Interfaces and Free Boundaries **9** (2007), 107–132

Asymptotic analysis of Mumford–Shah type energies in periodically perforated domains

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[Received 28 January 2006 and in revised form 13 September 2006]

We study the asymptotic limit of obstacle problems for Mumford–Shah type functionals with p-growth in periodically perforated domains *via* the Γ-convergence of the associated free-discontinuity energies. In the limit a non-trivial penalization term related to the 1-capacity of the reference hole appears if and only if the size of the perforation scales like $\varepsilon^{n/(n-1)}$, ε being its periodicity. We give two different formulations of the obstacle problem to include also perforations with Lebesgue measure zero.

1. Introduction

The aim of this paper is to study the limiting behavior of Mumford–Shah type functionals in periodically perforated domains. We express the obstacle constraint by two different formulations according to the "size" of the perforation, thus including $(n-1)$-dimensional sets. For both cases we identify the meaningful scaling yielding a non-trivial limit energy (see Theorems 3.1 and 4.1).

A model case for this kind of problem is the following: find the asymptotics as $\varepsilon \to 0$ of

$$\inf\left\{\int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_u) + \text{lower order terms} : u \in SBV(\Omega),\ u = 0 \text{ on } \mathbf{B}_\varepsilon \cup \partial\Omega\right\}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a given regular bounded open set, ∇u and S_u are, respectively, the approximate gradient and the set of approximate discontinuities of u (see Subsection 2.3), and $\mathbf{B}_\varepsilon = \Omega \cap \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{r_\varepsilon}(\underline{i}\varepsilon)$, with $B_{r_\varepsilon}(\underline{i}\varepsilon)$ the ball centered at $\underline{i}\varepsilon$ of radius $r_\varepsilon > 0$. This is the first step in studying *obstacle problems for free-discontinuity energies* which we are currently investigating [30].

The case in which the minimum problems (1.1) above are restricted to the Sobolev space $W^{1,p}$, $p > 1$, is classical and it has been object of much research since the pioneering works of Marchenko and Khruslov [31], Rauch and Taylor [34], [35] and Cioranescu and Murat [14]. A wide literature also deals with Neumann or Robin conditions on the boundary of the set of perforations (see [15], [13] and the books [12], [16] for a more exhaustive list of references).

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A typical phenomenon occurring in this context is that the limit problem is no longer related to an obstacle constraint and the limit energy to be minimized contains an extra term. The latter is a finite penalization keeping track of the local capacity density of the homogenizing obstacles (with the appropriate notion of capacity related to the Dirichlet type energy under consideration).

In order to deal with this *relaxation phenomenon*, De Giorgi, Dal Maso and Longo proposed in [27] an approach which was then carried out by many authors (see [9], [23], [3], [4], [20], [21], [33]). The method is based on abstract Γ-convergence arguments (see Section 2.2 for the definition and main properties of Γ-limits) for the associated Dirichlet energies and requires a deep study of some fine properties of Sobolev functions. It turns out that one can confine the analysis to the range $1 < p \leqslant n$ since for $p > n$ the convergence result is trivial. Moreover, also in case $1 < p \leqslant n$ a simple computation shows that there exists only one meaningful scaling of the *radius of the periodic perforation* r_ε depending on the space dimension n and on the exponent p: $r_\varepsilon \sim \varepsilon^{n/(n-p)}$ if $1 < p < n$, and $r_\varepsilon \sim e^{-\varepsilon^{-n}}$ if $p = n$.

A different method using direct Γ-convergence arguments was developed more recently in [2]. The main tool there is a joining lemma in varying domains (see Lemma 3.1 of [2]) which allows one to modify sequences of functions in the vicinity of the perforation set, reminiscent of a method proposed by De Giorgi to match boundary conditions.

Going back to our framework, in order to deal with problems (1.1) we introduce for any $p > 1$ the functionals $\mathcal{F}_\varepsilon : SBV(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_u), & u \in SBV(\Omega), u = 0 \ \mathcal{L}^n\text{-a.e. on } \mathbf{B}_\varepsilon, \\ +\infty, & \text{otherwise in } SBV(\Omega), \end{cases} \tag{1.2}$$

thus neglecting the boundary condition on the fixed boundary $\partial\Omega$ (we refer to Theorem 3.1 and Proposition 3.3 for the exact statement and the right functional framework). In Proposition 3.4 we show how to recover the case in which the boundary datum on $\partial\Omega$ is imposed.

Unlike the Sobolev setting, it turns out that for any $p > 1$ there exists only one meaningful scaling for the radius r_ε which depends only on the space dimension n. This is due to the enlarged domain of the problem allowing for *fractured configurations*, with a penalization on the site of fracture added. In terms of Γ-convergence a rigorous statement of this fact is the following (see Proposition 3.3): $(\mathcal{F}_\varepsilon)$ Γ-converges to the functional $\mathcal{F}$ given for any $u \in SBV(\Omega)$ by

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_u) + n\omega_n\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}) \tag{1.3}$$

with respect to the L^1 convergence, where the coefficient β is finite and different from 0 if and only if $r_\varepsilon \sim \varepsilon^{n/(n-1)}$. This result is achieved by studying the more general case of a unilateral constraint of the same type (see Theorem 3.1).

Similarly to the Sobolev case, the term $n\omega_n$ has a capacitary interpretation and it is related to the *functional capacity of degree 1* studied in detail in [29], [10]. Indeed, we prove the convergence result of Theorem 3.1 for a generic reference perforation set E replacing $n\omega_n$ in (1.3) with $\mathrm{C}_1(E_+)$, the 1-capacity of a suitable $\mathcal{L}^n$ representative of E (see Subsection 2.5 and Remark 3.2).

A heuristic motivation explaining the appearance of the capacitary term (and also the independence from p in the meaningful threshold) can be given by considering the energy of an optimizing sequence for a constant function $u \equiv \eta < 0$. The latter is obtained by modifying u itself in a neighborhood of the periodic perforation in order to satisfy the constraint. In such a neighborhood the transition between the values 0 and η is minimal, for Mumford–Shah type energies, on *totally fractured* configurations, since the contribution of the bulk term is of order strictly greater than that of the surface term (see Lemma 3.6). Moreover, since on piecewise constant functions the energy $\mathcal{F}_\varepsilon$ reduces to the perimeter of their level sets, one has to solve locally an obstacle problem for minimal surfaces taking also into account the effect of the vanishing size of the perforation. This is indeed the argument with which an upper bound for the Γ-limit is obtained for a generic SBV function (see Proposition 3.9).

To prove that the latter is actually an optimal bound, one reduces to a local picture and estimates in each ε-cell contained in Ω separately the contribution of the energy far and close to the perforation set. The first term accounts for the Mumford–Shah energy in the limit, while the second for the capacitary contribution (see Steps 1 and 2 of Lemma 3.5).

In Section 4 we consider reference perforation sets which may also have Lebesgue measure zero, the so called *thin obstacles* (see Theorem 4.1). In such a case formulation (1.2) of the obstacle condition is trivial and the constraint has to be imposed in a different way. As usual in this kind of problem (see [10]), this can be done by exploiting fine properties of the class of functions under consideration. In particular, for a function u in $BV(\Omega)$ the representative u^+ is defined $\mathcal{H}^{n-1}$-a.e. on Ω. By taking this into account, we prove that the family $(\mathcal{F}_\varepsilon)$, with $\mathcal{F}_\varepsilon : SBV(\Omega) \to [0, +\infty]$ given by

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, \mathrm{d}x + \mathcal{H}^{n-1}(S_u), & u \in SBV(\Omega), u^+ \geqslant 0 \ \mathcal{H}^{n-1} \text{-a.e. on } \mathbf{E}_\varepsilon, \\ +\infty, & \text{otherwise in } SBV(\Omega), \end{cases}$$

where $\mathbf{E}_\varepsilon = \Omega \cap \bigcup_{\underline{i} \in \mathbb{Z}^n} (\underline{i}\varepsilon + r_\varepsilon E)$, Γ-converges with respect to the L^1 convergence to the functional $\mathcal{F}$ equal for any $u \in SBV(\Omega)$ to

$$\mathcal{F}(u) = \int_\Omega |\nabla u|^p \, \mathrm{d}x + \mathcal{H}^{n-1}(S_u) + \mathrm{C}_1(E)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}) \tag{1.4}$$

(see Theorem 4.1). Due to the occurrence of a relaxation phenomenon the analysis of the capacitary contribution in the statement above requires a delicate argument founded on the theory of obstacle problems in the linear setting [26], [10], [11] (see Lemma 4.4).

This fact led us to distinguish two formulations of the obstacle problem, the one in Section 3 being more intuitive and less technically demanding than that of Section 4 (see Remark 4.2 for a comparison between Theorems 3.1 and 4.1).

Finally, in Section 5 we generalize the results obtained in the model case of the Mumford–Shah functional to a wider class of free-discontinuity energies (see Theorem 5.1).

2. Notation and preliminaries

2.1 *Basic notation*

In the following, Ω denotes a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and $\mathcal{H}^{n-1}(\partial\Omega) < +\infty$, with $n \geqslant 2$ a fixed integer. Given an open set $A \subseteq \mathbb{R}^n$ the family of its open subsets is denoted by $\mathcal{A}(A)$.

The symbol $B \triangle C$ stands for the symmetric difference $(B \setminus C) \cup (C \setminus B)$ of the sets B and C in $\mathbb{R}^n$.

As usual, B_1 denotes the open ball in $\mathbb{R}^n$ of radius 1 centered at the origin, and Q_1 the semi-open unit cube with side 1 centered at the origin, that is, $Q_1 = [-1/2, 1/2)^n$. For any set $E \subset \mathbb{R}^n$, $z \in \mathbb{R}^n$ and $r > 0$, we denote by $E_r(z)$ the set $z + rE$; in case $z = \underline{0}$ we simply write E_r for $E_r(\underline{0})$.

If $B, C \in \mathcal{A}(\Omega)$ and $\mathrm{dist}(B, C) = L > 0$, a *cut-off function between B and C* is any $\theta \in C^\infty(\overline{\Omega})$ with $0 \leqslant \theta \leqslant 1$ such that $\theta \equiv 1$ on B and $\theta \equiv 0$ on C. Moreover, we will assume that $|\nabla\theta| \leqslant c/L$.

We employ the standard notation $\overline{C}$ for the topological closure in $\mathbb{R}^n$ of the set C.

2.2 *Γ-convergence*

We recall the notion of Γ-convergence introduced by De Giorgi (see [22], [6]) in a generic metric space (X, d) endowed with the topology induced by d. A family of functionals $\mathcal{F}_\varepsilon : X \to [0, +\infty]$ *Γ-converges* to a functional $\mathcal{F} : X \to [0, +\infty]$ at $u \in X$, for short $\mathcal{F}(u) = \Gamma\text{-}\lim_\varepsilon \mathcal{F}_\varepsilon(u)$, if for every sequence (ε_j) of positive numbers decreasing to 0 the following two conditions hold:

(i) (*liminf inequality*) for any (u_j) converging to u in X, we have $\liminf_j \mathcal{F}_{\varepsilon_j}(u_j) \geqslant \mathcal{F}(u)$;
(ii) (*limsup inequality*) there exists (u_j) converging to u in X such that $\limsup_j \mathcal{F}_{\varepsilon_j}(u_j) \leqslant \mathcal{F}(u)$.

We say that $\mathcal{F}_\varepsilon$ *Γ-converges* to $\mathcal{F}$ (or $\mathcal{F} = \Gamma\text{-}\lim_\varepsilon \mathcal{F}_\varepsilon$) if $\mathcal{F}(u) = \Gamma\text{-}\lim_\varepsilon \mathcal{F}_\varepsilon(u)$ for all $u \in X$. We also define the *upper* and *lower Γ-limits* as

$$\Gamma\text{-}\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\},$$
$$\Gamma\text{-}\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\},$$

respectively, so that conditions (i) and (ii) are equivalent to $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon(u) = \Gamma\text{-}\liminf_\varepsilon \mathcal{F}_\varepsilon(u) = \mathcal{F}(u)$. Moreover, the functions $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon(\cdot)$ and $\Gamma\text{-}\liminf_\varepsilon \mathcal{F}_\varepsilon(\cdot)$ are lower semicontinuous.

One of the main reasons for the introduction of this notion is explained by the following fundamental theorem.

THEOREM 2.1 Let $\mathcal{F} = \Gamma\text{-}\lim_\varepsilon \mathcal{F}_\varepsilon$, and assume there exists a compact set $K \subset X$ such that $\inf_X \mathcal{F}_\varepsilon = \inf_K \mathcal{F}_\varepsilon$ for all ε. Then $\min_X \mathcal{F} = \lim_\varepsilon \inf_X \mathcal{F}_\varepsilon$ exists. Moreover, if (u_j) is a convergent sequence such that $\lim_j \mathcal{F}_{\varepsilon_j}(u_j) = \lim_j \inf_X \mathcal{F}_{\varepsilon_j}$ then its limit is a minimum point for $\mathcal{F}$.

