Crystalline curvature flow of planar networks[†]

G. BELLETTINI[‡]

*Dipartimento di Matematica, Università di Roma "Tor Vergata",
via della Ricerca Scientifica, 00133 Roma, Italy, and
INFN Laboratori Nazionali di Frascati, Italy*

M. CHERMISI[§]

*Dipartimento di Matematica, Università di Roma "Tor Vergata",
via della Ricerca Scientifica, 00133 Roma, Italy*

AND

M. NOVAGA[¶]

*Dipartimento di Matematica, Università di Pisa,
Largo Bruno Pontecorvo 5, 56127 Pisa, Italy*

We consider the evolution of a polycrystalline material with three or more phases, in the presence of an even crystalline anisotropy. We analyze existence, uniqueness, regularity and stability of the flow. In particular, if the flow becomes unstable at a finite time, we prove that an additional segment (or even an arc) at the triple junction may develop in order to decrease the energy and make the flow stable at subsequent times. We discuss some examples of collapsing situations that lead to changes of topology, such as the collision of two triple junctions.

2000 Mathematics Subject Classification: Primary 53C44; Secondary 74E10, 74E15.

Keywords: Motion by crystalline curvature of networks; polycrystalline materials.

1. Introduction

Several models in phase transitions treat phenomena in which two or more phases of the same material, or the same phase of a crystal with different orientations, can coexist without mixing. A curve or a surface Γ bounding different regions is called a surface boundary, or interface, and is moving in a nonequilibrium state. In some cases the motion of Γ does not depend on the physical situation in the various phases but only on its geometry, and is described by geometric equations relating, for instance, the normal velocity of the interface to its curvatures. The crystalline curvature flow in two dimensions is the formal gradient flow of the energy functional

$$\mathcal{F}_\varphi(\Gamma) := \int_\Gamma \varphi^o(\nu) \, \mathrm{d}\mathcal{H}^1, \tag{1.1}$$

[‡]E-mail: belletti@mat.uniroma2.it

[§]E-mail: chermisi@mat.uniroma2.it

[¶]E-mail: novaga@dm.unipi.it

where ν is a unit normal vector field to Γ and the energy density $\varphi^o : \mathbb{R}^2 \to [0, +\infty)$, sometimes called surface tension, is a crystalline (i.e. piecewise linear) norm. When φ^o is isotropic the energy functional (1.1) is proportional to the length of the interfaces and the resulting geometric parabolic equation is the curvature flow (at least in the simplest case when Γ is the boundary of an open set). However, when dealing with crystalline and polycrystalline materials, φ^o is anisotropic and neither smooth nor strictly elliptic; in addition, multi-phase boundaries with more than two phases occur.

To our knowledge, J. E. Taylor [31], [33], [34], [36] (see also [5] and [9]) was the first to introduce the notion of crystalline geometry and to determine the crystalline flow of curves with triple junctions, in particular to compute the motion of the triple junction. The analysis of the evolution of grain boundaries has been pursued also by other authors: see for instance [5]–[8], [11], [12], [23], [25], [26], [28]. See also [4], [6], [11], [22] for related physical models of crystal growth, and [1], [2], [10], [13], [14], [16], [17], [19]–[21], [24], [27], [29], [30] for related results.

In the present paper we consider the evolution of a polycrystalline material with three or more phases for a crystalline φ^o whose one-sublevel set $F_\varphi := \{\varphi^o \leqslant 1\}$ (the *Frank diagram*) is a regular polygon of n sides. The dual function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(\xi) := \sup\{\xi \cdot \eta : \varphi^o(\eta) \leqslant 1\}$ is crystalline too and $\mathcal{W}_\varphi := \{\varphi \leqslant 1\}$ is called the *Wulff shape*. We are particularly interested in the motion by crystalline curvature of special planar networks called elementary triods, namely regular three-phase boundaries given by the union of three Lipschitz curves, the interfaces, intersecting at a point called a triple junction. Each interface is the union of a segment of finite length and a half-line, corresponding to two consecutive sides of $\mathcal{W}_\varphi$.

We analyze local and global existence and stability of the flow. In general, the flow may become unstable at a finite time. If this occurs, we prove that at subsequent times a regular flow can be constructed, by adding a new segment (or even an arc with zero crystalline curvature) at the triple junction. In all flows we exhibit, the crystalline curvature remains bounded (even if a segment appears or disappears) and has a jump discontinuity at the time of instability only in the case of the disappearance of a segment. We also discuss some examples of collision of two triple junctions. These examples (as well as the local in time existence result) show one of the advantages of crystalline flows with respect, for instance, to the usual mean curvature flow: explicit computations can be performed to some extent, and in case of nonuniqueness, a comparison between the energies of different evolutions (difficult in the euclidean case) can be made.

The rigorous definition of crystalline curvature for networks has been introduced in [3]; we will see that the corresponding flow essentially agrees with the one suggested in [34]. Finally, we stress that Taylor already predicted the emergence of new edges and zero weighted curvature curves from a triple junction, wrote a computer program [32] adding such edges and approximations to the curves, and made explicit calculations [36] for determining them (see also the video quoted in [32]).

The plan of the paper is the following. In Section 2.1 we present some basic definitions and results from [3], where the crystalline curvature of partitions is computed through the first variation of $\mathcal{F}_\varphi$. The crystalline curvature is the tangential divergence of a vector field $N_{\min} : \Pi \to \mathbb{R}^2$ which minimizes the functional

$$\int_\Pi (\operatorname{div}_\tau N)^2 \varphi^o(\nu) \, d\mathcal{H}^1 \tag{1.2}$$

among all Cahn–Hoffman vector fields N on the elementary triod Π which satisfy the so-called balance condition at the triple junction q,

$$N_{|\Sigma_1}(q) + N_{|\Sigma_2}(q) + N_{|\Sigma_3}(q) = 0. \tag{1.3}$$

Such a minimizer $N_{\min}$ is *unique* (this is true, in general, only in two dimensions) and identifies the

direction along which the functional $\mathcal{F}_\varphi$ decreases most quickly. The balance condition (1.3) is the analog of the Herring condition (120 degrees condition) in the euclidean case. By the definition of a Cahn–Hoffman vector field, $N_{|\Sigma_j}(q) \in \partial \mathcal{W}_\varphi$. Any triplet of vectors $(X, Y, Z) \in (\partial \mathcal{W}_\varphi)^3$ satisfying $X + Y + Z = 0$ is called an admissible triplet. In Section 2.2 we introduce the notions of elementary, quasi-elementary, nonpolygonal and degenerate triod, and of configuration of an elementary triod. The regularity of an elementary triod is related to the regularity of each interface and to the balance condition (1.3). We also recall the notion of stability [3] and introduce the concept of stability region of a configuration. In Section 2.3 we give the definition of flow by crystalline curvature starting from an elementary triod which allows us to consider also initial data which may develop a new segment or an arc from the triple junction. In Section 2.4 we determine the geometry of an elementary triod, that is, the three angles at the triple junction q between the interfaces. These angles are determined by the balance condition (1.3) at q, which in turn is related to the existence of admissible triplets. We prove that any regular polygon $P_{\mathfrak{n}}$, $\mathfrak{n}$ even $\geqslant 6$, has a unique admissible triplet (X, Y, Z) once we fix one of the vectors of the triplet, for instance X, in $\partial P_{\mathfrak{n}}$. We also determine the range of all admissible triplets of $N_{\min}$ at q, and using this result we compute in Section 2.5 the crystalline curvature of the triod. Since $N_{\min}$ is unique and its values are fixed (up to a sign change) at the three vertices of the partitions, it follows that $N_{\min}$ is given on all the interfaces by linear interpolation. Thus, as shown in [3] in the case $n = 8$, it is possible to reduce the minimum problem (1.2) to a one-dimensional minimum problem. In the case of a partition consisting of two adjacent triple junctions, the solution $N_{\min}$ of (1.2) is completely determined by the values of two independent real variables. Since the Cahn–Hoffman vector fields have constant normal component, the crystalline curvature is simply the tangential derivative of the tangential component of $N_{\min}$, that is, a ratio of lengths. Finally, we establish which values of the lengths of the finite segments of Π provide stable triods (*stability region*).

In Section 3 we prove that there exists, locally in time, a unique stable regular flow starting from a stable regular initial datum. In Section 3.1 we show a case of global existence. The analysis of the long time behaviour requires the study of the stability region of each configuration. Stability is the ingredient that ensures that no additional segments develop at the triple junction during the flow. If the initial triod is unstable then an additional segment may develop in order to decrease the surface energy and make the evolved triods stable at positive times. In Section 4 we exhibit an example of this occurrence.

In Sections 5–7 we show that the flow becomes unstable at a finite time $\mathcal{T}$ and that at the subsequent times a regular flow can be constructed; in particular, a new segment (resp. an arc with zero crystalline curvature) develops at the triple junction in the flow of Theorem 5.1 (resp. of Theorem 6.1). In Theorem 7.1 we prove that the flow has two different behaviours depending on the initial datum Π. For a suitable choice of Π, we show that at $t = \mathcal{T}$ one of the three segments vanishes, its crystalline curvature remains bounded, the Cahn–Hoffman vector field $N_{\min}$ has a jump discontinuity and the triple junction translates along the remaining adjacent half-line in $[\mathcal{T}, +\infty)$. For the other choices of stable Π we prove that a curve appears from the triple junction, as in Section 6, with the difference that the adjacent segment now has positive φ-curvature and keeps on moving at subsequent times. Each of these flows has the property that all crystalline curvatures remain bounded.

In Section 8 we study the crystalline curvature flow starting from a *stable* φ-regular partition formed by *two* adjacent elementary triple junctions. We discuss some examples of collapsing situations that lead to changes of topology, such as for instance the collision of two triple junctions.

We present several candidates to continue the flow after the singularity (see Example 8.4). In Section 9 we introduce the notion of homothetic flow. We classify homothetic flows when $\mathfrak{n} = 6\mathfrak{m}$ and we show that the global flows studied in Section 3.1 converge to homothetic flows as $t \to +\infty$.

While the case $\mathfrak{n} = 6\mathfrak{m}$ is exhaustively studied, as is the case $\mathfrak{n} = 8$, the cases $\mathfrak{n} = 6\mathfrak{m} - 4$ and $\mathfrak{n} = 6\mathfrak{m} - 2$ have been studied in less detail. The reason is twofold: first of all, the cases $\mathfrak{n} = 6\mathfrak{m}$ and $\mathfrak{n} = 8$ present all the phenomena we are interested in (formation of new edges, emergence of curves and disappearance of edges at the triple junction); secondly, the analysis of the long time behaviour of all possible configurations can be described in a rather concise way, while in the general cases $\mathfrak{n} = 6\mathfrak{m} - 4$ and $\mathfrak{n} = 6\mathfrak{m} - 2$ it seems more complicated.

2. Preliminaries

In this paper, $\cdot$, $|\cdot|$ and $\mathcal{H}^1$ are respectively the euclidean canonical inner product, the euclidean norm and the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Points and vectors of $\mathbb{R}^2$ will be identified. Given two points $p, q \in \mathbb{R}^2$ we denote by pq the vector with initial point and end point at p and q respectively. Given two vectors $v, w \in \mathbb{R}^2$, we denote by $v^\perp$ the counterclockwise rotation of v through $\pi/2$ around the origin and by $\vartheta(v, w) \in [0, \pi]$ the angle between v and w. Given $f : (a, b) \to \mathbb{R}$ and $t \in (a, b)$, we denote by $f(t+)$ and $f(t-)$ respectively the right and left limits of f at t (if they exist).

Given a subset U of $\mathbb{R}^2$ we denote by $\text{int}(U)$, $\overline{U}$ and ∂U respectively the interior, closure and boundary of U. In particular, given a segment $S \subset \mathbb{R}^2$, we denote by $\text{int}(S)$ the relative interior of S. Given two parallel (possibly infinite) segments S_1, S_2, the *distance vector* of S_2 from S_1 is the vector having norm $\text{dist}(S_1, S_2)$ pointing from S_1 to S_2.

By a *Lipschitz curve with boundary* in $\mathbb{R}^2$ we mean a 1-dimensional bounded set $\Sigma \subset \mathbb{R}^2$ which can be written locally as a Lipschitz graph on an open interval of $\mathbb{R}$. Any Lipschitz function or vector field defined on Σ will be considered as defined up to $\partial \Sigma$. We denote by $\text{Lip}(\Sigma; \mathbb{R}^2)$ the set of all Lipschitz vector fields on Σ. Given a point $x \in \Sigma$ we denote by $\text{T}_x \Sigma$ the tangent line to Σ at x.

We denote by $\mathfrak{n}$, $\mathfrak{m}$ positive integers and by $P_\mathfrak{n}$ the regular polygon of $\mathfrak{n}$ ($\mathfrak{n}$ even) sides of length L inscribed in the unit circle centered at the origin of $\mathbb{R}^2$. $P_\mathfrak{n}$ has two horizontal sides and is oriented in clockwise sense.

2.1   *Crystalline curvature of regular partitions of* $\mathbb{R}^2$

In this section we present some basic notations and definitions from [3]. Let $\varphi : \mathbb{R}^2 \to [0, +\infty)$ be a crystalline anisotropy on $\mathbb{R}^2$ (i.e. an even piecewise linear convex function) satisfying $\mathcal{W}_\varphi = P_\mathfrak{n}$. We let φ^o be the dual function of φ, and we denote by T_φ and T_{φ^o} the multivalued mappings (duality mappings) defined as $T_\varphi(\xi) := \frac{1}{2}\partial(\varphi^2)(\xi)$ and $T_{\varphi^o}(\xi^o) := \frac{1}{2}\partial((\varphi^o)^2)(\xi^o)$ for all $\xi, \xi^o \in \mathbb{R}^2$, where ∂ denotes the usual subdifferential for convex functions. We observe that T_φ (resp. T_{φ^o}) is a maximal monotone operator which takes $\partial \mathcal{W}_\varphi$ (resp. ∂F_φ) onto ∂F_φ (resp. onto $\partial \mathcal{W}_\varphi$).

DEFINITION 2.1   Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz curve with boundary, $x \in \partial \Sigma$, assume that $\overline{\Sigma}$ admits tangent line $\text{T}_x(\overline{\Sigma})$ at x, and let $z \in \mathbb{R}^2 \setminus \text{T}_x(\overline{\Sigma})$. We define the vector $z^{\partial \Sigma} \in \mathbb{R}^2$ as the rotation through angle $\pi/2$ of the vector z in such a way that $z^{\partial \Sigma}$ points outwards from Σ.

That is, since we assume the existence of the tangent line to Σ at $x \in \partial \Sigma$, the vector $z^{\partial \Sigma}$ is required to have a nonzero component along the half-tangent line (to $\overline{\Sigma}$ at x) pointing outwards from Σ.

DEFINITION 2.2 A *partition* of $\mathbb{R}^2$ is a finite family $\{E_i\}_i$ of open subsets of $\mathbb{R}^2$ (called *phases*) such that $\bigcup_i \overline{E}_i = \mathbb{R}^2$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\partial E_i \cap \partial E_j$, when it is nonempty, is a Lipschitz curve with boundary, called an *interface*. By an *m-multiple junction* of $\{E_i\}$ ($m \geqslant 3$ a natural number) we mean a point q belonging to m distinct interfaces.

Given a partition $\{E_i\}$ of $\mathbb{R}^2$, we set

$$\Sigma_{ij} := \partial E_i \cap \partial E_j, \quad i \neq j, \quad \Gamma := \bigcup_{i,j} \Sigma_{ij}, \quad J := \bigcup_{i,j} \partial \Sigma_{ij}. \tag{2.1}$$

When we write Σ_{ij} we always assume that $i \neq j$ and $\Sigma_{ij} \neq \emptyset$. We denote by ν^{ij} an $\mathcal{H}^1$-a.e. defined euclidean unit normal to Σ_{ij} and we set $\nu_\varphi^{ij} := \nu^{ij}/\varphi^o(\nu^{ij})$. We denote by $\text{Lip}_{\nu,\varphi}(\Gamma; \mathbb{R}^2)$ the space of vector fields $N : \Gamma \rightarrow \mathbb{R}^2$ such that $N_{|\Sigma_{ij}} \in \text{Lip}(\Sigma_{ij}; \mathbb{R}^2)$ and $N_{|\Sigma_{ij}}(x) \in T_{\varphi^o}(\nu_\varphi^{ij}(x))$ for $\mathcal{H}^1$-almost every $x \in \Sigma_{ij}$. Set

$$\mathcal{N} := \left\{ N \in \text{Lip}_{\nu,\varphi}(\Gamma; \mathbb{R}^2) : \sum_{i,j}(N_{|\Sigma_{ij}})^{\partial \Sigma_{ij}} = 0 \text{ on } J \right\}. \tag{2.2}$$

The condition on J in (2.2) is usually called the *balance condition*.

DEFINITION 2.3 If $\mathcal{N} \neq \emptyset$, the partition $\{E_i\}$ is said to be *φ-regular* and any $N \in \mathcal{N}$ is called a *Cahn–Hoffman vector field* on Γ.

The following definition of φ-curvature is based on [3, Theorem 4.8] and the crucial fact that we are considering *planar* partitions: if $\{E_i\}_i$ is a φ-regular partition then the minimum problem

$$\min\left\{ \left[\int_\Gamma (\text{div}_\tau N)^2 \varphi^o(\nu) \, d\mathcal{H}^1 \right]^{1/2} : N \in \mathcal{N} \right\} \tag{2.3}$$

admits a *unique* solution which identifies the direction along which the functional (1.1) decreases most quickly. Let $N_{\min} : \Gamma \rightarrow \mathbb{R}^2$ be the solution of problem (2.3).

DEFINITION 2.4 Let $\{E_i\}$ be a φ-regular partition. We define the *φ-curvature* κ_φ of Γ as

$$\kappa_\varphi := \text{div}_\tau N_{\min}, \quad \text{a.e. on } \Gamma.$$

REMARK 2.5 Let $\Sigma = \partial E$ be a simple Lipschitz curve which admits a Lipschitz Cahn–Hoffman vector field, i.e. $N \in \text{Lip}(\Sigma; \mathbb{R}^2)$ with $N(x) \in T_{\varphi^o}(\nu_\varphi(x))$ for $\mathcal{H}^1$-a.e. $x \in \Sigma$, where $\nu_\varphi := \nu/\varphi^o(\nu)$ and ν is an $\mathcal{H}^1$-a.e. defined euclidean unit normal to Σ. It is easy to see that $\kappa_\varphi = 0$ on any nonflat arc γ contained in Σ since N on γ is constantly equal to a vertex of $\partial \mathcal{W}_\varphi$. Assume now that S is an open segment of length $L > 0$ contained in Σ. Denote by N_1, N_2 respectively the values of N at the initial and final point of S according to $\tau := -\nu^\perp$. The φ-curvature of S is zero if $L = +\infty$, while if $L < +\infty$,

$$\kappa_\varphi(p) = \frac{1}{L}(N_2 - N_1) \cdot \tau, \quad p \in S. \tag{2.4}$$

Hence S has constant φ-curvature which, setting $l := L\tau$, will be denoted by $\kappa_\varphi(l)$. Notice that κ_φ in (2.4) changes sign if we change the sign of ν.

For simplicity, in this paper we restrict ourselves to Wulff shapes $\mathcal{W}_\varphi$ having an even number of sides; dropping the central symmetry assumption on $\mathcal{W}_\varphi$ would require to take into account the orientation of the various phases, making the analysis more complicated. Again for simplicity, we assume that the Wulff shape is the same for each pair of adjacent phases. The main definitions (such

as Definitions 2.3, 2.4, 2.6) can also be given in the case of different Wulff shapes on different pairs of adjacent phases; however, the analysis (starting from Lemma 2.16) would become far more difficult, as well as the catalog of all possible configurations and phenomena.

2.2 *Elementary, quasi-elementary, nonpolygonal triods*

In this section we introduce the notions of elementary, quasi-elementary and nonpolygonal triod, and of configuration of an elementary triod, and we fix the orientation of a triod.

DEFINITION 2.6 When $\{E_1, E_2, E_3\}$ is a partition of $\mathbb{R}^2$ into three sets having only one 3-multiple junction, called a *triple junction* and denoted by q, the set Γ defined in (2.1) will be called a *triod*, and denoted by Π. If the partition is φ-regular, the triod is said to be φ-regular. For simplicity, $\Sigma_{12}, \Sigma_{23}, \Sigma_{13}$ will be denoted respectively by $\Sigma_1, \Sigma_2, \Sigma_3$, and correspondingly ν_φ^{ij} will be denoted by ν_φ^j. The *angles* of Π are the three angles at q between $\Sigma_1, \Sigma_2, \Sigma_3$ (see Figure 5).

REMARK 2.7 The notion of regularity in Definition 2.6 is essentially the same as given by J. E. Taylor in [36] when each Σ_j is polygonal. In the case of a φ-regular partition with $\Gamma = \bigcup_{i=1}^{5} \Sigma_i$, $J = \{q_1, q_2\}$, and $N \in \mathcal{N}$ as in Figure 1, the triplet of vectors

$$(N_{|\Sigma_1}(q_1)^{\partial\Sigma_1}, N_{|\Sigma_2}(q_1)^{\partial\Sigma_2}, N_{|\Sigma_3}(q_1)^{\partial\Sigma_3})$$

is the clockwise rotation of the triplet $(N_{|\Sigma_1}(q_1), N_{|\Sigma_2}(q_1), N_{|\Sigma_3}(q_1))$, while

$$(N_{|\Sigma_1}(q_2)^{\partial\Sigma_1}, N_{|\Sigma_4}(q_2)^{\partial\Sigma_4}, N_{|\Sigma_5}(q_2)^{\partial\Sigma_5})$$

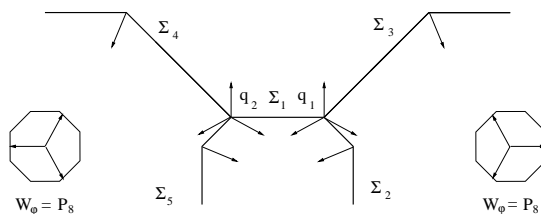is the counterclockwise rotation of $(N_{|\Sigma_1}(q_2), N_{|\Sigma_4}(q_2), N_{|\Sigma_5}(q_2))$.

FIG. 1. $(N_{|\Sigma_1}(q_1)^{\partial\Sigma_1}, N_{|\Sigma_2}(q_1)^{\partial\Sigma_2}, N_{|\Sigma_3}(q_1)^{\partial\Sigma_3})$ and $(N_{|\Sigma_1}(q_2)^{\partial\Sigma_1}, N_{|\Sigma_4}(q_2)^{\partial\Sigma_4}, N_{|\Sigma_5}(q_2)^{\partial\Sigma_5})$.

DEFINITION 2.8 Let $\Pi = \bigcup_{j=1}^{3} \Sigma_j$ be a φ-regular triod. We say that Π is *elementary* if

($\mathcal{E}$) each interface Σ_j is the union of a segment S_j of finite length $L_j > 0$ and a half-line R_j such that S_j and R_j correspond to two consecutive sides of $\mathcal{W}_\varphi$ (see Figure 2(i)).

We say that Π is *degenerate* if two interfaces satisfy ($\mathcal{E}$) and the remaining one $\Sigma_{\bar{k}}$ is a half-line.

We say that Π is *quasi-elementary* if two interfaces satisfy ($\mathcal{E}$) and the remaining one Σ_k is the union of two segments S_4 and S_k of finite lengths, $L_4 > 0$ and $L_k > 0$ respectively, and a half-line R_k such that S_4 and S_k, and S_k and R_k, correspond to two consecutive sides of $\mathcal{W}_\varphi$ (see Figure 2(ii))).

