2\pi c^\pm) \cos^2(2\pi x) \\ &\quad - \frac{\delta^3}{2}(2\pi)^{-1} \cos(2\pi c^\pm)(f \mp \rho) \cos^3(2\pi x) + O(\delta^3). \end{aligned}$$

Hence $u^\pm_{xx} + \delta \sin(2\pi u^\pm) \cos(2\pi x) + \delta^2 f = \pm\delta^2\rho + O(\delta^3)$ so that

$$\begin{aligned} u^+_{xx} + \delta \sin(2\pi u^+) \cos(2\pi x) + \delta^2 f &\geqslant \delta^2\rho - O(\delta^3) > 0, \\ u^-_{xx} + \delta \sin(2\pi u^-) \cos(2\pi x) + \delta^2 f &\leqslant -\delta^2\rho + O(\delta^3) < 0, \end{aligned}$$

i.e. the desired sub- and supersolutions exist. □

4.2 *Reaction-diffusion (Allen–Cahn) case*

This example is not constrained by the heterogeneity being weak and yet the exact values of the threshold force and the relevant constant in the time asymptotic (9) can be found. The computations are formal but yet illustrative. Consider the equation

$$v_t = v_{xx} - \frac{W'(v)}{2} + A\sin\left(\frac{x}{L}\right) + F \tag{60}$$

where $0 < A, F \ll 1$ and $W(\cdot)$ is chosen to be the the bilinear double-well potential

$$W(p) = (|p| - 1)^2 \quad \text{so that} \quad W'(p) = \begin{cases} 2(p-1) & \text{for } p > 0, \\ 2(p+1) & \text{for } p < 0. \end{cases}$$

We first look for *stationary solutions* of (60) with one transitional interface—there is one single value of z at which v changes its sign. Under this assumption, the solution v takes the form

$$\begin{aligned} v_{xx} - v + 1 + A\sin\left(\frac{x}{L}\right) + F &= 0 \quad \text{for } x \leqslant z, \\ v_{xx} - v - 1 + A\sin\left(\frac{x}{L}\right) + F &= 0 \quad \text{for } x \geqslant z, \\ v(z^-),\ v(z^+) &= 0, \\ v_x(z^-) &= v_x(z^+). \end{aligned}$$

In the above, we implicitly assume that $v > (<)\ 0$ for $x < (>)\ z$. Now v is explicitly given by

$$v(x) = \begin{cases} -e^{x-z} + \dfrac{L^2}{1+L^2} A\sin\left(\dfrac{x}{L}\right) + F + 1 & \text{for } x < z, \\ e^{-(x-z)} + \dfrac{L^2}{1+L^2} A\sin\left(\dfrac{x}{L}\right) + F - 1 & \text{for } x > z. \end{cases}$$

We also have the following expressions for the value of z and the slope of v at z:

$$\sin\left(\frac{z}{L}\right) = -\frac{F}{A}\frac{1+L^2}{L^2}, \tag{61}$$

$$v_x(z) = -1 + \frac{AL}{1+L^2}\cos\left(\frac{z}{L}\right). \tag{62}$$

From (61), we see that the threshold force is given by

$$F_* = \frac{AL^2}{1+L^2} < 1,$$

so that at the threshold, the spatial inhomogeneity still changes sign so that de-pinning is also nontrivial.

The above value of F_* coincides with the forcing which makes the stationary solution just lose its stability. This is illustrated as follows. The eigenvalue problem of the linearised operator for the stationary solution of (60) is written formally as

$$\varphi_{xx} - \frac{W''(v)}{2}\varphi = \lambda\varphi.$$

If we assume without loss of generality that $z = 0$, the above problem takes the form

$$\begin{aligned} \varphi_{xx} - \varphi &= \lambda\varphi \quad \text{for } x \neq 0, \\ \varphi_x(0^+) - \varphi_x(0^-) &= -\frac{2\varphi(0)}{|v_x(0)|}, \\ \varphi(0^+) &= \varphi(0^-). \end{aligned}$$

The principal eigenfunction $\varphi_p(x)$ (which is positive) and eigenvalue λ_p are given by

$$\varphi_p(x) = \begin{cases} e^{\sqrt{1+\lambda_p}x}, & x \leqslant 0, \\ e^{-\sqrt{1+\lambda_p}x}, & x \geqslant 0, \end{cases} \qquad \lambda_p = \frac{1}{|v_x(0)|^2} - 1.$$

Hence v is stable, i.e. $\lambda_p < 0$ if and only if $|v_x(0)| > 1$.

