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Optimal regularity for elliptic transmission problems including C^1 interfaces

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We prove an optimal regularity result for elliptic operators $-\nabla \cdot \mu\nabla : W_0^{1,q} \to W^{-1,q}$ for a $q > 3$ in the case when the coefficient function μ has a jump across a C^1 interface and is continuous elsewhere. A counterexample shows that the C^1 condition cannot be relaxed in general. Finally, we draw some conclusions for corresponding parabolic operators.

1. Introduction

This work is situated on the intersection of two mathematical questions: the first is the regularity of solutions of elliptic transmission problems (see, e.g., [38, 45, 47, 49, 22, 3, 4, 41, 15, 48, 37, 20, 16], and references therein). The other concerns the isomorphism property of elliptic operators $-\nabla \cdot \mu\nabla : X \to Y$ between suitable Banach spaces X, Y in the case of nonsmooth domains and/or discontinuous coefficient functions μ (see [7, 19, 27, 33, 48, 57, 12]). In particular, the latter question in connection with transmission problems for the spaces $X := W^{1,q}$, $Y := W^{-1,q}$ (with boundary conditions incorporated) has been treated in [27, 12, 43, 7] (see also [32] and references therein). All of these have in common that they transfer geometrical properties of the underlying domain or/and geometrical properties of the smoothness regions for the coefficient function to functional-analytic properties of the relevant spaces $W^{1,q}$ and $W^{-1,q}$. Exactly this is also the case in this paper; our aim is to prove a sharpened (and optimal) version of the results from [12, Ch. 4], namely:

THEOREM 1.1 Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. Further, let $\Omega_\circ \subset \Omega$ be another domain which is supposed to satisfy one of the following conditions:

(i) $\Omega_\circ$ is a C^1 domain which does not touch the boundary of Ω.
(ii) The dimension d equals 3, $\Omega_\circ$ is a Lipschitz domain, and $\partial\Omega_\circ \cap \Omega$ is a C^1 hypersurface. Moreover, $\partial\Omega$ and $\partial\Omega_\circ$ meet suitably (see the definition below).

Let μ be a function on Ω with values in the set of real, symmetric $d \times d$ matrices which is uniformly continuous on both $\Omega_\circ$ and $\Omega \setminus \bar{\Omega}_\circ$. Additionally, μ is supposed to satisfy the usual ellipticity condition

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\Omega} \inf_{\xi\in\mathbb{C}^d, \|\xi\|_{\mathbb{C}^d}=1} \mu(\mathbf{x})\xi \cdot \bar{\xi} > 0. \tag{1.1}$$

Then there is a $p > 3$ such that for every λ from the closed right complex half-plane,

$$-\nabla \cdot \mu\nabla + \lambda : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega) \tag{1.2}$$

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is a topological isomorphism for all $q \in]p', p[$. If Ω itself is also a C^1 domain and $\Omega_\circ$ satisfies (i), then p may be taken as ∞.

DEFINITION 1.2 We say that $\partial\Omega$ and $\partial\Omega_\circ$ *meet suitably* if for any point $\mathbf{x}$ from the boundary of $\partial\Omega \cap \partial\Omega_\circ$ within $\partial\Omega$ there is an open neighbourhood $\mathcal{U}_\mathbf{x}$ of $\mathbf{x}$ in $\mathbb{R}^3$ and a C^1 diffeomorphism $\Phi_\mathbf{x}$ from $\mathcal{U}_\mathbf{x}$ onto an open subset of $\mathbb{R}^3$ such that

- $\Phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega)$ equals an open bounded convex polyhedron $\mathcal{C}_\mathbf{x}$,
- $\Phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega \cap \partial\Omega_\circ) = \mathcal{C}_\mathbf{x} \cap \mathcal{H}_\mathbf{x}$, where $\mathcal{H}_\mathbf{x}$ is a plane which contains $\Phi_\mathbf{x}(\mathbf{x})$ and a point of $\mathcal{C}_\mathbf{x}$.

Note that our result is a certain complement to [19], where for 3D-problems with mixed boundary conditions, but without heterogeneities, isomorphism theorems within the $W^{1,q} \leftrightarrow W^{-1,q}$ scales are obtained. Furthermore, it is somewhat similar to the results of [41], where piecewise Hölder continuity of the first order derivatives is proved under slightly stronger assumptions on the data. Last but not least, Theorem 1.1 is related to the results of [14], where $W^{1,\infty}_{\text{loc}}$ regularity is proved for the solution if the right hand side is sufficiently regular.

Operators of type (1.2)—which may be seen as the principal part of the homogenized version of an elliptic operator with inhomogeneous Dirichlet data—are of fundamental significance in many application areas. This is the case not only in mechanics (see [40, Ch. IV.3]), thermodynamics [51], and electrodynamics [50] of heterogeneous media, but also in mining, multiphase flow and mathematical biology. Especially in biological models it often seems unavoidable to take into account heterogeneties (see [23] or [11] and references therein). Moreover, such operators are also of interest for the description of submicron devices by means of a Schrödinger operator in effective mass approximation (see for example [10, 55, 54, 42]). Here heterostructures are the determining features of many fundamental effects (see for instance [9, 34]). With ongoing miniaturization of electronic devices the resolution of material interfaces becomes ever more important, so that one definitely has to deal with discontinuous coefficient functions here. Moreover, a large amount of papers exist on the numerics of such problems (see e.g. [1, 31, 13, 53] and references therein).

The $W^{1,q}_0 \leftrightarrow W^{-1,q}$ setting is attractive for many problems for the following reasons: if the gradient of the solution belongs to a summability class q, larger than the space dimension d, then the solution is automatically Hölder continuous—which is often of use for auxiliary problems. By the way, in three dimensions this cannot be achieved within the $W^{s,2}$ scale because $W^{3/2,2}$ is a principal threshold in the case of jumping coefficients (see [48] for further results). Secondly, the result has far reaching consequences for the treatment of quasilinear parabolic equations in L^p spaces—as carried out in [43, 46]. Moreover, our elliptic regularity theorem, combined with a result from [8], also yields maximal parabolic regularity on $W^{-1,q}$.

Another important application of the information $q > d$ is the possibility of obtaining uniqueness results for associated nonlinear equations and systems (see for example [24, 25]). Of course, these things are most relevant in the "physical" space dimension 3. Last but not least, $W^{-1,q}$ is large enough to contain (suitable, say bounded) surface densities and even (not too singular) measures (see [58, Ch. 4]). In particular, this enables one to include prescribed jump conditions for the conormal derivative of the solution across the interface (see [13]).

The outline of the paper is as follows: First we introduce some notation. In the next section we prove Theorem 1.1. In Section 4 it is shown by a counterexample that if the C^1 condition on the subdomain is violated at only one point, then one loses the result completely. The last section is devoted to conclusions for corresponding parabolic operators, such as maximal parabolic regularity on $W^{-1,q}$.