We say that Π is *nonpolygonal* if two interfaces satisfy ($\mathcal{E}$) and the remaining one Σ_k is the union of a curve γ_4, a segment S_k of finite length $L_k > 0$ and a half-line R_k such that S_k and R_k correspond to two consecutive sides of $\mathcal{W}_\varphi$ (see Figure 2(iii)).
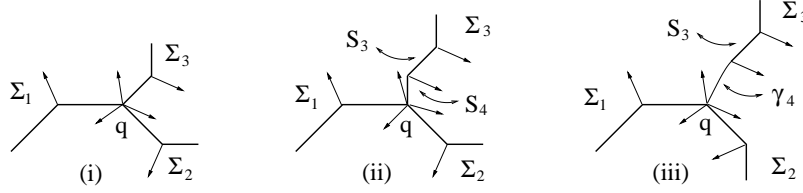
FIG. 2. (i) Elementary, (ii) quasi-elementary, (iii) nonpolygonal triod ($\mathcal{W}_\varphi = P_8$). Note that $\kappa_\varphi = 0$ on S_3 in (i) and (ii), $\kappa_\varphi < 0$ on S_4 in (ii), and $\kappa_\varphi = 0$ on γ_4 in (iii).

Given an elementary degenerate or quasi-elementary or nonpolygonal triod Π and $N \in \mathcal{N}$, we set $A_j := \overline{S}_j \cap \overline{R}_j$ for any $j = 1, 2, 3$ such that $R_j \neq \emptyset$, $A_4 := \overline{S}_4 \cap \overline{S}_k$ if Π is quasi-elementary, and $A_4 := \overline{\gamma}_4 \cap \overline{S}_k$ if Π is nonpolygonal. We denote by α_j the angle of Σ_j at A_j opposite to the region where $N(A_j)$ lies (see Figure 3). Notice that $\alpha_j \in \{\pi - \pi/(2\mathfrak{n}), \pi + \pi/(2\mathfrak{n})\}$.

Let ν be the $\mathcal{H}^1$-almost everywhere defined euclidean unit normal to Π oriented in such a way that $\nu_{|\text{int}(S_j)} \cdot N(A_j) > 0$. We set $\nu_j := \nu_{|\text{int}(S_j)}$, $\tau_j := -\nu_j^\perp$ and $l_j := L_j \tau_j$, for any $j = 1, 2, 3$, and also $j = 4$ if Π is quasi-elementary. Thus $\{\tau_j, \nu_j\}$ is a positively oriented basis of $\mathbb{R}^2$ and, without loss of generality, we assume that each l_j points towards q. We denote by $\kappa_\varphi(l_j)$ the φ-curvature of S_j.

For an elementary triod, we always assume that S_1 is horizontal and Σ_2 and Σ_3 are given in counterclockwise sense as in Figure 3. We denote by V_j, W_j the vertices of the side of $P_\mathfrak{n}$ (in clockwise sense) having ν_j as outer normal and by M_j the middle point of the segment $[V_j, W_j]$. Note that

$$\tau_1 \cdot \nu_3 = -\tau_1 \cdot \nu_2, \qquad \nu_1 \cdot \tau_3 = -\nu_1 \cdot \tau_2, \qquad \tau_1 \cdot \nu_3 = -\nu_1 \cdot \tau_3. \tag{2.5}$$

DEFINITION 2.9   Let Π, Π' be two elementary triods. We say that Π and Π' are *equivalent* (or *belong to the same configuration*) if they coincide after possible rescalings of their bounded edges and after a rotation. We denote by $[\Pi]$ the *configuration* of Π, i.e. the equivalence class of Π, and by $\mathfrak{C}$ the set of all possible configurations for elementary triods.

We also recall the notion of stability [3] and introduce the concept of stability region of a configuration.

DEFINITION 2.10   Let Π be a φ-regular triod. We say that Π is *stable* if $(N_{\min})_{|\Sigma_j}(q)$ is not a vertex of $\mathcal{W}_\varphi$ for any $j = 1, 2, 3$. We say that Π is *unstable* if it is not stable.

It follows that nonpolygonal triods are always unstable (see Remark 2.5). Elementary, degenerate and quasi-elementary triods can be either stable or unstable.

DEFINITION 2.11   Given a configuration $(\mathsf{e}) \in \mathfrak{C}$, the *stability region* of (e), denoted by $\mathcal{S}_\mathsf{e}$, is the set of all $(\Lambda_1, \Lambda_2, \Lambda_3) \in (0, +\infty)^3$ such that, if $\Pi \in (\mathsf{e})$ is an elementary triod with $|S_j| = \Lambda_j$ for any $j = 1, 2, 3$, then Π is stable. For $j_1, j_2, j_3 \in \{1, 2, 3\}$, $j_1 \neq j_2 \neq j_3 \neq j_1$, we let

$$\mathcal{S}_\mathsf{e}(j_2, j_3) := \left\{ \left(\frac{\Lambda_{j_1}}{\Lambda_{j_2}}, \frac{\Lambda_{j_1}}{\Lambda_{j_3}} \right) : (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathcal{S}_\mathsf{e} \right\}.$$

2.3   *Definition of crystalline flows of triods*

Our object is to provide a definition of φ-curvature flow allowing one to consider also initial data for which a new segment or a curve (with zero φ-curvature) can develop from the triple junction at time zero.

DEFINITION 2.12   Let $T > 0$ and Π be an elementary (resp. a degenerate) triod. For any $t \in [0, T)$, let $\Pi(t)$ be a φ-regular triod and $q(t)$ its triple junction. We say that $t \in [0, T) \mapsto \Pi(t)$ is a *φ-curvature flow* starting from $\Pi = \Pi(0)$ if for any $t \in (0, T)$:

(i) $\Pi(t)$ is either elementary or quasi-elementary or nonpolygonal (resp. degenerate);
(ii) for any $j = 1, 2, 3$, each $R_j(t)$ has zero normal velocity and each $S_j(t)$ is parallel to $S_j(0) = S_j$;
(iii) for each $j = 1, 2, 3$, and also $j = 4$ if $\Pi(t)$ is quasi-elementary, denoting by $h_j(t)$ the distance vector of the segment $S_j(t)$ from $S_j(0) = S_j$, we have $h_j \in C^1((0, T); v_j\mathbb{R})$ and

$$\begin{cases} \dfrac{\dot{h_j}(t)}{\varphi^o(v_j)} = -\kappa_\varphi(l_j(t))v_j, \\ h_j(0) = 0. \end{cases} \tag{2.6}$$

The flow is said to be *stable* if $\Pi(t)$ is stable for any $t \in (0, T)$.

REMARK 2.13   Since $\varphi^o(v_j)$ is a constant independent of $j \in \{1, 2, 3, 4\}$, the system in (2.7) is equivalent, up to rescaling in time, to

$$\dot{h_j}(t) = -\kappa_\varphi(l_j(t))v_j. \tag{2.7}$$

For simplicity, we will consider (2.7) in place of (2.6).

Note that, in Definition 2.12, Π is not required to be stable (even in the definition of stable flow). Let

$$h_j^v(t) := h_j(t) \cdot v_j \quad \text{for } j = 1, 2, 3, 4. \tag{2.8}$$

Then $h_j(t) = h_j^v(t)v_j$ and, with this notation, system (2.7) becomes

$$\begin{cases} \dot{h_j^v}(t) = -\kappa_\varphi(l_j(t)) = -\dfrac{1}{L_j(t)}[N_{\min|_{\Sigma_j(t)}}(q(t)) - N_{\min}(A_j(t))] \cdot \tau_j, \\ h_j^v(0) = 0. \end{cases} \tag{2.9}$$
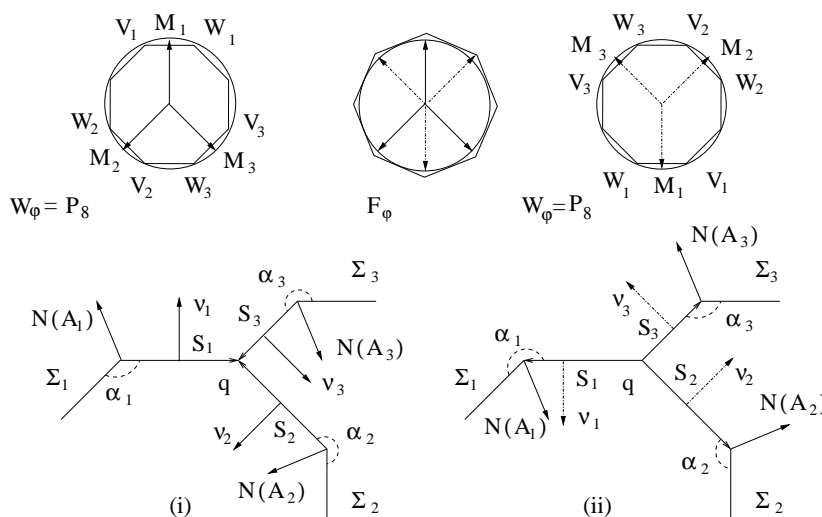

FIG. 3.   These triods have the same evolution according to system (2.9). Our convention is to take the orientation as in (i).

REMARK 2.14 We observe that $S_j(t)$ moves in the same direction of ν_j if and only if $\kappa_\varphi(l_j(t)) < 0$. Furthermore, system (2.9) is invariant under the change of the orientation of $\Pi(t)$ (see Figure 3).

2.4 *Geometry of elementary triods*

The angles of an elementary triod are given by the angles between the vectors ν_j's and are determined by the balance condition at q (see (2.2)), which, in turn, is related to the existence of admissible triplets.

DEFINITION 2.15 An *admissible triplet* is any triplet of vectors $(X, Y, Z) \in (\partial \mathcal{W}_\varphi)^3$ satisfying

$$X + Y + Z = 0. \tag{2.10}$$

LEMMA 2.16 Let $\psi : \mathbb{R}^2 \to [0, +\infty)$ be a Finsler norm on $\mathbb{R}^2$, i.e. an even one-homogeneous convex function for which there exists $c > 0$ such that $\psi(\xi) \geqslant c|\xi|$ for any $\xi \in \mathbb{R}^2$, and define $\mathcal{W}_\psi := \{\xi \in \mathbb{R}^2 : \psi(\xi) \leqslant 1\}$. Let $X \in \partial \mathcal{W}_\psi$. Then there exist two distinct vectors Y, Z in $\partial \mathcal{W}_\psi$ such that (X, Y, Z) is an admissible triplet. Moreover, if either $\mathcal{W}_\psi$ is strictly convex or for any segment $S \subset \partial \mathcal{W}_\psi$ parallel to $X \in \partial \mathcal{W}_\psi$ we have $|S| \leqslant |X|$, then the unordered pair $\{Y, Z\}$ is unique. Finally, if there exist $X_0 \in \partial \mathcal{W}_\psi$ and a segment $S \subset \partial \mathcal{W}_\psi$ parallel to X_0 with $|S| > |X_0|$, then there are infinitely many unordered pairs $\{Y, Z\}$ of distinct vectors in $\partial \mathcal{W}_\psi$ such that (X_0, Y, Z) is an admissible triplet.

Proof. Let $2h_M$ be the length of the orthogonal projection of $\mathcal{W}_\psi$ on $X^\perp \mathbb{R}$ and set $\hat{X} := X/|X|$. Define the multifunctions α_r and α_l as $\alpha_r(h) := (-h\hat{X}^\perp + X\mathbb{R}) \cap \partial \mathcal{W}_\psi$ and $\alpha_l(-h) := -\alpha_r(h)$ for any $h \in [0, h_M]$. It is easy to see that $\alpha_r(h)$ contains exactly two points for $h \neq h_M$ while $\alpha_r(h_M)$ can be either a point or a closed segment. Define the functions $\alpha_r^-, \alpha_l^- : [0, h_M] \to \mathbb{R}^2$ as $\alpha_l^-(-h) := \{Z \in \alpha_l(-h) : Z \cdot X \leqslant Y \cdot X, Y \in \alpha_l(-h)\}$ and $\alpha_r^-(h) := \{Z \in \alpha_r(h) : Z \cdot X \leqslant Y \cdot X, Y \in \alpha_r(h)\}$. Note that α_l^- and α_r^- are local parametrizations of $\partial \mathcal{W}_\psi$ which can be written with respect to the basis $(-\hat{X}^\perp, \hat{X})$ as $\alpha_l^-(-h) = (-h, \alpha_l^-(-h) \cdot \hat{X})$ and $\alpha_r^-(h) = (h, \alpha_r^-(h) \cdot \hat{X})$. Define now the function $\Phi : [0, h_M] \to \mathbb{R}$ as

$$\Phi(h) := \frac{1}{|X|}[\alpha_l^-(-h) + \alpha_r^+(h)] \cdot \hat{X}.$$

Then Φ is convex, since so are $h \mapsto \alpha_l^-(-h) \cdot \hat{X}$, $h \mapsto \alpha_r^-(h) \cdot \hat{X}$. Furthermore, $\Phi(0) = -2$, $\Phi(h_M) = 0$ if and only if $\alpha_r(h_M)$ is a singleton, while $\Phi(h_M) < 0$ if $\alpha_r(h_M)$ is a proper segment.

We divide the proof into two cases. First we observe that the existence of $h_* \in (0, h_M]$ with $\Phi(h_*) = -1$ implies that (2.10) is satisfied for $Y := \alpha_r(h_*)$ and $Z := \alpha_l(-h_*)$, and conversely, the existence of $X, Y \in \partial \mathcal{W}_\psi$ satisfying (2.10) implies that $\Phi(h_*) = -1$, where $h_* := \max\{Y \cdot \hat{X}, Z \cdot \hat{X}\}$.

Case 1. If either $\mathcal{W}_\psi$ is strictly convex (i.e. $\alpha_r(h_M)$ is a singleton) or $\alpha_r(h_M) \subset \partial \mathcal{W}_\psi$ is a segment parallel to X with length $|\alpha_r(h_M)| \leqslant |X|$, then $\Phi(h_M) \geqslant -1$ with equality holding if and only if $|S| = |X|$ (for instance if $\mathcal{W}_\psi = P_6$, see Figure 4). The convexity of Φ yields the existence of $h_* \in (0, h_M]$ with $\Phi(h_*) = -1$. Assume now that there exists $h^* \in (h_*, h_M]$ satisfying $\Phi(h^*) = \Phi(h_*) = -1$. Then, by the convexity of Φ, for every $\lambda \in (0, 1)$ we must have $\Phi((1-\lambda)h_* + \lambda h^*) = -1$, that is, $\partial \mathcal{W}_\psi$ should be flat along the direction $X^\perp$, but this contradicts the convexity of $\mathcal{W}_\psi$.
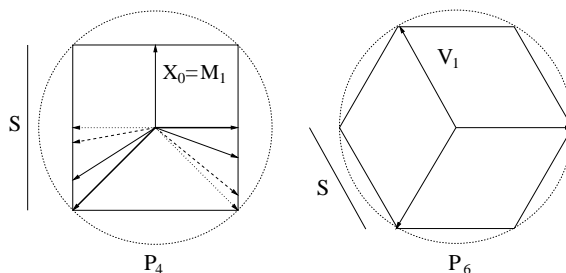
FIG. 4. P_4 admits infinitely many pairs $\{Y, Z\}$ satisfying $X_0 + Y + Z = 0$ with $X_0 = M_1$. P_6 has a unique pair all $X \in \partial P_6$.

Case 2. If $\alpha_r(h_M) \subset \partial \mathcal{W}_\psi$ is a segment parallel to X with length strictly greater than $|X|$, then $\Phi(h_M) < -1$ (for instance if $\mathcal{W}_\psi = P_4$, see Figure 4). Thus, we can find infinitely many pairs $\{Y, Z\}$ (as many as the points of a segment of length $|X|$) of distinct vectors in $\partial \mathcal{W}_\psi$ satisfying (2.10). □

REMARK 2.17 If $\mathcal{W}_\psi = P_4$ and $X_0 = M_1$ (see Figure 4), then $|S| = 2|X_0|$; hence there are infinitely many pairs $\{Y, Z\}$ of distinct vectors in ∂P_4 satisfying $X_0 + Y + Z = 0$. Moreover, any elementary triod has always two angles of $\pi/2$. If $\mathcal{W}_\psi = P_6$ and $X = V_1$ (see Figure 4), then $|S| = |V_1|$; hence for any $X \in \mathcal{W}_\psi$ there exists a unique unordered pair $\{Y, Z\}$ satisfying (2.10).

COROLLARY 2.18 Let $\mathfrak{n} \geqslant 6$. For any $X \in [V_1, W_1]$ there exist unique $Y = Y(X) \in [V_2, W_2]$ and $Z = Z(X) \in [V_3, W_3]$ such that (X, Y, Z) is an admissible triplet.

A direct computation yields the following result.

PROPOSITION 2.19 Let $\mathfrak{n} \in \mathbb{N}$, $\mathfrak{n} \geqslant 6$, $j = 2, 3$ and Π be elementary. Then

$$\vartheta(\nu_1, \nu_j) = \vartheta_{\mathfrak{n}} := \begin{cases} 2\pi/3, & \mathfrak{n} = 6m, \ m \geqslant 1, \\ (2\pi/3)(1 + 1/\mathfrak{n}), & \mathfrak{n} = 6m - 4, \ m \geqslant 2, \\ (2\pi/3)(1 - 1/\mathfrak{n}), & \mathfrak{n} = 6m - 2, \ m \geqslant 2. \end{cases} \tag{2.11}$$

Moreover, the cardinality of $\mathfrak{C}$ in Definition 2.9 is 4 if $\mathfrak{n} = 6m$, and 8 if $\mathfrak{n} \in \{6m - 4, 6m - 2\}$.

The angles of Π are strictly greater than $\pi/2$ and strictly less than π when $\mathfrak{n} \geqslant 6$ and $\mathfrak{n} \neq 8$. If $\mathfrak{n} = 8$ then $\vartheta(\nu_2, \nu_3) = \pi/2$. From Proposition 2.19, when $\mathfrak{n} \in \{6m - 4, 6m - 2\}$, there are eight different configurations which will be denoted by (a), (b), (c), (d), (a'), (b'), (c'), (d') (see Figure 5); when $\mathfrak{n} = 6m$, the four different configurations correspond to (a), (d), (a'), (d').

From Proposition 2.19 we deduce the following formulas which are used throughout the paper:

$$\tau_1 \cdot \tau_j = \nu_1 \cdot \nu_j = \cos \vartheta_{\mathfrak{n}}, \quad j = 2, 3, \tag{2.12}$$

$$\nu_1 \cdot \tau_2 = \tau_1 \cdot \nu_3 = \cos(\vartheta_{\mathfrak{n}} - \pi/2) = \sin \vartheta_{\mathfrak{n}}, \tag{2.13}$$

$$\tau_1 \cdot \nu_2 = \nu_1 \cdot \tau_3 = \cos(\vartheta_{\mathfrak{n}} + \pi/2) = -\sin \vartheta_{\mathfrak{n}}. \tag{2.14}$$

REMARK 2.20 (quasi-elementary and nonpolygonal triods) The angles of a quasi-elementary triod $\widetilde{\Pi}$ are still determined by the balance condition at q (see (2.2)) and are exactly equal to $\vartheta_{\mathfrak{n}}, \vartheta_{\mathfrak{n}}$ and $2\pi - 2\vartheta_{\mathfrak{n}}$, as in the case of an elementary triod. The notion of local configuration of $\widetilde{\Pi}$ at q can be introduced by considering the equivalence relation introduced in Definition 2.9 on

$$(S_4 \cup S_k) \cup \Sigma_{j_1} \cup \Sigma_{j_2}, \quad j_1, j_2 \in \{1, 2, 3\} \setminus \{k\}, \ j_1 \neq j_2,$$
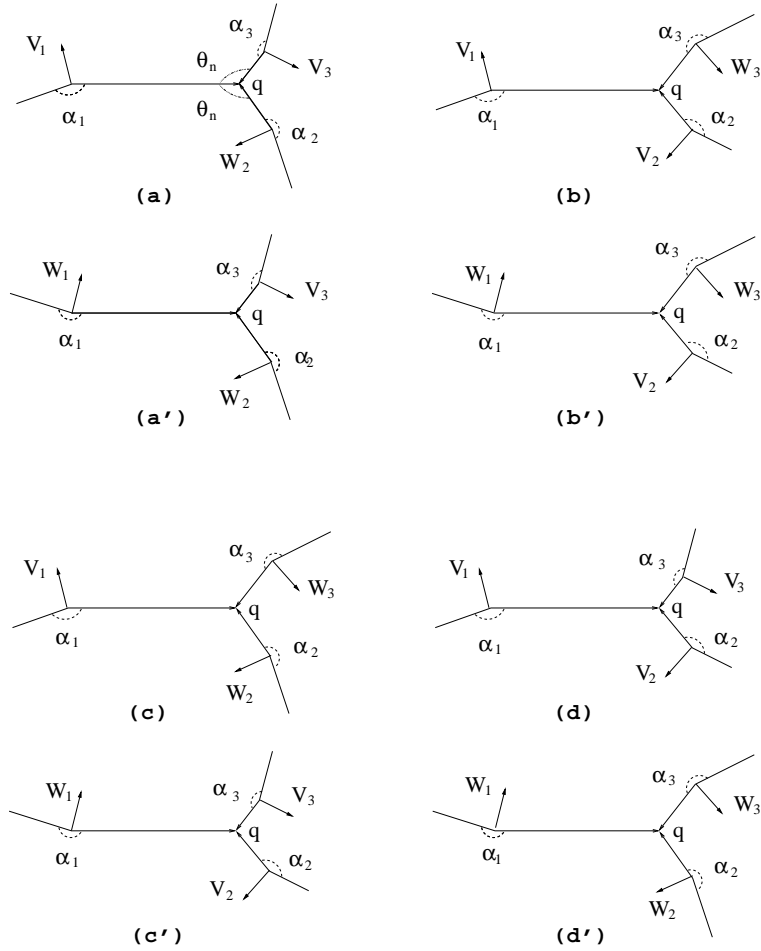
FIG. 5. Eight different configurations (up to rotations through $2\pi/\mathfrak{n}$) when $\mathfrak{n} \in \{6\mathfrak{m} - 4, 6\mathfrak{m} - 2\}$.

with k as in Definition 2.8 of quasi-elementary triod. The different local configurations of $\widetilde{\Pi}$ in q will be denoted by (a*), (b*), (c*), (d*), (a′*), (b′*), (c′*), (d′*). For nonpolygonal triods, only the angle between the interfaces Σ_{j_1} and Σ_{j_2}, $j_1, j_2 \in \{1, 2, 3\} \setminus \{k\}$, $j_1 \neq j_2$, with k as in Definition 2.8 of nonpolygonal triod, is known and equal to $\vartheta_{\mathfrak{n}}$.