2.3 *BV functions*

In this section we recall some basic definitions and results on sets of finite perimeter and BV, SBV and $GSBV$ functions. We will give precise references to the book [1] for all the results used throughout the paper.

Let $A \subseteq \mathbb{R}^n$ be an open set. For every $u \in L^1(A)$ and $x \in A$, we define

$$u^+(x) = \inf\{t \in \mathbb{R} : \lim_{r\to 0^+} r^{-n}\mathcal{L}^n(\{y \in B_r(x) : u(y) > t\}) = 0\},$$
$$u^-(x) = \sup\{t \in \mathbb{R} : \lim_{r\to 0^+} r^{-n}\mathcal{L}^n(\{y \in B_r(x) : u(y) < t\}) = 0\},$$

with the convention $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$. We remark that u^+, u^- are Borel functions uniquely determined by the $\mathcal{L}^n$-equivalence class of u. If $u^+(x) = u^-(x)$ the common value is denoted by $\tilde{u}(x)$ or $\text{ap-}\lim_{y\to x} u(y)$ and called the *approximate limit* of u at x.

Notice that for every $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ we have $(\chi_E)^+ = \chi_{E_+}$, where

$$E_+ = \{x \in \mathbb{R}^n : \limsup_{r \to 0^+} r^{-n} \mathcal{L}^n(E \cap B_r(x)) > 0\}.$$

Moreover, we have

$$\mathcal{L}^n(E \setminus D) = 0 \;\Leftrightarrow\; E_+ \subseteq D_+, \tag{2.1}$$

thus, by (2.1) above, E_+ is an $\mathcal{L}^n$ representative of E, i.e. $\mathcal{L}^n(E \triangle E_+) = 0$.

The set $S_u = \{x \in A : u^-(x) < u^+(x)\}$ is called the *set of approximate discontinuity points* of u and it is well known that $\mathcal{L}^n(S_u) = 0$. Let $x \in A \setminus S_u$ be such that $\tilde{u}(x) \in \mathbb{R}$. We say that u is *approximately differentiable* at x if there exists $L \in \mathbb{R}^n$ such that

$$\text{ap-}\lim_{y \to x} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{|y-x|} = 0. \tag{2.2}$$

If u is approximately differentiable at x, the vector L uniquely determined by (2.2) will be denoted by $\nabla u(x)$ and called the *approximate gradient* of u at x.

A function $u \in L^1(A)$ is said to be of *bounded variation* in A, for short $u \in BV(A)$, if its distributional derivative Du is an $\mathbb{R}^n$-valued finite Radon measure. If $u \in BV(A)$, denote by $D^a u$, $D^s u$ the absolutely continuous and singular parts of the Lebesgue decomposition of Du with respect to $\mathcal{L}^n \llcorner A$, respectively. Then u turns out to be approximately differentiable a.e. on A (Theorem 3.83 of [1]), S_u to be *countably* $\mathcal{H}^{n-1}$*-rectifiable* (see Theorem 3.78 of [1]), and the values $u^+(x)$, $u^-(x)$ are finite and specified $\mathcal{H}^{n-1}$-a.e. in A (see Remark 3.79 of [1]). Moreover,

$$D^a u = \nabla u \mathcal{L}^n \llcorner A, \qquad D^s u \llcorner S_u = (u^+ - u^-)\nu_u \, \mathcal{H}^{n-1} \llcorner S_u,$$

where $\nu_u \in \mathbb{S}^{n-1}$ is an orientation for S_u.

We say that an $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ is *of finite perimeter* in A if $\chi_E \in BV(A)$, and we call the total variation of χ_E in A the *perimeter* of E in A, denoting it by $\mathrm{Per}(E, A)$, or simply by $\mathrm{Per}(E)$ if $A \equiv \mathbb{R}^n$. It is well known that $D\chi_E = D\chi_E \llcorner \partial^* E = \nu_{\partial^* E} \mathcal{H}^{n-1} \llcorner \partial^* E$ (see Theorem 3.59 of [1]), where the countably $\mathcal{H}^{n-1}$-rectifiable set $\partial^* E$ is called the *essential boundary* of E and $\nu_{\partial^* E}$ is an orientation for it.

We recall that if A has Lipschitz boundary, then any $u \in BV(A)$ leaves an inner boundary trace on ∂A, which we denote by $\mathrm{tr}(u)$, and moreover $\mathrm{tr}(u) \in L^1(\partial A, \mathcal{H}^{n-1})$ (see Theorem 3.87 of [1]).

We say that $u \in BV(A)$ is a *special function of bounded variation* in A if $D^s u \equiv D^j u$ on A; we then write $u \in SBV(A)$. Moreover, $u \in SBV_{\mathrm{loc}}(A)$ if $u \in SBV(U)$ for every open subset $U \subset\subset A$.

We say that $u \in L^1(A)$ is a *generalized special function of bounded variation* in A, written $u \in GSBV(A)$, if for every $M > 0$ the truncated function $(u \wedge M) \vee (-M) \in SBV(A)$. Functions in $GSBV$ inherit many properties from BV functions: they are approximately differentiable a.e. on A, and S_u turns out to be countably $\mathcal{H}^{n-1}$-rectifiable (see Theorem 4.34 of [1]). The space $(G)SBV$ has been introduced by De Giorgi and Ambrosio [25] in connection with the weak formulation of the image segmentation model proposed by Mumford and Shah (see [32]). If $u \in GSBV(A)$ and $p \in (1, +\infty)$, the *Mumford–Shah energy* of u is defined as

$$MS_p(u) = \int_A |\nabla u|^p \, \mathrm{d}x + \mathcal{H}^{n-1}(S_u). \tag{2.3}$$

We recall the SBV compactness theorem due to Ambrosio in a form needed for our purposes (see Theorems 4.8 and Theorem 5.22 of [1]).

THEOREM 2.2 Let $(u_j) \subset SBV(A)$ and assume that for some $p \in (1, +\infty)$,

$$\sup_j (MS_p(u_j) + \|u_j\|_{L^\infty(A)}) < +\infty.$$

Then there exist a subsequence (u_{j_k}) and a function $u \in SBV(A)$ such that $u_{j_k} \to u$ a.e. in A, $\nabla u_{j_k} \to \nabla u$ weakly in $L^p(A; \mathbb{R}^n)$, $D^s u_{j_k} \llcorner S_{u_{j_k}} \to D^s u \llcorner S_u$ weak* in the sense of measures. Moreover, if $\psi : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$ satisfying $c_1 \leqslant \psi(\nu) \leqslant c_2$ for every $\nu \in \mathbb{S}^{n-1}$, with $c_1, c_2 > 0$, then

$$\int_{S_u} \psi(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1} \leqslant \liminf_k \int_{S_{u_{j_k}}} \psi(\nu_{u_{j_k}})\, \mathrm{d}\mathcal{H}^{n-1}.$$

Finally, in case $u \in GSBV(A)$ and $MS_p(u, A) < +\infty$ the values $u^+(x)$, $u^-(x)$ are finite and specified $\mathcal{H}^{n-1}$-a.e. in A (see Theorem 4.40 of [1]).

2.4 *Homogenization in SBV*

Here we collect the main results of [8] (see Proposition 2.1, Proposition 2.2 and Theorem 2.3 there) in a form which is convenient for our purposes.

Let $\varphi : \mathbb{R}^{2n} \to [0, +\infty)$ and $\psi : \mathbb{R}^{3n} \times \mathbb{S}^{n-1} \to [0, +\infty)$ be two Borel functions with $\psi(x, a, b, \nu) = \psi(x, b, a, -\nu)$ for every $(x, a, b, \nu) \in \mathbb{R}^{3n} \times \mathbb{S}^{n-1}$. Suppose that φ and ψ satisfy

(i) $\varphi(\cdot, \xi)$ is 1-periodic for every $\xi \in \mathbb{R}^n$, and there exist $c_1, c_2 > 0$ such that for every $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$,

$$c_1|\xi|^p \leqslant \varphi(x, \xi) \leqslant c_2(1 + |\xi|^p);$$

(ii) $\psi(\cdot, a, b, \nu)$ is 1-periodic for every $(a, b, \nu) \in \mathbb{R}^{2n} \times \mathbb{S}^{n-1}$, and there exist $c_3, c_4 > 0$ such that for every $(x, a, b, \nu) \in \mathbb{R}^{3n} \times \mathbb{S}^{n-1}$,

$$c_3(1 + |b - a|) \leqslant \psi(x, a, b, \nu) \leqslant c_4(1 + |b - a|);$$

(iii) there exists a continuous non-decreasing function $\omega : [0, +\infty) \to [0, +\infty)$, with $\omega(0) = 0$, and $L > 0$ such that $\omega(t) \leqslant Lt$ for $t \geqslant 1$ and

$$|\psi(x, a, b, \nu) - \psi(x, a_1, b_1, \nu)| \leqslant \omega(|a - a_1| + |b - b_1|)$$

for every $(x, a, b, \nu), (x, a_1, b_1, \nu) \in \mathbb{R}^{3n} \times \mathbb{S}^{n-1}$.

For every $\varepsilon > 0$, define $\mathcal{G}_\varepsilon : SBV(A) \times \mathcal{A}(A) \to [0, +\infty)$ by

$$\mathcal{G}_\varepsilon(u, U) = \int_U \varphi\left(\frac{x}{\varepsilon}, \nabla u\right) \mathrm{d}x + \int_{S_u \cap U} \psi\left(\frac{x}{\varepsilon}, u^+, u^-, \nu_u\right) \mathrm{d}\mathcal{H}^{n-1}. \tag{2.4}$$

Then we have

THEOREM 2.3 For every $U \in \mathcal{A}(A)$ the family $(\mathcal{G}_\varepsilon(\cdot, U))$ Γ-converges with respect to the L^1-convergence to the functional $\mathcal{G}_{\mathrm{hom}} : SBV(A) \times \mathcal{A}(A) \to [0, +\infty)$ defined by

$$\mathcal{G}_{\mathrm{hom}}(u, U) = \int_U \varphi_{\mathrm{hom}}(\nabla u)\, \mathrm{d}x + \int_{S_u \cap U} \psi_{\mathrm{hom}}(u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{n-1}, \tag{2.5}$$

where

1. $\varphi_{\hom} : \mathbb{R}^n \to [0, +\infty)$ is the convex function given by

$$\varphi_{\hom}(\xi) = \lim_{\varepsilon \to 0^+} \inf\left\{ \int_{Q_1} \varphi\left(\frac{x}{\varepsilon}, \nabla v + \xi\right) \mathrm{d}x : v \in W_0^{1,p}(Q_1) \right\}. \tag{2.6}$$

2. $\psi_{\hom} : \mathbb{R}^{2n} \times \mathbb{S}^{n-1} \to [0, +\infty)$ is the function given by

$$\psi_{\hom}(a, b, \nu) = \lim_{\varepsilon \to 0^+} \inf\left\{ \int_{S_v \cap Q^\nu} \psi\left(\frac{x}{\varepsilon}, v^+, v^-, \nu_v\right) \mathrm{d}\mathcal{H}^{n-1} : \right.$$
$$\left. v \in SBV(Q^\nu) \text{ with } \nabla v = 0 \text{ a.e.}, \operatorname{tr}(v) = \operatorname{tr}(v_{a,b,\nu}) \text{ on } \partial Q^\nu \right\}, \tag{2.7}$$

where Q^ν is any unit cube in $\mathbb{R}^n$ centered at the origin and with one face orthogonal to ν, and $v_{a,b,\nu}(x) = a\chi_{\{x : \langle x,\nu\rangle \geqslant 0\}}(x) + b\chi_{\{x : \langle x,\nu\rangle < 0\}}(x)$.

REMARK 2.4 In case $\varphi(x, \cdot)$ is convex for all $x \in \mathbb{R}^n$ formula (2.6) can be further specialized (see Theorem 14.7 of [7]) and reduces to a cell minimization formula

$$\varphi_{\hom}(\xi) = \min\left\{ \int_{Q_1} \varphi(x, \nabla v + \xi) \,\mathrm{d}x : v \in W_{\mathrm{per}}^{1,p}(Q_1) \right\}. \tag{2.8}$$

2.5 *Functional capacity of degree 1*

Let $\mathcal{Y}_1(\mathbb{R}^n)$ be the subspace of $L^{n/(n-1)}(\mathbb{R}^n)$ of functions with distributional derivative of function type. For any set $E \subseteq \mathbb{R}^n$ consider the quantity

$$\Gamma_1(E) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u| \,\mathrm{d}x : u \in \mathcal{Y}_1(\mathbb{R}^n),\ E \subset \operatorname{int}(\{x \in \mathbb{R}^n : u(x) \geqslant 1\}) \right\};$$

following Federer and Ziemer [29] we call it the *functional capacity of degree 1* of E. Actually, different minimization problems characterize it, in particular it can be expressed in terms of the perimeter of the sets containing E, as shown by the following proposition which summarizes the results of Section 4 of [29] and Theorem 2.1 of [10].