Next we will compute the constant A of (9).

A. Short time regularity estimates

This section provides some finite time regularity estimates for semilinear parabolic differential equations. The results are crude, but useful as an initial step to obtain more refined estimates. Here we only give the properties relevant to this paper, in particular in the weighted spaces $\mathcal{B}_{1,b}$ from (48).

PROPOSITION 21 Consider the equation

$$u_t = u_{xx} + f(u, u_x) + g(x, t), \qquad u(x, 0) = u_0(x). \tag{63}$$

Assume f is a C^1 function satisfying $|f(p, q)| \leqslant C(|p| + |q|)$, $\sup_{p,q}\{|f_p|, |f_q|\} < \infty$ and $\|g(\cdot, t)\|_{0,b} < \infty$ for all t. Then for all $T > 0$, there exists a $C(T)$ such that

$$\|u(\cdot, t)\|_{0,b} \leqslant C(T)\{\|u_0\|_{0,b} + \sup_{t\in[0,T]} \|g(\cdot, t)\|_{0,b}\}, \tag{64}$$

$$\|u(\cdot, t)\|_{1,b} \leqslant C(T)\{\|u_0\|_{0,b}/\sqrt{t} + \sup_{t\in[0,T]} \|g(\cdot, t)\|_{0,b}\}, \tag{65}$$

$$\|u(\cdot, t)\|_{1,b} \leqslant C(T)\{\|u_0\|_{1,b} + \sup_{t\in[0,T]} \|g(\cdot, t)\|_{0,b}\}. \tag{66}$$

If further, $\|g(\cdot, t)\|_{1,b} < \infty$, then

$$\|u_x(\cdot, t)\|_{1,b} \leqslant C(T)\{\|u_0\|_{1,b}/\sqrt{t} + \sup_{t\in[0,T]} \|g(\cdot, t)\|_{1,b}\}. \tag{67}$$

REMARK. The proof is omitted as the technique of proof can be found in many standard texts such as [12, 19]. But we remark that since f is globally Lipschitz, the results can be obtained by analysing the linear equation

$$u_t = u_{xx} + A(x, t)u + B(x, t)u_x + C(x, t)$$

with L^∞ coefficients. Standard iteration steps then produce the estimates (64)–(67) for the fully nonlinear equation (63).

The purpose of considering (63) is to analyse (44) which originates from (2). As the nonlinearity $W'(\cdot)$ satisfies $uW'(u) \geqslant cu^2$ for $|u|$ large enough, it follows that with the L^∞ source term $\delta(g(x) + F)$, the space $\{\|u\|_\infty < C\}$ is invariant under the flow of (2). In this setting, we can assume without loss of generality that the nonlinear term is globally Lipschitz. □

B. Spectral analysis of the Allen–Cahn operator

This section analyses the spectral and decay properties for the linear equation given by L_W (see (54)). The approach adopted here is based on [24]. For simplicity, we will suppress the subscript W.

Let $V_+ = -W''(1)$, $V_- = -W''(-1)$, and $V_* = \max V_\pm$. For a given complex number λ, consider the linear equation

$$(L - \lambda)\varphi = 0. \tag{68}$$

The parameter λ will be taken from

$$\Omega_{\alpha,\beta} = \{|\arg(\lambda - V_*)| < \pi - \alpha\} \cap \{|\lambda - V_*| > \beta\} \quad \text{for some } 0 < \alpha < \pi/2 \text{ and } \beta > 0. \tag{69}$$

The expressions $\gamma_\pm(\lambda) = \sqrt{\lambda - V_\pm}$ appear often in the analysis. They are taken to be single-valued and analytic in $\Omega_{0,0}$.

Equation (68) has solutions $\varphi_i(\lambda, x)$ and $\psi_i(\lambda, x)$ ($i = 1, 2$) such that $\Omega_{0,0} \ni \lambda \to \infty$; they satisfy (the prime $'$ refers to the x-derivative):

$$\text{for } x \geqslant 0: \quad \begin{cases} \varphi_1(\lambda, x) = e^{-\gamma_+(\lambda)x}(1 + O(|\lambda|^{-1/2})), \\ \varphi_1'(\lambda, x) = e^{-\gamma_+(\lambda)x}(-\gamma_+(\lambda) + O(1)), \\ \varphi_2(\lambda, x) = e^{\gamma_+(\lambda)x}(1 + O(|\lambda|^{-1/2})), \\ \varphi_2'(\lambda, x) = e^{\gamma_+(\lambda)x}(\gamma_+(\lambda) + O(1)), \end{cases} \tag{70}$$