We set $\delta := |V_1 - X(V_3)|$ if $\mathfrak{n} = 6\mathfrak{m} - 4$, $\delta := |W_1 - X(W_3)|$ if $\mathfrak{n} = 6\mathfrak{m} - 2$ (see Figure 6),

$$[a, b] := \begin{cases} [0, L], & \mathfrak{n} = 6\mathfrak{m}, \\ [\delta, L - \delta], & \mathfrak{n} = 6\mathfrak{m} - 4, 6\mathfrak{m} - 2, \end{cases} \qquad m := \begin{cases} 1, & \mathfrak{n} = 6\mathfrak{m}, \\ \delta/(L - 2\delta), & \mathfrak{n} = 6\mathfrak{m} - 4, 6\mathfrak{m} - 2, \end{cases}$$
(2.15)

and

$$q_y := \begin{cases} L, & \mathfrak{n} = 6\mathfrak{m}, \\ m(L - \delta), & \mathfrak{n} = 6\mathfrak{m} - 4, \\ L + m\delta, & \mathfrak{n} = 6\mathfrak{m} - 2, \end{cases} \qquad q_z := \begin{cases} 0, & \mathfrak{n} = 6\mathfrak{m}, \\ -m\delta, & \mathfrak{n} = 6\mathfrak{m} - 4, \\ L - m(L - \delta), & \mathfrak{n} = 6\mathfrak{m} - 2. \end{cases} \qquad (2.16)$$

Given an admissible triplet $(X, Y, Z) \in [V_1, W_1] \times [V_2, W_2] \times [V_3, W_3]$, we set

$$x := |V_1 - X|, \quad y := |W_2 - Y|, \quad z := |V_3 - Z|. \qquad (2.17)$$

The proof of the next result is omitted and follows by a direct computation.

PROPOSITION 2.21 If $\mathfrak{n} = 6\mathfrak{m} - 4$ then $\delta = |W_1 - X(W_2)| = |W_2 - Y(V_3)| = |V_3 - Z(W_2)|$. If $\mathfrak{n} = 6\mathfrak{m} - 2$ then $\delta = |V_1 - X(V_2)| = |V_2 - Y(W_3)| = |W_3 - Z(V_2)|$. Furthermore,

$$2\delta = (1 - \cos\vartheta_{\mathfrak{n}})^{-1}L, \quad m = -(2\cos\vartheta_{\mathfrak{n}})^{-1}, \quad \mathfrak{n} \in \{6\mathfrak{m} - 4, 6\mathfrak{m} - 2\}. \qquad (2.18)$$

Finally,

$$y = y(x) := -mx + q_y, \quad z = z(x) := mx + q_z, \quad x \in [a, b], \mathfrak{n} \geqslant 6. \qquad (2.19)$$

2.5 *Crystalline curvature of elementary triods*

In this section we compute the φ-curvatures $\kappa_\varphi(l_1)$, $\kappa_\varphi(l_2)$ and $\kappa_\varphi(l_3)$ (see Definition 2.4) of an elementary triod Π. Each configuration gives rise to a different vector field $N_{\min} : \Pi \to \mathbb{R}^2$. Since in two dimensions the value of $N_{\min}$ is fixed (up to sign) at each vertex A_j, the value of $N_{\min|\Sigma_j}$ at q uniquely determines $N_{\min}$ on Σ_j simply by linear interpolation. Hence, we can restrict the minimum problem (2.3) to the class of vector fields $N \in \mathcal{N}$ which are given by linear interpolation on each Σ_j. From Proposition 2.21, the admissible triplet $(N_{|\Sigma_1}(q), N_{|\Sigma_2}(q), N_{|\Sigma_3}(q))$ is uniquely associated with $(x, y(x), z(x))$ satisfying (2.19). Hence, we can rewrite the functional in (2.3) as a function of x. The problem of finding $N_{\min}$ in (2.3) reduces to the problem

$$\min_{x \in [a,b]} f(x), \quad f(x) := \int_\Pi (\operatorname{div}_\tau N)^2 \varphi^o(\nu) \, d\mathcal{H}^1 = \alpha x^2 + \beta x + \gamma, \qquad (2.20)$$

where α, β, γ are coefficients depending on the configuration of Π.

Let $x_{\min}$ be the minimizer of (2.20), $y_{\min} := y(x_{\min})$ and $z_{\min} := z(x_{\min})$. The stability of an elementary triod is equivalent to the condition

$$x_{\min} \in (a, b).$$

PROPOSITION 2.22 If $\Pi \in$ (d) then $x_{\min} = a$, where a is defined as in (2.15). If $\Pi \in \{(\text{a}), (\text{b}), (\text{c})\}$ is stable then

$$x_{\min}^{(\text{a})}(L_1, L_2, L_3) = m\left(\frac{q_y}{L_2} - \frac{q_z}{L_3}\right)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right]^{-1}, \qquad (2.21)$$

$$x_{\min}^{(\text{b})}(L_1, L_2, L_3) = m\left(-\frac{L - q_y}{L_2} + \frac{L - q_z}{L_3}\right)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right]^{-1}, \qquad (2.22)$$

$$x_{\min}^{(\text{c})}(L_1, L_2, L_3) = m\left(\frac{q_y}{L_2} + \frac{L - q_z}{L_3}\right)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right]^{-1}, \qquad (2.23)$$

where m, q_y, q_z are given by (2.15) and (2.16).

Proof. Let $N \in \mathcal{N}$ be given by linear interpolation on each Σ_j, let $(X, Y, Z) := (N_{|\Sigma_1}(q), N_{|\Sigma_2}(q), N_{|\Sigma_3}(q))$, and let x, y, z be as in (2.17). We observe that $\operatorname{div}_\tau N_{|\Sigma_1}$, $\operatorname{div}_\tau N_{|\Sigma_2}$, $\operatorname{div}_\tau N_{|\Sigma_3}$ are constant and given as in Table 1 after replacing $x_{\min}, y_{\min}, z_{\min}$ with x, y, z, and

$$f(x) = (\operatorname{div}_\tau N_{|\Sigma_1})^2 L_1 \varphi^o(\nu_1) + (\operatorname{div}_\tau N_{|\Sigma_2})^2 L_2 \varphi^o(\nu_2) + (\operatorname{div}_\tau N_{|\Sigma_3})^2 L_3 \varphi^o(\nu_3).$$

TABLE 1
Angles α_j and φ-curvatures $\kappa_\varphi(l_j)$ of an elementary triod

	α_1/π	α_2/π	α_3/π	$\kappa_\varphi(l_1)$	$\kappa_\varphi(l_2)$	$\kappa_\varphi(l_3)$
(a)	$1 - 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$x_{\min}/L_1$	$-y_{\min}/L_2$	$z_{\min}/L_3$
(b)	$1 - 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$x_{\min}/L_1$	$(L - y_{\min})/L_2$	$-(L - z_{\min})/L_3$
(c)	$1 - 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$x_{\min}/L_1$	$-y_{\min}/L_2$	$-(L - z_{\min})/L_3$
(d)	$1 - 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$a/L_1, a = x_{\min}$	$(L - y_{\min})/L_2$	$z_{\min}/L_3$
(a$'$)	$1 + 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$-(L - x_{\min})/L_1$	$-y_{\min}/L_2$	$z_{\min}/L_3$
(b$'$)	$1 + 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$-(L - x_{\min})/L_1$	$(L - y_{\min})/L_2$	$-(L - z_{\min})/L_3$
(c$'$)	$1 + 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$1 - 2/\mathfrak{n}$	$-(L - x_{\min})/L_1$	$(L - y_{\min})/L_2$	$z_{\min}/L_3$
(d$'$)	$1 + 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$1 + 2/\mathfrak{n}$	$-(L - b)/L_1, b = x_{\min}$	$-y_{\min}/L_2$	$-(L - z_{\min})/L_3$

Furthermore, $\varphi^o(v_1) = \varphi^o(v_2) = \varphi^o(v_3)$. In the case of configuration (d), since $f(x)$ is an increasing function of $x \in [a, b]$, it follows that the minimizer is given by $x_{\min} = a$ and the first assertion follows. In the other cases, since

$$\alpha = \varphi^o(v_1)\left[\frac{1}{L_1} + m^2\left(\frac{1}{L_2} + \frac{1}{L_3}\right)\right] > 0,$$

it follows that $x_{\min} = -\beta/(2\alpha) \in (a, b)$. Formulas (2.21)–(2.23) follow since

$$\beta = \begin{cases} -2m\varphi^o(v_1)\left(\dfrac{q_y}{L_2} - \dfrac{q_z}{L_3}\right) & \text{in configuration (a),} \\[2mm] -2m\varphi^o(v_1)\left(-\dfrac{L - q_y}{L_2} + \dfrac{L - q_z}{L_3}\right) & \text{in configuration (b),} \\[2mm] -2m\varphi^o(v_1)\left(\dfrac{q_y}{L_2} + \dfrac{L - q_z}{L_3}\right) & \text{in configuration (c).} \end{cases} \qquad \square$$

REMARK 2.23 Since (a$'$), (b$'$), (c$'$) and (d$'$) are respectively symmetric to (a), (b), (c) and (d) with respect to the l_1-axis, we can derive the expression of $x_{\min}$ for configurations (a$'$), (b$'$), (c$'$) and (d$'$) from those of (a), (b), (c) and (d) using the mirror law:

$$x_{\min}^{(e)}(L_1, L_2, L_3) \in [a, b] \mapsto x_{\min}^{(e')}(L_1, L_2, L_3) \equiv L - x_{\min}^{(e)}(L_1, L_3, L_2) \in [a, b]. \qquad (2.24)$$

Since $(x_{\min}, y_{\min}, z_{\min})$ identifies $N_{\min}$ at q, $\kappa_\varphi(l_j)$ is explicitly determined for each configuration, as shown in Table 1.

REMARK 2.24 When $\mathfrak{n} = 6m$, Π is unstable if and only if

$$(N_{\min|\Sigma_1}(q), N_{\min|\Sigma_2}(q), N_{\min|\Sigma_3}(q)) \in \{(V_1, V_2, V_3), (W_1, W_2, W_3)\}, \quad \text{i.e.} \quad x_{\min} \in \{0, L\}$$

(see Figure 6). When $\mathfrak{n} = 6m-4$ (resp. $\mathfrak{n} = 6m-2$), Π is unstable if and only if either $N_{\min|\Sigma_3}(q) = V_3$ or $N_{\min|\Sigma_2}(q) = W_2$ (resp. either $N_{\min|\Sigma_2}(q) = V_2$ or $N_{\min|\Sigma_3}(q) = W_3$), i.e. either $x_{\min} = \delta$ or $x_{\min} = L - \delta$ (see Figure 6). In particular, configurations (d) and (d$'$) are always unstable.
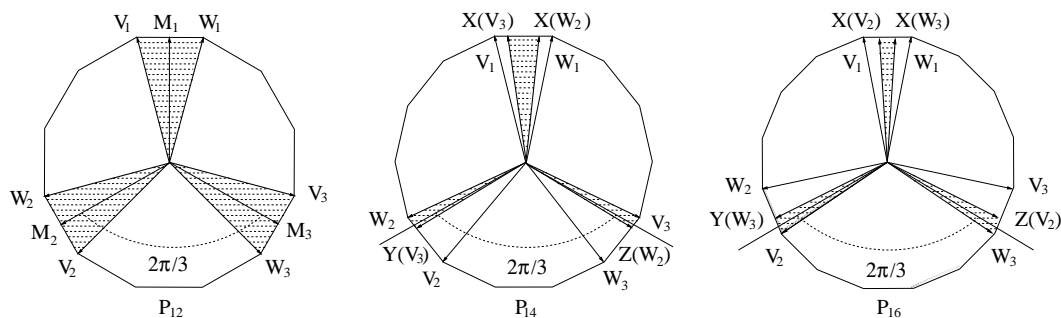
FIG. 6. Wulff shapes $\mathcal{W}_\varphi = P_{12}$, $\mathcal{W}_\varphi = P_{14}$ ($\mathfrak{n} = 6\mathfrak{m} - 4$) and $\mathcal{W}_\varphi = P_{16}$ ($\mathfrak{n} = 6\mathfrak{m} - 2$) and relative ranges of all admissible triplets (filled regions).

REMARK 2.25 If $\widetilde{\Pi}$ is a quasi-elementary triod, the solution $N_{\min}$ of the minimum problem (2.3) with Γ replaced by $\widetilde{\Pi}$ is still determined as the solution of the minimum problem (2.20) with Π replaced by $\widetilde{\Pi}$. Hence, formulas in Table 1 and in Proposition 2.22 still hold with (a), (b), (c), (d) replaced by (a*), (b*), (c*), (d*) and with l_1, l_2, l_3 suitably replaced by l_{j_1}, l_{j_2}, l_4, $j_1, j_2 \in \{1, 2, 3\} \setminus \{k\}$, k as in Definition 2.8 of quasi-elementary triod.

3. Short time existence and uniqueness of a crystalline flow

Before proving the short time existence result (Theorem 3.3) we need to understand the relations between the distances $h_j^\nu(t)$ (see (2.8) and Definition 2.12) and the lengths $L_j(t)$ (see Definition 2.8), in order to write the left hand side of system (2.9) as a function of $\dot{L}_k(t)$.

PROPOSITION 3.1 Let Π be an elementary triod and $T > 0$. Assume that $t \in [0, T] \mapsto \Pi(t) \in [\Pi]$ is a flow starting from Π which satisfies (ii) of Definition 2.12 and that the distance vector $h_j(t)$ of the segment $S_j(t)$ from $S_j(0) = S_j$ satisfies $h_j \in C^0((0, T); \nu_j \mathbb{R})$. Then, defining $h_j^\nu(t)$ as in (2.8), we have

$$
\begin{cases}
L_1(t) = L_1 + \cot\alpha_1\, h_1^\nu(t) + \dfrac{h_k^\nu(t) - \nu_1 \cdot \nu_k h_1^\nu(t)}{\tau_1 \cdot \nu_k}, & k = 2, 3, \\[2ex]
L_2(t) = L_2 + \cot\alpha_2\, h_2^\nu(t) + \dfrac{h_1^\nu(t) - \nu_1 \cdot \nu_2 h_2^\nu(t)}{\nu_1 \cdot \tau_2}, \\[2ex]
L_3(t) = L_3 + \cot\alpha_3\, h_3^\nu(t) + \dfrac{h_1^\nu(t) - \nu_1 \cdot \nu_3 h_3^\nu(t)}{\nu_1 \cdot \tau_3},
\end{cases}
\tag{3.1}
$$

$$
h_2^\nu(t) + h_3^\nu(t) = 2\tau_1 \cdot \tau_3 h_1^\nu(t),
\tag{3.2}
$$

and

$$
\mathrm{rank}
\begin{pmatrix}
\cot\alpha_1 - \dfrac{\nu_1 \cdot \nu_2}{\tau_1 \cdot \nu_2} & \dfrac{1}{\tau_1 \cdot \nu_2} & L_1(t) - L_1 \\[2ex]
\dfrac{1}{\nu_1 \cdot \tau_2} & \cot\alpha_2 - \dfrac{\nu_1 \cdot \nu_2}{\nu_1 \cdot \tau_2} & L_2(t) - L_2 \\[2ex]
2\tau_1 \cdot \tau_3 \cot\alpha_3 + \dfrac{1 - 2(\nu_1 \cdot \nu_3)^2}{\nu_1 \cdot \tau_3} & -\cot\alpha_3 + \dfrac{\nu_1 \cdot \nu_3}{\nu_1 \cdot \tau_3} & L_3(t) - L_3
\end{pmatrix}
= 2.
\tag{3.3}
$$

Conversely, for any $j = 1, 2, 3$, let $L_j : [0, T) \to (0, +\infty)$ and $h_j^\nu : [0, T) \to \mathbb{R}$ be continuous functions satisfying $L_j(0) = L_j$, $h_j^\nu(0) = 0$ and (3.1)–(3.3). If $\Pi(t) \in [\Pi]$ is the elementary triod having $|S_j(t)| = L_j(t)$ and $h_j(t) := h_j^\nu(t)\nu_j$, then $t \in [0, T) \mapsto \Pi(t)$ is a flow starting from Π which satisfies (ii) of Definition 2.12 and $h_j(t)$ is the distance vector of $S_j(t)$ from $S_j(0) = S_j$.

Proof. From

$$qq(t) = qq(t) \cdot \nu_j \nu_j + qq(t) \cdot \tau_j \tau_j = h_j^\nu(t)\nu_j + [L_j(t) - L_j - \cot\alpha_j \, h_j^\nu(t)]\tau_j, \tag{3.4}$$

we get, for $j = 2, 3$,

$$h_1^\nu(t) = \nu_1 \cdot \nu_j h_j^\nu(t) + \nu_1 \cdot \tau_j [L_j(t) - L_j - \cot\alpha_j \, h_j^\nu(t)], \tag{3.5}$$

$$L_1(t) = L_1 + \cot\alpha_1 \, h_1^\nu(t) + \tau_1 \cdot \nu_j h_j^\nu(t) + \tau_1 \cdot \tau_j [L_j(t) - L_j - \cot\alpha_j \, h_j^\nu(t)], \tag{3.6}$$

Thus, by (3.5),

$$L_j(t) - L_j - \cot\alpha_j \, h_j^\nu(t) = \frac{h_1^\nu(t) - \nu_1 \cdot \nu_j h_j^\nu(t)}{\nu_1 \cdot \tau_j}, \quad j = 2, 3, \tag{3.7}$$

and the second and third equalities in (3.1) are proved.

Inserting (3.7) in (3.6), subtracting the resulting equations, and using (2.12) and (2.5) yields

$$\tau_1 \cdot \nu_3 [h_2^\nu(t) + h_3^\nu(t)] + 2\frac{\tau_1 \cdot \tau_3}{\nu_1 \cdot \tau_3} h_1^\nu(t) - \frac{(\nu_1 \cdot \nu_3)^2}{\nu_1 \cdot \tau_3}[h_2^\nu(t) + h_3^\nu(t)] = 0.$$

Hence, by (2.5),

$$\frac{(\tau_1 \cdot \nu_3)^2 + (\nu_1 \cdot \nu_3)^2}{\tau_1 \cdot \nu_3}[h_2^\nu(t) + h_3^\nu(t)] - 2\frac{\tau_1 \cdot \tau_3}{\tau_1 \cdot \nu_3} h_1^\nu(t) = 0,$$

which proves (3.2).

Similarly, inserting (3.7) in (3.6), adding the resulting equations, and using (2.12) and (2.5) yields

$$2L_1(t) = 2L_1 + 2\cot\alpha_1 \, h_1^\nu(t) + \tau_1 \cdot \nu_3 [h_3^\nu(t) - h_2^\nu(t)] - \frac{(\nu_1 \cdot \nu_3)^2}{\nu_1 \cdot \tau_3}[h_3^\nu(t) - h_2^\nu(t)]$$

$$= 2L_1 + 2\cot\alpha_1 \, h_1^\nu(t) + \frac{1}{\tau_1 \cdot \nu_3}[h_3^\nu(t) - h_2^\nu(t)]. \tag{3.8}$$

Substituting (3.2) into (3.8) gives the first equality in (3.1). Now, systems (3.1) and (3.2) imply

$$\begin{pmatrix} L_1(t) - L_1 \\ L_2(t) - L_2 \\ L_3(t) - L_3 \end{pmatrix} = \begin{pmatrix} \cot\alpha_1 - \dfrac{\nu_1 \cdot \nu_2}{\tau_1 \cdot \nu_2} & \dfrac{1}{\tau_1 \cdot \nu_2} & 0 \\[2ex] \dfrac{1}{\nu_1 \cdot \tau_2} & \cot\alpha_2 - \dfrac{\nu_1 \cdot \nu_2}{\nu_1 \cdot \tau_2} & 0 \\[2ex] 2\tau_1 \cdot \tau_3 \cot\alpha_3 + \dfrac{1 - 2(\nu_1 \cdot \nu_3)^2}{\nu_1 \cdot \tau_3} & -\cot\alpha_3 + \dfrac{\nu_1 \cdot \nu_3}{\nu_1 \cdot \tau_3} & 0 \end{pmatrix} \begin{pmatrix} h_1^\nu(t) \\ h_2^\nu(t) \\ h_3^\nu(t) \end{pmatrix}. \tag{3.9}$$

If we show that system (3.9) always has rank 2, then, h_j^ν being solutions of (3.1)–(3.2), the condition (3.3) follows. Let $A_{1,2}$ be the 2×2 matrix given by the first two rows and the first two columns of the

matrix in (3.9). If $\cot \alpha_1 = \cot \alpha_2$, i.e. $\Pi \in \{(b), (d), (a'), (d')\}$, then $\det A_{1,2} = \cot^2 \alpha_1 + 1 \neq 0$. If $\cot \alpha_1 = -\cot \alpha_2$, i.e. $\Pi \in \{(a), (c), (b'), (c')\}$, then

$$\det A_{1,2} = \frac{1}{\sin^2 \vartheta_{\mathfrak{n}}} \left(1 - \frac{\sin^2(\alpha_1 + \vartheta_{\mathfrak{n}})}{\sin^2 \alpha_1} \right).$$

Using Table 1, one checks that $\det A_{1,2} \neq 0$ if either $\Pi \in \{(a), (c)\}$ and $\mathfrak{n} \neq 6$, or $\Pi \in \{(b'), (c')\}$ and $\mathfrak{n} \notin \{10, 12, 14\}$. In the remaining cases one can check similarly that the minors given by the first and third rows (or the second and third rows) of the matrix in (3.9) are not zero. Then (3.3) follows.

The converse follows by construction. $\square$

REMARK 3.2   System (3.1) is not symmetric under permutations of the indices $1, 2, 3$, unless $\mathfrak{n} = 6\mathfrak{m}$. This is due to the fact that for $\mathfrak{n} \neq 6\mathfrak{m}$ only two of the angles at the triple junction are equal. Finally, notice that a flow starting from Π which satisfies (ii) of Definition 2.12 with $\Pi(t)$ elementary in $[0, T)$ has two degrees of freedom.

THEOREM 3.3   Let Π be elementary and stable. Then there exist $T > 0$ and a unique stable φ-curvature flow $t \in [0, T) \mapsto \Pi(t)$ starting from Π with $[\Pi(t)] = [\Pi]$ for any $t \in [0, T)$. Moreover, $h_j \in C^\infty([0, T))$ for all $j = 1, 2, 3$.