PROPOSITION 2.5 Let $E \subseteq \mathbb{R}^n$ and let

$$\mathrm{C}_1(E) = \inf\left\{ \int_{\mathbb{R}^n} |\nabla u| \,\mathrm{d}x : u \in W^{1,1}(\mathbb{R}^n),\ u^+ \geqslant 1\ \mathcal{H}^{n-1}\text{-a.e. on } E \right\},$$
$$\gamma(E) = \inf\{\|Du\|(\mathbb{R}^n) : u \in BV(\mathbb{R}^n),\ u^+ \geqslant 1\ \mathcal{H}^{n-1}\text{-a.e. on } E\},$$
$$\delta(E) = \inf\{\mathrm{Per}(D) : D \text{ is } \mathcal{L}^n\text{-measurable},\ \mathcal{L}^n(D) < +\infty,\ \mathcal{H}^{n-1}(E \setminus D_+) = 0\}. \tag{2.9}$$

Then $\Gamma_1(E) = \mathrm{C}_1(E) = \gamma(E) = \delta(E)$.

The existence of extremals for the variational problems above fails for many sets E with $\mathrm{C}_1(E) < +\infty$ (e.g. if E is a line segment in $\mathbb{R}^2$). A sufficient condition ensuring existence of minimizers for the formulation (2.9) was proposed in Section 4 of [29] (see also Theorems 3.3 and 3.4, Chapter IV of [26]). Here we recall the result and its proof for the readers' convenience.

PROPOSITION 2.6 For every $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ with $\mathrm{C}_1(E) < +\infty$,

(a) $\mathrm{C}_1(E_+) \leqslant \mathrm{C}_1(E)$;

(b) problem (2.9) for E_+ always has a solution and

$$\mathrm{C}_1(E_+) = \min\{\mathrm{Per}(D) : D \text{ is } \mathcal{L}^n\text{-measurable}, \ \mathcal{L}^n(D) < +\infty, \ \mathcal{L}^n(E \setminus D) = 0\}. \tag{2.10}$$

Moreover, if $\mathcal{H}^{n-1}(E \setminus E_+) = 0$ then $\mathrm{C}_1(E_+) = \mathrm{C}_1(E)$ and problem (2.9) for E has a solution.

Proof. Let (D_j) be a minimizing sequence in problem (2.9) for E. Then by the isoperimetric inequality (see Theorem 3.46 of [1]) $\sup_j(\mathcal{L}^n(D_j) + \mathrm{Per}(D_j)) < +\infty$. The BV compactness theorem (see Theorem 3.23 of [1]) in turn implies the existence of a subsequence (not relabeled for convenience) and a set D with finite perimeter in $\mathbb{R}^n$ such that $\chi_{D_j} \to \chi_D$ in $L^1(\mathbb{R}^n)$. Thus $\mathcal{L}^n(E \setminus D) = 0$, and by taking into account (2.1) we have $E_+ \subseteq D_+$. Hence, D is admissible in problem (2.9) for E_+, i.e. $\mathcal{H}^{n-1}(E_+ \setminus D_+) = 0$, and so (a) is established since

$$\mathrm{C}_1(E) = \liminf_j \mathrm{Per}(D_j) \geqslant \mathrm{Per}(D) \geqslant \mathrm{C}_1(E_+).$$

Obviously the same argument applied to a minimizing sequence of $\mathrm{C}_1(E_+)$ provides a set D admissible for such a problem which is then a minimizer. Finally, characterization (2.10) holds true. □

Sligthly abusing the terminology introduced by De Giorgi in [24], [26] we call the sets satisfying $\mathcal{H}^{n-1}(E \setminus E_+) = 0$ *thick*. Indeed, De Giorgi's original definition required the stronger condition $E \subseteq E_+$.

In general, one can determine the relaxed problem associated to $\mathrm{C}_1(\cdot)$ by using *De Giorgi's measure* σ introduced in Chapter IV of [26] to study non-parametric minimal surface problems with obstacles. For any set $E \subseteq \mathbb{R}^n$, σ is the regular Borel measure given by

$$\sigma(E) = \sup_{\varepsilon>0}(\inf\{\mathrm{Per}(D) + \mathcal{L}^n(D)/\varepsilon : D \text{ is } \mathcal{L}^n\text{-measurable}, \ \mathcal{H}^{n-1}(E \setminus D_+) = 0\}). \tag{2.11}$$

We are now able to state the relaxation Theorem 7.1 of [10] in a form needed for our purposes (see also Theorem 3.4, Chapter IV of [26]).

THEOREM 2.7 For any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$,

$$\begin{aligned}\mathrm{C}_1(E) &= \min\left\{\|Du\|(\mathbb{R}^n) + \int_{\mathbb{R}^n}[(\chi_E - u^+) \vee 0]\,\mathrm{d}\sigma : u \in BV(\mathbb{R}^n)\right\}\\ &= \min\{\mathrm{Per}(D) + \sigma(E \setminus D_+) : D \text{ is } \mathcal{L}^n\text{-measurable}, \ \mathcal{L}^n(D) < +\infty\}.\end{aligned} \tag{2.12}$$

Finally, we recall that the set function $\mathrm{C}_1(\cdot)$ is *positively* $(n-1)$*-homogeneous*, that is, for any set $E \subseteq \mathbb{R}^n$ and $r > 0$ we have $\mathrm{C}_1(E_r) = r^{n-1}\mathrm{C}_1(E)$ (see [36]); moreover (see [29]),

$$\mathrm{C}_1(E) = 0 \ \Leftrightarrow \ \mathcal{H}^{n-1}(E) = 0.$$

REMARK 2.8 For any bounded set E it is easy to prove that $\mathrm{C}_1(E) < +\infty$. Moreover, if E is contained in the interior of a bounded convex set C, one can restrict the class of competing sets in the capacitary problem for E to those contained in C.

Indeed, by using the formulation (2.9), given a test set D, consider $D' = D \cap C$. Then D' has finite perimeter and, since $E \subset \mathrm{int}(C)$ and $C_+ = \overline{C}$, we have $\mathcal{H}^{n-1}(E \setminus D'_+) = \mathcal{H}^{n-1}(E \setminus (D \cap C)_+) = \mathcal{H}^{n-1}(E \setminus D_+) = 0$. If Π_C denotes the projection onto the convex set C, then $\mathcal{H}^{n-1}(\Pi_C(D \cap (\mathbb{R}^n \setminus C))) \leqslant \mathrm{Per}(D \cap (\mathbb{R}^n \setminus C))$. Hence,

$$\begin{aligned}\mathrm{Per}(D') &\leqslant \mathcal{H}^{n-1}(\Pi_C(D \setminus \mathrm{int}(C))) + \mathcal{H}^{n-1}(\partial^* D \cap \mathrm{int}(C))\\ &\leqslant \mathrm{Per}(D \setminus \mathrm{int}(C)) + \mathcal{H}^{n-1}(\partial^* D \cap \mathrm{int}(C)) \leqslant \mathrm{Per}(D).\end{aligned}$$

3. Obstacle constraint imposed in the $\mathcal{L}^n$ sense

Given an $\mathcal{L}^n$-measurable set $E \subseteq \overline{Q}_1$, for any $\varepsilon > 0$ let $r_\varepsilon \in (0, \varepsilon)$ and $\mathbf{E}_\varepsilon = \Omega \cap \bigcup_{\underline{i} \in \mathbb{Z}^n} E_{r_\varepsilon}(\underline{i}\varepsilon)$. Consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} MS_p(u), & u \in GSBV(\Omega),\ u \geqslant 0\ \mathcal{L}^n\text{-a.e. on } \mathbf{E}_\varepsilon, \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases} \tag{3.1}$$

Moreover, denote by $\mathcal{F}_\varepsilon(\cdot, A)$ its *localized version*, obtained by replacing in (3.1) above the domain of integration Ω with any open subset $A \in \mathcal{A}(\Omega)$.

The same convention will also be applied to the localized version of the Mumford–Shah energy (2.3), dropping the set dependence in case $A \equiv \Omega$.

THEOREM 3.1 Let E be an $\mathcal{L}^n$-measurable set and assume that $r_\varepsilon/\varepsilon^{n/(n-1)} \to \beta \in [0, +\infty)$ as $\varepsilon \to 0^+$. Then $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F} : L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}(u) = \begin{cases} MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases} \tag{3.2}$$

with respect to the L^1 convergence.

REMARK 3.2 It is worth noting that definition (3.1) of $\mathcal{F}_\varepsilon$ is not affected if we replace E with any other set G in its $\mathcal{L}^n$-equivalence class. For instance, it would not be restrictive to assume the perforation set E to be thick in the statement of Theorem 3.1, that is, to change E to E_+.

The reason why the representative E_+ is selected in the limit process is the minimality property

$$\mathrm{C}_1(E_+) = \min\{\mathrm{C}_1(G) : G \text{ is } \mathcal{L}^n\text{-measurable},\ \mathcal{L}^n(E \triangle G) = 0\},$$

as follows from Proposition 2.6(a). A further motivation will be discussed in Section 4 (see Theorem 4.1 and Remark 4.2 for details).

Before giving a proof of Theorem 3.1 we state the results mentioned in the introduction concerning the bilateral obstacle case and when a boundary datum on $\partial\Omega$ is imposed. Both their proofs will be given after that of Theorem 3.1, since they share many ideas and techniques developed for that theorem as well as use part of its results.

PROPOSITION 3.3 Let $\mathcal{F}'_\varepsilon$ be defined as $\mathcal{F}_\varepsilon$ with the unilateral positivity condition on $\mathbf{E}_\varepsilon$ in definition (3.1) replaced with $u = 0$ $\mathcal{L}^n$-a.e. on $\mathbf{E}_\varepsilon$. Then $(\mathcal{F}'_\varepsilon)$ Γ-converges to $\mathcal{F}' : L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}'(u) = \begin{cases} MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases} \tag{3.3}$$

with respect to the L^1 convergence.

We now consider the case in which a Dirichlet boundary datum is imposed on $\partial\Omega$. For the sake of simplicity we assume in what follows the additional hypothesis that Ω has C^2 boundary, although this condition might be weakened (see for instance Section 8 of [8]).

We introduce for any $\varepsilon > 0$ the "boundary" functionals $\mathcal{D}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{D}_\varepsilon(u) = \begin{cases} \mathcal{F}'_\varepsilon(u), & u \in GSBV(\Omega),\ \mathrm{tr}(u) = 0 \text{ on } \partial\Omega, \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases}$$

and state the following convergence result.

PROPOSITION 3.4 *$(\mathcal{D}_\varepsilon)$ Γ-converges with respect to the L^1 convergence to $\mathcal{D} : L^1(\Omega) \to [0, +\infty]$ given by*

$$\mathcal{D}(u) = \begin{cases} \mathcal{F}'(u) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \mathrm{tr}(u)(x) \neq 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases}$$

Notice that if we consider lower order terms converging in a suitable sense (see Proposition 6.20 of [22]), for instance fidelity terms or linear perburations, Proposition 3.4 and Theorem 2.1 imply the convergence of problems (1.1) mentioned in the introduction to

$$\min\{\mathcal{D}(u) + \text{lower order terms} : u \in GSBV(\Omega)\}.$$

Theorem 3.1 will be a consequence of Propositions 3.7 and 3.9 below in which we show separately the liminf and limsup inequalities, respectively. Proposition 3.7 will easily follow from Lemma 3.5 below in which we treat the case of sequences bounded in L^∞.

LEMMA 3.5 *For every sequence $u_\varepsilon \to u$ in $L^1(\Omega)$ such that $\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)} < +\infty$,*

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant \mathcal{F}(u).$$

Proof. We may suppose $\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}) > 0$ and $\mathcal{L}^n(E) > 0$, since otherwise the statement is trivial. Moreover, it is not restrictive to assume $\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) = \lim_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$. Hence, Ambrosio's SBV closure and compactness Theorem 2.2 implies that $u \in SBV(\Omega)$ and also $\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) = \liminf_\varepsilon MS_p(u_\varepsilon) \geqslant MS_p(u)$.

Note that the L^1 convergence assumption implies that for $\mathcal{L}^1$-a.e. $\eta < 0$ and for any $A \in \mathcal{A}(\Omega)$,

$$\lim_\varepsilon \mathcal{L}^n(\{x \in A : u_\varepsilon(x) < \eta\} \triangle \{x \in A : u(x) < \eta\}) = 0. \tag{3.4}$$

For every $\eta < 0$ we are going to prove that

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) \geqslant MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < \eta\}). \tag{3.5}$$

Once (3.5) is established the assertion follows by letting $\eta \to 0^-$.