$$\text{for } x \leqslant 0: \quad \begin{cases} \psi_1(\lambda, x) = e^{\gamma_-(\lambda)x}(1 + O(|\lambda|^{-1/2})), \\ \psi_1'(\lambda, x) = e^{\gamma_-(\lambda)x}(\gamma_-(\lambda) + O(1)), \\ \psi_2(\lambda, x) = e^{-\gamma_-(\lambda)x}(1 + O(|\lambda|^{-1/2})), \\ \psi_2'(\lambda, x) = e^{-\gamma_-(\lambda)x}(-\gamma_-(\lambda) + O(1)). \end{cases} \tag{71}$$

In addition, the functions φ_i and ψ_i can be related in the following way:

$$\varphi_i(\lambda, x) = A_i(\lambda)\psi_1(\lambda, x) + B_i(\lambda)\psi_2(\lambda, x), \tag{72}$$

$$\psi_i(\lambda, x) = C_i(\lambda)\varphi_1(\lambda, x) + D_i(\lambda)\varphi_2(\lambda, x), \tag{73}$$

where the coefficients A_i, B_i, C_i and D_i are analytic functions of $\lambda \in \Omega_{0,0}$ and can all be shown to be uniformly bounded as $\Omega_{0,0} \ni \lambda \to \infty$.

Recall the definitions (47) and (48) of the weighted function spaces $\mathcal{B}_{0,b}$ and $\mathcal{B}_{1,b}$. We have the following main estimates in these spaces.

LEMMA 22 Given $0 < \alpha < \pi/2$, $\beta > 0$ and the domain $\Omega_{\alpha,\beta}$, let

$$0 < b < \tfrac{1}{2}\min\{\Re\gamma_+(\lambda), \Re\gamma_-(\lambda) : \lambda \in \Omega_{\alpha,\beta}\}. \tag{74}$$

Then the following two statements hold.

(I) For any $\lambda \in \Omega_{\alpha,\beta}$, if the Wronskian

$$W(\varphi_1, \psi_1, \lambda) = \varphi_1(x, \lambda)\psi_1'(x, \lambda) - \varphi_1'(x, \lambda)\psi_1(x, \lambda) \neq 0,$$

then the resolvent operator $(L - \lambda)^{-1}$ exists as a bounded operator on $\mathcal{B}_{1,b}$.

(II) There exists a constant $C(\alpha,\beta)$ such that for any large enough $\lambda \in \Omega_{\alpha,\beta}$, the resolvent operator $(L-\lambda)^{-1}$ exists and the following estimates hold:

$$\|(L-\lambda)^{-1}g\|_{0,b} \leqslant \frac{C(\alpha,\beta)}{|\lambda|}\|g\|_{0,b}, \tag{75}$$

$$\|(L-\lambda)^{-1}g\|_{1,b} \leqslant \frac{C(\alpha,\beta)}{\sqrt{|\lambda|}}\|g\|_{0,b}, \tag{76}$$

$$\|(L-\lambda)^{-1}g\|_{1,b} \leqslant \frac{C(\alpha,\beta)}{|\lambda|}\|g\|_{1,b}. \tag{77}$$

Proof. The proof relies on estimating the Green's function of the resolvent:

$$G(x,y,\lambda) = \frac{1}{W(\varphi_1,\psi_1,\lambda)}\begin{cases}\psi_1(y,\lambda)\varphi_1(x,\lambda), & x>y,\\ \varphi_1(y,\lambda)\psi_1(x,\lambda), & x<y.\end{cases}$$

Note that W is independent of x. Hence $G(x,y,\lambda)$ is well defined if $W \neq 0$. We will only prove statement (II) (75, 76, 77), statement (I) follows immediately.