Proof.   We assume $\Pi \in (a)$, the proof in the other configurations being similar. Let $x_{\min}^{(a)}$ be defined as in (2.21). For any $w := (w_1, w_2, w_3) \in (0, +\infty)^3$, we set $G(w) := x_{\min}^{(a)}(w)$ and define the vector field $F = (F_1, F_2, F_3) : (0, +\infty)^3 \to \mathbb{R}^3$ as

$$F_1(w) := -\left(\cot \alpha_1 - \frac{\nu_1 \cdot \nu_3}{\tau_1 \cdot \nu_3} \right) \frac{G(w)}{w_1} - \frac{1}{\tau_1 \cdot \nu_3} \frac{mG(w) + q_z}{w_3},$$

$$F_2(w) := -\frac{1}{\nu_1 \cdot \tau_2} \frac{G(w)}{w_1} + \left(\cot \alpha_2 - \frac{\nu_1 \cdot \nu_2}{\nu_1 \cdot \tau_2} \right) \frac{-mG(w) + q_y}{w_2}$$

$$F_3(w) := -\frac{1}{\nu_1 \cdot \tau_3} \frac{G(w)}{w_1} - \left(\cot \alpha_3 - \frac{\nu_1 \cdot \nu_3}{\nu_1 \cdot \tau_3} \right) \frac{mG(w) + q_z}{w_3}.$$

Notice that F is obtained by differentiating with respect to t the right hand side of system (3.1) and by replacing the $\dot{h}_j^\nu$'s (in (2.9)) with the expressions in G and w_j (where we use Table 1, (2.21) and (2.19)). Consider the Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)), \\ w(0) = (L_1, L_2, L_3) \in (0, +\infty)^3. \end{cases} \tag{3.10}$$

Since F is C^∞ in $(0, +\infty)^3$, there exists a unique solution $w \in C^\infty([0, T); (0, +\infty)^3)$ of (3.10) for some $T > 0$. Denote by $\Pi(t)$ the elementary triod belonging to configuration (a) and having $|S_j(t)| := w_j(t)$ for any $j = 1, 2, 3$. Define $x(t) := x_{\min}^{(a)}(w(t))$, $y(t) := -mx(t) + q_y$, $z(t) := mx(t) + q_z$. By construction, for any $t \in [0, T)$, $x(t)$ is the solution of the minimum problem (2.20) with Π replaced by $\Pi(t)$, and $N_{\min}(t)$, the solution of (2.3) with Γ replaced by $\Pi(t)$, is determined by $(x(t), y(t), z(t))$. Thus

$$\kappa_\varphi(l_1(t)) = \frac{x(t)}{w_1(t)}, \qquad \kappa_\varphi(l_2(t)) = -\frac{y(t)}{w_2(t)}, \qquad \kappa_\varphi(l_3(t)) = \frac{z(t)}{w_3(t)},$$

where $l_j(t) := w_j(t)\tau_j$. Since $w_j \in C^\infty([0, T))$, possibly reducing $T > 0$, we have $x(t) \in (a, b)$ for any $t \in (0, T)$. Therefore, $\Pi(t)$ is elementary and stable for any $t \in [0, T)$, and $\kappa_\varphi(l_j(\cdot)) \in C^\infty([0, T))$ for any $j = 1, 2, 3$. Defining, for any $j = 1, 2, 3$,

$$h_j(t) := h_j^\nu(t)\nu_j, \quad h_j^\nu(t) := -\int_0^t \kappa_\varphi(l_j(s)) \, ds, \quad t \in [0, T), \tag{3.11}$$

we get a flow satisfying (2.7). To prove (ii) of Definition 2.12, in view of (the converse part of) Proposition 3.1 it is sufficient to show that (3.1)–(3.3) are satisfied. Denote by f_t the function in (2.20) where Π is replaced by $\Pi(t)$. Equality (3.2) follows from

$$0 = \frac{df_t(x)}{dx} = 2\varphi^o(\nu_1)\left[\frac{x}{w_1(t)} + \frac{y(x)}{w_2(t)}\frac{dy}{dx} + \frac{z(x)}{w_3(t)}\frac{dz}{dx}\right] = 2\varphi^o(\nu_1)[-\dot{h}_1^\nu(t) - m(\dot{h}_2^\nu(t) + \dot{h}_3^\nu(t))]$$

and from $-m = (2\tau_1 \cdot \tau_3)^{-1}$ (see (2.15) and (2.18)). Integrating (3.10) yields (3.1). Finally, (3.3) follows from (3.2), since

$$w_2(t) - L_2 = \left(2\tau_1 \cdot \tau_2 \cot\alpha_2 + \frac{1 - 2(\nu_1 \cdot \nu_2)^2}{\nu_1 \cdot \tau_2}\right)h_1^\nu(t) + \left(-\cot\alpha_2 + \frac{\nu_1 \cdot \nu_2}{\nu_1 \cdot \tau_2}\right)h_3^\nu(t).$$

Uniqueness of the flow follows by uniqueness of w_j and h_j^ν. $\qquad\square$

COROLLARY 3.4 If Π is degenerate and stable, then there exist $T > 0$ and a unique stable φ-curvature flow $t \in [0, T) \mapsto \Pi(t)$ starting from Π with $\Pi(t) \in [\Pi]$ degenerate for any $t \in [0, T)$. Moreover, $h_{j_1}^\nu, h_{j_2}^\nu \in C^\infty([0, T))$, $j_1, j_2 \in \{1, 2, 3\} \setminus \{\bar{k}\}$, $\bar{k}$ as in Definition 2.8 of degenerate triod.

Proof. As in the proof of Theorem 3.3 we obtain the (two) nondegenerate lengths w_{j_1}, w_{j_2} as solutions of a system of two ordinary differential equations, and the assertion follows by the same arguments. $\qquad\square$

DEFINITION 3.5 Let Π be elementary and stable. We define $\mathcal{T} = \mathcal{T}(\Pi)$ as the supremum of all $T > 0$ for which there exists a unique stable φ-curvature flow $t \in [0, T) \mapsto \Pi(t)$ starting from Π.

COROLLARY 3.6 Let Π be elementary and stable, and $t \in [0, \mathcal{T}) \mapsto \Pi(t) \in [\Pi]$ be the stable φ-curvature flow starting from Π. Then

$$-2\tau_1 \cdot \tau_3 \kappa_\varphi(l_1(t)) + \kappa_\varphi(l_2(t)) + \kappa_\varphi(l_3(t)) = 0, \quad \forall t \in [0, \mathcal{T}). \tag{3.12}$$

Proof. This follows by differentiating (3.2) with respect to t and using (2.9). $\qquad\square$

Condition (3.12) is related to the geometry of the triod near the triple junction and is equivalent to stability of the triod. In particular, if $\mathfrak{n} = 6\mathfrak{m}$, we have $\tau_1 \cdot \tau_3 = -1/2$, hence the sum of the φ-curvatures at the triple junction is zero (as in the euclidean case).

If we use (3.2), (2.13) and (2.5), for any $t \in [0, \mathcal{T})$ and all $\mathfrak{n} \geq 6$, system (3.1) can be written as follows:

$$L_1(t) - L_1 = [\cot\alpha_1 + \cot\vartheta_\mathfrak{n}]h_1^\nu(t) - \frac{1}{\sin\vartheta_\mathfrak{n}}h_2^\nu(t) \tag{3.13}$$

$$= [\cot\alpha_1 - \cot\vartheta_\mathfrak{n}]h_1^\nu(t) + \frac{1}{\sin\vartheta_\mathfrak{n}}h_3^\nu(t), \tag{3.14}$$

$$L_2(t) - L_2 = \frac{1}{\sin\vartheta_{\mathfrak{n}}} h_1^{\nu}(t) + [\cot\alpha_2 - \cot\vartheta_{\mathfrak{n}}] h_2^{\nu}(t) \tag{3.15}$$

$$= 2\cos\vartheta_{\mathfrak{n}}[\cot\alpha_2 - \cot(2\vartheta_{\mathfrak{n}})] h_1^{\nu}(t) - [\cot\alpha_2 - \cot\vartheta_{\mathfrak{n}}] h_3^{\nu}(t) \tag{3.16}$$

$$= [\cot\alpha_2 - \cot(2\vartheta_{\mathfrak{n}})] h_2^{\nu}(t) + \frac{1}{\sin(2\vartheta_{\mathfrak{n}})} h_3^{\nu}(t), \tag{3.17}$$

$$L_3(t) - L_3 = \frac{1}{\sin\vartheta_{\mathfrak{n}}} h_1^{\nu}(t) + [\cot\alpha_3 + \cot\vartheta_{\mathfrak{n}}] h_3^{\nu}(t) \tag{3.18}$$

$$= \frac{1}{\sin(2\vartheta_{\mathfrak{n}})} h_2^{\nu}(t) + [\cot\alpha_3 + \cot(2\vartheta_{\mathfrak{n}})] h_3^{\nu}(t). \tag{3.19}$$

The following proposition describes some useful qualitative properties of the flow.

PROPOSITION 3.7 Let Π be elementary and stable, and $t \in [0, \mathcal{T}) \mapsto \Pi(t) \in [\Pi]$ be the stable φ-curvature flow starting from Π. Denote by j_1 and j_2 the two indices for which the two angles of the triod at A_{j_1} and A_{j_2}, from the side of their common phase, are both larger than π (for instance, $j_1 = 2$, $j_2 = 3$ in Figure 7(i), $j_1 = 1$, $j_2 = 3$ in Figure 7(ii), $j_1 = 1$, $j_2 = 3$ in Figure 7(iii)). Then

  (i) L_{j_1} and L_{j_2} are nondecreasing in $[0, \mathcal{T})$;
 (ii) $\sup_j \sup_{t \in [0, \mathcal{T})} |\kappa_{\varphi}(l_j(t))| < +\infty$;
(iii) $\sup_j \sup_{t \in [0, \mathcal{T})} |\dot{L}_j(t)| < +\infty$;
 (iv) $L_{j_3} \in \mathrm{Lip}((0, \mathcal{T}))$, $j_3 \neq j_1, j_2$;
  (v) if $\mathcal{T} < +\infty$ then $L_j(\mathcal{T}-) < +\infty$ for any $j = 1, 2, 3$.
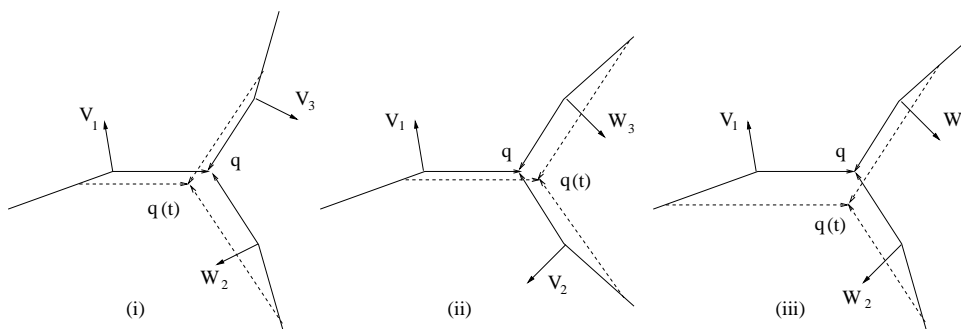

FIG. 7. Stable φ-curvature flow from a stable elementary triod Π (see Theorem 3.3): (i) $\Pi \in$ (a), (ii) $\Pi \in$ (b), (iii) $\Pi \in$ (c).

Proof. We show assertion (i) for configurations (a), (b) and (c), the cases (a′), (b′), (c′) being similar (interchange $L_2(t)$ with $L_3(t)$). Differentiating (3.14), (3.17) and (3.18) with respect to t, using $\cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}$, (2.11) and Table 1, we see that $j_1 = 2$, $j_2 = 3$, $\dot{L}_{j_1}(t) > 0$, $\dot{L}_{j_2}(t) > 0$ in configurations (a), and $j_1 = 1$, $j_2 = 3$, $\dot{L}_{j_1}(t) > 0$, $\dot{L}_{j_2}(t) > 0$ in configuration (b) and (c). Assertion (ii) follows from (i) since, by Table 1, $h_{j_1}^{\nu}(t)$ and $h_{j_2}^{\nu}(t)$ are bounded and, by (3.2), $\dot{h}_{j_3}^{\nu}(t)$ is bounded. In view of (3.1), (ii) implies (iii). Conclusions (iv) and (v) follow from (iii). □

For any $t \in (0, \mathcal{T})$, for simplicity, we denote by $x(t)$ the solution of the minimum problem (2.20) with Π replaced by $\Pi(t)$ and $y(t) := -mx(t) + q_y$, $z(t) := mx(t) + q_z$. Thanks to Proposition 3.7, $L_j(\mathcal{T}-)$, $x(\mathcal{T}-)$ and $\kappa_{\varphi}(l_j(\mathcal{T}-))$ are well defined. We denote by $\Pi(\mathcal{T})$ the elementary triod

satisfying $[\Pi(\mathcal{T})] = [\Pi]$ and $|S_j| = L_j(\mathcal{T}-)$ for any $j = 1, 2, 3$. Finally, $N_{\min}(\mathcal{T})$ denotes the solution of (2.3) with Γ replaced by $\Pi(\mathcal{T})$.

REMARK 3.8 We conclude from Proposition 3.7 that either $L_{j_3}(\mathcal{T}-) = 0$ or $\Pi(\mathcal{T})$ is unstable or both of these occurrences happen at time $t = \mathcal{T}$. Finally, $L_{j_3}(\mathcal{T}-) = 0$ is not equivalent to $|\kappa_\varphi(l_{j_3}(\mathcal{T}-))| = +\infty$, but whenever S_{j_3} disappears in a stable flow, its φ-curvature remains bounded.

3.1 A case of global existence

In general, $\mathcal{T}$ is finite and the flow develops a singularity at time $t = \mathcal{T}$. The following result shows that in a specific case the flow is global, i.e. $\mathcal{T} = +\infty$. For all $\mathfrak{n} = 6\mathfrak{m} \geqslant 12$, set

$$u_\infty := \frac{v_\infty}{r}, \qquad v_\infty := 1 + \sqrt{3}\sin\alpha_1 - \cos\alpha_1, \qquad r := \left(\frac{2}{\sqrt{3}} + \frac{1}{\sin\alpha_1}\right)\left(-\cot\alpha_1 - \frac{1}{\sqrt{3}}\right)^{-1}. \quad (3.20)$$

THEOREM 3.9 Let $\mathfrak{n} = 6\mathfrak{m}$ and let $\Pi \in$ (a) be stable. Then $\mathcal{T} = +\infty$.

(i) If $\mathfrak{n} \geqslant 12$ then $\lim_{t\to\infty} L_j(t) = +\infty$ and $\lim_{t\to\infty} \kappa_\varphi(l_j(t)) = 0$ for any $j = 1, 2, 3$. Furthermore,

$$\lim_{t\to\infty} \frac{L_2(t)}{L_3(t)} = u_\infty, \qquad \lim_{t\to\infty} \frac{L_2(t)}{L_1(t)} = v_\infty, \qquad \lim_{t\to\infty} x(t) = x(\infty)$$

where u_∞, v_∞ are as in (3.20) and $x(\infty) := l(u_\infty + v_\infty + 1)^{-1} \in (0, L)$.

(ii) If $\mathfrak{n} = 6$ then $\lim_{t\to\infty} L_1(t) = 0$, $\lim_{t\to\infty} L_2(t) = L_1 + L_2$, $\lim_{t\to\infty} L_3(t) = +\infty$, and $\lim_{t\to\infty} x_{\min}(t) = 0$. Furthermore,

$$\lim_{t\to\infty} \kappa_\varphi(l_3(t)) = 0, \qquad \lim_{t\to\infty} \kappa_\varphi(l_2(t)) = -\lim_{t\to\infty} \kappa_\varphi(l_1(t)) = \frac{l}{L_2 + L_1}.$$

The analysis of the long time behaviour requires the following lemma (recall Definition 2.11), which in particular shows that if $\mathfrak{n} = 6\mathfrak{m}$ then the stability region $\mathcal{S}_a$ is the whole of $(0, +\infty)^3$.

LEMMA 3.10 For all $\mathfrak{n} \geqslant 6$, $\mathcal{S}_d = \mathcal{S}_{d'} = \emptyset$. If $\mathfrak{n} = 6\mathfrak{m}$ we have $\mathcal{S}_a(3, 1) = \mathcal{S}_{a'}(2, 1) = (0, +\infty)^2$. Moreover, if m is defined as in (2.15), then for $\mathfrak{n} = 6\mathfrak{m} - 4$,

$$\mathcal{S}_a(3, 1) = \mathcal{S}_{a'}(2, 1) = \{(u, v) \in (0, +\infty)^2 : v < m\},$$

$$\mathcal{S}_b(2, 1) = \mathcal{S}_{b'}(3, 1) = \left\{(u, v) \in (0, +\infty)^2 : m\frac{L - \delta - Lu}{L - \delta} < v < \frac{L - (L - \delta)u}{L - 2\delta}\right\},$$

$$\mathcal{S}_c(2, 1) = \mathcal{S}_{c'}(3, 1) = \left\{(u, v) \in (0, +\infty)^2 : m < v < \frac{L + \delta u}{L - 2\delta}\right\},$$

while for $\mathfrak{n} = 6\mathfrak{m} - 2$,

$$\mathcal{S}_a(3, 1) = \mathcal{S}_{a'}(2, 1) = \left\{(u, v) \in (0, +\infty)^2 : m\frac{L - \delta - Lu}{L - \delta} < v < \frac{L - (L - \delta)u}{L - 2\delta}\right\},$$

$$\mathcal{S}_b(2, 1) = \mathcal{S}_{b'}(3, 1) = \{(u, v) \in (0, +\infty)^2 : v < m\},$$

$$\mathcal{S}_c(3, 1) = \mathcal{S}_{c'}(2, 1) = \left\{(u, v) \in (0, +\infty)^2 : m < v < \frac{L + \delta u}{L - 2\delta}\right\}.$$

Proof. Using Proposition 2.22, and (2.24) for configurations (a′), (b′), (c′), the conclusion follows by imposing $x_{\min} \in (a, b)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 3.9. Let $x_{\min}^{(\mathrm{a})}$ be as in (2.21). For any $t \in [0, \mathcal{T})$ define $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t) := x_{\min}^{(\mathrm{a})}(L_1(t), L_2(t), L_3(t))$. Set

$$d(t) := \Big(\sum_{i<j} L_i(t) L_j(t) \Big)^{-1}.$$

By Table 1, (2.15) and (2.16), system (2.9) reads

$$\begin{cases} \dot{h}_1^\nu(t) = -\dfrac{x(t)}{L_1(t)} = -L d(t) L_3(t), \\[2mm] \dot{h}_2^\nu(t) = \dfrac{y(t)}{L_2(t)} = L d(t)(L_1(t) + L_3(t)), \\[2mm] \dot{h}_3^\nu(t) = -\dfrac{z(t)}{L_3(t)} = -L d(t) L_1(t). \end{cases} \qquad (3.21)$$

Set

$$A_1 := \Big(\cot \alpha_1 - \frac{1}{\sqrt{3}} \Big) \frac{\cot \alpha_1 + \sqrt{3}}{\cot^2 \alpha_1 + 1}, \quad A_3 := \Big(\cot \alpha_1 + \frac{1}{\sqrt{3}} \Big) \frac{\cot \alpha_1 - \sqrt{3}}{\cot^2 \alpha_1 + 1}.$$

Then condition (3.3) becomes

$$L_2(t) = A_1 L_1(t) + A_3 L_3(t) + (L_2 - A_1 L_1 - A_3 L_3)(0), \quad t \in [0, \mathcal{T}). \qquad (3.22)$$

Differentiating (3.14) and (3.18) with respect to t and using (3.21) yields

$$\dot{L}_1 = L d(t) f_1, \quad \dot{L}_3 = L d(t) f_3, \qquad (3.23)$$

where

$$f_1 := -\frac{2}{\sqrt{3}} L_1 - \Big(\cot \alpha_1 + \frac{1}{\sqrt{3}} \Big) L_3, \quad f_3 := -\Big(\cot \alpha_1 - \frac{1}{\sqrt{3}} \Big) L_1 + \frac{2}{\sqrt{3}} L_3.$$

Thus $dL_1/f_1 = dL_3/f_3$, and if we substitute $L_3 = R L_1$, we obtain

$$p_1 dL_1 + p_2 dR = 0, \qquad (3.24)$$

with

$$p_1 := \Big(\cot \alpha_1 + \frac{1}{\sqrt{3}} \Big) R^2 + \frac{4}{\sqrt{3}} R - \Big(\cot \alpha_1 - \frac{1}{\sqrt{3}} \Big), \quad p_2 := \Big[\Big(\cot \alpha_1 + \frac{1}{\sqrt{3}} \Big) R + \frac{2}{\sqrt{3}} \Big] L_1.$$

Let us show (i). Integrating (3.24) yields

$$\log L_1 + \frac{1}{2} \int \Big(\frac{1}{R - \bar{R}} + \frac{1}{R - r} \Big) dR = 0$$

where $r > 0$ is defined as in (3.20) and

$$\bar{R} := -\frac{2/\sqrt{3} - 1/\sin \alpha_1}{\cot \alpha_1 + 1/\sqrt{3}} < 0.$$

Thus

$$L_1^2(t)\left(\frac{L_3}{L_1}(t) - \bar{R}\right)\left(\frac{L_3}{L_1}(t) - r\right) = C, \quad C := (L_3 - \bar{R}L_1)(L_3 - rL_1). \tag{3.25}$$

CLAIM. There exists $t_0 \in [0, \mathcal{T})$ such that $f_1(t) > 0$ (and hence $\dot{L}_1 > 0$) for any $t \in [t_0, \mathcal{T})$.

If f_1 is positive at time $t = 0$ then it is so at the subsequent times since, by (3.14) and (3.18),

$$\dot{f}_1(t) = -\dot{h}_3^\nu(\cot^2 \alpha_1 + 1) > 0, \quad t \in [0, \mathcal{T}). \tag{3.26}$$

We proceed by contradiction: assume that $f_1 \leqslant 0$ in $[0, \mathcal{T})$, i.e.

$$L_1(t) \geqslant M_0 L_3(t), \quad t \in [0, \mathcal{T}), \quad M_0 := -\frac{\sqrt{3}}{2}\left(\cot \alpha_1 + \frac{1}{\sqrt{3}}\right) \geqslant 1. \tag{3.27}$$

Then, from (3.23), L_1 is decreasing in $[0, \mathcal{T})$ but bounded below by $M_0 L_3$. From Propositions 3.10 and 3.7(v), we conclude that $\mathcal{T} = +\infty$. Note that $A_1 \geqslant 0$ and $A_3 > 0$ if $\mathfrak{n} \geqslant 12$. Set

$$M_1 := \frac{LL_3(0)}{[(A_3/M_0)L_1(0) + L_2(0) - A_3 L_3(0)]\frac{M_0+1}{M_0}L_1(0) + L_1(0)^2/M_0} > 0.$$

From (3.21), (3.22) and (3.27) we get $-\dot{h}_3^\nu \geqslant M_1$, and from (3.26), $\dot{f}_1 \geqslant M_1(\cot^2 \alpha_1 + 1)$ in $[0, +\infty)$. Thus $-(\cot \alpha_1 + 1/\sqrt{3})L_3(t) \geqslant (2/\sqrt{3})L_1(t) + M_1(\cot^2 \alpha_1 + 1)t + f_1(0)$ for any $t \in [0, \infty)$. Hence, letting $t \to +\infty$ we get $L_3 \to +\infty$, and from (3.27), $L_1 \to +\infty$. This gives a contradiction since, by our assumption, L_1 is decreasing on $[0, +\infty)$. The Claim is proved.

Since (3.27) holds when L_1 is decreasing, $\mathcal{T} = +\infty$ follows from Proposition 3.7 and formula (2.21). Let us show $\lim_{t\to\infty} L_1(t) = +\infty$. Assume that $\lim_{t\to\infty} L_1(t) < +\infty$. Then from (3.25) and (3.22) we get respectively $\lim_{t\to\infty} L_3(t) < +\infty$ and $\lim_{t\to\infty} L_2(t) < +\infty$. But this implies $\lim_{t\to\infty} \dot{L}_j(t) = 0$ for any $j = 1, 2, 3$. Differentiating (3.16) with respect to t and recalling that $\dot{h}_1^\nu \leqslant 0$ and $\dot{h}_3^\nu \leqslant 0$, we get $\lim_{t\to\infty} \dot{h}_1^\nu(t) = \lim_{t\to\infty} \dot{h}_3^\nu(t) = 0$, and consequently $\lim_{t\to\infty} \dot{h}_2^\nu(t) = 0$, which gives a contradiction with system (3.21).

Finally, we also get $\lim_{t\to\infty} L_3(t) = +\infty$ and $\lim_{t\to\infty} L_2(t) = +\infty$ respectively from $f_1 > 0$ on $[t_0, +\infty)$ and (3.22). Again (3.22), (3.25) and (2.21) yield the conclusion since $A_1 + A_3 r = v_\infty$.