Since by Ambrosio's lower semicontinuity Theorem 2.2, for any $A \in \mathcal{A}(\Omega)$ we have

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, A) \geqslant \mathcal{H}^{n-1}(S_u \cap A), \tag{3.6}$$

in order to prove (3.5), it suffices to show that for any $A \in \mathcal{A}(\Omega)$,

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, A) \geqslant \int_A |\nabla u|^p \,\mathrm{d}x + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in A : u(x) < \eta\}). \tag{3.7}$$

Indeed, taking (3.7) for granted, inequality (3.5) follows from standard measure-theoretic arguments by taking into account that the two quantities on the right hand side of (3.6), (3.7) are mutually orthogonal measures and the left hand side term is a superadditive set function defined on $\mathcal{A}(\Omega)$ (for details see Proposition 1.16 of [5]).

Fix $A \in \mathcal{A}(\Omega)$ and choose η for which (3.4) holds for the open set A. Moreover, define $V = \{x \in A : u(x) < \eta\}$, and assume that $\mathcal{L}^n(\{x \in A : u(x) < \eta\}) > 0$, since otherwise (3.7) is trivial. For $k \in \mathbb{N}$ fixed we consider the following splitting of the energies:[1]

$$\mathcal{F}_\varepsilon(u_\varepsilon, A) = MS_p\Big(u_\varepsilon, A \setminus \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\Big) + MS_p\Big(u_\varepsilon, A \cap \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\Big). \tag{3.8}$$

We will now estimate separately the two terms on the right hand side of (3.8) showing that the first contributes to the gradient energy (Step 1) while the latter provides the capacitary term of (3.7) (Step 2).

Step 1: Gradient estimate. We prove that

$$\lim_k \Big(\liminf_\varepsilon MS_p\Big(u_\varepsilon, A \setminus \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\Big)\Big) \geqslant \int_A |\nabla u|^p \,\mathrm{d}x. \tag{3.9}$$

In order to match the assumptions of Theorem 2.3, fix a parameter $\gamma > 0$ and consider the auxiliary (localized) functionals $\mathcal{G}_\varepsilon^{\gamma,k} : SBV(A) \times \mathcal{A}(A) \to [0, +\infty)$ defined as

$$\mathcal{G}_\varepsilon^{\gamma,k}(v, U) = \int_U \varphi^{\gamma,k}\Big(\frac{x}{\varepsilon}, \nabla v\Big)\,\mathrm{d}x + \int_{S_v\cap U} \psi^{\gamma,k}\Big(\frac{x}{\varepsilon}, v^+, v^-, \nu_v\Big)\,\mathrm{d}\mathcal{H}^{n-1},$$

where $\varphi^{\gamma,k}(x,\xi) = a^{\gamma,k}(x)|\xi|^p$ for $(x,\xi) \in \mathbb{R}^{2n}$, $\psi^{\gamma,k}(x,a,b,\nu) = a^{\gamma,k}(x) + \gamma|b-a|$ for $(x,a,b,\nu) \in \mathbb{R}^{3n} \times \mathbb{S}^{n-1}$, and $a^{\gamma,k}$ is the (Borel) 1-periodic function defined by

$$a^{\gamma,k}(x) = \begin{cases} 1, & x \in Q_1 \setminus B_{3/(4k)}, \\ \gamma, & x \in B_{3/(4k)}. \end{cases}$$

Since $\sup_\varepsilon MS_p(u_\varepsilon) < +\infty$, for a positive constant c we get

$$\limsup_\varepsilon \int_{S_{u_\varepsilon}\cap A} |u_\varepsilon^+ - u_\varepsilon^-|\,\mathrm{d}\mathcal{H}^{n-1} \leqslant 2\sup_\varepsilon(\|u_\varepsilon\|_{L^\infty(\Omega)}\mathcal{H}^{n-1}(S_{u_\varepsilon})) \leqslant c,$$

and

$$\liminf_\varepsilon MS_p\Big(u_\varepsilon, A \setminus \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\Big) \geqslant \liminf_\varepsilon \mathcal{G}_\varepsilon^{\gamma,k}(u_\varepsilon, A) - c\gamma. \tag{3.10}$$

For every $U \in \mathcal{A}(A)$ the family $(\mathcal{G}_\varepsilon^{\gamma,k}(\cdot, U))$ satisfies the assumptions of Theorem 2.3, and thus it Γ-converges to the functional $\mathcal{G}_{\mathrm{hom}}^{\gamma,k}(\cdot, U)$ defined in (2.5) of Theorem 2.3. Hence, to prove Step 1 it suffices to estimate the volume density $\varphi_{\mathrm{hom}}^{\gamma,k}$ of $\mathcal{G}_{\mathrm{hom}}^{\gamma,k}$ since (3.10) can be rewritten as

$$\liminf_\varepsilon MS_p\Big(u_\varepsilon, A \setminus \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\Big) \geqslant \mathcal{G}_{\mathrm{hom}}^{\gamma,k}(u, A) - c\gamma \geqslant \int_A \varphi_{\mathrm{hom}}^{\gamma,k}(\nabla u)\,\mathrm{d}x - c\gamma. \tag{3.11}$$

[1] The choice of the coefficient 3/4 in the radius of the balls in (3.8) is arbitrary and could be replaced with any $t \in (0,1)$. Indeed, since $r_\varepsilon = o(\varepsilon)$ the set E_{r_ε} is contained in $B_{t\varepsilon}$ for any $t \in (0,1)$.

We claim that, with fixed $\gamma > 0$, for every $\xi \in \mathbb{R}^n$ we have

$$\lim_k \varphi_{\text{hom}}^{\gamma,k}(\xi) = \sup_k \varphi_{\text{hom}}^{\gamma,k}(\xi) = |\xi|^p. \tag{3.12}$$

Once (3.12) is established, (3.9) follows from (3.11) by letting first $k \to +\infty$ and using the monotone convergence theorem, and then $\gamma \to 0^+$.

In order to prove (3.12) we take advantage of (2.4). Indeed, with fixed $\xi \in \mathbb{R}^n$, we prove that the Γ-limit (as $k \to +\infty$) in the L^1 strong topology of the sequence $\mathcal{A}^{\gamma,k} : W^{1,p}_{\text{per}}(Q_1) \to [0, +\infty]$ with

$$\mathcal{A}^{\gamma,k}(v) = \int_{Q_1} a^{\gamma,k}(x)|\nabla v + \xi|^p \,\mathrm{d}x$$

is given by

$$\mathcal{A}(v) = \int_{Q_1} |\nabla v + \xi|^p \,\mathrm{d}x.$$

Notice that by definition $\min_{W^{1,p}_{\text{per}}(Q_1)} \mathcal{A}^{\gamma,k} = \varphi_{\text{hom}}^{\gamma,k}(\xi)$ and by Jensen's inequality $\min_{W^{1,p}_{\text{per}}(Q_1)} \mathcal{A} = |\xi|^p$. Moreover, for any fixed $\gamma > 0$ the sequence $(\mathcal{A}^{\gamma,k})$ is equi-coercive in $L^1(Q_1)$, so that we may apply Theorem 2.1 to deduce (3.12).

Finally, we establish the claimed Γ-limit concerning $(\mathcal{A}^{\gamma,k})$.

The limsup inequality is trivial, since the recovery sequence for any given $v \in W^{1,p}_{\text{per}}(Q_1)$ is provided by the function itself thanks to Lebesgue's dominated convergence theorem. Indeed, $a^{\gamma,k} \to 1$ in $L^1(Q_1)$ and $0 \leqslant a^{\gamma,k}(x) \leqslant 1$ for every $x \in Q_1$.

To prove the liminf inequality it suffices to note that for every $(v_k) \subset W^{1,p}_{\text{per}}(Q_1)$ such that $v_k \to v$ in $L^1(Q_1)$ and $\liminf_k \mathcal{A}^{\gamma,k}(v_k) < +\infty$, actually (v_k) converges to v weakly in $W^{1,p}(Q_1)$. Hence, for every $\delta > 0$ we have

$$\begin{aligned}\liminf_k \mathcal{A}^{\gamma,k}(v_k) &\geqslant \liminf_k \int_{Q_1 \setminus \overline{B}_\delta} a^{\gamma,k}(x)|\nabla v_k + \xi|^p \,\mathrm{d}x \\ &= \liminf_k \int_{Q_1 \setminus \overline{B}_\delta} |\nabla v_k + \xi|^p \,\mathrm{d}x \geqslant \int_{Q_1 \setminus \overline{B}_\delta} |\nabla v + \xi|^p \,\mathrm{d}x,\end{aligned}$$

and the conclusion follows by letting $\delta \to 0^+$.

Step 2: Capacitary estimate. We prove that

$$\liminf_\varepsilon MS_p\Big(u_\varepsilon, A \cap \bigcup_{\underline{i} \in \mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\Big) \geqslant \mathrm{C}_1(E_+)\beta^{n-1}\Big(\mathcal{L}^n(V) - \frac{1}{k^{n+1}}\Big). \tag{3.13}$$

Choose an open set $W \subseteq A$ such that $W \supseteq V$ and $\mathcal{L}^n(W \setminus V) \leqslant 1/(2k^{2(n+1)})$. By (3.4) the set $\{x \in A : u_\varepsilon(x) \geqslant \eta\} \cap V$ has vanishing $\mathcal{L}^n$-measure so for ε sufficiently small we have

$$\mathcal{L}^n(\{x \in A : u_\varepsilon(x) \geqslant \eta\} \cap V) \leqslant \frac{1}{2k^{2(n+1)}}.$$

Set $U_\varepsilon = \{x \in W : u_\varepsilon(x) \geqslant \eta\}$. Then for ε small enough,

$$\mathcal{L}^n(U_\varepsilon) \leqslant \mathcal{L}^n(U_\varepsilon \cap V) + \mathcal{L}^n(U_\varepsilon \cap (W \setminus V)) \leqslant \frac{1}{k^{2(n+1)}}.$$

Let

$$\mathcal{W}_\varepsilon = \{\underline{i} \in \mathbb{Z}^n : Q_\varepsilon(\underline{i}\varepsilon) \subset\subset W\},$$

and consider

$$\mathcal{I}_\varepsilon^k = \{\underline{i} \in \mathcal{W}_\varepsilon : \mathcal{L}^n(U_\varepsilon \cap Q_\varepsilon(\underline{i}\varepsilon)) \leqslant \varepsilon^n/k^{n+1}\}.$$

The set of indices $\mathcal{I}_\varepsilon^k$ identifies those cells for which the contribution to the capacitary term can be estimated up to an error infinitesimal as $k \to +\infty$.

Let us first show that $\mathcal{I}_\varepsilon^k$ nearly exhausts $\mathcal{W}_\varepsilon$. Indeed, we have

$$\frac{1}{k^{2(n+1)}} \geqslant \mathcal{L}^n(U_\varepsilon) \geqslant \sum_{\underline{i}\in\mathcal{W}_\varepsilon} \mathcal{L}^n(U_\varepsilon \cap Q_\varepsilon(\underline{i}\varepsilon)) \geqslant \#(\mathcal{W}_\varepsilon \setminus \mathcal{I}_\varepsilon^k)\frac{\varepsilon^n}{k^{n+1}},$$

from which we deduce $\#(\mathcal{W}_\varepsilon \setminus \mathcal{I}_\varepsilon^k) \leqslant 1/(k^{n+1}\varepsilon^n)$. Moreover, if we set $\rho_\varepsilon = 3\varepsilon/(4k)$, the very definition of $\mathcal{I}_\varepsilon^k$ also yields

$$\mathcal{L}^n(U_\varepsilon \cap B_{\rho_\varepsilon}(\underline{i}\varepsilon)) \leqslant \frac{2^n}{w_n k}\mathcal{L}^n(B_{\rho_\varepsilon}(\underline{i}\varepsilon)), \tag{3.14}$$

and a simple translation argument shows that for any such index $\underline{i} \in \mathcal{I}_\varepsilon^k$ we have

$$MS_p(u_\varepsilon, B_{\rho_\varepsilon}(\underline{i}\varepsilon)) \geqslant m_\varepsilon(\eta) = \inf\Big\{MS_p(v, B_{\rho_\varepsilon}) : v \in SBV(B_{\rho_\varepsilon}),$$
$$v \geqslant 0 \text{ a.e. on } E_{r_\varepsilon},\ \mathcal{L}^n(\{x \in B_{\rho_\varepsilon} : v(x) \geqslant \eta\}) \leqslant \frac{2^n}{\omega_n k}\mathcal{L}^n(B_{\rho_\varepsilon})\Big\}.$$

It is clear that if we restrict the class of admissible functions v in the definition of $m_\varepsilon(\eta)$ above to simple functions assuming values in $\{0, \eta\}$, we have, by (2.10),

$$m_\varepsilon(\eta) \leqslant \mathrm{C}_1((E_+)_{r_\varepsilon}) = \mathrm{C}_1(E_+)r_\varepsilon^{n-1}.$$

Next we want to estimate $m_\varepsilon(\eta)$ from below, more precisely we prove

$$\lim_\varepsilon r_\varepsilon^{1-n} m_\varepsilon(\eta) = \mathrm{C}_1(E_+). \tag{3.15}$$

To do that we need the following result.