2. Notations

The real scalar product $\sum_{j=1}^{d} x_j y_j$ of two vectors $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{C}^d$ is denoted by $\mathbf{x} \cdot \mathbf{y}$. Throughout this paper, Ω and Λ are always domains in $\mathbb{R}^d$. For the definition of a Lipschitz domain and a domain with Lipschitz boundary we refer the reader primarily to [26, Ch. 1.2] (see also [56, Ch. 1.2]). If X is a complex Banach space, then $L^\infty(\Lambda; X)$ denotes the space of Lebesgue measurable, essentially bounded functions on Λ with values in X. $W^{1,q}(\Lambda)$ stands for the usual (complex) Sobolev space on the set Λ (see [26] or [52]). Further, we use the symbol $W_0^{1,q}(\Lambda)$ for the closure of $\{v|_\Lambda : v \in C_0^\infty(\mathbb{R}^d), \operatorname{supp} v \subset \Lambda\}$ in $W^{1,q}(\Lambda)$. $W^{-1,q'}(\Lambda)$ denotes the dual to $W_0^{1,q}(\Lambda)$; here and in what follows, q' always denotes the adjoint exponent $q' := q/(q-1)$. If ρ is a Lebesgue measurable, essentially bounded function on the domain Λ, taking its values in the set of real, symmetric $d \times d$ matrices, then we define

$$-\nabla \cdot \rho \nabla : W_0^{1,2}(\Lambda) \to W^{-1,2}(\Lambda) \tag{2.1}$$

by

$$\langle -\nabla \cdot \rho \nabla v, w \rangle := \int_\Lambda \rho \nabla v \cdot \nabla w \, \mathrm{d}\mathbf{x}, \qquad v, w \in W_0^{1,2}(\Lambda). \tag{2.2}$$

Here and in the following, $\langle \cdot, \cdot \rangle$ always denotes the dual pairing between $W_0^{1,2}$ and $W^{-1,2}$. The maximal restriction of $-\nabla \cdot \rho \nabla$ to any of the spaces $W^{-1,q}(\Lambda)$ ($q > 2$) will be denoted by the same symbol. The norm in a Banach space X will always be indicated by $\| \cdot \|_X$. For two Banach spaces X and Y we denote the space of bounded linear operators from X into Y by $\mathcal{B}(X; Y)$. If $X = Y$, then we abbreviate $\mathcal{B}(X)$.

3. Proof of Theorem 1.1

Let us briefly outline the proof; it rests heavily on nontrivial regularity results for adequate model problems within the same scale of spaces. We begin by collecting results of this type which are already known and afterwards establish some technical prerequisites. In the second subsection we first prove a regularity result for another model situation, namely for an operator $-\nabla \cdot \sigma \nabla + 1$ on $\mathbb{R}^d$, where σ equals a (real, symmetric, positive definite) $d \times d$ matrix on a half-space and another $d \times d$ matrix on the complementing half-space (see Theorem 3.11 below). Afterwards the Jerison–Kenig result concerning the Dirichlet Laplacian on domains with Lipschitz boundary is generalized to divergence operators with uniformly continuous coefficient function. The proof itself is then carried out via some localization procedure which permits us to reduce the considerations to the constituting model constellations.

3.1 *Known results and preliminaries*

Two cornerstones for all what follows are the two results below:

PROPOSITION 3.1 (see [39, pp. 156–157] and [2, Ch. 15]) Let Λ be bounded and have a C^1 boundary. If ρ is a function on Λ with values in the set of real $d \times d$ matrices which is elliptic and uniformly continuous, then $-\nabla \cdot \rho \nabla : W_0^{1,q}(\Lambda) \to W^{-1,q}(\Lambda)$ is a topological isomorphism for any $q \in \,]1, \infty[$.

PROPOSITION 3.2 ([33]) If $\Lambda \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, then there is a number $q > 3$ such that the Dirichlet Laplacian provides a topological isomorphism between $W_0^{1,q}(\Lambda)$ and $W^{-1,q}(\Lambda)$.

For the proof of assertion (ii) of Theorem 1.1 we employ

PROPOSITION 3.3 ([21]) Assume that $\mathcal{C} \subset \mathbb{R}^3$ is a (bounded, open) convex polyhedron and that $\mathcal{H} \subset \mathbb{R}^3$ is a plane which intersects $\mathcal{C}$. Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be the two components of $\mathcal{C} \setminus \mathcal{H}$, and let ρ be a function on $\mathcal{C}$, constant on $\mathcal{C}_+$ and $\mathcal{C}_-$, and whose values are two real, symmetric, positive definite 3×3 matrices there. Then there is a $q > 3$ such that

$$-\nabla \cdot \rho \nabla : W_0^{1,q}(\mathcal{C}) \to W^{-1,q}(\mathcal{C})$$

is a topological isomorphism.

Additionally, the following scaling argument is required:

LEMMA 3.4 Let $\mathcal{C} \subset \mathbb{R}^3$ be a bounded, open, convex set whose closure contains $\mathbf{0}$. Assume that ρ is a bounded, measurable, elliptic coefficient function on $\mathcal{C}$, taking its values in the set of real, symmetric 3×3 matrices and which additionally satisfies $\rho(\alpha \mathbf{x}) = \rho(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{C}$ and $\alpha \in \,]0, 1[$. For any $\alpha \in \,]0, 1]$ equip the space $W_0^{1,q}(\alpha\mathcal{C})$ with the norm $\psi \mapsto \left(\int_{\alpha\mathcal{C}} |\nabla\psi|^q \,\mathrm{d}x\right)^{1/q}$. Then

$$\|(-\nabla \cdot \rho|_{\alpha\mathcal{C}} \nabla)^{-1}\|_{\mathcal{B}(W^{-1,q}(\alpha\mathcal{C}); W_0^{1,q}(\alpha\mathcal{C}))} = \|(-\nabla \cdot \rho \nabla)^{-1}\|_{\mathcal{B}(W^{-1,q}(\mathcal{C}); W_0^{1,q}(\mathcal{C}))}. \tag{3.1}$$

Proof. One checks that for $q \in [1, \infty[$ and $\alpha \in \,]0, 1[$ the mapping

$$T_{q,\alpha} : W_0^{1,q}(\mathcal{C}) \ni \psi \mapsto \alpha^{1-3/q} \psi(\alpha^{-1}(\cdot))$$

is an isometric isomorphism from $W_0^{1,q}(\mathcal{C})$ onto $W_0^{1,q}(\alpha\mathcal{C})$. Then one verifies the identity

$$T^*_{q',\alpha}(-\nabla \cdot \rho|_{\alpha\mathcal{C}} \nabla) T_{q,\alpha} = -\nabla \cdot \rho \nabla. \qquad \square$$

Further, we need the following interpolation result:

LEMMA 3.5 If $\Lambda \subset \mathbb{R}^d$ is a Lipschitz domain, then one has the interpolation identities

$$[W_0^{1,p_1}(\Lambda), W_0^{1,p_2}(\Lambda)]_\theta = W_0^{1,p}(\Lambda) \quad \text{and} \quad [W^{-1,p_1}(\Lambda), W^{-1,p_2}(\Lambda)]_\theta = W^{-1,p}(\Lambda)$$

if $p_1, p_2 \in \,]1, \infty[$ and $1/p = (1-\theta)/p_1 + \theta/p_2$.