Let us show (ii). If $\mathfrak{n} = 6$ then $\alpha_1 = \vartheta_\mathfrak{n} = 2\pi/3$. Thus, (3.22) reduces to

$$L_2(t) = -L_1(t) + S_0, \quad t \in [0, \mathcal{T}), \quad S_0 := L_1 + L_2, \tag{3.28}$$

and (3.24) to $(2R+1)dL_1 + L_1 dR = 0$ so that integrating yields $\log(L_1\sqrt{2R+1}) = C$. Hence

$$L_3(t) = \frac{1}{2}\left(\frac{C_0}{L_1(t)} - L_1(t)\right), \quad t \in [0, \mathcal{T}), \quad C_0 := L_1(L_1 + 2L_3). \tag{3.29}$$

Using (3.21), (3.29), (3.28), we can rewrite the first equation in (3.23) as $\dot{L}_1 = 4LL_1^2/\sqrt{3}g(L_1)$, where $g(s) := 2s^3 - S_0 s^2 - C_0 S_0$. Given $\Lambda \in (0, L_1)$, let $T \in (0, \mathcal{T})$ be the time that the solution needs to achieve the value Λ. Then we get

$$\left[s^2 - S_0 s + C_0 S_0 \frac{1}{s}\right]_{L_1}^{\Lambda} = \int_{L_1}^{\Lambda}\left(2s - S_0 - \frac{C_0 S_0}{s^2}\right)ds = \frac{4}{\sqrt{3}}LT. \tag{3.30}$$

From (3.30) we discover that $T < \mathcal{T}$ for any $\Lambda \in (0, L_1)$. Hence $\Lambda = 0$ and $\mathcal{T} = +\infty$. Further, $L_1(t)$ being strictly decreasing in $[0, +\infty)$ (from (3.23)), we have $L_1 > 0$ in $[0, +\infty)$, and from (3.30) we get $\lim_{t\uparrow\infty} L_1(t) = 0$. $\qquad \square$

4. Configurations (d) and (d′): development of a new segment

In this section we assume $\mathfrak{n} = 8$ and $\Pi \in$ (d). From Proposition 2.22, Π is unstable with $x_{\min} = \delta$, i.e. $N_{\min} = (X(V_3), Y(V_3), V_3)$ (see Figure 8), and from Table 1, $\kappa_\varphi(l_1) = \delta/L_1$, $\kappa_\varphi(l_2) = (L - \delta)/L_2$, $\kappa_\varphi(l_3) = 0$. Since $x_{\min}$ tends to be smaller than δ and the constraint $N_{\min|\Sigma_j}(q) \in T_{\varphi^o}(v_\varphi^j)$ cannot be violated, the appearance of a vertical segment at q is forced during the flow, as explained in the following result.
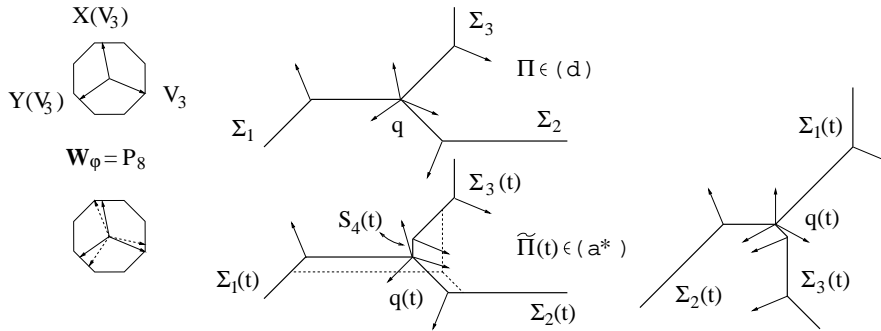
FIG. 8. Development of a vertical segment $S_4(t)$. The rotated quasi-elementary triod on the right will be used in the proof of Theorem 4.1.

THEOREM 4.1 Let $\mathfrak{n} = 8$ and $\Pi \in$ (d). Then there exist $T > 0$ and a stable φ-curvature flow $t \in [0, T) \mapsto \widetilde{\Pi}(t)$ starting from Π. More precisely,

$$\widetilde{\Pi}(t) = \Sigma_1(t) \cup \Sigma_2(t) \cup (S_4(t) \cup \Sigma_3(t)) \in (\mathrm{a}^*), \quad \forall t \in (0, T), \tag{4.1}$$

and if we define

$$\widetilde{x}(t) := x_{\min}^{(\mathrm{a})}(L_2(t), L_4(t), L_1(t)), \quad \widetilde{y}(t) := -m\widetilde{x}(t) + q_y, \quad \widetilde{z}(t) := m\widetilde{x}(t) + q_z, \tag{4.2}$$

with $x_{\min}^{(\mathrm{a})}$ as in (2.21) and m, q_y, q_z as in (2.15), (2.16), then $\kappa_\varphi(l_3(t)) = 0$,

$$\kappa_\varphi(l_1(t)) = \frac{\widetilde{z}(t)}{L_1(t)}, \quad \kappa_\varphi(l_2(t)) = \frac{\widetilde{x}(t)}{L_2(t)}, \quad \kappa_\varphi(l_4(t)) = -\frac{\widetilde{y}(t)}{L_4(t)}. \tag{4.3}$$

Finally, $\kappa_\varphi(l_j(\cdot)) \in C^\infty((0, T))$ for any $j = 1, 2, 3, 4$, and

$$\lim_{t \downarrow 0} \kappa_\varphi(l_1(t)) = \frac{\delta}{L_1}, \quad \lim_{t \downarrow 0} \kappa_\varphi(l_2(t)) = \frac{L - \delta}{L_2}, \quad \lim_{t \downarrow 0} \kappa_\varphi(l_4(t)) = -\frac{\delta}{L_1} + 2\tau_1 \cdot \tau_2 \frac{L - \delta}{L_2} < 0. \tag{4.4}$$

The idea of the proof is to consider the φ-curvature flow starting from the rotated quasi-elementary triod $\widetilde{\Pi}$ in Figure 8, with singular initial datum $(L_2, 0, L_1)$. Notice that, from Lemma 3.10, $(L_2, 0, L_1)$ belongs to the boundary of the stability region $\mathcal{S}_{\mathrm{a}}$.

Proof. Set $G(w) := x_{\min}^{(\mathrm{a})}(w)$ for $w := (w_1, w_2, w_3) \in (0, +\infty)^3$ with $x_{\min}^{(\mathrm{a})}$ as in (2.21), and define the vector field $F = (F_1, F_2, F_3) \in C^\infty((0, +\infty)^3; \mathbb{R}^3)$ as

$$F_1(w) := \frac{1}{\sin\vartheta_\mathfrak{n}} \frac{mG(w) + q_z}{w_3}, \quad F_2(w) = F_3(w) := \frac{-mG(w) + q_y}{w_2} + \frac{mG(w) + q_z}{w_3}, \tag{4.5}$$

where m, q_y, q_z are defined as in (2.15) and (2.16). Notice that F is obtained by differentiating with respect to t the right hand side of (3.14), (3.17), (3.19) and by replacing the $\dot{h}_j^\nu$'s (in (2.9)) with the expressions in G and w_j (where we used Table 1, (2.21) and (2.19)). Despite the appearance of w_2 in the denominators of F_2, the presence of G ensures that F and all its partial derivatives are bounded in $(0, +\infty) \times \{w_2 = 0\} \times (0, +\infty)$. Thus the Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)), \\ w(0) = (L_2, 0, L_1), \end{cases} \tag{4.6}$$

admits a unique solution $(\Lambda_1, \Lambda_2, \Lambda_3) \in C^\infty([0, T); (0, +\infty)^3)$ for some $T > 0$. Set $L_1(t) := \Lambda_3(t)$, $L_2(t) := \Lambda_1(t)$, $L_4(t) := \Lambda_2(t)$ for any $t \in [0, T)$. Let us show that $L_4 > 0$ in $[0, T)$. In order to do that, one checks that $\ddot{L}_4(0) > 0$ as a consequence of

$$\dot{L}_4(t) = \frac{q_y - mG(L_2(t), L_4(t), L_1(t))}{L_4(t)} + \frac{mG(L_2(t), L_4(t), L_1(t)) + q_z}{L_1(t)}$$

and

$$\lim_{w_2 \to 0} G(w_1, w_2, w_3) = L - \delta, \quad \lim_{w_2 \to 0} F_2(w_1, w_2, w_3) = 0 \, (\lim_{t \to 0} \dot{L}_4(t) = 0), \quad w_1, w_3 \in (0, +\infty).$$

Hence, we conclude that L_4 is increasing in a neighbourhood of 0, say $[0, T)$.

Let $\widetilde{\Pi}(t)$ be the quasi-elementary triod defined in (4.1) and having $|S_j(t)| = L_j(t)$, $j = 1, 2, 3, 4$. Define $\widetilde{x}(t) := G(L_2(t), L_4(t), L_1(t))$ and $\widetilde{y}(t), \widetilde{z}(t)$ as in (4.2). By construction, for any $t \in (0, T)$, the solution $N_{\min}(t)$ of (2.3) with Γ replaced by $\widetilde{\Pi}(t)$ is determined by $(\widetilde{x}(t), \widetilde{y}(t), \widetilde{z}(t))$. Since $L_j \in C^\infty([0, T))$, possibly reducing $T > 0$, we have $\widetilde{x} \in (\delta, L - \delta)$ in $(0, T)$. Therefore, $\widetilde{\Pi}$ is stable in $(0, T)$, $\kappa_\varphi(l_j(\cdot)) \in C^\infty([0, T))$ for any $j = 1, 2, 3$, and (4.3) holds. Defining, for any $j = 1, 2, 3$, $h_j^\nu : [0, T) \to \mathbb{R}$ as in (3.11) and reasoning as in the last part of Theorem 3.3 we get a stable φ-curvature flow starting from Π. Since L_1, L_2, L_4 are monotone functions in $(0, T)$, the limits in (4.4) exist and their values follow from $\lim_{t \to 0} \widetilde{x}(t) = L - \delta$ and $\lim_{t \to 0} L_4(t) = 0$. $\quad\square$

REMARK 4.2 The flow $t \mapsto \widetilde{\Pi}(t)$ of Theorem 4.1 is the unique stable flow starting from Π. Indeed, if $t \mapsto \Pi'(t)$ is a stable flow starting from Π then, from Proposition 2.22, $\Pi'(t) \notin$ (d) and, as $x_{\min} = \delta$, i.e. $N_{\min\lfloor \Sigma_3} = V_3$ (see Figure 8), $\Pi'(t)$ must be quasi-elementary with $\Pi'(t) \in$ (a*). We expect that $t \mapsto \widetilde{\Pi}(t)$ is also unique among all regular flows starting from Π.

5. The case $\mathfrak{n} = 8$ and $\Pi \in$ (b): development of a new segment

In this section we prove that at time $t = \mathcal{T} \in (0, +\infty)$ the flow starting from a stable triod $\Pi \in$ (b) becomes unstable and a vertical segment develops in order to decrease the energy functional and make the flow stable at subsequent times.

THEOREM 5.1 Let $\mathfrak{n} = 8$ and let $\Pi \in$ (b) be stable. Then $\mathcal{T} < +\infty$ and $N_{\min}(\mathcal{T}) = (X(V_3), Y(V_3), V_3)$. Furthermore, there exist $\mathcal{T}_1 \in (\mathcal{T}, +\infty]$ and a stable φ-curvature flow $t \in [\mathcal{T}, \mathcal{T}_1) \mapsto \widetilde{\Pi}(t)$ starting from $\Pi(\mathcal{T})$. More precisely, for any $t \in (\mathcal{T}, \mathcal{T}_1)$, $\widetilde{\Pi}(t)$ is the quasi-elementary triod defined as in (4.1), $\kappa_\varphi(l_3(t)) = -L/L_3(t)$ and (4.3) holds. Finally, $\kappa_\varphi(l_j(\cdot)) \in C^\infty((\mathcal{T}, \mathcal{T}_1))$ for any $j = 1, 2, 3, 4$, and (4.4) holds with $t \downarrow 0$ replaced by $t \downarrow \mathcal{T}$.

The idea of the proof is that at the finite time $t = \mathcal{T}$ the solution reaches the boundary of the stability region, and at the same time an infinitesimal segment appears; then the flow is continued by arguing as in Theorem 4.1.

Proof. For any $t \in [0, \mathcal{T})$ define $x(t) := x_{\min}^{(b)}(L_1(t), L_2(t), L_3(t))$, $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t)$. Then system (2.9) reads

$$\dot{h}_1^v(t) = -\frac{x(t)}{L_1(t)}, \quad \dot{h}_2^v(t) = -\frac{L - y(t)}{L_2(t)}, \quad \dot{h}_3^v(t) = \frac{L - z(t)}{L_3(t)}, \tag{5.1}$$

system (3.1) reads

$$L_1(t) = L_1 + \sqrt{2}h_3^v(t), \quad L_2(t) = L_2 - h_2^v(t) - h_3^v(t), \quad L_3(t) = L_3 + h_2^v(t) + h_3^v(t), \tag{5.2}$$

while (3.2) and (3.3) become

$$-\sqrt{2}h_1^v(t) = h_2^v(t) + h_3^v(t), \quad L_3(t) - L_3 = L_2 - L_2(t). \tag{5.3}$$

Define $u(t) := L_3(t)/L_2(t)$ and $v(t) := L_3(t)/L_1(t)$. Differentiating (5.2) with respect to $t \in (0, \mathcal{T})$ and using (2.22), (2.19) and $m = \sqrt{2}/2$, we obtain

$$\dot{u} = \frac{L - 2\delta}{L_3^2} \frac{uv(1 + u)(D - Eu)}{\delta^2 + \delta^2 u + (L - 2\delta)^2 v}, \quad \dot{v} = \frac{(L - 2\delta)(L - \delta)}{\delta L_3^2} \frac{v^2(A - Bu - Cv)}{\delta^2 + \delta^2 u + (L - 2\delta)^2 v}, \tag{5.4}$$

where $A := \delta(L - \delta)$, $B := \delta L$, $C := (L - \delta)(L - 2\delta)$, $D := (L - \delta)^2$, $E := L^2 - 3\delta L + \delta^2$. Recall that the stability region $\mathcal{S}_b(2, 1)$ is given by Lemma 3.10 (see Figure 9). Notice that $A - Bu - Cv \leqslant 0$ is equivalent to $x_{\min} \leqslant L - \delta$ and $D - Eu > 0$ for any $(u, v) \in \mathcal{S}_b(2, 1)$. Thus $\dot{u} > 0$ and $\dot{v} < 0$ in $\mathcal{S}_b(2, 1)$. From (5.4) we get

$$\frac{dv}{du} = \frac{L - \delta}{\delta} \frac{v(A - Bu - Cv)}{u(u + 1)(D - Eu)} > -\frac{L - \delta}{\delta} \frac{v}{u(u + 1)}, \tag{5.5}$$

since $-\frac{A - Bu - Cv}{D - Eu} \leqslant 1$ for any $(u, v) \in \mathcal{S}_b(2, 1)$. For any $(u_0, v_0) \in \mathcal{S}_b(2, 1)$ we have

$$v(u) \geqslant v_0 \left(\frac{u_0}{1 + u_0} \frac{u + 1}{u}\right)^{(L - \delta)/\delta} \geqslant v_0 \left(\frac{u_0}{1 + u_0}\right)^{(L - \delta)/\delta}.$$
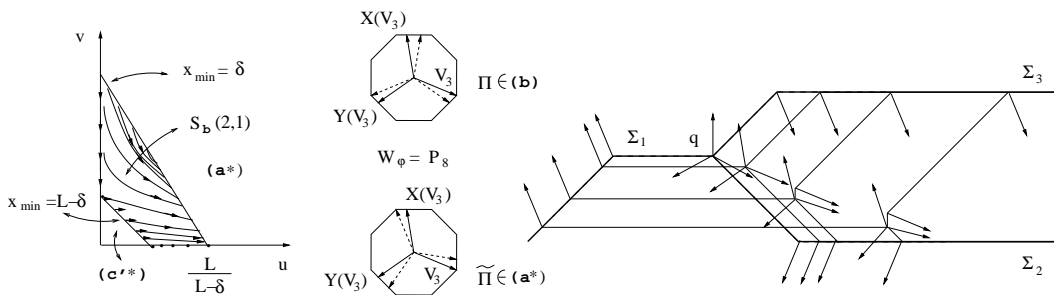

FIG. 9. φ-curvature flow starting from a stable $\Pi \in \{(b)\}$ (see Theorem 5.1): at time $t = \mathcal{T}$ the flow becomes unstable and a vertical segment develops in order to make the triod stable.

From Proposition 3.7 we know that $L_3(\cdot)$ is nondecreasing; hence, from (5.3) it follows that $L_2(\cdot)$ is nonincreasing, so that $0 < L_3(0) \leqslant L_3(t)$, $L_2(t) \leqslant L_2(0)$ for any $t \in (0, \mathcal{T})$. Since $L_3(t)/L_2(t) \leqslant L/(l - \delta)$ it follows that $(L - \delta)L_3(t) \leqslant LL_2(0)$ and $LL_2(t) \geqslant (L - \delta)L_3(0)$ for any $t \in (0, \mathcal{T})$. It follows that $L_1(\mathcal{T}-) - L_1 < +\infty$, since $L_3(t)/L_1(t)$ is bounded from below. Furthermore, since by (5.2) and (5.1),

$$\dot{L}_2(t) \leqslant -\dot{h}_2^v(t) \leqslant \frac{L}{L_2(t)}, \qquad \dot{L}_1(t) \geqslant -\sqrt{2}\dot{h}_2^v(t) \geqslant \sqrt{2}\frac{L - \delta}{L_2(t)},$$

it follows that $\mathcal{T} < +\infty$, $x(\mathcal{T}-) = \delta$ and

$$\kappa_\varphi(l_1(\mathcal{T}-)) = -\frac{\delta}{L_1(\mathcal{T}-)}, \qquad \kappa_\varphi(l_2(\mathcal{T}-)) = -\frac{L - \delta}{L_2(\mathcal{T}-)}, \qquad \kappa_\varphi(l_3(\mathcal{T}-)) = \frac{L}{L_3(\mathcal{T}-)}.$$

We proceed as in the proof of Theorem 4.1, with the difference that now $\kappa_\varphi(l_3(\mathcal{T}-)) > 0$, so that a system of four ODEs is required. For any $\widehat{w} := (w_1, w_2, w_3) \in (0, +\infty)^3$ we set $G(\widehat{w}) := x_{\min}^{(a)}(\widehat{w})$ with $x_{\min}^{(a)}$ as in (2.21). For any $w := (\widehat{w}, w_4) \in (0, +\infty)^4$ we define the vector field $F \in C^\infty((0, +\infty)^4; \mathbb{R}^4)$ as

$$F_1(w) := -\sqrt{2}\,\frac{mG(\widehat{w}) + q_z}{w_3}, \qquad\qquad F_2(w) := F_3(w) - \sqrt{2}\,\frac{L}{w_4},$$
$$F_3(w) := \frac{q_y - mG(\widehat{w})}{w_2} + \frac{mG(\widehat{w}) + q_z}{w_3}, \qquad F_4(w) := 2\frac{L}{w_4} - \sqrt{2}\,\frac{q_y - mG(\widehat{w})}{w_2}, \tag{5.6}$$

where m, q_y, q_z are given by (2.15) and (2.16). Since F and all its partial derivatives are bounded in $(0, +\infty) \times \{w_2 = 0\} \times (0, +\infty)^2$, the Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)), \\ w(\mathcal{T}) = (L_2(\mathcal{T}), 0, L_1(\mathcal{T}), L_3(\mathcal{T})), \end{cases} \tag{5.7}$$

admits a unique solution $(w_1, w_2, w_3, w_4) \in C^\infty([\mathcal{T}, \mathcal{T}_1); (0, +\infty)^4)$ for $\mathcal{T}_1 \in (\mathcal{T}, +\infty]$. For any $t \in [\mathcal{T}, \mathcal{T}_1)$, set $L_1(t) := \Lambda_3(t)$, $L_2(t) := \Lambda_1(t)$, $L_3(t) := \Lambda_4(t)$ and $L_4(t) := \Lambda_2(t)$. As in the proof of Theorem 4.1, we find that $\ddot{L}_4(0) > 0$ as a consequence of

$$\dot{L}_4(t) = \frac{q_y - mG(L_2(t), L_4(t), L_1(t))}{L_4(t)} + \frac{mG(L_2(t), L_4(t), L_1(t)) + q_z}{L_1(t)} - \sqrt{2}\frac{L}{L_3(t)}$$

and

$$\lim_{w_2 \to 0} G(w_1, w_2, w_3) = L - \delta, \qquad \lim_{w_2 \to 0} F_2(w_1, w_2, w_3, w_4) = 0 \quad (\lim_{t \to 0} \dot{L}_4(t) = 0),$$
$$w_1, w_3, w_4 \in (0, +\infty).$$

The conclusion follows by the same argument once we define, for any $t \in [\mathcal{T}, \mathcal{T}_1)$, the quasi-elementary triod $\widetilde{\Pi}(t)$ as in (4.1) with $|S_j(t)| := L_j(t)$, $\widetilde{x}(t) := G(L_2(t), L_4(t), L_1(t))$, $h_j(t) := h_j^v(t)\nu_j$ and $h_j^v(t)$ as in (3.11) with

$$\lim_{t \downarrow \mathcal{T}} h_j^v(t) = h_j^v(\mathcal{T}), \qquad j = 1, 2, 3, \qquad h_4^v(\mathcal{T}) = 0.$$

We notice that condition (ii) of Definition 2.12 holds since, similarly to Proposition 3.1, one can show that (ii) is satisfied for $t \in (\mathcal{T}, \mathcal{T}_1) \mapsto \widetilde{\Pi}(t)$ if and only if system (5.7) with $w(t) = (L_2(t), L_4(t), L_1(t), L_3(t))$ holds in $(\mathcal{T}, \mathcal{T}_1)$ and $\widetilde{\Pi}(t)$ is stable for any $t \in (\mathcal{T}, \mathcal{T}_1)$. $\qquad\square$

REMARK 5.2   The flow $t \mapsto \widetilde{\Pi}(t)$ of Theorem 5.1 is the unique stable flow starting from $\Pi(\mathcal{T})$. Indeed, assume that $t \mapsto \Pi'(t)$ is a stable flow starting from Π. If $\Pi'(t) \in$ (b) and u, v are defined as in the proof of Theorem 5.1 then $u(t), v(t) \notin \mathcal{S}_{\mathrm{b}}(2, 1)$, which gives a contradiction. Since any nonpolygonal triod is unstable and since $x_{\min} = \delta$, i.e. $N_{\min|\Sigma_3} = V_3$ (see Figure 9), $\Pi'(t)$ must be quasi-elementary with $\Pi'(t) \in$ (a*). We expect that $t \mapsto \widetilde{\Pi}(t)$ is also unique among all regular flows starting from $\Pi(\mathcal{T})$.

6. The case $\mathfrak{n} = 6\mathfrak{m} - 4$ and $\Pi \in$ (a): development of a curve

In this section we prove that at time $t = \mathcal{T} \in (0, +\infty)$ the flow starting from a stable triod $\Pi \in$ (a) becomes unstable and a curve develops from the triple junction at subsequent times.

THEOREM 6.1   Let $\mathfrak{n} = 6\mathfrak{m} - 4$ and let $\Pi \in$ (a) be stable. Then $\mathcal{T} < +\infty$ and there exists a φ-curvature flow $t \in [\mathcal{T}, +\infty) \mapsto \Pi(t)$ starting from $\Pi(\mathcal{T})$. Moreover, for any $t \in [\mathcal{T}, +\infty)$, the triod $\Pi(t)$ is nonpolygonal and unstable with $N_{\min}(t) = (X(V_3), Y(V_3), V_3)$. Finally, $\kappa_\varphi(l_3(t)) = 0$, and

$$\kappa_\varphi(l_1(t)) = \frac{\delta}{L_1(t)}, \qquad \kappa_\varphi(l_2(t)) = -\frac{\delta}{L_2(t)}, \tag{6.1}$$

$$\lim_{t \to \infty} L_1(t) = \lim_{t \to \infty} L_2(t) = +\infty, \qquad \lim_{t \to \infty} \frac{L_2(t)}{L_1(t)} = 1. \tag{6.2}$$

Proof. For any $t \in [0, \mathcal{T})$ define $x(t) = x_{\min}^{(a)}(L_1(t), L_2(t), L_3(t))$, $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t)$. Then system (2.9) reads

$$\dot{h}_1^v(t) = -\frac{x(t)}{L_1(t)}, \qquad \dot{h}_2^v(t) = \frac{y(t)}{L_2(t)}, \qquad \dot{h}_3^v(t) = -\frac{z(t)}{L_3(t)}. \tag{6.3}$$

Step 1. L_1, L_2, L_3 are strictly positive and bounded in $[0, \mathcal{T}]$.