LEMMA 3.6 Let $H \subset \mathbb{R}^n$ be a bounded $\mathcal{L}^n$-measurable thick set, and $v_\varepsilon \in SBV(B_{R_\varepsilon})$, $R_\varepsilon \to +\infty$, be such that

(i) $v_\varepsilon \geqslant 0$ a.e. on H, $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(B_{R_\varepsilon})} < +\infty$,
(ii) $\lim_\varepsilon \|\nabla v_\varepsilon\|_{L^p(B_{R_\varepsilon})} = 0$, $\limsup_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) \leqslant \mathrm{C}_1(H)$,
(iii) $\sup_\varepsilon \|Dv_\varepsilon\|(B_{R_\varepsilon}) < +\infty$,
(iv) there exists $\zeta < 0$ such that $\mathcal{L}^n(\{x \in B_{R_\varepsilon} : v_\varepsilon(x) \geqslant \zeta\}) < \frac{1}{2}\mathcal{L}^n(B_{R_\varepsilon})$.

Then $\lim_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) = \mathrm{C}_1(H)$. Moreover, for every subsequence (v_{ε_m}) there exist $(v_{\varepsilon_{m_j}})$ and $v \in SBV_{\mathrm{loc}}(\mathbb{R}^n)$ such that $v_{\varepsilon_{m_j}} \to v$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $v \geqslant 0$ a.e. on H, $v = \sum_{s\in I} a_i \chi_{E_i}$, where I is a finite set, E_i has finite perimeter, $a_i \in \mathbb{R}$, and $\mathcal{H}^{n-1}(S_v) = \mathrm{C}_1(H)$.

Proof of Lemma 3.6. First note that by assumption (ii) it is sufficient to show that

$$\liminf_{\varepsilon} \mathcal{H}^{n-1}(S_{v_\varepsilon}) \geqslant \mathrm{C}_1(H).$$

Denote by (v_{ε_m}) a sequence for which $\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) = \lim_m \mathcal{H}^{n-1}(S_{v_{\varepsilon_m}})$. Ambrosio' SBV compactness and lower semicontinuity Theorem 2.2 applied on every ball B_R, $R > 0$, and an obvious diagonalization argument ensure the existence of a subsequence $(v_{\varepsilon_{m_j}}) \subseteq (v_{\varepsilon_m})$, and of $v \in SBV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^n)$ such that $v_{\varepsilon_{m_j}} \to v$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\nabla v = 0$ a.e. in $\mathbb{R}^n$ and

$$\mathcal{H}^{n-1}(S_v) \leqslant \lim_j \mathcal{H}^{n-1}(S_{v_{\varepsilon_{m_j}}}) \leqslant \mathrm{C}_1(H).$$

For the sake of simplicity in the rest of the proof we set $v_j = v_{\varepsilon_{m_j}}$ and $R_j = R_{\varepsilon_{m_j}}$.

The BV coarea formula (see Theorem 3.40 of [1]) and the mean value theorem provide $t_j \in (\zeta, \zeta/2)$ such that

$$\begin{aligned} \|Dv_j\|(B_{R_j}) &\geqslant \int_\zeta^{\zeta/2} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) > t\}, B_{R_j})\,\mathrm{d}t \\ &\geqslant \frac{|\zeta|}{2}\mathrm{Per}(\{x \in B_{R_j} : v_j(x) > t_j\}, B_{R_j}) \geqslant \frac{|\zeta|}{2} c\mathcal{L}^n(\{x \in B_{R_j} : v_j(x) > t_j\})^{1-1/n} \\ &\geqslant \frac{|\zeta|}{2} c\mathcal{L}^n(\{x \in B_{R_j} : v_j(x) > \zeta/2\})^{1-1/n}, \end{aligned}$$

where in the third inequality we have used assumption (iv) and the relative isoperimetric inequality in balls (see Remark 3.50 of [1]). Hence, (iii) gives $\sup_j \mathcal{L}^n(\{x \in B_{R_j} : v_j(x) > \zeta/2\}) < +\infty$, so that the L^1_{loc} convergence implies $\mathcal{L}^n(\{x \in \mathbb{R}^n : v(x) > \zeta/2\}) < +\infty$ as well as $v \geqslant 0$ a.e. on H.

As $v \in SBV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^n)$ with $\nabla v = 0$ a.e. on $\mathbb{R}^n$ and $\mathcal{H}^{n-1}(S_v) < +\infty$, we have the decomposition $v = \sum_{i \geqslant 0} a_i \chi_{\Sigma_i}$, with Σ_i a set with finite perimeter for every i, and the equality $2\mathcal{H}^{n-1}(S_v) = \sum_{i\geqslant 0} \mathrm{Per}(\Sigma_i)$ holds true (see Theorem 4.23 of [1]).

Since $\{x \in \mathbb{R}^n : v(x) \geqslant 0\} = \bigcup_{r=1}^s \Sigma_{i_r}$ for some i_r, we have $\mathrm{Per}(\bigcup_{r=1}^s \Sigma_{i_r}) \leqslant \mathcal{H}^{n-1}(S_v) \leqslant \mathrm{C}_1(H)$. Moreover, since H is a thick obstacle, $\bigcup_{r=1}^s \Sigma_{i_r}$ has finite perimeter and $\bigcup_{r=1}^s \Sigma_{i_r} \supseteq H$, we have $\mathcal{H}^{n-1}(H \setminus (\bigcup_{r=1}^s \Sigma_{i_r})_+) = 0$. Thus, $\chi_{\bigcup_{r=1}^s \Sigma_{i_r}}$ is a test function for the capacitary problem on H, which implies $\mathrm{Per}(\bigcup_{r=1}^s \Sigma_{i_r}) = \mathrm{C}_1(H)$.

Finally, if $\Sigma = \bigcup_{i \neq i_r} \Sigma_i$ it is easy to prove that there exists an index $t \geqslant 1$, with $a_t \neq a_{i_r}$ for every r, such that $v = \sum_{r=1}^s a_{i_r}\chi_{\Sigma_{i_r}} + a_t \chi_\Sigma$. □

Let us go back to the proof of inequality (3.15). Given w_ε such that $MS_p(w_\varepsilon, B_{\rho_\varepsilon}) \leqslant m_\varepsilon(\eta) + r_\varepsilon^n$, let us check that the family $v_\varepsilon(x) = w_\varepsilon(r_\varepsilon x)$, $x \in B_{R_\varepsilon}$, where $R_\varepsilon = \rho_\varepsilon / r_\varepsilon$, satisfies the assumptions of Lemma 3.6 above with $H = E_+$. Indeed, (i) is trivially satisfied, while (ii) holds true since by scaling

$$\frac{MS_p(w_\varepsilon, B_{\rho_\varepsilon})}{r_\varepsilon^{n-1}} = r_\varepsilon^{1-p} \int_{B_{R_\varepsilon}} |\nabla v_\varepsilon|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_{v_\varepsilon}) \leqslant \mathrm{C}_1(E_+) + r_\varepsilon. \tag{3.16}$$

Moreover, (3.16) and Hölder's inequality yield

$$\int_{B_{R_\varepsilon}} |\nabla v_\varepsilon| \,\mathrm{d}x \leqslant R_\varepsilon^{n-n/p} \|\nabla v_\varepsilon\|_{L^p(B_{R_\varepsilon})} \leqslant \left(\frac{3\varepsilon}{4kr_\varepsilon^{1-1/n}}\right)^{n-n/p} (\mathrm{C}_1(E_+) + r_\varepsilon)^{1/p}, \tag{3.17}$$

so that $\sup_\varepsilon \|Dv_\varepsilon\|(B_{R_\varepsilon}) < +\infty$, and (iii) is satisfied too. Finally, (iv) easily follows from (3.14) for $k \geqslant 2^{n+2}$, hence Lemma 3.6 implies (3.15).

To conclude fix $W' \subset\subset W$ and notice that for ε small, $W' \subset \bigcup_{\underline{i}\in\mathcal{W}_\varepsilon} Q_\varepsilon(\underline{i}\varepsilon)$. Then

$$\liminf_\varepsilon \sum_{\underline{i}\in\mathcal{I}^k_\varepsilon} MS_p(u_\varepsilon, B_{\rho_\varepsilon}(\underline{i}\varepsilon)) \geqslant \liminf_\varepsilon m_\varepsilon(\eta)\#\mathcal{I}^k_\varepsilon \geqslant \beta^{n-1}\lim_\varepsilon \frac{m_\varepsilon(\eta)}{r_\varepsilon^{n-1}}\left(\varepsilon^n\#\mathcal{W}_\varepsilon - \frac{1}{k^{n+1}}\right)$$

$$\geqslant \beta^{n-1}\mathrm{C}_1(E_+)\left(\mathcal{L}^n(W') - \frac{1}{k^{n+1}}\right). \tag{3.18}$$

To get (3.13), it remains to take the supremum over the sets $W' \subset\subset W$ and to recall that $W \supseteq V$.

Step 3: Estimate (3.7). We finally obtain (3.7) by combining Step 1 and Step 2, and by letting $k \to +\infty$ in (3.8), i.e.

$$\liminf_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, A) \geqslant \liminf_k\left(\liminf_\varepsilon MS_p\left(u_\varepsilon, A \setminus \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\right)\right)$$

$$+ \liminf_k\left(\liminf_\varepsilon MS_p\left(u_\varepsilon, A \cap \bigcup_{\underline{i}\in\mathbb{Z}^n} B_{3\varepsilon/(4k)}(\underline{i}\varepsilon)\right)\right)$$

$$\geqslant \int_A |\nabla u|^p\,\mathrm{d}x + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(V). \qquad \square$$

The lower bound inequality in the general case is an easy consequence of a standard truncation argument.

PROPOSITION 3.7 Under the hypotheses of Theorem 3.1, for every $u \in L^1(\Omega)$,

$$\Gamma\text{-}\liminf_\varepsilon \mathcal{F}_\varepsilon(u) \geqslant \mathcal{F}(u),$$

where $\mathcal{F}$ is defined in (3.2).

Proof. The assertion follows directly from Lemma 3.5 once one notices that the energies $\mathcal{F}_\varepsilon$, $\mathcal{F}$ are decreasing by truncation and the Mumford–Shah functional is continuous along such sequences. More precisely, if $v \in L^1(\Omega)$ and $N > 0$, denote $(v \wedge N) \vee (-N)$ by v^N; then v satisfies the same constraint as v^N does, $MS_p(v^N) \leqslant MS_p(v)$, and $MS_p(v^N) \to MS_p(v)$ as $N \to +\infty$. $\square$

REMARK 3.8 As a consequence of Step 2 in Lemma 3.5 above, in case $\varepsilon^{n/(n-1)} = o(r_\varepsilon)$, that is, $\beta = +\infty$, the Γ-limit of $(\mathcal{F}_\varepsilon)$ equals $MS_p(u)$ if $u \in GSBV(\Omega)$, $u \geqslant 0$ $\mathcal{L}^n$-a.e. on Ω, and $+\infty$ otherwise in $L^1(\Omega)$. This follows directly from (3.18).

Let us now conclude the proof of Theorem 3.1 and prove the upper bound inequality. We introduce the notation

$$\mathcal{U}_\rho(A) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < \rho\}$$

for $\rho > 0$ and $A \subseteq \mathbb{R}^n$.

PROPOSITION 3.9 Under the hypotheses of Theorem 3.1, for every $u \in L^1(\Omega)$,

$$\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon(u) \leqslant \mathcal{F}(u), \tag{3.19}$$

where $\mathcal{F}$ is defined in (3.2).

Proof. Let $u \in GSBV(\Omega)$ be such that $\mathcal{F}(u) < +\infty$; otherwise the inequality is trivial. We first prove the Γ-lim sup inequality under the following additional assumptions:

(a) $u \in SBV(\Omega)$, $\mathcal{H}^{n-1}(\overline{S}_u \setminus S_u) = 0$, $u \in W^{k,\infty}(\Omega \setminus \overline{S}_u)$ for any $k \in \mathbb{N}$, and $\overline{S}_u \subseteq \bigcup_{j=1}^N \Sigma_j$ where Σ_j are $(n-1)$-simplexes;

(b) the set $\{x \in \Omega : u(x) < 0\}$ has finite perimeter in Ω, and $\{x \in \Omega \setminus \overline{S}_u : u(x) = 0\}$ is an $(n-1)$-dimensional smooth manifold in $\Omega \setminus \overline{S}_u$.