Proof. Continuation outside Λ by zero defines a continuous coretraction from $W_0^{1,q}(\Lambda)$ into $W^{1,q}(\mathbb{R}^d)$, where the restriction is the retraction. Thus, the first identity follows from the $\mathbb{R}^d$ case (see [52]). The second is implied by the first and duality for complex interpolation (see [52, Ch. 1.11.3]). $\square$

REMARK 3.6 From Lemma 3.5 the following may be deduced (see [52, Ch. 1.9.3]): If $\mathfrak{A}$ is a bounded subset of $\mathcal{B}(W^{-1,2}(\Lambda); W_0^{1,2}(\Lambda))$ and

$$\sup_{A \in \mathfrak{A}} \|A\|_{\mathcal{B}(W^{-1,q}(\Lambda); W_0^{1,q}(\Lambda))} < \infty$$

for one $q > 2$, then

$$\sup_{t \in [2,q]} \sup_{A \in \mathfrak{A}} \|A\|_{\mathcal{B}(W^{-1,t}(\Lambda); W_0^{1,t}(\Lambda))} < \infty.$$

LEMMA 3.7 Assume $q \in [1, \infty[$. Then the norm of the mapping

$$L^\infty(\Lambda; \mathcal{B}(\mathbb{C}^d)) \ni \rho \mapsto \nabla \cdot \rho \nabla \in \mathcal{B}(W_0^{1,q}(\Lambda); W^{-1,q}(\Lambda))$$

does not exceed 1. If $\lambda \in \mathbb{C}$ and ω is a coefficient function on Λ which satisfies

$$\|\omega\|_{L^\infty(\Lambda;\mathcal{B}(\mathbb{C}^d))} \|(-\nabla \cdot \rho\nabla + \lambda)^{-1}\|_{\mathcal{B}(W^{-1,p}(\Lambda); W_0^{1,p}(\Lambda))} \leqslant 1/2,$$

then

$$\|(-\nabla \cdot (\rho + \omega)\nabla + \lambda)^{-1}\|_{\mathcal{B}(W^{-1,p}(\Lambda); W_0^{1,p}(\Lambda))} \leqslant 2\|(-\nabla \cdot \rho\nabla + \lambda)^{-1}\|_{\mathcal{B}(W^{-1,p}(\Lambda); W_0^{1,p}(\Lambda))}. \tag{3.2}$$

Proof. The first assertion is implied by Hölder's inequality. The proof of the second follows from the first and a classical perturbation theorem (see [35, Ch. IV.1.4, Thm. 1.16]). □

REMARK 3.8 The lemma makes it clear that the L^∞ norm on the space of coefficient functions is adequate to control the bounded invertibility for divergence operators within the $W^{1,q} \leftrightarrow W^{-1,q}$ context. Most of what follows heavily rests upon this fact.

Next we present a localization principle which is similar to that proved in [27] for the Laplacian. In essence, this will permit us to deduce the isomorphism property (1.2) from the same property for suitable local model constellations.

LEMMA 3.9 Let $\Lambda \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\mathcal{O} \subset \mathbb{R}^d$ be open such that $\Lambda_\bullet := \Lambda \cap \mathcal{O}$ is again a Lipschitz domain. Fix an arbitrary function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \eta \subset \mathcal{O}$. Let $\rho_\bullet$ denote the restriction of the coefficient function ρ to $\Lambda_\bullet$. Assume $\lambda \geqslant 0$ and $u \in W_0^{1,2}(\Lambda)$ to be the solution of

$$-\nabla \cdot \rho\nabla u + \lambda u = f \in W^{-1,2}(\Lambda). \tag{3.3}$$

Then the following holds true:

(i) The linear form $f_\bullet : w \mapsto \langle f, \widetilde{\eta w}\rangle$ (where $\widetilde{\eta w}$ means the extension by zero to the whole Λ) is well defined and continuous on $W_0^{1,r'}(\Lambda_\bullet)$ whenever $f \in W^{-1,r}(\Lambda)$.

(ii) Let T_u denote the linear form $w \mapsto \int_{\Lambda_\bullet} u\rho_\bullet \nabla\eta \cdot \nabla w \, \mathrm{d}\mathbf{x}$ on $W_0^{1,2}(\Lambda_\bullet)$. If $u \in W^{1,r}(\Lambda)$, then $-\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla\eta|_{\Lambda_\bullet} + T_u \in W^{-1,s}(\Lambda_\bullet)$, where $s = s(r)$ is given by

$$s = \begin{cases} \dfrac{rd}{d-r} & \text{if } r \in [2, d[, \\ \text{any (large) positive number} & \text{if } r \geqslant d. \end{cases} \tag{3.4}$$

(iii) $v := \eta u|_{\Lambda_\bullet} \in W_0^{1,2}(\Lambda_\bullet)$ satisfies

$$-\nabla \cdot \rho_\bullet \nabla v + \lambda v = -\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla\eta|_{\Lambda_\bullet} + T_u + f_\bullet. \tag{3.5}$$

Proof. (i) The mapping $f \mapsto f_\bullet$ is the adjoint to $w \mapsto \widetilde{\eta w}$ which maps $W_0^{1,r'}(\Lambda_\bullet)$ continuously into $W_0^{1,r'}(\Lambda)$.

(ii) The case $r \geqslant d$ may be reduced by the embedding $W^{1,r}(\Lambda) \hookrightarrow W^{1,d-\epsilon}(\Lambda)$ to the case $r < d$; we treat the latter: clearly, $\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla \eta \in L^r(\Lambda_\bullet)$, which gives by Sobolev embedding and duality $\rho_\bullet \nabla u|_{\Lambda_\bullet} \cdot \nabla \eta|_{\Lambda_\bullet} \in W^{-1,\frac{rd}{d-r}}(\Lambda_\bullet)$ for $r \in [2, d[$. Concerning T_u, we will show that it is a continuous linear form on $W_0^{1,(\frac{rd}{d-r})'}(\Lambda_\bullet)$: one can estimate

$$|\langle T_u, w\rangle| \leqslant \|u\|_{L^{\frac{rd}{d-r}}(\Lambda_\bullet)} \|\rho\|_{L^\infty(\Lambda;\mathcal{B}(\mathbb{C}^d))} \|\nabla \eta\|_{L^\infty(\Lambda_\bullet)} \|\nabla w\|_{L^{(\frac{rd}{d-r})'}(\Lambda_\bullet)}. \tag{3.6}$$

Using again Sobolev embedding, the right hand side of (3.6) may be estimated by

$$c\|u\|_{W^{1,r}(\Lambda_\bullet)} \|\rho\|_{L^\infty(\Lambda;\mathcal{B}(\mathbb{C}^d))} \|\nabla \eta\|_{L^\infty(\Lambda_\bullet)} \|w\|_{W^{1,(\frac{rd}{d-r})'}(\Lambda_\bullet)}.$$