From Lemma 3.10, we know that $(u, v) \in \mathcal{S}_{\mathrm{a}}(3, 1) = \{(u, v) \in (0, +\infty)^2 : v < -(2 \cos \vartheta_\mathfrak{n})^{-1}\}$ in $[0, \mathcal{T})$, where $u(t) := L_2(t)/L_3(t)$ and $v(t) := L_2(t)/L_1(t)$, i.e.

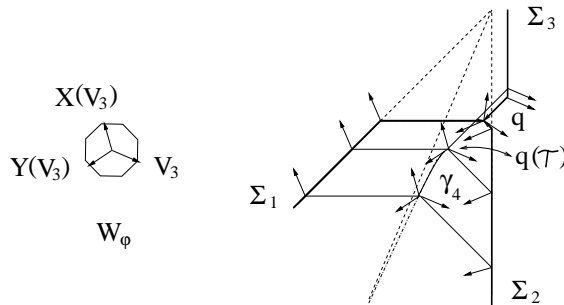$$-2 \cos \vartheta_\mathfrak{n} L_2(t) < L_1(t), \quad t \in [0, \mathcal{T}). \tag{6.4}$$

FIG. 10. φ-curvature flow starting from a stable $\Pi \in$ (a) for $\mathfrak{n} = 8$ (see Theorem 6.1): at time $t = \mathcal{T}$ the flow becomes unstable and a curve γ_4 of zero φ-curvature develops from the triple junction.
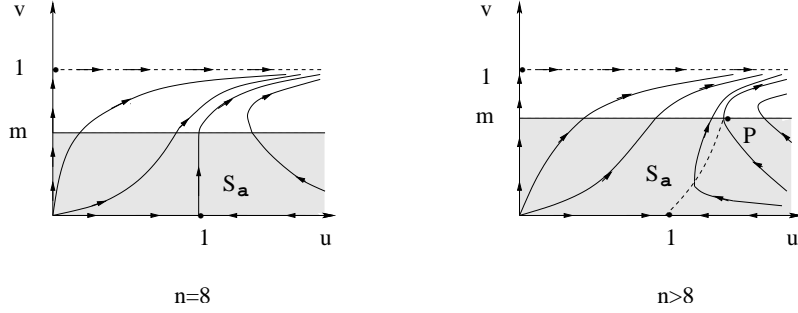
FIG. 11. Flow lines diagram in the variables (u, v) corresponding to φ-curvature flows starting from stable $\Pi \in$ (a) (see Theorem 6.1).

Set

$$M_0 := \frac{\cot\alpha_1 - \cot\vartheta_{\mathfrak{n}}}{-2\cos\vartheta_{\mathfrak{n}}(\cot\alpha_1 + \cot(2\vartheta_{\mathfrak{n}}))} > 0.$$

From (3.14), (3.16), $\cot\alpha_1 = -\cot\alpha_2 < 0$, $\dot{h}_3^v < 0$, $\cos\vartheta_n < 0$, we deduce $\dot{L}_1(t) \leqslant (\cot\alpha_1 - \cot\vartheta_{\mathfrak{n}})\dot{h}_1^v(t) \leqslant M_0\dot{L}_2(t)$ for any $t \in [0, \mathcal{T})$. Hence, using (6.4), we obtain

$$L_1(t)\left(1 + \frac{M_0}{2\cos\vartheta_{\mathfrak{n}}}\right) \leqslant L_1 - M_0 L_2, \quad t \in [0, \mathcal{T}).$$

Now we observe that $1 + M_0/(2\cos\vartheta_{\mathfrak{n}}) > 0$ and $L_1 - M_0 L_2 > 0$. Indeed, $L_1 - M_0 L_2 \geqslant (-2\cos\vartheta_{\mathfrak{n}} - M_0)L_2$ and using Table 1 and (2.11) we get

$$-2\cos\vartheta_{\mathfrak{n}} - M_0 = \frac{(1 - 4\cos^2\vartheta_{\mathfrak{n}})\cot\alpha_1 + \frac{-2\cos^2\vartheta_{\mathfrak{n}}}{\sin(2\vartheta_{\mathfrak{n}})}(2\cos(2\vartheta_{\mathfrak{n}}) + 1)}{2\cos\vartheta_{\mathfrak{n}}(\cot\alpha_1 + \cot(2\vartheta_{\mathfrak{n}}))}$$

$$= \frac{(1 - 4\cos^2\vartheta_{\mathfrak{n}})(\cot\alpha_1 + \cot\vartheta_{\mathfrak{n}})}{2\cos\vartheta_{\mathfrak{n}}(\cot\alpha_1 + \cot(2\vartheta_{\mathfrak{n}}))} > 0 \quad \forall\mathfrak{n} \geqslant 8,$$

since $1 - 4\cos^2\vartheta_{\mathfrak{n}} < 0$, $\cot\alpha_1 + \cot\vartheta_{\mathfrak{n}} < 0$ and $\cot\alpha_1 + \cot(2\vartheta_{\mathfrak{n}}) < 0$. Thus L_1 is bounded in $[0, \mathcal{T})$, and from (6.4) and (3.3), so are L_2 and L_3.

Step 2. $L_3(t) - L_3 \geqslant C\sqrt{L_1^2 - 2(\cot\alpha_1 - \cot\vartheta_{\mathfrak{n}})(L - \delta)t}$, $t \in [0, \mathcal{T})$, for some constant $C > 0$.
   This follows since, using (3.13), (3.18) and (6.3), we have

$$\dot{L}_1(t) \leqslant -(\cot\alpha_1 + \cot\vartheta_{\mathfrak{n}})\frac{L - \delta}{L_1(t)}, \quad \dot{L}_3(t) \geqslant \frac{\delta}{\sin\vartheta_{\mathfrak{n}} L_1(t)}, \quad t \in [0, \mathcal{T}).$$

From Steps 1 and 2, we get $\mathcal{T} < +\infty$, and hence $x(\mathcal{T}-) = \delta$.
   For any $w := (w_1, w_2) \in (0, +\infty)^2$ we define the vector field $F = (F_1, F_2) \in C^\infty((0, +\infty)^2; \mathbb{R}^2)$ as

$$F_1(w) := -\delta\frac{\cot\alpha_1 + \cot\vartheta_{\mathfrak{n}}}{w_1} - \frac{\delta}{w_2\sin\vartheta_{\mathfrak{n}}}, \quad F_2(w) := \frac{-\delta}{w_1\sin\vartheta_{\mathfrak{n}}} - \delta\frac{\cot\alpha_1 + \cot\vartheta_{\mathfrak{n}}}{w_2}. \quad (6.5)$$

The Cauchy problem

$$\begin{cases} \dot{w}(t) = F(w(t)), \\ w(\mathcal{T}) = (L_1(\mathcal{T}), L_2(\mathcal{T})), \end{cases} \tag{6.6}$$

admits a unique solution $(w_1, w_2) \in C^\infty([\mathcal{T}, T); (0, +\infty)^2)$ for some $T \in (\mathcal{T}, +\infty]$. Notice that F is obtained by differentiating with respect to t the right hand side of (3.13), (3.15) and by replacing $\dot{h}_1^v$ and $\dot{h}_2^v$ respectively by $-\delta/w_1$ and δ/w_2. For any $t \in (\mathcal{T}, T)$ and $j = 1, 2$ we denote by $\Sigma_j(t)$ the interface of an elementary triod having $S_j(t)$ parallel to $S_j(\mathcal{T})$ with one of the endpoints in $R_j(\mathcal{T})$ and $L_j(t) := |S_j(t)| = w_j(t)$. It follows from Proposition 3.1 that condition (ii) of Definition 2.12 is satisfied with $\emptyset \neq \overline{S_1(t)} \cap \overline{S_2(t)} =: q(t)$ and furthermore (6.1) holds in $[\mathcal{T}, T)$. For any $j = 1, 2$ define $h_j^v : (\mathcal{T}, T) \to \mathbb{R}$ as in (3.11) with $\lim_{t \downarrow \mathcal{T}} h_j^v(t) = h_j^v(\mathcal{T}-)$. Then $t \in (\mathcal{T}, T) \mapsto \Sigma_1(t) \cup \Sigma_2(t)$ is a flow starting from $\Sigma_1(\mathcal{T}) \cup \Sigma_2(\mathcal{T})$ which satisfies (i)–(iii) of Definition 2.12.

Step 3.   We have $T = +\infty$.

Since $L_1(t) = L_2(t)$ is the solution of the system (6.6) with initial datum $L_1(\mathcal{T}) = L_2(\mathcal{T})$, we have $v(t) := L_2(t)/L_1(t) < 1$ for any $t \in [\mathcal{T}, T)$. Moreover, v is increasing since

$$\dot{v}(t) = \frac{\delta(\cot\alpha_1 + \cot\vartheta_n)}{L_2^2} v(v^2 - 1) > 0. \tag{6.7}$$

Set

$$c_0 := -\frac{\sin\alpha_1}{\sin(\alpha_1 + \vartheta_n)} = \left(\frac{L - 2\delta}{\delta} + \frac{\sin(\alpha_1 - \vartheta_n)}{\sin\alpha_1}\right)^{-1} \leqslant \frac{\delta}{L - 2\delta} = m.$$

Then L_1 and L_2 are increasing in $[\mathcal{T}, T)$, since from (6.5) it follows that $\dot{L}_1(t) > 0$ if and only if $v(t) > c_0 \in (0, m)$ while $\dot{L}_2(t) > 0$ if and only if $v(t) < 1/c_0 \in (1, +\infty)$. Substituting $w_1 = L_1$ and $w_2 = vL_1 (= L_2)$ in the second equation in (6.6) and solving in L_1, v yields

$$(1 + v)^{a_1}(1 - v)^{a_2} = C_{\mathcal{T}}/L_1, \tag{6.8}$$

where $a_1 := (1 - c_0)/2, a_2 := (1 + c_0)/2$ and $C_{\mathcal{T}} := (L_1(\mathcal{T}) + L_2(\mathcal{T}))^{a_1}(L_1(\mathcal{T}) - L_2(\mathcal{T}))^{a_2}$. If, by contradiction, the maximal time of existence T is finite then, using the first equation in (6.6) (with $w_1 = L_1$ and $w_2 = L_2$) and $L_2 < L_1$ in $[\mathcal{T}, T)$, we get

$$\dot{L}_1(t) \leqslant -\frac{\delta}{L_1(t)}\left(\cot\alpha_1 + \cot\vartheta_n + \frac{1}{\sin\vartheta_n}\right) = -\frac{\delta}{L_1(t)}\left[\cot\alpha_1 + \cot\left(\frac{\vartheta_n}{2}\right)\right], \quad t \in [\mathcal{T}, T). \tag{6.9}$$

Integrating (6.9) gives $L_1(T) \leqslant \sqrt{L_1(\mathcal{T})^2 - 2\delta[\cot\alpha_1 + \cot(\vartheta_n/2)](T - \mathcal{T})} < +\infty$, which contradicts the maximality of T. Hence, $T = +\infty$. From (6.9) and (6.8), (6.2) follows.

Step 4.   For any $t \in (\mathcal{T}, +\infty)$ let $\gamma_4(t)$ be the curve which has initial point in $q(\mathcal{T})$ and is created by the motion of $q(s) := \overline{S_1(s)} \cap \overline{S_2(s)}$ for $s \in (\mathcal{T}, t)$. Then $\gamma(t)$ is φ-regular for any $t \in (\mathcal{T}, \infty)$.

Let $(X(t), Y(t))$ be the component of $qq(t)$ with respect to the (τ_1, ν_1)-axis. Then, from (3.4), we get $\dot{X} = \dot{h}_1^v(t) \cot\vartheta_n - \dot{h}_2^v(t)/\sin\vartheta_n$, $\dot{Y} = \dot{h}_1^v$, and the slope of the tangent to the curve with respect to the (τ_1, ν_1)-axis is given by

$$\mathcal{K}(t) = \frac{\dot{Y}}{\dot{X}} = \frac{\dot{h}_1^v(t)}{\dot{h}_1^v(t)\cot\vartheta_n - \dot{h}_2^v(t)/\sin\vartheta_n} = \left(\cot\alpha_1 + \frac{1}{v(t)\sin\vartheta_n}\right)^{-1}.$$

Thus, γ_4 and Σ_3 join in a C^1 fashion since $\mathcal{K}(\mathcal{T}) = \tan(\pi - \vartheta_{\mathfrak{n}}) > 0$. Furthermore, γ_4 is concave in $[\mathcal{T}, +\infty)$ since from (6.7), $\dot{\mathcal{K}} = \dot{v}\mathcal{K}^2/(v^2 \sin \vartheta_{\mathfrak{n}}) > 0$, $\dot{X} = -\delta/(L_1\mathcal{K}) < 0$ and $d^2Y/dX^2 = \dot{\mathcal{K}}/\dot{X} < 0$. Finally,

$$\lim_{t \to \infty} \mathcal{K}(t) = \left(\cot \vartheta_{\mathfrak{n}} + \frac{1}{\sin \vartheta_{\mathfrak{n}}} \right)^{-1} = \tan\left(\frac{\vartheta_{\mathfrak{n}}}{2} \right) < \tan\left(\pi + \frac{2\pi}{n} - \vartheta_{\mathfrak{n}} \right),$$

where the right hand side gives the slope of a segment parallel to R_3.

Step 5. Conclusion of the proof.

Let $\Sigma_3(t) := \gamma_4(t) \cup \Sigma_3(\mathcal{T})$ and $\Pi(t) := \Sigma_1(t) \cup \Sigma_2(t) \cup \Sigma_3(t)$ for any $t \in (\mathcal{T}, +\infty)$. Then $\Pi(t)$ is nonpolygonal, unstable, and $t \in [\mathcal{T}, +\infty) \mapsto \Pi(t)$ is a φ-curvature flow starting from $\Pi(\mathcal{T})$. □

REMARK 6.2 We expect that the flow of Theorem 5.1 is the unique regular flow starting from $\Pi(\mathcal{T})$. Notice that if $x_{\min} < \delta$ in some open interval contained in $(\mathcal{T}, \mathcal{T} + \sigma)$ for some $\sigma > 0$, then, in view of the constraint $N_{\min|\Sigma_j}(q) \in T_{\varphi^o}(v_\varphi^j)$, at time $t = \mathcal{T}$ a new segment S_4 should appear in Σ_3 in such a way that $\Sigma_1(t) \cup \Sigma_2(t) \cup (S_4(t) \cup S_3(t)) \in (\text{a}')$ for any $t \in (\mathcal{T}, \mathcal{T} + \sigma)$ but this would give an unstable triod with $N_{\min} = (X(V_3), Y(V_3), V_3)$ in $(\mathcal{T}, \mathcal{T} + \sigma)$ since $L_2/L_1 < 1$ in $[\mathcal{T}, \mathcal{T} + \sigma)$ and

$$\mathcal{S}_{\text{a}'}\left(\frac{L_1}{L_4}, \frac{L_1}{L_2} \right) = \left\{ \left(\frac{L_1}{L_4}, \frac{L_1}{L_2} \right) \in (0, +\infty)^2 : \frac{L_1}{L_2} < \frac{\delta}{L - 2\delta} \right\},$$

a contradiction. Hence $x_{\min} = \delta$ in $[\mathcal{T}, \mathcal{T} + \sigma)$.

7. The case $\mathfrak{n} = 8$ and $\Pi \in (\text{c})$: disappearance of a segment

In this section we show that the flow has two different behaviours depending on the initial datum $\Pi \in (\text{c})$. For a suitable choice of Π, we show that one of the three segments vanishes at $t = \mathcal{T}$, its φ-curvature remains bounded, the Cahn–Hoffman vector field $N_{\min}$ has a jump discontinuity at $q(\mathcal{T})$ on each Σ_j and the triple junction translates along the remaining adjacent half-line in $[\mathcal{T}, +\infty)$. For the other choices of stable $\Pi \in (\text{c})$ we prove that at time $t = \mathcal{T} \in (0, +\infty)$ the flow becomes unstable, a curve appears from the triple junction, as in Section 6, with the difference that the adjacent segment now has positive φ-curvature and keeps on moving at subsequent times.

In the following theorem we denote by $x_{\min}^{(\text{b})}(\Lambda_1, +\infty, \Lambda_3)$ the limit of $x_{\min}^{(\text{b})}(\Lambda_1, \Lambda_2, \Lambda_3)$ as $\Lambda_2 \to +\infty$, where $x_{\min}^{(\text{b})}$ is defined as in (2.22).

THEOREM 7.1 Let $\mathfrak{n} = 8$ and let $\Pi \in (\text{c})$ be stable. Then $\mathcal{T} < +\infty$ and $\Pi(\mathcal{T})$ is unstable. Moreover, there exists a curve γ tangent to the line $\{x_{\min}(u, v) = \delta\}$ at $(u_2, v_2) = P_2$ (see Figure 12) which divides $\mathcal{S}_{\text{c}}(2, 1)$ into two disjoint regions $U_{\text{c}} := \{(\bar{u}, \bar{v}) \in \mathcal{S}_{\text{c}}(2, 1) : \bar{u} > u_2, \bar{v} > v_\gamma := \gamma \cap \{u = \bar{u}\}\}$ and $B_{\text{c}} := \mathcal{S}_{\text{c}}(2, 1) \setminus U_{\text{c}}$ such that

(i) If $(u(0), v(0)) \in B_{\text{c}}$ then $N_{\min}(\mathcal{T}-) = (X(V_2), W_2, Z(W_2))$, i.e. $x(\mathcal{T}-) = L-\delta$, $L_2(\mathcal{T}-) = 0$, $\kappa_\varphi(l_2(\mathcal{T}-)) = 0$ and $L_3(\mathcal{T}-)/L_1(\mathcal{T}-) = \sqrt{2}/2$. Furthermore, there exists a stable φ-curvature flow in $(\mathcal{T}, +\infty)$ starting from $\Pi(\mathcal{T})$ with

$$\kappa_\varphi(l_1(t)) = \frac{L - \widetilde{z}(t)}{L_1(t)}, \quad \kappa_\varphi(l_3(t)) = -\frac{L - \widetilde{x}(t)}{L_3(t)}, \quad t \in (\mathcal{T}, +\infty), \tag{7.1}$$

where

$$\widetilde{x}(t) := x_{\min}^{(b)}(L_3(t), +\infty, L_1(t)) = \frac{\delta(L-\delta)^2 L_3(t)}{\delta^2 L_3(t) + (L-2\delta)^2 L_1(t)}, \quad \widetilde{z}(t) = m\widetilde{x}(t) + q_z.$$

Finally, the triple junction translates along $R_2(\mathcal{T})$ in $(\mathcal{T}, +\infty)$ and

$$\lim_{t\uparrow\infty} \widetilde{x}(t) = L - \delta, \quad \lim_{t\uparrow\infty} \frac{L_3(t)}{L_1(t)} = \sqrt{2}, \quad \lim_{t\downarrow\mathcal{T}} \widetilde{x}(t) = \delta\frac{(1+\sqrt{2})^2}{1+2\sqrt{2}} \in (\delta, L - \delta).$$

In particular, $N_{\min}(\mathcal{T}-) \neq N_{\min}(\mathcal{T}+)$ (see Figure 13(i)).

(ii) If $(u(0), v(0)) \in U_c$ then $N_{\min}(\mathcal{T}) = (X(V_3), Y(V_3), V_3)$. Furthermore, there exist $\mathcal{T}_2 \in (\mathcal{T}, +\infty]$ and a φ-curvature flow in $[\mathcal{T}, \mathcal{T}_2)$ starting from $\Pi(\mathcal{T})$. Furthermore, for any $t \in [\mathcal{T}, \mathcal{T}_2)$, $\Pi(t)$ is nonpolygonal, $N_{\min}(t) = (X(V_3), Y(V_3), V_3)$, $\kappa_\varphi(l_3(t)) = -L/L_3(t)$ and (6.1) holds (see Figure 13(ii)).
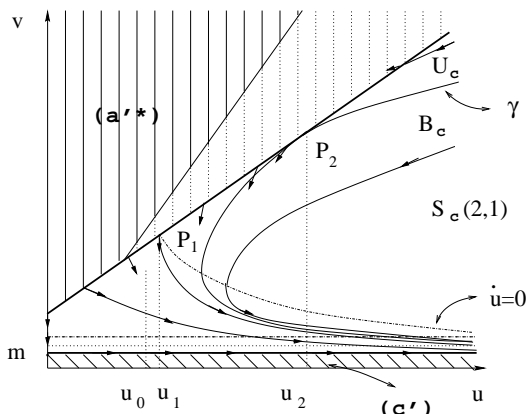
FIG. 12. The white region is the stability region $\mathcal{S}_c(2, 1)$. Flow lines diagram of system (7.5) corresponding to φ-curvature flows starting from stable $\Pi \in \{(\text{c})\}$ for $\mathfrak{n} = 8$ (see Theorem 7.1); $x_{\min}(\mathcal{T}) \in \{\delta, L - \delta\}$.
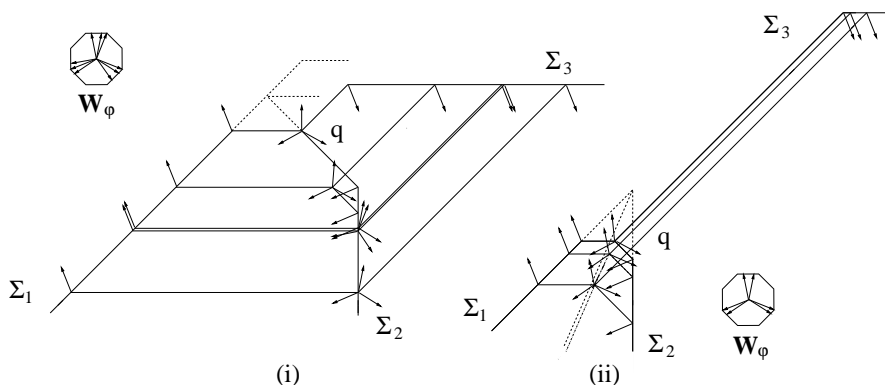

FIG. 13. φ-curvature flow starting from a stable $\Pi \in \{(\text{c})\}$ (see Theorem 7.1): (i) the segment S_2 has zero length at time $t = \mathcal{T}$ and $x(\mathcal{T}) = L - \delta$; (ii) $x(\mathcal{T}) = \delta$ and a curve develops from the triple junction for $t > \mathcal{T}$.