By (2.10) and Remark 2.8 we choose a set $D \subseteq Q_1$ of finite perimeter with $\mathrm{C}_1(E_+) = \mathrm{Per}(D)$ and $\mathcal{H}^{n-1}(E_+ \setminus D_+) = 0$, which of course implies $\mathcal{L}^n(E \setminus D) = 0$.

Define $\mathcal{J} = \{\underline{i} \in \mathbb{Z}^n : \mathcal{L}^n(D_{r_\varepsilon}(\underline{i}\varepsilon) \cap \{x \in \Omega : u(x) < 0\}) > 0\}$, $\mathbf{D}_\varepsilon = \bigcup_{\underline{i} \in \mathcal{J}} D_{r_\varepsilon}(\underline{i}\varepsilon)$, and define $u_\varepsilon \in L^1(\Omega)$ as $u_\varepsilon = u\chi_{\Omega\setminus\mathbf{D}_\varepsilon}$. Then $u_\varepsilon \in SBV(\Omega)$ and by construction $u_\varepsilon \geqslant 0$ $\mathcal{L}^n$-a.e. on $\mathbf{D}_\varepsilon$, actually $u_\varepsilon = 0$ $\mathcal{L}^n$-a.e. on $\mathbf{D}_\varepsilon$. Since $\mathcal{L}^n(\mathbf{D}_\varepsilon) \leqslant \#(\mathcal{J}) r_\varepsilon^n \mathcal{L}^n(D) \leqslant c r_\varepsilon$, we have $u_\varepsilon \to u$ in $L^1(\Omega)$, and a direct computation shows

$$\begin{aligned}\mathcal{F}_\varepsilon(u_\varepsilon) &\leqslant \int_{\Omega\setminus\mathbf{D}_\varepsilon} |\nabla u|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_u \setminus \mathbf{D}_\varepsilon) + \mathrm{Per}(\mathbf{D}_\varepsilon)\\ &\leqslant \int_{\Omega} |\nabla u|^p \,\mathrm{d}x + \mathcal{H}^{n-1}(S_u) + \#(\mathcal{J}) r_\varepsilon^{n-1}\mathrm{Per}(D)\\ &\leqslant MS_p(u) + \mathrm{C}_1(E_+)\frac{r_\varepsilon^{n-1}}{\varepsilon^n}\mathcal{L}^n(\mathcal{U}_{\sqrt{n}\varepsilon}(\{x \in \Omega : u(x) < 0\})). \qquad (3.20)\end{aligned}$$

In the last inequality we used the fact that $\#(\mathcal{J})\varepsilon^n = \mathcal{L}^n(\bigcup_{\underline{i}\in\mathcal{J}} Q_\varepsilon(\underline{i}\varepsilon))$ and $\bigcup_{\underline{i}\in\mathcal{J}} Q_\varepsilon(\underline{i}\varepsilon) \subseteq \mathcal{U}_{\sqrt{n}\varepsilon}(\{x \in \Omega : u(x) < 0\})$. To estimate the Lebesgue measure in the last term of (3.20) we use the equality

$$\bigcap_{\varepsilon>0} \mathcal{U}_{\sqrt{n}\varepsilon}(\{x \in \Omega : u(x) < 0\}) = \overline{\{x \in \Omega : u(x) < 0\}},$$

so that

$$\lim_{\varepsilon\to 0^+} \mathcal{L}^n(\mathcal{U}_{\sqrt{n}\varepsilon}(\{x \in \Omega : u(x) < 0\})) = \mathcal{L}^n(\overline{\{x \in \Omega : u(x) < 0\}}).$$

By passing to the limsup as $\varepsilon \to 0^+$ in (3.20) we get

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon) \leqslant MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\overline{\{x \in \Omega : u(x) < 0\}}).$$

To obtain (3.19) it suffices to notice that

$$\mathcal{L}^n(\overline{\{x \in \Omega : u(x) < 0\}} \setminus \{x \in \Omega : u(x) < 0\}) = 0$$

thanks to (a), (b) and the regularity of $\partial\Omega$.

We now remove assumption (b). In order to do that it suffices to note that by applying Sard's lemma to u on $\Omega \setminus \overline{S}_u$ and by the BV coarea formula (see Theorem 3.40 of [1]), we can find a sequence $\eta_k \to 0^-$ such that for any $k \in \mathbb{N}$ the functions $u - \eta_k$ satisfy (b). Hence, the previous step implies

$$\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}_\varepsilon(u - \eta_k) \leqslant \mathcal{F}(u - \eta_k) \leqslant \mathcal{F}(u),$$

and the upper bound inequality for u follows by letting $\eta_k \to 0^-$ and by taking into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon$.

For a general function $u \in GSBV(\Omega)$ we use a density result with respect to Mumford–Shah type energies and in $L^1(\Omega)$ for functions satisfying (a), proved in [17] (see also [18] for a more general statement).

Now, consider (u_j) satisfying (a) and such that $u_j \to u$ in $L^1(\Omega)$ and $MS_p(u_j) \to MS_p(u)$, and let $\eta_k \to 0^-$ be such that $\mathcal{L}^n(\{x \in \Omega : u_j(x) < \eta_k\}) \to \mathcal{L}^n(\{x \in \Omega : u(x) < \eta_k\})$ as $j \to +\infty$ for every $k \in \mathbb{N}$. Then, by using for every $j \in \mathbb{N}$ the identity $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon(u_j - \eta_k) = \mathcal{F}(u_j - \eta_k)$, and the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon$, we infer that

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}_\varepsilon(u - \eta_k) &\leqslant \lim_j \mathcal{F}(u_j - \eta_k)\\ &= \lim_j (MS_p(u_j) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u_j(x) < \eta_k\}))\\ &= MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < \eta_k\}) \leqslant \mathcal{F}(u).\end{aligned}$$

Passing to the liminf as $k \to +\infty$ and taking again into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}_\varepsilon$ we conclude the proof. □

REMARK 3.10 It is clear from the proof of Proposition 3.9 that in the regime $r_\varepsilon = o(\varepsilon^{n/(n-1)})$, that is, $\beta = 0$, the Γ-limit of $(\mathcal{F}_\varepsilon)$ is trivial and identically equal to MS_p.

We now provide the proof of the bilateral obstacle case contained in Proposition 3.3.

Proof of Proposition 3.3. Lower bound: First notice that for every $A \in \mathcal{A}(\Omega)$, $\varepsilon > 0$ and $u \in L^1(\Omega)$ we have

$$\mathcal{F}'_\varepsilon(u, A) \geqslant \mathcal{F}_\varepsilon(u, A), \qquad \mathcal{F}'_\varepsilon(u, A) \geqslant \mathcal{F}_\varepsilon(-u, A). \tag{3.21}$$

Hence, given (u_ε) converging to u in $L^1(\Omega)$, by applying Proposition 3.7 to the right hand sides in (3.21), we get

$$\liminf_\varepsilon \mathcal{F}'_\varepsilon(u_\varepsilon, A) \geqslant \mathcal{F}(u, A) = MS_p(u, A) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in A : u(x) < 0\}) \tag{3.22}$$

and

$$\liminf_\varepsilon \mathcal{F}'_\varepsilon(u_\varepsilon, A) \geqslant \mathcal{F}(-u, A) = MS_p(u, A) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in A : u(x) > 0\}). \tag{3.23}$$

In particular, this entails $u \in GSBV(\Omega)$ provided $\liminf_\varepsilon \mathcal{F}'(u_\varepsilon) < +\infty$. Moreover, the usual measure-theoretic arguments imply the lower bound inequality. Indeed, the second terms in the sums on the right hand sides of (3.22), (3.23) are mutually orthogonal measures and the left hand side term is a superadditive set function defined on $\mathcal{A}(\Omega)$ (for details see Proposition 1.16 of [5]).

Upper bound: We construct a recovery sequence for any $u \in GSBV(\Omega)$ such that $\mathcal{F}'(u) < +\infty$. Moreover, we may assume $\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}) > 0$, the result being trivial otherwise.

We keep the notation of Proposition 3.9, and first prove the limsup inequality under the additional assumptions (a) and (b) with the set $\{x \in \Omega : u(x) \neq 0\}$ playing the role of $\{x \in \Omega : u(x) < 0\}$ there.

Supposing this, we may perform the very same construction of Proposition 3.9 substituting the 0 sub-level set of u with $\{x \in \Omega : u(x) \neq 0\}$. Indeed, the recovery sequence $(u_\varepsilon) \subset SBV(\Omega)$ built up there is such that $u_\varepsilon = 0$ $\mathcal{L}^n$-a.e. on $\mathbf{E}_\varepsilon$. Hence, using the same arguments one achieves

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leqslant MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) \neq 0\}) + o(1),$$

and passing to the limsup as $\varepsilon \to 0^+$ we get the desired inequality.

We now remove the regularity assumption (b) on the set $\{x \in \Omega : u(x) \neq 0\}$. To do this, argue as in Proposition 3.9 and consider a positive sequence (η_k) such that $\eta_k \to 0^+$ as $k \to +\infty$ and both the sets $\{x \in \Omega : u(x) > \eta_k\}$, $\{x \in \Omega : u(x) < -\eta_k\}$ satisfy (b). Let $u^k \in SBV(\Omega)$ be defined as $u^k = (u \vee \eta_k) + (u \wedge (-\eta_k))$. Notice that $|u(x)| \leqslant \eta_k \Leftrightarrow u^k(x) = 0$, $u(x) \geqslant \eta_k \Rightarrow u^k(x) = u(x) - \eta_k$, $u(x) \leqslant -\eta_k \Rightarrow u^k(x) = u(x) + \eta_k$, and $\mathcal{H}^{n-1}(S_{u^k} \setminus S_u) = 0$. Clearly $u^k \to u$ in $L^1(\Omega)$ and

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}'_\varepsilon(u^k) &\leqslant \mathcal{F}'(u^k) = MS_p(u^k) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u^k(x) \neq 0\}) \\ &\leqslant MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : |u(x)| > \eta_k\}).\end{aligned}$$

Passing to the liminf as $k \to +\infty$ and taking into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}'_\varepsilon$, we get the desired inequality.

To finish the proof for any $u \in GSBV(\Omega)$ consider a sequence (u_j) satisfying (a) and such that $u_j \to u$ in $L^1(\Omega)$ and $MS_p(u_j) \to MS_p(u)$ (see Theorem 3.9 of [17]), and let $\eta_k \to 0^+$ be such that $\mathcal{L}^n(\{x \in \Omega : |u_j(x)| > \eta_k\}) \to \mathcal{L}^n(\{x \in \Omega : |u(x)| > \eta_k\})$ as $j \to +\infty$ for every $k \in \mathbb{N}$. Since $(u_j)^k \to u^k$ in $L^1(\Omega)$, arguing as in the last step of Proposition 3.9 we infer that

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon} \mathcal{F}'_\varepsilon(u^k) &\leqslant \liminf_{j} \mathcal{F}'((u_j)^k) \\ &\leqslant \lim_{j}(MS_p(u_j) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : |u_j(x)| > \eta_k\})) \\ &= MS_p(u) + \mathrm{C}_1(E_+)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : |u(x)| > \eta_k\}) \leqslant \mathcal{F}'(u).\end{aligned}$$

Passing to the liminf as $k \to +\infty$ and taking again into account the lower semicontinuity of $\Gamma\text{-}\limsup_\varepsilon \mathcal{F}'_\varepsilon$ completes the proof. □

Finally, we handle the case in which Dirichlet boundary conditions are imposed.

Proof of Proposition 3.4. Lower bound: The lower bound inequality can be easily derived from Proposition 3.3. Given $v \in GSBV(\Omega)$ denote by $\tilde{v}$ the function extending v by 0 on $\mathbb{R}^n \setminus \Omega$. Then $\tilde{v} \in GSBV(\mathbb{R}^n)$ and for any open set $\Omega' \supset\supset \Omega$ with Lipschitz boundary, we have

$$MS_p(\tilde{v}, \Omega') = MS_p(v, \Omega) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \mathrm{tr}(v)(x) \neq 0\})$$

(see Theorems 3.84 and 3.87 of [1]).