(iii) For every $w \in W_0^{1,2}(\Lambda_\bullet)$ we have

$$\langle -\nabla \cdot \rho_\bullet \nabla v + \lambda v, w\rangle = \int_{\Lambda_\bullet} \rho_\bullet \nabla(\eta u) \cdot \nabla w \, \mathrm{d}\mathbf{x} + \lambda \int_{\Lambda_\bullet} \eta u w \, \mathrm{d}\mathbf{x}$$
$$= -\int_{\Lambda_\bullet} w \rho_\bullet \nabla u \cdot \nabla \eta \, \mathrm{d}\mathbf{x} + \int_{\Lambda_\bullet} u \rho_\bullet \nabla \eta \cdot \nabla w \, \mathrm{d}\mathbf{x} + \int_{\Lambda} \rho \nabla u \cdot \nabla(\widetilde{\eta w}) \, \mathrm{d}\mathbf{x} + \lambda \int_{\Lambda} u \widetilde{\eta w} \, \mathrm{d}\mathbf{x},$$

which gives the assertion. □

Further, we need the following technical lemma, the proof of which can be found in [36, Remark 2.1.3]:

LEMMA 3.10 Let Ω be a domain with Lipschitz boundary. Then for any $\mathbf{x} \in \partial\Omega$ and any neighbourhood of $\mathbf{x}$ there is a (possibly) smaller open neighbourhood $\mathcal{V}_\mathbf{x}$ of $\mathbf{x}$ such that $\Omega \cap \mathcal{V}_\mathbf{x}$ is a domain with Lipschitz boundary.

3.2 *Core of the proof*

THEOREM 3.11 Let σ be a coefficient function on $\mathbb{R}^d$ which equals a real, symmetric, positive definite $d \times d$ matrix σ^- on $\mathbb{R}^d_- = \{\mathbf{x} \in \mathbb{R}^d : x_d < 0\}$ and another real, symmetric, positive definite $d \times d$ matrix σ^+ on $\mathbb{R}^d_+ = \{\mathbf{x} \in \mathbb{R}^d : x_d > 0\}$. Then $-\nabla \cdot \sigma \nabla + 1$ provides a topological isomorphism between $W^{1,q}(\mathbb{R}^d)$ and $W^{-1,q}(\mathbb{R}^d)$ for all $q \in]1, \infty[$.

Proof. Let $\mathbf{x} = (x', x_d) \in \mathbb{R}^d$, $x' \in \mathbb{R}^{d-1}$, and $\partial_i = \partial_{x_i}$, $1 \leqslant i \leqslant d$. Moreover, we identify $\{\mathbf{x} \in \mathbb{R}^d : x_d = 0\}$ with $\mathbb{R}^{d-1}$. It is sufficient to prove that the unique solution $u \in W^{1,2}(\mathbb{R}^d)$ for each of the equations

$$-\nabla \cdot \sigma \nabla u + u = f, \qquad f \in L^q(\mathbb{R}^d), \quad 2 < q < \infty, \tag{3.7}$$

$$-\nabla \cdot \sigma \nabla u + u = \partial_i f, \qquad f \in L^q(\mathbb{R}^d), \quad 2 < q < \infty, \tag{3.8}$$

$i \in \{1, \ldots, d\}$, belongs to $W^{1,q}(\mathbb{R}^d)$. To do this, it is enough to show the estimate

$$\|u\|_{W^{1,q}(\mathbb{R}^d)} \leqslant c\|f\|_{L^q(\mathbb{R}^d)}, \quad f \in \tilde{C}^\infty, \tag{3.9}$$

where c denotes a generic positive constant and $\tilde{C}^\infty$ stands for the dense subset of $L^q(\mathbb{R}^d)$ defined by

$$\tilde{C}^\infty = \{\psi \in C_0^\infty(\mathbb{R}^d) : \psi = 0 \text{ in some neighbourhood of } \mathbb{R}^{d-1}\}.$$

Applying classical elliptic theory of transmission problems (see, e.g., [47]) to the equation

$$-\nabla \cdot \sigma \nabla v + v = f, \quad f \in \tilde{C}^\infty, \tag{3.10}$$

we obtain the inequality

$$\|v\|_{W^{2,q}(\mathbb{R}^d_- \cup \mathbb{R}^d_+)} \leqslant c\|f\|_{L^q(\mathbb{R}^d)}. \tag{3.11}$$

This ensures (3.9) in the case of (3.7). We now establish (3.9) in the case of (3.8) and the transversal derivative ∂_d; the proof for the tangential derivatives is immediate. Looking for the solution of (3.8) with $i = d$ in the form $u = \partial_d v + w$, we observe that w has to satisfy the following transmission problem:

$$\begin{gathered} -\nabla \cdot \sigma^\pm \nabla w^\pm + w^\pm = 0 \quad \text{in } \mathbb{R}^d_\pm, \quad [w] = -[\partial_d v] =: g, \\ [\partial_{\nu,\sigma} w] = -[\partial_{\nu,\sigma} \partial_d v] =: h, \end{gathered} \tag{3.12}$$

where $w^\pm = w|_{\mathbb{R}^d_\pm}$, $[w] = (w^- - w^+)|_{\mathbb{R}^{d-1}}$ and

$$[\partial_{\nu,\sigma} w] = (\sigma^- \nu \cdot \nabla w^- - \sigma^+ \nu \cdot \nabla w^+)|_{\mathbb{R}^{d-1}}, \quad \nu = (0, \dots, 0, 1).$$

Since $v^\pm$ satisfy the homogeneous differential equations near $\mathbb{R}^{d-1}$, the term $[\partial_{\nu,\sigma} \partial_d v]$ is a linear combination of $\partial_j \partial_d v^\pm|_{\mathbb{R}^{d-1}}$ for $j = 1, \dots, d-1$. Thus, by the trace theorem and the continuity of differentiation in tangential direction, we conclude from (3.11) that

$$\|[\partial_d v]\|_{W^{1-1/q,q}(\mathbb{R}^{d-1})} + \|[\partial_{\nu,\sigma} \partial_d v]\|_{W^{-1/q,q}(\mathbb{R}^{d-1})} \leqslant c\|f\|_{L^q(\mathbb{R}^d)}. \tag{3.13}$$

We refer to [52, Ch. 2] for the required properties of Sobolev spaces.