Proof. For any $t \in [0, \mathcal{T})$ we define $x(t) = x_{\min}^{(c)}(L_1(t), L_2(t), L_3(t))$, $y(t)$ and $z(t)$ as in (2.19) with x replaced by $x(t)$. Then system (2.9) reads

$$\dot{h}_1^v(t) = -\frac{x(t)}{L_1(t)}, \qquad \dot{h}_2^v(t) = \frac{y(t)}{L_2(t)}, \qquad \dot{h}_3^v(t) = \frac{L - z(t)}{L_3(t)}, \tag{7.2}$$

system (3.1) reads

$$L_1(t) = L_1 + \sqrt{2}h_3^v(t), \qquad L_2(t) = L_2 + h_2^v(t) - h_3^v(t), \qquad L_3(t) = L_3 + h_2^v(t) + h_3^v(t), \tag{7.3}$$

while (3.2) and (3.3) become

$$-\sqrt{2}h_1^v(t) = h_2^v + h_3^v, \qquad \sqrt{2}(L_1(t) - L_1) = L_3(t) - L_2(t) + L_2 - L_3. \tag{7.4}$$

Define $u(t) := L_3(t)/L_2(t)$ and $v(t) := L_3(t)/L_1(t)$. Differentiating (7.3) with respect to time and using (2.23), (2.19), and $m = \sqrt{2}/2$, we obtain, for any $t \in [0, \mathcal{T})$,

$$\begin{aligned}
\dot{u} &= \frac{L - \delta}{L_3^2} \frac{u[2\delta^2 u^2 + v(L - 2\delta)(-\delta u^2 + Lu + L - \delta)]}{\delta^2 + \delta^2 u + (L - 2\delta)^2 v}, \\
\dot{v} &= \frac{(L - 2\delta)(L - \delta)^2}{L_3^2} \frac{v^2(1 - \sqrt{2}v)}{\delta^2 + \delta^2 u + (L - 2\delta)^2 v}.
\end{aligned} \tag{7.5}$$

Recall that the stability region $\mathcal{S}_c(2, 1)$ is given by Lemma 3.10 (see Figure 12). It follows that $\dot{v} \leqslant 0$ in $\overline{\mathcal{S}_c(2, 1)}$ with equality holding only if $v = m$ (i.e. $\{x_{\min}(u, v) = L - \delta\}$). Notice that $-\delta u^2 + Lu + L - \delta < 0$ for $u > u_0 := (L + \sqrt{L^2 + 4\delta(L - \delta)})/(2\delta)$. Moreover, $\dot{u} \leqslant 0$ if and only if $u > u_0$ and

$$v \geqslant \frac{\sqrt{2}\delta u^2}{\delta u^2 - Lu - L + \delta} \tag{7.6}$$

or, more precisely, if and only if (7.6) holds for any $u > u_1$, where $u_1 := (L + \delta + \sqrt{(L + \delta)^2 + 4\delta L})/(2\delta)$ is the intersection point of the line $\{x_{\min}(u, v) = \delta\}$ and the curve of points satisfying $\dot{u} = 0$. Since the condition

$$\left(\frac{\dot{v}}{\dot{u}}\right)_{|\{x_{\min}(u,v)=\delta\}} < \frac{\delta}{L - 2\delta}, \qquad u > u_1,$$

is satisfied if and only if $g(u) := -\delta^3 u^3 + 2\delta^2 Lu^2 + \delta L(2L - \delta)u + L^2(L - \delta) < 0$, that is, for any $u > u_2$, $u_2 > u_1$ (for $g(u_0) > 0$), it follows that the trajectories of solutions of system (7.5) intersect the line $\{x_{\min}(u, v) = \delta\}$ for $u < u_2$. Denote by P_2 the point belonging to the line $\{x_{\min}(u, v) = \delta\}$ having u-coordinate equal to u_2. Let $\gamma \subset \mathcal{S}_c$ be the flow line tangent to $\{x_{\min}(u, v) = \delta\}$ at P_2. Then γ decomposes $\mathcal{S}_c(2, 1)$ into B_c and U_c.

Let us first prove (ii). If $(u(0), v(0)) \in U_c$, then the trajectory of the solution of (7.5) intersects the line $\{x_{\min}(u, v) = \delta\}$ at $\mathcal{T}$. It is clear that $\mathcal{T} < +\infty$ since any $L_j(\mathcal{T})$ is bounded and

$$\dot{L}_1(t) \leqslant -2\dot{h}_1^v(t) \leqslant 2\frac{L - \delta}{L_1(t)}, \qquad \dot{L}_3(t) = -\sqrt{2}\dot{h}_1^v(t) \geqslant \frac{\delta}{L_1(t)}.$$

Let $(w_1, w_2) \in C^\infty([\mathcal{T}, T); (0, +\infty)^2)$ for some $T \in (\mathcal{T}, +\infty]$ be the solution of (6.6) with F defined as in (6.5). For any $t \in (\mathcal{T}, T)$ and $j = 1, 2$ we define $\Sigma_j(t)$, h_j^v and γ_4 as in Theorem 6.1.

Then, by the same argument, (6.1) holds in $[\mathcal{T}, T)$, $t \in (\mathcal{T}, T) \mapsto \Sigma_1(t) \cup \Sigma_2(t)$ is a flow starting from $\Sigma_1(\mathcal{T}) \cup \Sigma_2(\mathcal{T})$ which satisfies (i)–(iii) of Definition 2.12, $T = +\infty$ and γ_4 is concave and φ-regular in $[\mathcal{T}, +\infty)$. Let $t \in [\mathcal{T}, +\infty) \mapsto \gamma_4(\infty) \cup S_3(t) \cup R_3(t)$ be the φ-curvature flow starting from $\gamma_4(\infty) \cup (S_3 \cup R_3)(\mathcal{T})$. Since $\dot{h}_3^v(t) = -L/L_3(t) \in C^\infty([\mathcal{T}, +\infty))$ and from (7.4) we have

$$\sqrt{2}|\dot{h}_1^v(\mathcal{T})| = -\sqrt{2}\dot{h}_1^v(\mathcal{T}) = \dot{h}_2^v(\mathcal{T}) + \dot{h}_3^v(\mathcal{T}) > \dot{h}_3^v(\mathcal{T}) = |\dot{h}_3^v(\mathcal{T})|,$$

it follows that $|\dot{h}_3^v(t)| \leqslant \sqrt{2}|\dot{h}_1^v(t)|$ (that is, $v(t) \geqslant \sqrt{2}/2$) for any t in a neighbourhood of $\mathcal{T}$, say $(\mathcal{T}, \mathcal{T}_2)$. We conclude that the normal velocity $\dot{h}_3^v$ of $S_3(t)$ is smaller than the ones of $S_1(t)$ and $S_3(t)$. Thus, setting $\Sigma_3(t) := \gamma_4(t) \cup (S_3 \cup R_3)(t)$ and $\Pi(t) := \Sigma_1(t) \cup \Sigma_2(t) \cup \Sigma_3(t)$ for any $t \in (\mathcal{T}, \mathcal{T}_2)$, we conclude that the triod $\Pi(t)$ is φ-regular and unstable in $[\mathcal{T}, \mathcal{T}_2)$ and $t \in [\mathcal{T}, \mathcal{T}_2) \mapsto \Pi(t)$ is the unique φ-curvature flow starting from $\Pi(\mathcal{T})$.

Let us prove (i). Given $\sigma > 0$, set

$$B_c(\sigma) := \{(u, v) \in B_c : 2\delta^2 u^2 + v(L - 2\delta)(-\delta u^2 + Lu + L - \delta) > \sigma, u > u_1\}.$$

Notice that $B_c(\sigma) = \{(u, v) \in B_c : \dot{u}(u, v) > \sigma, u > u_1\}$. Without loss of generality, assume $(u(0), v(0)) \in B_c(\sigma)$. Then $(u(t), v(t)) \in B_c(\sigma)$ for any $t \in [0, \mathcal{T})$. For any $(u, v) \in B_c(\sigma)$ we deduce the estimates

$$-c_1 \frac{\sqrt{2}v - 1}{\sigma u} \leqslant \frac{\dot{v}}{\dot{u}} \leqslant -c_2 \frac{\sqrt{2}v - 1}{u^3}$$

with $c_1 := (L - 2\delta)(L - \delta)v_1^2$ and $c_2 := (L - 2\delta)(L - \delta)/(4\delta^2)$. Integrating yields

$$\frac{\sqrt{2}}{2}(1 + u^{-\sqrt{2}c_1/\sigma}) \leqslant v \leqslant \frac{\sqrt{2}}{2}(1 + e^{c_2/(\sqrt{2}u^2)}),$$

and hence $\lim_{t \to \mathcal{T}} u(t) = +\infty$, $\lim_{t \to \mathcal{T}} v(t) = \sqrt{2}/2$. Thus the first part of the assertion follows from (2.23) and (2.19).

Let us show $\mathcal{T} < +\infty$. Let $\varepsilon > 0$ and assume $v(0) \leqslant \sqrt{2} - \varepsilon$. Then $v(t) \leqslant \sqrt{2} - \varepsilon$ for any $t \in [0, \mathcal{T})$, and $\dot{L}_3(t) = \sqrt{2}x/L_1(t) \leqslant CL_3(t)$, where $C := (2 - \sqrt{2}\varepsilon)(l - \delta)$. It follows that

$$L_3(t) \leqslant \sqrt{L_3^2 + 2Ct}. \tag{7.7}$$

Using (7.3) and (7.2) we get

$$\dot{L}_2(t) = -\sqrt{2}\dot{h}_1^v(t) - 2\dot{h}_3^v(t) \leqslant \sqrt{2}\frac{L - \delta}{L_3(t)}\left(\frac{L_3(t)}{L_1(t)} - \sqrt{2}\right) \leqslant -\sqrt{2}\varepsilon \frac{L - \delta}{L_3(t)},$$

and inserting (7.7) and integrating yields

$$L_2(t) \leqslant L_2^2 - \frac{\sqrt{2}\varepsilon(L - \delta)}{C}\sqrt{L_3^2 + 2Ct}.$$

We conclude that $\mathcal{T} < +\infty$ and, from Proposition 3.7(v), also $L_1(\mathcal{T}-), L_3(\mathcal{T}-) < +\infty$. Moreover, $\Pi(\mathcal{T}-) \in (b')$ is degenerate. Notice that the φ-curvature flow starting from $\Pi(\mathcal{T}-)$ can be described as the φ-curvature flow starting from $\widetilde{\Pi} \in (b)$ obtained by a rotation and a symmetry with respect the L_3-axis from Π, i.e. $\widetilde{L}_2 = +\infty$ and $\widetilde{L}_3 = \sqrt{2}\widetilde{L}_1$. By the mirror law (2.24), $\Pi(\mathcal{T}-)$ is

stable, the φ-curvature flow starting from $\Pi(\mathcal{T}-)$ satisfies system (2.7) with φ-curvatures in (7.1) and

$$\lim_{t \to \infty} \frac{\widetilde{L}_3(t)}{\widetilde{L}_1(t)} = \lim_{t \to \infty} \frac{L_1(t)}{L_3(t)} = \frac{\sqrt{2}}{2}.$$

Finally, since $\delta \dot{h}_1^\nu = -(L - \delta)\dot{h}_3^\nu$ in $[\mathcal{T}, +\infty)$, the triple junction translates along $R_2(\mathcal{T})$. $\square$

REMARK 7.2 We believe that the flows of Theorem 7.1(i) and (ii) are the unique regular flows starting from $\Pi(\mathcal{T})$.

8. Adjacent triple junctions: solutions after the collision

In this section we fix $\mathcal{W}_\varphi = P_8$ and consider the φ-curvature flow starting from a *stable* φ-regular partition, denoted by Υ, consisting of two adjacent elementary triple junctions q_1 and q_2. Given a Cahn–Hoffman vector field N on Υ as in Figure 14, we set

$$(X_1, Y_1, Z_1) := (N_{|\Sigma_1}(q_1), N_{|\Sigma_2}(q_1), N_{|\Sigma_3}(q_1)),$$
$$(X_2, Y_2, Z_2) := (N_{|\Sigma_1}(q_2), N_{|\Sigma_4}(q_2), N_{|\Sigma_5}(q_2)),$$

and $x_j := |V_1 - N_{|\Sigma_1}(q_j)|$, $y_j := y(x_j)$, $z_j := z(x_j)$, $j = 1, 2$, where y and z are defined in (2.19). From Proposition 2.21 we know that the admissible triplet (X_j, Y_j, Z_j) is uniquely associated with (x_j, y_j, z_j). Upon noticing that we can restrict the minimum (2.3) to vector fields which are linear on each Σ_i and satisfy the required constraints, the problem of finding $N_{\min}$ in Definition 2.4 reduces [3] to the following minimum problem:

$$\min_{(x_1, x_2) \in [\delta, L-\delta]^2} f(x_1, x_2),$$

where

$$f(x_1, x_2) = \int_\Upsilon (\mathrm{div}_\tau N)^2 \varphi^o(\nu) \, \mathrm{d}\mathcal{H}^1 = \sigma_1 x_1^2 + \sigma_2 x_2^2 + \sigma_{12} x_1 x_2 + \beta_1 x_1 + \beta_2 x_2 + \gamma,$$

and $\sigma_1, \sigma_2, \sigma_{12}, \beta_1, \beta_2, \gamma$ are coefficients depending on the configuration we are analyzing.

We say that Υ is *stable* if $N_{\min|\Sigma_j}(q_k)$ is not a vertex of $\mathcal{W}_\varphi$ for any $j = 1, 2, 3$ and $k = 1, 2$. We say that Υ is *unstable* if it is not stable. The stability of Υ is equivalent to

$$(x_{1\min}, x_{2\min}) \in (\delta, L - \delta)^2.$$

Notice that if Υ is stable then

$$x_{1\min} = \frac{\sigma_{12}\beta_2 - 2\sigma_2\beta_1}{4\sigma_1\sigma_2 - \sigma_{12}^2}, \qquad x_{2\min} = \frac{\sigma_{12}\beta_1 - 2\sigma_1\beta_2}{4\sigma_1\sigma_2 - \sigma_{12}^2}. \qquad (8.1)$$

From now on, $x_{1\min}, x_{2\min}$ will be denoted simply by $\overline{x}_1, \overline{x}_2$ and we set $\overline{y}_j := y(\overline{x}_j), \overline{z}_j := z(\overline{x}_j)$.

The discussion which values of L_i, $i = 1, \ldots, 5$, provide a stable Υ simplifies only in the case of adjacent triple junctions which either belong to the same symmetry classes (i.e. $L_2 = L_4$ and $L_3 = L_5$, see Figure 14(i)) or are symmetric with respect to the axis orthogonal to Σ_1 at its middle point (see Figure 14(ii)), leading respectively to $\overline{x}_1 = \overline{x}_2$ and $\overline{x}_1 = L - \overline{x}_2$.

Let $T > 0$ and let us introduce the orientation of Υ as in the comment after Definition 2.8. We say that $t \in [0, T) \mapsto \Upsilon(t)$ is a φ-curvature flow starting from Υ if $\Upsilon(t)$ is a φ-regular partition consisting of two adjacent elementary triple junctions $q_1(t)$ and $q_2(t)$, and conditions (ii)–(iii) of Definition 2.12 hold for any $j = 1, 2, 3, 4, 5$. If the adjacent triple junctions of Υ belong to the same symmetry classes or are symmetric with respect to the axis orthogonal to Σ_1 then, arguing

as in Theorem 3.3, one can show that there exists a unique stable φ-curvature flow starting from a stable Υ.

When Υ is not stable, at least one of the two triple junctions is not stable. If in addition the gradient of f on $\partial[\delta, L - \delta]^2$ points inside $[\delta, L - \delta]^2$, then the appearance of a new edge from one of the two triple junctions (or from both) is forced during the subsequent crystalline flow.

The following example shows that the collision phenomenon occurs and a quadrijunction forms.

EXAMPLE 8.1   Consider the partition of Figure 14(i) with $L_1 > 0$, $L_2 = L_4 > 0$ and $L_3 = L_5 > 0$. In this case f reads

$$f(x_1, x_2) = \varphi^o(\nu_1) \left(\frac{(x_1 - x_2)^2}{L_1} + \frac{y_1^2}{L_2} + \frac{z_1^2}{L_3} + \frac{y_2^2}{L_4} + \frac{z_2^2}{L_5} \right), \tag{8.2}$$

so that

$$\sigma_1 = \frac{1}{L_1} + m \left(\frac{1}{L_2} + \frac{1}{L_3} \right) > 0, \quad \sigma_2 = \frac{1}{L_1} + m \left(\frac{1}{L_4} + \frac{1}{L_5} \right) > 0, \quad \sigma_{12} = -\frac{2}{L_1},$$
$$\beta_1 = 2m \left(\frac{-q_y}{L_2} + \frac{q_z}{L_3} \right), \qquad \beta_2 = 2m \left(\frac{-q_y}{L_4} + \frac{q_z}{L_5} \right), \tag{8.3}$$

where m, q_y, q_z are defined in (2.15) and (2.16). Thus, the triod is *always stable* since by (8.1),

$$\bar{x}_1 = \bar{x}_2 = \frac{-\beta_1}{2\sigma_1 + \sigma_{12}} = \frac{\delta L_2 + (L - \delta)L_3}{L_2 + L_3} \in (\delta, L - \delta). \tag{8.4}$$

The evolution equations are given by

$$\dot{h}_1^\nu = 0, \quad \dot{h}_2^\nu = \frac{\bar{y}_1}{L_2}, \quad \dot{h}_3^\nu = -\frac{\bar{z}_1}{L_3}, \quad \dot{h}_4^\nu = -\frac{\bar{y}_2}{L_4}, \quad \dot{h}_5^\nu = \frac{\bar{z}_2}{L_5},$$

so that the triple junctions move along Σ_1 until they collide at the middle point at a finite time.
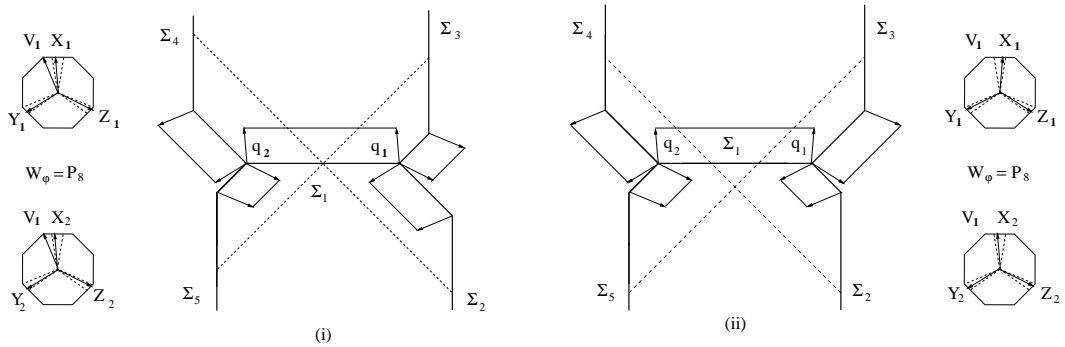

FIG. 14. Example 8.1: collision of two adjacent triple junctions; in (i) we have $L_2 = L_4$, $L_3 = L_5$ and $\kappa_\varphi = 0$ on Σ_1.

Consider now the partition of Figure 14(ii) with $L_1 > 0$, $L_2 = L_5 > 0$ and $L_3 = L_4 > 0$. Formulas (8.2) and (8.3) still hold, and using (8.1), we get

$$\bar{x}_1 = L - \bar{x}_2 = \frac{\frac{L}{m^2 L_1} + \frac{L - \delta}{L_2} + \frac{\delta}{L_3}}{\frac{2}{m^2 L_1} + \frac{1}{L_2} + \frac{1}{L_3}} \in (\delta, L - \delta).$$

Therefore, the triod is *always stable* and the evolution equations are given by

$$\dot{h}_1^v = -\frac{\overline{x}_1 - \overline{x}_2}{L_1} = \frac{L - 2\overline{x}_1}{L_1}, \quad \dot{h}_2^v = \frac{\overline{y}_1}{L_2}, \quad \dot{h}_3^v = -\frac{\overline{z}_1}{L_3}, \quad \dot{h}_4^v = -\frac{\overline{y}_2}{L_4}, \quad \dot{h}_5^v = \frac{\overline{z}_2}{L_5}.$$

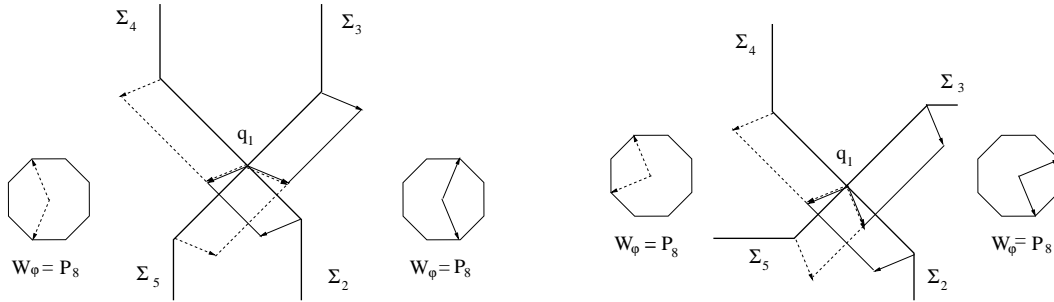If in addition $L_3 > L_2$, the triple junctions move as shown in Figure 14(ii) until they collide at a finite time and a quadrijunction forms.

FIG. 15. These quadrijunctions are φ-regular with $\kappa_\varphi = 0$ and unstable, i.e. $N_{\min | \Sigma_j}(q_1)$ is a vertex of $\mathcal{W}_\varphi$ for some $j = 2, 3, 4, 5$.

In general, it is not clear what happens after the collision. In a special case the solution can be continued in a "natural" way (see Example 8.4 below). A quadrijunction $\varXi$ as in Figure 15 is φ-regular with $\kappa_\varphi = 0$ and unstable, i.e. $N_{\min | \Sigma_j}(q_1)$ is a vertex of $\mathcal{W}_\varphi$ for some $j \in \{2, 3, 4, 5\}$. Indeed, the minimizer $N_{\min}$ of (2.3) with Γ replaced by $\varXi$ which satisfies $\sum_{j=2}^{5}(N_{\min | \Sigma_j})^{\partial \Sigma_j} = 0$ is given in Figure 15. Finally, $t \mapsto \varXi$ is a stationary φ-curvature flow starting from $\varXi$, i.e. $\varXi$ does not move.

(i)   (ii)
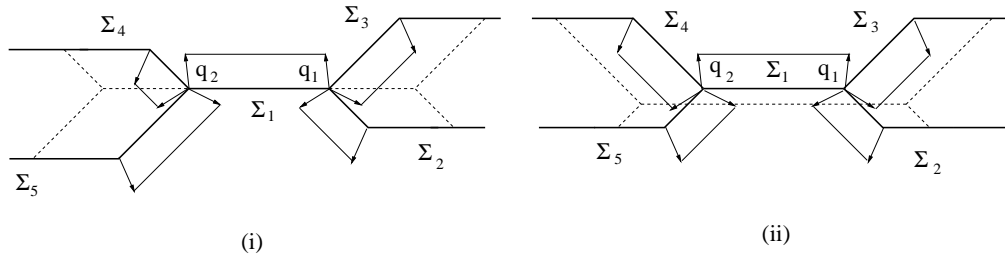
FIG. 16. Example 8.2: these configurations are stable only for suitable choices of L_j, $j = 1, \ldots, 5$.

The following example concerns the stability of the partitions given in Figure 16 (see (8.6)), and will be used to construct the flow after the collision of two triple junctions (see Example 8.1).