Given $(u_\varepsilon) \subset GSBV(\Omega)$ with $\mathrm{tr}(u_\varepsilon) = 0$ on $\partial\Omega$ and converging to u in $L^1(\Omega)$, $(\tilde{u}_\varepsilon)$ converges to $\tilde{u}$ in $L^1(\Omega')$, and applying Proposition 3.3 with Ω replaced by Ω', we have

$$\begin{aligned}\liminf_{\varepsilon} \mathcal{D}_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon} \mathcal{F}'_\varepsilon(\tilde{u}_\varepsilon, \Omega') \geqslant \mathcal{F}'(\tilde{u}, \Omega') \\ &= \mathcal{F}'(u, \Omega) + \mathcal{H}^{n-1}(\{x \in \partial\Omega : \mathrm{tr}(u)(x) \neq 0\}) = \mathcal{D}(u).\end{aligned}$$

Upper bound: It remains to prove the upper bound inequality. First note that a recovery sequence for $u \in GSBV(\Omega)$ with $\mathrm{tr}(u) = 0$ on $\partial\Omega$ is the one constructed when no boundary condition is imposed in Proposition 3.3 above.

Given a generic function $u \in GSBV(\Omega)$, it is possible to find a sequence $(u_j) \subset GSBV(\Omega)$ with $\mathrm{tr}(u_j) = 0$ on $\partial\Omega$ and converging to u in $L^1(\Omega)$ such that $\lim_j \mathcal{D}(u_j) = \mathcal{D}(u)$. Then the result follows by the lower semicontinuity of $\Gamma\text{-}\limsup \mathcal{D}_\varepsilon$.

This sequence can be obtained by modifying u in a suitable neighborhood of the boundary in which the distance function is regular. Fix a sequence of positive numbers r_j tending to 0 and define $d(x) = \operatorname{dist}(x, \partial\Omega)$. Set

$$u_j(x) = \begin{cases} u(x) & \text{if } 2r_j < d(x), \\ u(x + (d(x) - 2r_j)\nabla d(x)) & \text{if } r_j < d(x) < 2r_j, \\ 0 & \text{if } 0 < d(x) < r_j. \end{cases}$$

It can be easily checked that

$$MS_p(u_j, \Omega) \leqslant MS_p(u, \Omega) + cMS_p(u, \{x \in \Omega : r_j < d(x) < 2r_j\}) \\ + \mathcal{H}^{n-1}(\{x \in \Omega : d(x) = r_j,\ \operatorname{tr}(u)(x - r_j\nabla d(x)) \neq 0\})$$

for a positive constant c not depending on j. Defining $\varphi_j : \partial\Omega \to \Omega$ as $\varphi_j(y) = y + r_j\nabla d(y)$ (with a slight abuse of notation $\nabla d(y)$ denotes the inner normal to $\partial\Omega$ at y), we have

$$\{x \in \Omega : d(x) = r_j, \operatorname{tr}(u)(x - r_j\nabla d(x)) \neq 0\} = \varphi_j(\{y \in \partial\Omega : \operatorname{tr}(u)(y) \neq 0\}).$$

The conclusion then follows from the fact that $\mathcal{H}^{n-1}(\varphi_j(H)) \leqslant (\operatorname{Lip}\varphi_j)^{n-1}\mathcal{H}^{n-1}(H)$ for any set H, and $\operatorname{Lip}\varphi_j \to 1$ as $j \to +\infty$. □

4. The case of thin obstacles

In this section we show how to deal with a general reference perforation set, including *thin obstacles*, i.e. sets with Lebesgue measure zero. To consider (non-trivial) thin obstacle problems it is clearly necessary to express the constraint in a different form. To do that it suffices to recall that if $u \in GSBV(\Omega)$ and $MS_p(u) < +\infty$ then the values $u^+(x)$, $u^-(x)$ are finite and specified for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ (see Theorem 4.40 of [1]).

Given an $\mathcal{H}^{n-1}$-measurable set $T \subseteq \overline{Q}_1$, for any $\varepsilon > 0$ let $r_\varepsilon \in (0, \varepsilon)$ and let $\mathbf{T}_\varepsilon = \Omega \cap \bigcup_{\underline{i}\in\mathbb{Z}^n} T_{r_\varepsilon}(\underline{i}\varepsilon)$. Consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} MS_p(u), & u \in GSBV(\Omega),\ u^+ \geqslant 0\ \mathcal{H}^{n-1}\text{-a.e. on } \mathbf{T}_\varepsilon, \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases} \tag{4.1}$$

The asymptotic analysis of $(\mathcal{F}_\varepsilon)$ takes advantage of the ideas and techniques developed in Section 3. The main difference is in the proof of Lemma 4.4, the counterpart of Lemma 3.6 in this framework, for which substantial changes are required. This is not an accidental or merely technical fact: we want to point out that Lemma 4.4 relies on the deep relaxation results contained in Chapter IV of [26] and in [10] (see Theorem 2.7).

THEOREM 4.1 Let T be an $\mathcal{H}^{n-1}$-measurable set, and assume that $r_\varepsilon/\varepsilon^{n/(n-1)} \to \beta \in [0, +\infty)$ as $\varepsilon \to 0^+$. Then $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F} : L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}(u) = \begin{cases} MS_p(u) + \mathrm{C}_1(T)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases} \tag{4.2}$$

with respect to the L^1 convergence.

REMARK 4.2 Let us point out how Theorem 3.1 can be recovered from Theorem 4.1 above. Notice that given a set E the following equivalence holds:

$$u \geqslant 0 \ \mathcal{L}^n\text{-a.e. on } E \ \Leftrightarrow \ u^+ \geqslant 0 \ \mathcal{H}^{n-1}\text{-a.e. on } E_+, \tag{4.3}$$

so that one can rephrase the unilateral obstacle condition in the sense of $\mathcal{L}^n$ on E with the more precise $\mathcal{H}^{n-1}$ meaning exactly on E_+. Roughly speaking, the equivalence in (4.3) means that the constraint for u intended in the $\mathcal{L}^n$ sense is active, for a suitable representative, only on the $\mathcal{L}^n$ measure-theoretic closure of E, and thus it is neglected on lower dimensional parts of the set. By taking this into account, the functionals in the statement of Theorem 3.1 can be rewritten as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} MS_p(u), & u \in GSBV(\Omega),\ u^+ \geqslant 0\ \mathcal{H}^{n-1}\text{-a.e. on } (\mathbf{E}_+)_\varepsilon, \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases}$$

Theorem 3.1 then follows by applying Theorem 4.1 with $T = E_+$.

REMARK 4.3 It is worth noting that a priori the functional $\mathcal{F}_\varepsilon$ in (4.1) may not be L^1 lower semicontinuous. More generally, given an $\mathcal{H}^{n-1}$-measurable set $H \subseteq \Omega$, one can study the lower semicontinuity properties of the functional $G : L^1(\Omega) \to [0, +\infty]$ defined as

$$G(u) = \begin{cases} MS_p(u), & u \in GSBV(\Omega),\ u^+ \geqslant 0\ \mathcal{H}^{n-1}\text{-a.e. on } H, \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases}$$

In a forthcoming paper (see [30]) we will prove that the lower semicontinuous envelope of G in the L^1 topology is given by

$$\mathrm{sc}^-(G)(u) = \begin{cases} MS_p(u) + \frac{1}{2}\sigma(\{x \in H \cap S_u : u^+(x) < 0\}) & \\ \qquad + \sigma(\{x \in H \setminus S_u : u^+(x) < 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases}$$

where σ is the measure defined in (2.11). Thus, by taking into account Theorem 4.1, well known results (see Proposition 6.11 of [22]) yield the Γ-convergence to the functional $\mathcal{F}$ of (4.2) of the energies

$$\mathrm{sc}^-(\mathcal{F}_\varepsilon)(u) = \begin{cases} MS_p(u) + \frac{1}{2}\sigma(\{x \in \mathbf{T}_\varepsilon \cap S_u : u^+(x) < 0\}) & \\ \qquad + \sigma(\{x \in \mathbf{T}_\varepsilon \setminus S_u : u^+(x) < 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases}$$

As a further development of this research we are investigating the compactness and integral representation properties of Mumford–Shah type energies with general obstacle constraints, as those established by Dal Maso in the Sobolev setting ([20], see also [10], [11]).

We now turn to the proof of Theorem 4.1. As already stated, we will point out only the changes needed in the proof of Theorem 3.1 in order to reach the conclusion. The set T in Theorem 4.1 plays the same role of E_+ in Lemma 3.5. Moreover, we keep the same notation used in Section 3 to which obviously we refer.

Proof of Theorem 4.1. Lower bound: The proof of Lemma 3.5 goes through until the capacitary estimate (3.13) of Step 2, assuming $\mathcal{H}^{n-1}(E) > 0$, as otherwise the statement is trivial. The latter is now a consequence of Lemma 4.4 below. Taking this for granted, to prove the lower bound inequality for sequences bounded in L^∞ it suffices to verify that the blow-up functions v_ε satisfy the assumptions of Lemma 4.4. The same arguments used in Theorem 3.1 ensure (i) and (iii), while (ii) follows by (3.17) taking into account that the right hand side in that formula is bounded as a function of ε and infinitesimal as $k \to +\infty$.

Finally, the truncation argument of Proposition 3.7 needs no change, so that the lower bound is established.

Upper bound: The same argument of Proposition 3.9 works upon replacing in the construction of the recovery sequence, a minimizing set D with a minimizing sequence for the capacitary problem for T. □

The statement of Lemma 4.4 below is given in a slightly more general framework than needed in our context.

LEMMA 4.4 Let H be a bounded $\mathcal{H}^{n-1}$-measurable set with $\mathcal{H}^{n-1}(H) > 0$, $N \in \mathbb{N}$, $N \geqslant 4$, and $v_\varepsilon \in BV(B_{R_\varepsilon})$, $R_\varepsilon \to +\infty$, be such that

(i) $v_\varepsilon^+ \geqslant 0$ $\mathcal{H}^{n-1}$-a.e. on H, $\sup_\varepsilon \|v_\varepsilon\|_{L^\infty(B_{R_\varepsilon})} < +\infty$,
(ii) $\sup_\varepsilon \|Dv_\varepsilon\|(B_{R_\varepsilon} \setminus S_{v_\varepsilon}) < 1/N$,
(iii) there exists $\zeta < 0$ such that $\mathcal{L}^n(\{x \in B_{R_\varepsilon} : v_\varepsilon(x) \geqslant \zeta\}) < \frac{1}{2}\mathcal{L}^n(B_{R_\varepsilon})$.

Then there exists a positive constant $c = c(\zeta)$ such that $\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) \geqslant \mathrm{C}_1(H) - c/\sqrt{N}$.

Proof. It is not restrictive to assume $\liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) < +\infty$, otherwise the statement is trivial. Let (v_{ε_j}) be such that $\lim_j \mathcal{H}^{n-1}(S_{v_{\varepsilon_j}}) = \liminf_\varepsilon \mathcal{H}^{n-1}(S_{v_\varepsilon}) < +\infty$; for simplicity for the rest of the proof we set $v_j = v_{\varepsilon_j}$ and $R_j = R_{\varepsilon_j}$.

Step 1: For any open set A, for any $v \in BV(A)$ satisfying $v^+ \geqslant 0$ $\mathcal{H}^{n-1}$-a.e. on H, with $H \subseteq A$, and for any $\delta < 0$, there exists $\eta \in (\delta, 0)$ for which $\mathcal{L}^n(\{x \in A : v(x) \geqslant \eta\}) > 0$, $\mathrm{Per}(\{x \in A : v(x) \geqslant \eta\}, A) < +\infty$ *and* $\mathcal{H}^{n-1}(H \setminus \{x \in A : v(x) \geqslant \eta\}_+) = 0$.

Let us first prove that there exists $t \in (\delta, 0)$ for which the corresponding super-level set has positive measure. Arguing by contradiction, if there were $\delta_0 < 0$ such that $\mathcal{L}^n(\{x \in A : v(x) \geqslant t\}) = 0$ for every $t \in (\delta_o, 0)$, then the very definition of v^+ would give $v^+(x) \leqslant \delta_o$ $\mathcal{H}^{n-1}$-a.e. on H, which is clearly a contradiction since $v^+ \geqslant 0$ $\mathcal{H}^{n-1}$-a.e. on H and $\mathcal{H}^{n-1}(H) > 0$.

Moreover, since $\{x \in A : v(x) \geqslant \eta\} \supseteq \{x \in A : v(x) \geqslant t\}$ if $\eta < t$, and $\mathrm{Per}(\{x \in A : v(x) \geqslant \eta\}, A) < +\infty$ for $\mathcal{L}^1$-a.e. $\eta \in \mathbb{R}$ we get $\{x \in A : v(x) \geqslant \eta\}_+ \supseteq \{x \in A : v^+(x) \geqslant t\}$ for $\mathcal{L}^1$-a.e. $\eta \in \mathbb{R}$, $\eta < t$. Since $\{x \in A : v(x) \geqslant \eta\}_+ \supseteq \{x \in A : v^+(x) \geqslant t\} \supseteq \{x \in A : v^+(x) \geqslant 0\}$, it is clear that we can find $\eta \in (\delta, t)$ for which all the required conditions are satisfied.