To prove (3.9), in view of (3.11) and (3.13), it now suffices to show that the solution of (3.12) satisfies

$$\|w\|_{W^{1,q}(\mathbb{R}^d_- \cup \mathbb{R}^d_+)} \leqslant c(\|h\|_{W^{-1/q,q}(\mathbb{R}^{d-1})} + \|g\|_{W^{1-1/q,q}(\mathbb{R}^{d-1})}). \tag{3.14}$$

We will reduce (3.14) to well known continuity properties of Poisson operators (see [28]), the symbols of which can be calculated explicitly. In order to do so, we solve (3.12) by taking partial Fourier transform with respect to x' denoted by $\mathcal{F}u = \mathcal{F}u(\xi', x_d)$ for a function u on $\mathbb{R}^d$, with $\mathcal{F}^{-1}$ being the inverse transform. We set

$$B^\pm = (\sigma^\pm_{ij})^{d-1}_{i,j=1}, \quad a^\pm = (\sigma^\pm_{1d}, \dots, \sigma^\pm_{d-1,d}), \quad b^\pm = \sigma^\pm_{dd},$$

where $\sigma^\pm_{ij}$ are the entries of the matrices $\sigma^\pm$. Applying the partial Fourier transform to (3.12), we obtain

$$\begin{gathered} (-b^\pm \partial_d^2 + 2ia^\pm \cdot \xi' \partial_d + B^\pm \xi' \cdot \xi' + 1)\mathcal{F}w^\pm(\xi', x_d) = 0 \quad \text{in } \mathbb{R}^d_\pm, \\ \mathcal{F}w^-(\xi', 0) - \mathcal{F}w^+(\xi', 0) = \mathcal{F}g(\xi'), \\ (b^- \partial_d - ia^- \cdot \xi')\mathcal{F}w^-(\xi', 0) - (b^+ \partial_d - ia^+ \cdot \xi')\mathcal{F}w^+(\xi', 0) = \mathcal{F}h(\xi'). \end{gathered} \tag{3.15}$$

Ignoring the exponentially increasing solutions of the homogeneous differential equations in (3.15), we have

$$\mathcal{F}w^\pm(\xi', x_d) = C^\pm(\xi') \exp(\mp x_d(\mathcal{A}^\pm(\xi') + ia^\pm \cdot \xi')/b^\pm) \tag{3.16}$$

with $\mathcal{A}^{\pm}(\xi') = (b^{\pm}(1+B^{\pm}\xi'\cdot\xi') - (a^{\pm}\cdot\xi')^2)^{1/2}$. Then we determine $C^{\pm}(\xi')$ from the transmission conditions in (3.15),

$$C^{-}(\xi') - C^{+}(\xi') = \mathcal{F}g(\xi'),$$
$$\mathcal{A}^{-}(\xi')C^{-}(\xi') + \mathcal{A}^{+}(\xi')C^{+}(\xi') = \mathcal{F}h(\xi'),$$

which gives

$$C^{\pm} = (\mathcal{A}^{-} + \mathcal{A}^{+})^{-1}\mathcal{F}h \mp \mathcal{A}^{\mp}(\mathcal{A}^{-} + \mathcal{A}^{+})^{-1}\mathcal{F}g. \tag{3.17}$$

Note that the ellipticity of $\nabla\cdot\sigma\nabla$ implies the lower bound

$$\mathcal{A}^{\pm}(\xi') \geqslant c\langle\xi'\rangle, \qquad \langle\xi'\rangle = (1+|\xi'|^2)^{1/2}.$$

We will only prove the corresponding estimate (3.14) for the upper half-space since the proof for $\mathbb{R}^d_-$ is completely analogous. From (3.16) and (3.17) we obtain the representation

$$w(x', x_d) = \mathcal{F}^{-1}k_1(\xi', x_d)\mathcal{F}h(\xi') + \mathcal{F}^{-1}k_2(\xi', x_d)\mathcal{F}g(\xi') =: \mathcal{K}_1 h + \mathcal{K}_2 g \tag{3.18}$$

for $x_d > 0$. Here $\mathcal{K}_1, \mathcal{K}_2$ are Poisson operators with the symbols

$$\begin{aligned} k_1(\xi', x_d) &= (\mathcal{A}^{-}(\xi') + \mathcal{A}^{+}(\xi'))^{-1}\exp\{-x_d(\mathcal{A}^{+}(\xi') + ia^{+}\cdot\xi')\},\\ k_2(\xi', x_d) &= -\mathcal{A}^{-}(\xi')k_1(\xi', x_d). \end{aligned} \tag{3.19}$$

Using (3.19) and the expressions for $\mathcal{A}^{\pm}$, it is not difficult to check that k_1 is a symbol of order -1, i.e., it satisfies the estimates

$$\|x_d^m\partial_d^n\partial_{\xi'}^{\alpha}k_1(\xi', \cdot)\|_{L^2(\mathbb{R}^+)} \leqslant c_{mn\alpha}\langle\xi'\rangle^{-3/2-|\alpha|-m+n} \tag{3.20}$$

for all $\xi' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}^+$, $m, n \in \mathbb{N}$ and all multi-indices α. Analogously, k_2 is a symbol of order 0, i.e., the $-3/2$ in the exponent of $\langle\xi'\rangle$ in (3.20) has to be replaced by $-1/2$. Therefore, from [28, Thm. 3.1] we obtain the continuity of the operators

$$\mathcal{K}_1 : W^{s-1/q,q}(\mathbb{R}^{d-1}) \to W^{s+1,q}(\mathbb{R}^d_+), \quad \mathcal{K}_2 : W^{s-1/q,q}(\mathbb{R}^{d-1}) \to W^{s,q}(\mathbb{R}^d_+)$$

for all $s \in \mathbb{Z}$. In particular, together with (3.18) this implies that the $W^{1,q}$ norm of w on $\mathbb{R}^d_+$ can be estimated by the right hand side of (3.14). □

Next we want to show the assertion of Theorem 1.1 if the coefficient function is uniformly continuous on the whole domain. This will be needed later on as a tool for the general situation.

THEOREM 3.12 Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and ρ a real, symmetric-valued, uniformly continuous coefficient function on Ω, elliptic in the sense of (1.1). Then there is a $p > 3$ such that

$$\sup_{\mathbf{x}\in\bar{\Omega}} \|(-\nabla\cdot\rho(\mathbf{x})\nabla)^{-1}\|_{\mathcal{B}(W^{-1,p}(\Omega);W_0^{1,p}(\Omega))} < \infty.$$

For all $q \in [2, p]$,

$$-\nabla\cdot\rho\nabla : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega) \tag{3.21}$$

is a topological isomorphism.