EXAMPLE 8.2 Consider the partition of Figure 16(i) with $L_1 > 0$, $L_2 = L_4 > 0$ and $L_3 = L_5 > 0$. In this case

$$f(x_1, x_2) = \varphi^o(\nu_1)\left(\frac{(x_1 - x_2)^2}{L_1} + \frac{(L - y_1)^2}{L_2} + \frac{(L - z_1)^2}{L_3} + \frac{(L - y_2)^2}{L_4} + \frac{(L - z_2)^2}{L_5}\right), \quad (8.5)$$

so that $\sigma_1, \sigma_2, \sigma_{12}$ are given as in (8.3) and

$$\beta_1 = 2m\left(\frac{L - q_y}{L_2} - \frac{L - q_z}{L_3}\right), \quad \beta_2 = 2m\left(\frac{L - q_y}{L_4} - \frac{L - q_z}{L_5}\right).$$

Furthermore, since the first two equalities in (8.4) hold, we have

$$\overline{x}_1 = \overline{x}_2 = \frac{(L-\delta)^2 L_2 - (L^2 - 3\delta L + \delta^2)L_3}{\delta(L_2 + L_3)} \in (\delta, L - \delta) \Leftrightarrow \frac{L_3}{L_2} \in \left(\frac{L-\delta}{L}, \frac{L}{L-\delta} \right). \quad (8.6)$$

The evolution equations are given by

$$\dot{h}_1^\nu = 0, \quad \dot{h}_2^\nu = -\frac{L - \overline{y}_1}{L_2}, \quad \dot{h}_3^\nu = \frac{L - \overline{z}_1}{L_3}, \quad \dot{h}_4^\nu = \frac{L - \overline{y}_2}{L_4}, \quad \dot{h}_5^\nu = -\frac{L - \overline{z}_2}{L_5}.$$

We observe that, by symmetry, $\dot{h}_4^\nu = -\dot{h}_2^\nu$ and $\dot{h}_5^\nu = -\dot{h}_3^\nu$, and by direct computations, $\dot{h}_2^\nu = -(2L - \delta)/(L_2 + L_3) = -\dot{h}_3^\nu$. Hence, if we assume that the initial partition is stable, the flow is stable in the whole of $[0, +\infty)$ with the triple junctions translating in opposite directions along Σ_1.

   Consider now the partition of Figure 16(ii) with $L_1 > 0$, $L_2 = L_5 > 0$ and $L_3 = L_4 > 0$. Since (8.5) still holds, the expressions of $\sigma_1, \sigma_2, \sigma_{12}, \beta_1, \beta_2$ are the same. From (8.1) it follows that

$$\overline{x}_1 = L - \overline{x}_2 = \frac{\frac{L(L-2\delta)^2}{L_1} - \frac{\delta(L^2 - 3\delta L + \delta^2)}{L_2} + \frac{\delta(L-\delta)^2}{L_3}}{\frac{2(L-2\delta)^2}{L_1} + \frac{\delta^2}{L_2} + \frac{\delta^2}{L_3}},$$

so that

$$\overline{x}_1 > \delta \Leftrightarrow \frac{L_3}{L_1} > \frac{m}{L - 2\delta} \left(-L + (L - \delta)\frac{L_3}{L_2} \right),$$

$$\overline{x}_1 < L - \delta \Leftrightarrow \frac{L_3}{L_1} > \frac{m}{L - 2\delta} \left(L - \delta - \delta\frac{L_3}{L_2} \right).$$

Therefore, if the initial partition is stable, the evolution equations are given by

$$\dot{h}_1^\nu = \frac{L - 2\overline{x}_1}{L_1}, \quad \dot{h}_2^\nu = -\frac{L - \overline{y}_1}{L_2}, \quad \dot{h}_3^\nu = \frac{L - \overline{z}_1}{L_3}, \quad \dot{h}_4^\nu = \frac{L - \overline{y}_2}{L_4}, \quad \dot{h}_5^\nu = -\frac{L - \overline{z}_2}{L_5}. \quad (8.7)$$

If in addition $L_3 > L_2$, the triple junctions move (for small times) as shown in Figure 14(ii).

REMARK 8.3   From the computations in Example 8.2, it follows that the first partition is stable for any $L_1 > 0$ while the second is stable provided that L_1 is small enough. In particular, if $L_1 = 0$ then $\overline{x}_1 = \overline{x}_2 = L/2$. Hence, in Example 8.2 we have constructed *stable φ-curvature flows starting from a quadrijunction*.

   In the following example we construct flows after the collision of two triple junctions.

EXAMPLE 8.4   Consider the partition of Figure 17(i) with $L_1 > 0$, $L_2 = L_4 = L_3 = L_5 > 0$. As shown in Example 8.1 there exists a finite time $T_0 > 0$ such that $L_1(T_0) = 0$. From Remark 8.3, there exists at least one stable φ-curvature flow starting from the quadrijunction $\Upsilon(T_0)$ as shown in Figure 17(i). This is not the only stable φ-curvature flow starting from $\Upsilon(T_0)$. There are other candidates to continue the flow after the singularity: the stable flows shown in Figures 17(ii) and (iii) and the stationary flow $t \in [T_0, +\infty) \mapsto \Upsilon(T_0)$. The latter flow is not stable and has the largest energy (1.1) among the four flows.

   Some explicit comparisons between the energies of the different evolutions can be made. For instance, if we denote by $\Upsilon_{(i)}(t)$ (resp. $\Upsilon_{(ii)}(t)$) the partition in Figure 17(i) (resp. Figure 17(ii)) at time t, then $\mathcal{F}_\varphi(\Upsilon_{(ii)}(t)) < \mathcal{F}_\varphi(\Upsilon_{(i)}(t))$ for any $t > T_0$, since $\dot{h}_{3(i)}^\nu(t) = (2l - \delta)/(2L_3(T_0)) < L/L_3(t) = \dot{h}_{3(ii)}^\nu(t)$.
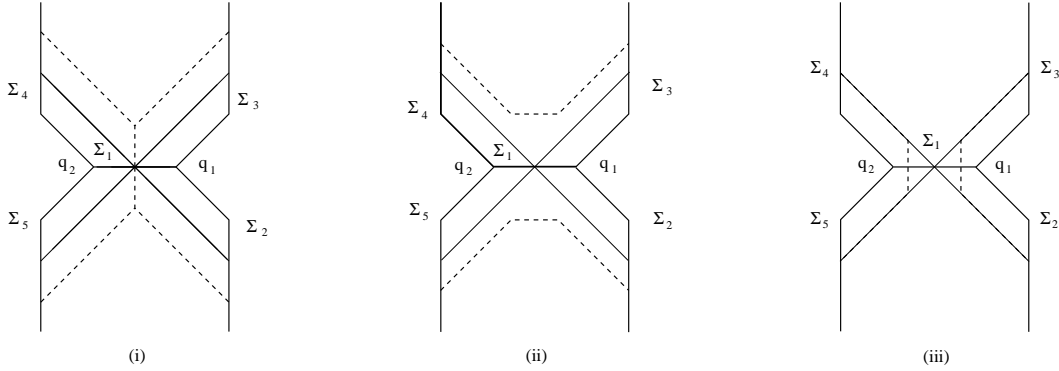
FIG. 17. Example 8.4: existence of three stable flows *after* the collision of two adjacent triple junctions. Another candidate to continue the flow after the singularity is the stationary (not stable) flow $t \in [T_0, +\infty) \mapsto \Upsilon(T_0)$. The dotted partition in (i) is the 90° rotation of the partition in Figure 16(i) with $L_2 = L_3 = L_4 = L_5$.

Furthermore, notice that in the case of the flow in Figure 17(i), $\kappa_\varphi(l_j(T_0-)) = \kappa_\varphi(l_j(T_0+))$ for any $j = 2, 3, 4, 5$, while in the other cases $\kappa_\varphi(l_j(T_0-)) \neq \kappa_\varphi(l_j(T_0+))$ for any $j = 2, 3, 4, 5$.

We believe that a selection of the "most natural" evolution between the three flows in Figure 17 cannot probably be done if one considers the evolution of interfaces without looking at the phases, i.e. without looking at the interfaces as the "boundaries" of their interior.

9. Homothetic flows and asymptotic convergence ($\mathfrak{n} = 6\mathfrak{m}$)

In this section we introduce the notion of homothetic flows and we assume $\mathfrak{n} = 6\mathfrak{m}$. In the case of curves the homothetic flows by crystalline curvature have been studied in [29], [15].

DEFINITION 9.1 Let Π be elementary. We say that $t \in [0, +\infty) \mapsto \Pi(t)$ is a *homothetic flow* starting from $\Pi = \Pi(0)$ if there exists $\lambda \in C^0([0, +\infty))$, $\lambda(0) = 1$, such that $\Pi(t) = \lambda(t)\Pi(0) + qq(t)$. If $\lambda \equiv 1$ we say that the flow is *translating*; if in addition $qq(t) = q$ the flow is *stationary*.

REMARK 9.2 $t \in [0, +\infty) \mapsto \Pi(t)$ is homothetic if and only if $L_j(t)/L_i(t) = L_j(0)/L_i(0)$ for any $i, j = 1, 2, 3$. The flow is stationary whenever an elementary triod has two of the segments S_j of infinite length.

We now characterize all homothetic flows for $\mathfrak{n} = 6\mathfrak{m}$. If $\mathfrak{n} = 6$ we will consider the following limit cases of degenerate triod:

(i) $\Pi \in$ (a) with $L_1 = 0$ and $L_3 = +\infty$ (see Figure 18(i));
(ii) $\Pi \in$ (a) with $L_2 = 0$ and $L_3 = +\infty$ (see Figure 18(ii)).

THEOREM 9.3 Let $\mathfrak{n} = 6\mathfrak{m}$. If $\Pi \in$ (d), the flow is stationary for any choice of $L_1, L_2, L_3 \in (0, +\infty]$. If $\Pi \in$ (a), the flow is homothetic if and only if one of the following holds:

(i) $L_2 = +\infty$ and $L_1, L_3 \in (0, +\infty]$. The flow is stationary and $\Pi(t)$ is unstable for any $t \in [0, +\infty)$.
(ii) $L_1 = +\infty$ and $L_2 = L_3$ (see Figures 18(iv) and 19(ii)). The flow is stable.
(iii) $\mathfrak{n} \geqslant 12$ and $L_3 = +\infty$, $L_1 = L_2$ (see Figure 19(i)). The flow is stable.
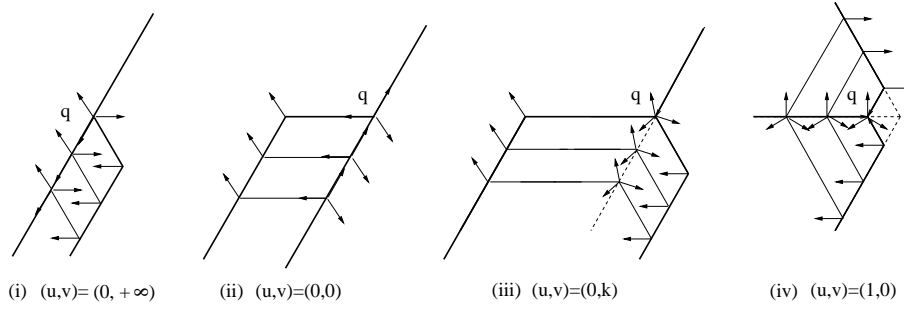
FIG. 18. The case $\mathfrak{n} = 6$. The flows (i), (ii) and (iii) are translating; (iv) is homothetic.
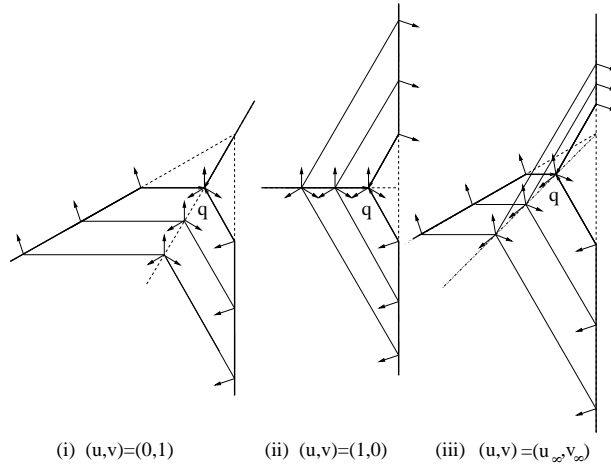

FIG. 19. The case $\mathfrak{n} = 12$. The flows (i), (ii) and (iii) are homothetic.

(iv) $\mathfrak{n} = 6$ and $L_3 = +\infty$, $(L_1, L_2) \in [0, +\infty)^2 \setminus \{(0, 0)\}$ (see Figure 18(i), (ii), (iii)). The flow is translating. The triod $\Pi(t)$ is unstable for any $t \in [0, +\infty)$ if either $L_1 = 0$, $L_2 \in (0, +\infty)$, or $L_2 = 0$, $L_1 \in (0, +\infty)$ (resp. (i) and (ii) in Figure 18); in the other cases, the flow is stable.

(v) $\mathfrak{n} \geqslant 12$ and $L_2 = u_\infty L_3$, $L_2 = v_\infty L_1$ (see Figure 19(iii)), where u_∞, v_∞ are defined in (3.20). The flow is stable.

Proof. If $\Pi \in$ (d) then $\dot{h}_j^v(t) = 0$ for any $t \in [0, +\infty)$, $j = 1, 2, 3$ (see Table 1).

Assume $\Pi \in$ (a). From (3.21), (i) follows since $x(t) = 0$ and $\dot{h}_j^v(t) = 0$ for any $t \in [0, +\infty)$, $j = 1, 2, 3$.

Now let $u(t) := L_2(t)/L_3(t)$ and $v := L_2(t)/L_1(t)$. Recalling that $\cot \alpha_1 = -\cot \alpha_2 = \cot \alpha_3$ and $\vartheta_\mathfrak{n} = 2\pi/3$, from (3.21), (3.13), (3.15) and (3.18), we obtain

$$
\begin{aligned}
\dot{u} &= \frac{L}{L_2^2} \frac{u[-(\cot \alpha_1 - 1/\sqrt{3})u - (\cot \alpha_1 + 1/\sqrt{3})v - (2/\sqrt{3})uv + (\cot \alpha_1 - 1/\sqrt{3})u^2]}{u + v + 1}, \\
\dot{v} &= \frac{L}{L_2^2} \frac{v[-(\cot \alpha_1 - 1/\sqrt{3})u - (\cot \alpha_1 + 1/\sqrt{3})v + (2/\sqrt{3})uv + (\cot \alpha_1 + 1/\sqrt{3})v^2]}{u + v + 1}.
\end{aligned}
\tag{9.1}
$$

From Remark 9.2 it follows that all homothetic flows are constant solutions (u, v) of system (9.1). We look for solutions of (9.1) of the form $v/u = K$, K constant. Imposing $\dot{v} = K\dot{u}$ we obtain three possibilities:

(1) $K = 0$; that gives (ii), since $\dot{h}_1^v(t) = 0$, $\dot{h}_2^v(t) = -\dot{h}_3^v(t) = L/(L_2(t) + L_3(t))$ and (3.17), (3.19) hold.

(2) $u = 0$; that gives (iii) and (iv), since $\dot{h}_3^v(t) = 0$, $\dot{h}_2^v(t) = -\dot{h}_1^v(t) = L/(L_1(t) + L_2(t))$ and (3.13), (3.15) hold. Notice that if $\mathfrak{n} = 6$ then $\dot{L}_1 = \dot{L}_2 = 0$ while if $\mathfrak{n} \geqslant 12$ then $\dot{L}_1 = \dot{L}_2 > 0$.

(3) $K = r$ and $v = v_\infty$, $u = u_\infty$; that gives (v). Indeed, equating the brackets on the right hand sides of (9.1) we get

$$-\left(\cot\alpha_1 + \frac{1}{\sqrt{3}}\right)v^2 - \frac{4}{\sqrt{3}}uv + \left(\cot\alpha_1 - \frac{1}{\sqrt{3}}\right)u^2 = 0,$$

and thus $v = ru$ for $\mathfrak{n} \geqslant 12$ and $u = 0$ for $\mathfrak{n} = 6$ (observe that if $\mathfrak{n} \geqslant 12$ then $\cot\alpha_1 + 1/\sqrt{3} \leqslant 0$ and equality holds if and only if $\mathfrak{n} = 6$). Hence, if $\mathfrak{n} \geqslant 12$, substituting $v = ru$ in $\dot{u} = 0$ yields the conclusion, i.e. $u_\infty = (1 + \sqrt{3}\sin\alpha_1 - \cos\alpha_1)/r$.

The converse follows by construction. $\qquad\square$

REMARK 9.4 The φ-curvature flows in Theorem 3.9 converge to homothetic flows, i.e. if $\mathfrak{n} \geqslant 12$ (resp. $\mathfrak{n} = 6$), then the limit triod satisfies (v) (resp. (iv) with $L_1 = 0$) of Lemma 9.3 (see Figure 19(iii), resp. Figure 18(i)).

REFERENCES

1. BELLETTINI, G., NOVAGA, M., & PAOLINI, M. Facet-breaking for three-dimensional crystals evolving by mean curvature. *Interfaces Free Bound.* **1** (1999), 39–55. Zbl 0934.49023 MR 1865105

2. BELLETTINI, G., NOVAGA, M., & PAOLINI, M. On a crystalline variational problem, part I: first variation and global L^∞-regularity. *Arch. Ration. Mech. Anal.* **157** (2001), 165–191. Zbl 0976.58016 MR 1826964

3. BELLETTINI, G., NOVAGA, M., & RIEY, G. First variation of anisotropic energies and crystalline mean curvature for partitions. *Interfaces Free Bound.* **5** (2003), 331–356. Zbl 1066.49028 MR 1998618

4. CAHN, J. W. Stability, microstructural evolution, grain growth, and coarsening in a two-dimensional two-phase microstructure. *Acta Metall. Mater.* **39** (1991), 2189–2199.

5. CAHN, J. W., HANDWERKER, C. A., & TAYLOR, J. E. Geometric models of crystal growth. *Acta Metall. Mater.* **40** (1992), 1443–1474.

6. CAHN, J. W., & HOFFMAN, D. W. A vector thermodynamics for anisotropic surfaces. I. Fundamentals and applications to plane surface junctions. *Surface Sci.* **31** (1972), 368–388.

7. CAHN, J. W., & KALONJI, G. Symmetry of grain boundary trijunctions. *Symposium on Atomic Scale Imperfections in Solids*, Phys. Chem. Solids **55** (1994), 1017–1022.

8. CAHN, J. W., & VAN VLECK, E. Quadrijunctions do not stop two-dimensional grain growth. *Scripta Mater.* **34** (1996), 909–912.

9. CARABALLO, D. G. A variational scheme for the evolution of polycrystals by curvature. Princeton University Ph.D. thesis (1997).

10. FUKUI, T., & GIGA, Y. Motion of a graph by nonsmooth weighted curvature. *World Congress of Nonlinear Analysts '92*, Vol. I–IV, de Gruyter (1996), 47–56. Zbl 0860.35061 MR 1389060

11. GARCKE, H., & NESTLER, B. A mathematical model for grain growth in thin metallic films. *Math. Models Methods Appl. Sci.* **10** (2000), 895–921. Zbl pre01882736 MR 1771511

12. GARCKE, H., NESTLER, B., & STOTH, B. A multiphase field concept: numerical simulations of moving phase boundaries and multiple junctions. *SIAM J. Appl. Math.* **60** (2000), 295–315. Zbl 0942.35095 MR 1740846

13. GIGA, M. H., & GIGA, Y. Evolving graphs by singular weighted curvature. *Arch. Ration. Mech. Anal.* **141** (1998), 117–198. Zbl 0896.35069 MR 1615520

14. GIGA, M. H., & GIGA, Y. Stability for evolving graphs by nonlocal weighted curvature. *Comm. Partial Differential Equations* **24** (1999), 109–184. Zbl 0924.35177 MR 1671993

15. GIGA, M. H., GIGA, Y., & HONTANI, H. Selfsimilar expanding solutions in a sector for a crystalline flow. *SIAM J. Math. Anal.* **37** (2006), 1207–1226. Zbl 1093.74048 MR 2192293

16. GIGA, Y., & GURTIN, M. E. A comparison theorem for crystalline evolution in the plane. *Quart. Appl. Math.* **54** (1996), 727–737. Zbl 0862.35047 MR 1417236

17. GIGA, Y., GURTIN, M. E., & MATIAS, J. On the dynamics of crystalline motion. *Japan J. Indust. Appl. Math.* **15** (1998), 7–50. MR 1610305

18. GIGA, Y., & RYBKA, P. Existence of self-similar evolution of crystals grown from supersaturated vapor. *Interfaces Free Bound.* **6** (2004), 405–421. Zbl 1075.35118 MR 2111563

19. GIGA, Y., & RYBKA, P. Stability of facets of self-similar motion of a crystal. *Adv. Differential Equations* **10** (2005), 601–634. MR 2133647

20. GIGA, Y., & RYBKA, P. Faceted crystals grown from solution—a Stefan type problem with a singular interfacial energy. To appear.

21. GIRÃO, P. M. Convergence of a crystalline algorithm for the motion of a simple closed convex curve by weighted curvature. *SIAM J. Numer. Anal.* **32** (1995), 886–899. Zbl 0830.65150 MR 1335660

22. GURTIN, M. E. *Thermodynamics of Evolving Phase Boundaries in the Plane*. Oxford Univ. Press, New York, 1993. Zbl 0787.73004 MR 1402243

23. HERRING, C. Surface tension as a motivation for sintering. *The Physics of Powder Metallurgy*, McGraw-Hill, New York (1951), 143–179. MR 1770892

24. ISHII, K., & SONER, H. S. Regularity and convergence of crystalline motion. *SIAM J. Math. Anal.* **30** (1998), 19–37. Zbl 0963.35082 MR 1646732

25. KINDERLEHRER, D., & LIU, C. Evolution of grain boundaries. *Math. Models Methods Appl. Sci.* **11** (2001), 713–729. Zbl 1036.74041 MR 1833000

26. KINDERLEHRER, D., LIU, C., MANOLACHE, F., & TA'ASAN, S. Remarks about analysis and simulation of grain boundary systems. *Proc. 3-rd International Conference of Grain Growth* (Pittsburg, PA, 1998).

27. MORGAN, F., FRENCH, G., & GREENLEAF, S. Wulff clusters in $\mathbb{R}^2$. *J. Geom. Anal.* **8** (1998), 97–115. Zbl 0934.49024 MR 1704570

28. MULLINS, W. Two-dimensional motion of idealized grain boundaries. *J. Appl. Phys.* **27** (1956), 900–904. MR 0078836

29. STANCU, A. Uniqueness of self-similar solutions for a crystalline flow. *Indiana Univ. Math. J.* **45** (1996), 1157–1173. Zbl 0873.35034 MR 1444481

30. STANCU, A. Asymptotic behavior of solutions to a crystalline flow. *Hokkaido Math. J.* **27** (1998), 303–320. Zbl 0989.53042 MR 1637988

31. TAYLOR, J. E. Crystalline variational problems. *Bull. Amer. Math. Soc.* **84** (1978), 568–588. Zbl 0392.49022 MR 0493671

32. TAYLOR, J. E. Motion by crystalline curvature. *Computing Optimal Geometries*, J. E. Taylor (ed.), Selected Lectures in Mathematics, Amer. Math. Soc. (1991), 63–65, and accompanying video.

33. TAYLOR, J. E. Mean curvature and weighted mean curvature. *Acta Metall. Mater.* **40** (1992), 1475–1485.

34. TAYLOR, J. E. Motion of curves by crystalline curvature, including triple junctions and boundary points. *Differential Geometry*, Proc. Sympos. Pure Math. 54 (part 1), Amer. Meth. Soc. (1993), 417–438. Zbl 0823.49028 MR 1216599

35. TAYLOR, J. E. The motion of multi-phase junctions under perscribed phase-boundary velocities. *J. Differential Equations* **119** (1995), 109–136. Zbl 0835.73006 MR 1334489

36. TAYLOR, J. E. A variational approach to crystalline triple-junction motion. *J. Statist. Phys.* **95** (1999), 1221–1244. Zbl 0952.74015 MR 1712448