Step 2: There exist $w_j \in SBV(B_{R_j})$ and $\eta_j \in (\zeta, 0)$, with ζ as in assumption (iii)*, such that $\nabla w_j = 0$ $\mathcal{L}^n$-a.e. on B_{R_j}, $\mathcal{H}^{n-1}(H \setminus \{x \in B_{R_j} : w_j(x) \geqslant \eta_j\}_+) = 0$, $\sup_j \mathcal{L}^n(\{x \in B_{R_j} : w_j(x) \geqslant \eta_j\}) < +\infty$, $\sup_j \mathrm{Per}(\{x \in B_{R_j} : w_j(x) \geqslant \eta_j\}, B_{R_j}) < +\infty$, and*

$$\liminf_j \mathcal{H}^{n-1}(S_{v_j}) \geqslant \liminf_j \mathcal{H}^{n-1}(S_{w_j}) - c/\sqrt{N}.$$

Let $C = \sup_j \|v_j\|_{L^\infty(B_{R_j})}$ and $k_N = [\sqrt{N}]$. Then apply the BV coarea formula to get

$$\|Dv_j\|(B_{R_j} \setminus S_{v_j}) = \sum_{i=0}^{k_N-1} \int_{\alpha_i}^{\alpha_{i+1}} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geqslant t\}, B_{R_j} \setminus S_{v_j})\, \mathrm{d}t,$$

where $\alpha_0 = -C$, $\alpha_{i+1} = \alpha_i + 2C/k_N$ for $0 \leqslant i \leqslant k_N - 1$. Let $0 \leqslant r \leqslant k_N - 1$ be such that $\zeta \in (\alpha_{r-1}, \alpha_r]$; and first assume that $\alpha_{r+1} \leqslant 0$. For every $0 \leqslant i \leqslant k_N - 1$ by the mean value theorem we may find $t_i^j \in (\alpha_i, \alpha_{i+1})$ such that

$$\frac{2C}{k_N}\mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geqslant t_i^j\}, B_{R_j} \setminus S_{v_j}) \leqslant \int_{\alpha_i}^{\alpha_{i+1}} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geqslant t\}, B_{R_j} \setminus S_{v_j})\, \mathrm{d}t. \quad (4.4)$$

Let $0 \leqslant s \leqslant k_N - 1$ be such that $0 \in (t_s^j, t_{s+1}^j]$, and note that $t_s^j \in (\zeta, 0)$ since $\alpha_{r+1} \leqslant 0$ implies $t_s^j \geqslant t_r^j > \alpha_r$. Consider $\eta_j \in (t_s^j, 0)$ provided by Step 1 and the sets $\Sigma_i^j = \{x \in B_{R_j} : t_i^j \leqslant v_j(x) < t_{i+1}^j\}$, then define the function $w_j : B_{R_j} \to \mathbb{R}$ as $w_j(x) = \eta_j$ if $x \in \Sigma_s^j$ and $w_j(x) = t_i^j$ if $x \in \Sigma_i^j$, $0 \leqslant i \leqslant k_N - 1$ and $i \neq s$.

Clearly, Σ_i^j being of finite perimeter in B_{R_j}, we have $w_j \in SBV(B_{R_j})$ with $\nabla w_j = 0$ $\mathcal{L}^n$-a.e. on B_{R_j}, $S_{w_j} \subseteq \bigcup_{i=0}^{k_N-1} \partial^* \Sigma_i^j$, and $\mathcal{H}^{n-1}(H \setminus \{x \in B_{R_j} : w_j(x) \geqslant \eta_j\}_+) = 0$ since by the choice of $\eta_j \in (t_s^j, 0)$,

$$\{x \in B_{R_j} : w_j(x) \geqslant \eta_j\} = \{x \in B_{R_j} : v_j(x) \geqslant t_s^j\} \supseteq \{x \in B_{R_j} : v_j(x) \geqslant \eta_j\}.$$

Moreover, by assumption (ii) and the definition of k_N,

$$\begin{aligned}\mathcal{H}^{n-1}(S_{w_j}) &\leqslant \mathcal{H}^{n-1}(S_{v_j}) + \sum_{i=0}^{k_N-1} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geqslant t_i^j\}, B_{R_j} \setminus S_{v_j}) \\ &\leqslant \mathcal{H}^{n-1}(S_{v_j}) + \frac{k_N}{2C}\|Dv_j\|(B_{R_j} \setminus S_{v_j}) \leqslant \mathcal{H}^{n-1}(S_{v_j}) + \frac{c}{\sqrt{N}}.\end{aligned}$$

Finally, the relative isoperimetric inequality in balls (see Remark 3.50 of [1]), the choices $\zeta < t_s^j < \eta_j < 0$ and assumption (iii) imply that $\sup_j \mathcal{L}^n(\{x \in B_{R_j} : w_j(x) \geqslant \eta_j\}) < +\infty$.

In case $\zeta \in (\alpha_{r-1}, \alpha_r]$ with $\alpha_{r+1} > 0$, this construction fails since t_s^j might not satisfy $t_s^j > \zeta$. Nevertheless, this case can be handled by slightly modifying the choice of the t_i^j's. Indeed, choose t_i^j as in (4.4) for $i \notin \{r-1, r\}$, choose $t_r^j \in (\zeta, 0)$ such that

$$|\zeta|\, \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geqslant t_r^j\}, B_{R_j} \setminus S_{v_j}) \leqslant \int_{\alpha_{r-1}}^{\alpha_{r+1}} \mathrm{Per}(\{x \in B_{R_j} : v_j(x) \geqslant t\}, B_{R_j} \setminus S_{v_j})\, \mathrm{d}t,$$

and set $t_{r-1}^j = t_r^j$. Let $\eta_j \in (t_r^j, 0)$ be provided by Step 1, and define $w_j : B_{R_j} \to \mathbb{R}$ as $w_j(x) = \eta_j$ if $x \in \Sigma_r^j$, and $w_j(x) = t_i^j$ if $x \in \Sigma_i^j$, $0 \leqslant i \leqslant k_N - 1$ and $i \notin \{r-1, r\}$. Notice that $\Sigma_{r-1}^j = \emptyset$. The same arguments exploited before entail that (w_j) still satisfies the statement of Step 2.

Step 3: Conclusion. Set $\Sigma_j = \{x \in B_{R_j} : w_j(x) \geqslant \eta_j\}$. Then $\mathcal{H}^{n-1}(S_{w_j}) \geqslant \mathrm{Per}(\Sigma_j, B_{R_j})$ (see Theorem 4.23 of [1]), which, together with Step 2, yields

$$+\infty > \lim_j \mathcal{H}^{n-1}(S_{v_j}) \geqslant \liminf_j \mathcal{H}^{n-1}(S_{w_j}) - \frac{c}{\sqrt{N}} \geqslant \liminf_j \mathrm{Per}(\Sigma_j, B_{R_j}) - \frac{c}{\sqrt{N}}. \quad (4.5)$$

By applying the BV compactness theorem we may extract a subsequence (not relabeled for convenience) and find a set Σ with locally finite perimeter in $\mathbb{R}^n$ such that $\chi_{\Sigma_j} \to \chi_\Sigma$ in $L^1_{\rm loc}(\mathbb{R}^n)$. By Theorem 6.1 of [10] (see also Chapter IV of [26]) for every $R > 0$ we get

$$\liminf_j \operatorname{Per}(\Sigma_j, B_R) \geqslant \operatorname{Per}(\Sigma, B_R) + \sigma((H \setminus \Sigma_+) \cap B_R), \tag{4.6}$$

so that by combining (4.5) and (4.6), and by passing to the supremum over R, Σ has finite perimeter in $\mathbb{R}^n$ and moreover

$$\lim_j \mathcal{H}^{n-1}(S_{v_j}) \geqslant \operatorname{Per}(\Sigma) + \sigma(H \setminus \Sigma_+) - c/\sqrt{N}.$$

The assertion now follows by taking into account (2.12) of Theorem 2.7. □

5. Further results

In the previous sections we have described the asymptotic behavior of the Mumford–Shah energy in periodically perforated domains. In the present section we extend the results of Sections 3 and 4 to more general free-discontinuity energies. We limit ourselves to stating and giving the hints of the proof of the generalization of Theorem 4.1 in this setting, since the analogues of Propositions 3.3 and 3.4 are trivial.

Let $p > 1$ and $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$ be continuous functions such that

(a) φ is convex, and there exist constants $c_1, c_3 > 0$ and $c_2 \in \mathbb{R}$ such that for every $\xi \in \mathbb{R}^n$,

$$c_1|\xi|^p - c_2 \leqslant \varphi(\xi) \leqslant c_3(|\xi|^p + 1);$$

(b) ψ is a norm on $\mathbb{R}^n$, and there exist constants $c_4, c_5 > 0$ such that for every $\nu \in \mathbb{S}^{n-1}$,

$$c_4 \leqslant \psi(\nu) \leqslant c_5.$$

Analogously to the case in which ψ is the euclidean norm one can define an anisotropic capacity as follows: for any set $E \subseteq \mathbb{R}^n$ let

$$C_\psi(E) = \inf\left\{\int_{\partial^* D} \psi(\nu_{\partial^* D})\, \mathrm{d}\mathcal{H}^{n-1} : D \text{ is } \mathcal{L}^n\text{-measurable},\ \mathcal{L}^n(D) < +\infty,\ \mathcal{H}^{n-1}(E \setminus D_+) = 0\right\}.$$

Different characterizations of C_ψ, similar to those of Proposition 2.5, can be deduced from Theorem 6.1 of [10].

Given an $\mathcal{H}^{n-1}$ measurable set $T \subseteq \overline{Q}_1$, for any $\varepsilon > 0$ let $r_\varepsilon \in (0, \varepsilon)$ and let $\mathbf{T}_\varepsilon = \Omega \cap \bigcup_{\underline{i} \in \mathbb{Z}^n} T_{r_\varepsilon}(\underline{i}\varepsilon)$. Consider the functional $\mathcal{F}_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined as

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \varphi(\nabla u)\, \mathrm{d}x + \int_{S_u} \psi(\nu_u)\, \mathrm{d}\mathcal{H}^{n-1}, & u \in GSBV(\Omega),\ u^+ \geqslant 0\ \mathcal{H}^{n-1}\text{-a.e. on } \mathbf{T}_\varepsilon, \\ +\infty, & \text{otherwise in } L^1(\Omega). \end{cases} \tag{5.1}$$

We are now in a position to state the following result whose proof is just a technical adjustment of those of Theorems 3.1 and 4.1.

THEOREM 5.1 Let T be an $\mathcal{H}^{n-1}$-measurable set, and assume that $r_\varepsilon/\varepsilon^{n/(n-1)} \to \beta \in [0, +\infty)$ as $\varepsilon \to 0^+$. Suppose that φ and ψ satisfy assumptions (a) and (b) above. Then $(\mathcal{F}_\varepsilon)$ Γ-converges to $\mathcal{F} : L^1(\Omega) \to [0, +\infty]$ defined by

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_\Omega \varphi(\nabla u)\,\mathrm{d}x + \int_{S_u} \psi(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \\ \quad + C_\psi(T)\beta^{n-1}\mathcal{L}^n(\{x \in \Omega : u(x) < 0\}), & u \in GSBV(\Omega), \\ +\infty, & \text{otherwise in } L^1(\Omega), \end{cases}$$

with respect to the L^1 convergence.

Proof. Lower bound: We first point out that estimate (3.6) in Lemma 3.5 follows directly by Theorem 2.2. Moreover, in order to get a gradient estimate as in (3.9) of Step 1, the same argument developed there can be repeated with the function $|\cdot|^p$ replaced by φ, and taking into account the convexity of φ and assumption (a).

Furthermore, a capacitary estimate as in (3.13) of Step 2 follows by replacing in the statement of Lemma 4.4 the total variation of a BV function with the anisotropic variation

$$\int_\Omega \psi\left(\frac{\mathrm{d}Du}{\mathrm{d}\|Du\|}\right)\mathrm{d}\|Du\|, \tag{5.2}$$

and in its conclusion C_1 with C_ψ. Indeed, thanks to assumption (b), one can use for the anisotropic variation in (5.2) suitable versions of the BV coarea formula (see Lemma 2.4 of [19]) and of the relaxation result for energies with linear growth with obstacles (see Theorem 7.1 of [10]).

Finally, the truncation argument of Proposition 3.7 can be carried out with only minor changes.

Upper bound: The proof of Proposition 3.9 works upon replacing, in the construction of the recovery sequence, a minimizing set D for the usual capacity with a minimizing sequence for the anisotropic capacitary problem for T related to ψ. □

Acknowledgements

The authors would like to thank Andrea Braides for proposing the problem and for some fruitful discussions on it. They also acknowledge the valuable comments of the anonymous referee on early versions of the manuscript.

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