Proof. If the first assertion were not true, then there would be a sequence $\{\mathbf{x}_n\}_n$ from $\bar{\Omega}$ converging to $\mathbf{x}_0 \in \bar{\Omega}$ and a sequence $\{p_n\}_n$ with $p_n > 3$, $\lim p_n = 3$ such that

$$\lim_{n\to\infty} \|(-\nabla \cdot \rho(\mathbf{x}_n)\nabla)^{-1}\|_{\mathcal{B}(W^{-1,p_n}(\Omega);W_0^{1,p_n}(\Omega))} = \infty. \tag{3.22}$$

If one transforms $-\nabla \cdot \rho(\mathbf{x}_0)\nabla$ with respect to the coordinate transform $(\rho(\mathbf{x}_0))^{-1/2}$, then one ends up with a multiple of the Dirichlet Laplacian on $(\rho(\mathbf{x}_0))^{-1/2}\Omega$. Using the equivalent characterization of domains with Lipschitz boundary by the uniform cone condition (see [26, Thm. 1.2.2.2]) one verifies that $(\rho(\mathbf{x}_0))^{-1/2}\Omega$ is also a domain with Lipschitz boundary. Then, by Proposition 3.2, there is number $p_0 > 3$ such that the Dirichlet Laplacian is a topological isomorphism between $W_0^{1,p_0}((\rho(\mathbf{x}_0))^{-1/2}\Omega)$ and $W^{-1,p_0}((\rho(\mathbf{x}_0))^{-1/2}\Omega)$. It is not hard to see that this carries over to $-\nabla \cdot \rho(\mathbf{x}_0)\nabla : W_0^{1,p_0}(\Omega) \to W^{-1,p_0}(\Omega)$. But then, due to the continuity of ρ and Lemma 3.7, the set $\{(\nabla \cdot \rho(\mathbf{x}_n)\nabla)^{-1} : n > n_0\}$ is bounded in $\mathcal{B}(W^{-1,p_0}(\Omega); W_0^{1,p_0}(\Omega))$ for suffciently large n_0. Additionally, the set is bounded in $\mathcal{B}(W^{-1,2}(\Omega); W_0^{1,2}(\Omega))$ by Lax–Milgram because the matrices $\rho(\mathbf{x}_n)$ have a common ellipticity constant. Taking into account Remark 3.6, this yields a contradiction to (3.22). We prove the second statement, first for $q = p$. In this case (3.21) is injective by Lax–Milgram; by the open mapping theorem it suffices to show that it is also surjective. Choose for every point $\mathbf{x} \in \bar{\Omega}$ a ball $B_\mathbf{x}$ around $\mathbf{x}$ with radius $R_\mathbf{x}$ such that for $\mathbf{y} \in B_\mathbf{x} \cap \bar{\Omega}$,

$$\|\rho(\mathbf{y}) - \rho(\mathbf{x})\|_{\mathcal{B}(\mathbb{R}^d)} \leqslant \frac{1}{2}\Big(\sup_{t\in[2,p]} \sup_{\mathbf{z}\in\bar{\Omega}} \|(-\nabla \cdot \rho(\mathbf{z})\nabla + 1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\Omega);W_0^{1,t}(\Omega))}\Big)^{-1}. \tag{3.23}$$

This radius $R_\mathbf{x}$ is indeed nonzero due to (i) and Remark 3.6. We choose a finite subcovering $B_{\mathbf{x}_1}, \dots, B_{\mathbf{x}_m}$ for $\bar{\Omega}$. Let $\eta_1, \dots, \eta_m$ be a partition of unity on $\bar{\Omega}$ subordinate to this subcovering. Assume now $f \in W^{-1,p}(\Omega)$ and let u be a solution of $-\nabla \cdot \rho\nabla u = f$. By the Lax–Milgram lemma u must be from $W_0^{1,2}(\Omega)$. Putting $\mathcal{O} := \bigcup_{l=1}^m B_{\mathbf{x}_l}$ we get, from Lemma 3.9,

$$-\nabla \cdot \rho\nabla(\eta_l u) = g_l, \tag{3.24}$$

where g_l is from $W^{-1,\min(s(2),p)}(\Omega)$. We now set $t := \min(s(2), p)$ and define for every $l \in \{1, \dots, m\}$ a modified coefficient function ρ_l on Ω as follows:

$$\rho_l(\mathbf{y}) = \begin{cases} \rho(\mathbf{y}) & \text{if } \mathbf{y} \in B_{\mathbf{x}_l} \cap \Omega, \\ \rho(\mathbf{x}_l) & \text{elsewhere on } \Omega. \end{cases} \tag{3.25}$$

Because $\eta_l u$ has its support in $B_{\mathbf{x}_l}$, it satisfies besides (3.24) also the equation

$$-\nabla \cdot \rho_l\nabla(\eta_l u) = g_l. \tag{3.26}$$

We will now show that $g_l \in W^{-1,t}(\Omega)$ implies $\eta_l u \in W_0^{1,t}(\Omega)$. We rewrite (3.26) as

$$-\nabla \cdot \rho(\mathbf{x}_l)\nabla(\eta_l u) + \nabla \cdot [\rho(\mathbf{x}_l) - \rho_l]\nabla(\eta_l u) = g_l.$$

Taking into account (3.23) and Lemma 3.7 we see that $-\nabla \cdot \rho_l\nabla : W_0^{1,t}(\Omega) \to W^{-1,t}(\Omega)$ is boundedly invertible. Thus, each $\eta_l u$ must be from $W_0^{1,t}(\Omega)$, which gives $u \in W_0^{1,t}(\Omega)$. Repeating these considerations with the improved information on the integrability exponent of ∇u—each time using Lemma 3.9—one, after finitely many steps, ends up with $u \in W_0^{1,p}(\Omega)$. Hence, (3.21) is surjective, which proves the assertion for $q = p$. The numbers from $[2, p[$ are obtained from Remark 3.6. □

COROLLARY 3.13 The isomorphy property also holds for $-\nabla \cdot \rho\nabla + \lambda$ with the same range for q, if $\Re\lambda \geqslant 0$.

Proof. The resolvent of $-\nabla \cdot \rho\nabla$ is compact and, due to Lax–Milgram, no λ with $\Re\lambda \leqslant 0$ is an eigenvalue. □

Now we have all the occurring model situations at hand. The next result will provide the asserted regularity when the problem is restricted to (suitable) neighbourhoods of the boundary points. Let us first introduce the following notation: we denote by $\mathcal{E}$ the open unit cube in $\mathbb{R}^d$, while $\mathcal{E}_-$, $\mathcal{E}_+$ are used as symbols for the lower and upper open half cubes, respectively. Finally, we denote by $\mathcal{P}$ the plate which separates $\mathcal{E}_-$ and $\mathcal{E}_+$, $\mathcal{P} := \mathcal{E} \cap \{\mathbf{x} : x_d = 0\}$.

LEMMA 3.14 Under the assumptions of Theorem 1.1 for any $\mathbf{x} \in \partial\Omega$ there is a neighbourhood $\mathcal{O}_\mathbf{x}$ and a $q = q_\mathbf{x} > 3$ such that $\mathcal{O}_\mathbf{x} \cap \Omega$ is a Lipschitz domain and

$$-\nabla \cdot \mu\nabla + 1 : W_0^{1,q}(\mathcal{O}_\mathbf{x} \cap \Omega) \to W^{-1,q}(\mathcal{O}_\mathbf{x} \cap \Omega) \tag{3.27}$$

is a topological isomorphism. If Ω is a C^1 domain and $\Omega_\circ$ has positive distance to the boundary, then q may be taken arbitrarily large.