Using the estimates for φ and ψ, it is possible to show that as $\lambda \to \infty$:

$$W(\varphi_1,\psi_1,\lambda) = -(\gamma_+(\lambda)+\gamma_-(\lambda)) + O(1) = O(\sqrt{\lambda})$$

Hence if $\lambda \in \Omega_{\alpha,\beta}$ and is large enough, $W(\lambda)$ does not vanish and thus G is well defined. In this case, the function $f = (L-\lambda)^{-1}g$ can be written as

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} G(x,y,\lambda)g(y)\,\mathrm{d}y \\ &= \frac{1}{W(\lambda)}\left\{\int_{-\infty}^{x}\varphi_1(x,\lambda)\psi_1(y,\lambda)g(y)\,\mathrm{d}y + \int_{x}^{\infty}\psi_1(x,\lambda)\varphi_1(y,\lambda)g(y)\,\mathrm{d}y\right\}. \end{aligned} \tag{78}$$

Suppose $\|g\|_{1,b} < \infty$. We will estimate $\|f\|_{0,b}$ as follows. Let $x > 0$. The first term of (78) is bounded by

$$\begin{aligned} &\left|\frac{\varphi_1(x,\lambda)}{W(\lambda)}\right|\left\{\int_{-\infty}^{0}|\psi_1(y,\lambda)g(y)e^{-by}e^{by}|\,\mathrm{d}y + \int_{0}^{x}|\psi_1(y,\lambda)g(y)e^{by}e^{-by}|\,\mathrm{d}y\right\} \\ &\quad\leqslant \left|\frac{\varphi_1(x,\lambda)}{W(\lambda)}\right|\left\{\int_{-\infty}^{0}|\psi_1(y,\lambda)e^{by}|\,\mathrm{d}y + \int_{0}^{x}|\psi_1(y,\lambda)e^{-by}|\,\mathrm{d}y\right\}\|g\|_{0,b} \\ &\quad\leqslant C\left|\frac{e^{-\Re\gamma_+(\lambda)x}}{W(\lambda)}\right|\left\{\int_{-\infty}^{0}e^{\Re\gamma_-(\lambda)y}e^{by}\,\mathrm{d}y + \int_{0}^{x}D_1(\lambda)e^{\Re\gamma_+(\lambda)y}e^{-by}\,\mathrm{d}y\right\}\|g\|_{0,b} \quad \text{(by (73))} \\ &\quad\leqslant C\left|\frac{e^{-\Re\gamma_+(\lambda)x}}{W(\lambda)}\right|\left\{\frac{1}{\Re\gamma_-(\lambda)+b} + \frac{e^{(\Re\gamma_+(\lambda)-b)x}-1}{\Re\gamma_+(\lambda)-b}\right\}\|g\|_{0,b} \leqslant \frac{Ce^{-bx}}{|\lambda|}\|g\|_{0,b} \end{aligned}$$

where we have used the fact that $D_1(\lambda)$ is bounded. The second term of (78) is estimated as

$$\begin{aligned} \left|\frac{\psi_1(x,\lambda)}{W(\lambda)}\right|\int_{x}^{\infty}|\varphi_1(y,\lambda)g(y)e^{by}e^{-by}|\,\mathrm{d}y &\leqslant C\left|\frac{e^{\Re\gamma_+(\lambda)x}}{W(\lambda)}\right|\int_{x}^{\infty}e^{-(\Re\gamma_+(\lambda)+b)y}\,\mathrm{d}y\|g\|_{1,b} \\ &\leqslant \frac{Ce^{-bx}}{|\lambda|}\|g\|_{0,b}. \end{aligned}$$

Similar estimates can be obtained for $x < 0$. Thus (75) is proved.

Estimate (76) is proved in exactly the same way. Taking the derivative brings down a $\sqrt{|\lambda|}$. Estimate (77) is proved by the following observation:

$$(f_x)_{xx} + (V(x) + \lambda) f_x = g_x + V'(x) f,$$

i.e.

$$\|f_x\|_{0,b} \leqslant \frac{C\|g\|_{1,b}}{|\lambda|} + \|f\|_{0,b} \leqslant \frac{C\|g\|_{1,b}}{|\lambda|} + \frac{C\|g\|_{0,b}}{|\lambda|} \leqslant \frac{C\|g\|_{1,b}}{|\lambda|}. \qquad \square$$

Next we introduce the concept of eigenvalues of L.

DEFINITION 23 A complex number λ is called an *eigenvalue* of L if there exists a bounded function $\varphi \not\equiv 0$ such that $(L - \lambda)\varphi = 0$.

A complex number λ is said to belong to the *resolvent set* of L if $(L - \lambda)^{-1}$ exists as a bounded operator on the space of bounded functions, i.e. $\mathcal{B}_{0,0}$.

With the above, we have the following lemma (which basically summarises the results of [24, Lemma 5.3 and Theorem 5.5]).

LEMMA 24 1. The eigenvalues of L are real. Furthermore, those eigenvalues in the interval (V_*, ∞) are discrete and actually confined in $(V_*, \|V\|_\infty)$ and can only cluster at V_*.
2. Since $m'(x) > 0$ is an eigenfunction with respect to $\lambda = 0$, the operator L has no positive eigenvalues. The previous statement then implies that there is a spectral gap, i.e. there is a positive distance between the principal eigenvalue, 0, and the the second largest eigenvalue.
3. There are $0 < \alpha < \pi/2$ and $\beta > 0$ such that $(L - \lambda)^{-1}$ exists for all $\lambda \in \Omega_{\alpha,\beta}$. Hence there exist $C(\alpha, \beta)$ and $b > 0$ such that the estimates (75), (76) and (77) hold for all $\lambda \in \Omega_{\alpha,\beta}$.

We now state and prove a linear decay estimate for the semigroup of (54).

THEOREM 25 Let $v(x, t)$ be the solution of

$$v_t = v_{xx} - W''(m(x))v, \qquad v(x, 0) = v_0, \tag{79}$$

where $v_0 \in \mathcal{B}_{1,b}$ and $\int_{-\infty}^{\infty} v_0(x)m'(x)\,\mathrm{d}x = 0$. Then there exist $D \geqslant 1$ and $\omega > 0$ such that for all $t > 0$,

$$\|v(t)\|_{1,b} \leqslant D\frac{e^{-\omega t}}{\sqrt{t}}\|v_0\|_{0,b}, \tag{80}$$

$$\|v(t)\|_{1,b} \leqslant De^{-\omega t}\|v_0\|_{1,b}. \tag{81}$$

Proof. From Lemma 24(3), we can find $\omega > 0$ and $\pi/2 < \gamma < \pi$ such that the spectrum of L minus zero lies completely to the left of the complex curve

$$C_{\omega,\gamma} = \{\lambda = -\omega + \rho e^{\pm i\gamma} : 0 < \rho < \infty\}. \tag{82}$$

Hence the solution $v(t)$ can be written as the following contour integral:

$$v(t) = \frac{1}{2\pi i}\int_{C_{\omega,\gamma}} e^{t\lambda}(\lambda - L)^{-1} v_0\,\mathrm{d}\lambda.$$

Choose b with

$$0 < b < \tfrac{1}{2}\min\{\Re\gamma_+(\lambda), \Re\gamma_-(\lambda) : \lambda \in C_{\omega,\gamma}\}. \tag{83}$$

We now estimate the solution $v(t)$ as follows:

$$\|v(t)\|_{1,b} \leqslant \frac{1}{2\pi}\int_{C_{\omega,\gamma}} |e^{t\lambda}|\|(\lambda - L)^{-1}\|\,\|v_0\|_{0,b}\,|\mathrm{d}\lambda| \leqslant \frac{C(\alpha,\beta)}{2\pi}\left(\int_{C_{\omega,\gamma}} \frac{|e^{t\lambda}|}{\sqrt{|\lambda|}}\,|\mathrm{d}\lambda|\right)\|v_0\|_{0,b}.$$

The last integral is bounded by

$$\begin{aligned}\int_{C_{\omega,\gamma}} \frac{|e^{t\lambda}|}{\sqrt{\lambda}}\,|\mathrm{d}\lambda| &\leqslant 2\int_0^\infty \frac{e^{-\omega t}e^{t\rho\cos\gamma}\,\mathrm{d}\rho}{[(\rho\cos\gamma-\omega)^2+\rho^2\sin^2\gamma]^{1/4}} \leqslant 2\int_0^\infty \frac{e^{-\omega t}e^{t\rho\cos\gamma}\,\mathrm{d}\rho}{[\omega^2+\rho^2\sin^2\gamma]^{1/4}}\\ &\leqslant De^{-\omega t}\int_0^\infty \frac{e^{t\rho\cos\gamma}\,\mathrm{d}\rho}{\sqrt{\rho\sin\gamma}} \leqslant D\frac{e^{-\omega t}}{\sqrt{t}},\end{aligned}$$

leading to (80).

Finally, by parabolic regularity, for any $T > 0$, there exists $C(T)$ such that

$$\|v(t)\|_{1,b} \leqslant C(T)\|v_0\|_{1,b} \quad \text{for } 0 < t < T.$$

Hence by choosing a larger D and a smaller ω (depending on T), the conclusion (81) also follows. □

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