Proof. First we consider case (i) in Theorem 1.1. For any $\mathbf{x} \in \partial\Omega$ let $\mathcal{O}_\mathbf{x}$ be an open neighbourhood which satisfies the following two conditions:

(I) $\mathcal{O}_\mathbf{x} \cap \bar{\Omega}_\circ = \emptyset$.
(II) If Ω is C^1, then $\mathcal{A}_\mathbf{x} := \mathcal{O}_\mathbf{x} \cap \Omega$ is C^1; and if Ω has a Lipschitz boundary, then $\mathcal{A}_\mathbf{x} := \mathcal{O}_\mathbf{x} \cap \Omega$ has a Lipschitz boundary.

The existence of such a neighbourhood is almost obvious in the C^1 case and follows from Lemma 3.10 in the other case. Thus Corollary 3.13 implies the assertion; in particular, q may be chosen arbitrarily large if Ω is C^1 (see Proposition 3.1).

In case (ii) one cannot treat all the points from $\partial\Omega$ together, but has to divide $\partial\Omega$ into three (disjoint) subsets the points of which have to be treated separately:

(a) $\partial\Omega \setminus \partial\Omega_\circ$,
(b) the inner points of $\partial\Omega \cap \partial\Omega_\circ$ within $\partial\Omega$,
(c) the boundary points of $\partial\Omega \cap \partial\Omega_\circ$ within $\partial\Omega$.

(a) If $\mathbf{x} \in \partial\Omega \setminus \partial\Omega_\circ$, then there is an open neighbourhood $\mathcal{W}_\mathbf{x}$ of $\mathbf{x}$ such that $\mathcal{W}_\mathbf{x} \cap \Omega$ does not intersect $\bar{\Omega}_\circ$. Namely, if this were not the case, then $\mathbf{x}$ would be an accumulation point of $\bar{\Omega}_\circ$, and hence, would belong to $\bar{\Omega}_\circ$. Because $\mathbf{x}$ is not in $\Omega_\circ$ this would mean $\mathbf{x} \in \partial\Omega_\circ$, which is not the case. Thus, by possibly shrinking $\mathcal{W}_\mathbf{x}$ according to Lemma 3.10, $\mathbf{x}$ can be treated as in (i).

Let us now show that in case (b) one can find a neighbourhood $\mathcal{O}_\mathbf{x}$ of $\mathbf{x}$ such that $\mathcal{O}_\mathbf{x} \cap \Omega = \mathcal{O}_\mathbf{x} \cap \Omega_\circ$ and $\mathcal{O}_\mathbf{x} \cap \Omega$ is a domain with Lipschitz boundary. First we construct an open neighbourhood $\mathcal{M}_\mathbf{x}$ of $\mathbf{x}$ with $\mathcal{M}_\mathbf{x} \cap \Omega = \mathcal{M}_\mathbf{x} \cap \Omega_\circ$. Namely, because Ω is a Lipschitz domain (see [26, Ch. 1.2] or [56, Ch. I.2.3]) there is an open neighbourhood $\mathcal{W}_\mathbf{x}$ of $\mathbf{x}$ and a bi-Lipschitz map $\Psi_\mathbf{x} : \mathcal{W}_\mathbf{x} \to \mathcal{E}$ such that $\Psi_\mathbf{x}(\mathbf{x}) = \mathbf{0}$, $\Psi_\mathbf{x}(\Omega \cap \mathcal{W}_\mathbf{x}) = \mathcal{E}_+$ and $\Psi_\mathbf{x}(\partial\Omega \cap \mathcal{W}_\mathbf{x}) = \mathcal{P}$. Because $\mathbf{x}$ was an inner point of $\partial\Omega \cap \partial\Omega_\circ$, there is a positive number $r_\mathbf{x}$ such that $r_\mathbf{x}\mathcal{P} \subset \Psi(\partial\Omega \cap \partial\Omega_\circ) \subset \Psi(\partial\Omega_\circ)$. But, by assumption, $\Omega_\circ$ itself is a Lipschitz domain; thus there is $s_\mathbf{x} \in]0, r_\mathbf{x}]$ such that

$$\Psi_\mathbf{x}(\partial\Omega_\circ) \cap s_\mathbf{x}\mathcal{E} = s_\mathbf{x}\mathcal{P}. \tag{3.28}$$

Now we define $\mathcal{M}_{\mathbf{x}} := \Psi_{\mathbf{x}}^{-1}(s_{\mathbf{x}}\mathcal{E})$ and write

$$\mathcal{M}_{\mathbf{x}} \cap \Omega = (\mathcal{M}_{\mathbf{x}} \cap \Omega_\circ) \cup (\mathcal{M}_{\mathbf{x}} \cap \Omega \cap \partial\Omega_\circ) \cup (\mathcal{M}_{\mathbf{x}} \cap (\Omega \setminus \bar{\Omega}_\circ)). \tag{3.29}$$

From the definition of $\mathcal{M}_{\mathbf{x}}$ and (3.28) it is clear that $\mathcal{M}_{\mathbf{x}} \cap \Omega \cap \partial\Omega_\circ$ is empty. Thus, (3.29) reduces to

$$\mathcal{M}_{\mathbf{x}} \cap \Omega = (\mathcal{M}_{\mathbf{x}} \cap \Omega_\circ) \cup (\mathcal{M}_{\mathbf{x}} \cap (\Omega \setminus \bar{\Omega}_\circ)). \tag{3.30}$$

But $\mathcal{M}_{\mathbf{x}} \cap \Omega$, being a continuous image of a connected set, is itself connected. Thus, one of the (open) sets on the right hand side of (3.30) must be empty, which is definitely not true of $\mathcal{M}_{\mathbf{x}} \cap \Omega_\circ$. This gives $\mathcal{M}_{\mathbf{x}} \cap \Omega = \mathcal{M}_{\mathbf{x}} \cap \Omega_\circ$. Due to Lemma 3.10 we may pass to a neighbourhood $\mathcal{O}_{\mathbf{x}} \subset \mathcal{M}_{\mathbf{x}}$ for which $\mathcal{O}_{\mathbf{x}} \cap \Omega$ is additionally a domain with Lipschitz boundary. Hence, the coefficient function is also uniformly continuous on $\mathcal{O}_{\mathbf{x}} \cap \Omega$ and one can again argue by Corollary 3.13. It remains to consider case (c): assume that **x** is a boundary point of $\partial\Omega \cap \partial\Omega_\circ$ within $\partial\Omega$. Then, by assumption, there is an open neighbourhood $\mathcal{U}_{\mathbf{x}}$, a C^1 diffeomorphism $\Phi_{\mathbf{x}}$, a convex polyhedron $\mathcal{C}_{\mathbf{x}}$ and a plane $\mathcal{H}_{\mathbf{x}}$ which together satisfy the conditions of Definition 1.2. Modulo a translation we may additionally assume $\Phi_{\mathbf{x}}(\mathbf{x}) = \mathbf{0} \in \mathbb{R}^3$. Let $\rho_{\mathbf{x}}$ be the coefficient function on $\mathcal{C}_{\mathbf{x}}$ which is induced by $\mu|_{\mathcal{U}_{\mathbf{x}} \cap \Omega}$ under the mapping $\Phi_{\mathbf{x}}$. If $\mathcal{C}_{\mathbf{x}}^+$ and $\mathcal{C}_{\mathbf{x}}^-$ are the two components of $\mathcal{C}_{\mathbf{x}} \setminus \mathcal{H}_{\mathbf{x}}$, then $\rho_{\mathbf{x}}$ is uniformly continuous on both of them. Define the matrices

$$\rho_{\mathbf{x}}^+ := \lim_{\mathbf{y}\to 0,\, \mathbf{y}\in\mathcal{C}_+} \rho_{\mathbf{x}}(\mathbf{y}) \quad \text{and} \quad \rho_{\mathbf{x}}^- := \lim_{\mathbf{y}\to 0,\, \mathbf{y}\in\mathcal{C}_-} \rho_{\mathbf{x}}(\mathbf{y}) \tag{3.31}$$

and the coefficient function $\tilde{\rho}_{\mathbf{x}}$ on $\mathcal{C}_{\mathbf{x}}$ by

$$\tilde{\rho}_{\mathbf{x}} := \begin{cases} \rho_{\mathbf{x}}^+ & \text{on } \mathcal{C}_{\mathbf{x}}^+, \\ \rho_{\mathbf{x}}^- & \text{on } \mathcal{C}_{\mathbf{x}}^-. \end{cases} \tag{3.32}$$

Let $\alpha = \alpha_{\mathbf{x}} \in \,]0, 1[$ and $q > 3$ be such that

$$\operatorname*{ess\,sup}_{\mathbf{y}\in\alpha\mathcal{C}_{\mathbf{x}}} \|\rho_{\mathbf{x}}(\mathbf{y}) - \tilde{\rho}_{\mathbf{x}}(\mathbf{y})\|_{\mathcal{B}(\mathbb{C}^3)} \|(-\nabla\cdot\tilde{\rho}_{\mathbf{x}}\nabla)^{-1}\|_{\mathcal{B}(W^{-1,q}(\mathcal{C}_{\mathbf{x}});\, W_0^{1,q}(\mathcal{C}_{\mathbf{x}}))} \leqslant 1/2.$$

This is possible due to Proposition 3.3, (3.31) and (3.32). In view of Lemma 3.4 then also

$$\operatorname*{ess\,sup}_{\mathbf{y}\in\alpha\mathcal{C}_{\mathbf{x}}} \|\rho_{\mathbf{x}}(\mathbf{y}) - \tilde{\rho}_{\mathbf{x}}(\mathbf{y})\|_{\mathcal{B}(\mathbb{C}^3)} \|(-\nabla\cdot\tilde{\rho}_{\mathbf{x}}\nabla)^{-1}\|_{\mathcal{B}(W^{-1,q}(\alpha\mathcal{C}_{\mathbf{x}});\, W_0^{1,q}(\alpha_{\mathbf{x}}\mathcal{C}_{\mathbf{x}}))} \leqslant 1/2.$$

Lemma 3.7 implies that

$$-\nabla\cdot\rho_{\mathbf{x}}\nabla : W_0^{1,q}(\alpha\mathcal{C}_{\mathbf{x}}) \to W^{-1,q}(\alpha\mathcal{C}_{\mathbf{x}}) \tag{3.33}$$

is also a topological isomorphism. Let $\mathcal{Q}_{\mathbf{x}}$ be a cube, centred at $\mathbf{0} \in \mathbb{R}^3$, with the properties

$$\mathcal{Q}_{\mathbf{x}} \subset \Phi_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}}), \qquad \mathcal{Q}_{\mathbf{x}} \cap (\mathcal{C}_{\mathbf{x}} \setminus \alpha\mathcal{C}_{\mathbf{x}}) = \emptyset, \tag{3.34}$$

and such that $\alpha\mathcal{C}_{\mathbf{x}} \cup \mathcal{Q}_{\mathbf{x}}$ is a Lipschitz domain. (As $\mathcal{C}_{\mathbf{x}}$ is a convex polyhedron, with **0** being one of its boundary points, it is not hard to see that any sufficiently small cube $\mathcal{Q}_{\mathbf{x}}$ does this job.) If one defines $\mathcal{O}_{\mathbf{x}} := \Phi_{\mathbf{x}}^{-1}(\alpha\mathcal{C}_{\mathbf{x}} \cup \mathcal{Q}_{\mathbf{x}})$, then $\mathcal{O}_{\mathbf{x}}$ is also a Lipschitz domain (see [26, Ch. 1.2, Lem. 1.2.1.3]). Moreover, in view of $\alpha\mathcal{C}_{\mathbf{x}} \subset \mathcal{C}_{\mathbf{x}} = \Phi_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}} \cap \Omega)$, (3.34) and the injectivity of $\Phi_{\mathbf{x}}$ one has

$$\begin{aligned}\Phi_{\mathbf{x}}(\mathcal{O}_{\mathbf{x}} \cap \Omega) &= \Phi_{\mathbf{x}}(\Phi_{\mathbf{x}}^{-1}(\alpha\mathcal{C}_{\mathbf{x}} \cup \mathcal{Q}_{\mathbf{x}}) \cap (\mathcal{U}_{\mathbf{x}} \cap \Omega)) \\ &= (\alpha\mathcal{C}_{\mathbf{x}} \cup \mathcal{Q}_{\mathbf{x}}) \cap \Phi_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}} \cap \Omega) = (\alpha\mathcal{C}_{\mathbf{x}} \cup \mathcal{Q}_{\mathbf{x}}) \cap \mathcal{C}_{\mathbf{x}} = (\alpha\mathcal{C}_{\mathbf{x}} \cap \mathcal{C}_{\mathbf{x}}) \cup (\mathcal{Q}_{\mathbf{x}} \cap \mathcal{C}_{\mathbf{x}}) = \alpha\mathcal{C}_{\mathbf{x}}.\end{aligned}$$

This and the isomorphism property of (3.33) imply that

$$-\nabla\cdot\mu\nabla : W_0^{1,q}(\mathcal{O}_{\mathbf{x}}\cap\Omega)\to W^{-1,q}(\mathcal{O}_{\mathbf{x}}\cap\Omega)$$

is also a topological isomorphism. But the resolvent is compact and -1 is obviously not an eigenvalue, hence $\mathcal{O}_{\mathbf{x}}$ also satisfies the assertion of Lemma 3.14. □

We now come to the proof of Theorem 1.1, first restricting the considerations to the case $q > 2$. For these q, (1.2) is injective by the Lax–Milgram lemma. Hence, by the open mapping theorem it suffices to show that for the asserted q's and any $f\in W^{-1,q}(\Omega)$ the solution u of $-\nabla\cdot\mu\nabla u+u = f$ belongs to $W_0^{1,q}(\Omega)$. Let $\{\mathcal{O}_{\mathbf{x}}\}_{\mathbf{x}\in\partial\Omega}$ be a system of open sets from the foregoing lemma and $\mathcal{O}_{\mathbf{x}_1},\dots,\mathcal{O}_{\mathbf{x}_k}$ be a finite subcovering of $\partial\Omega$. Let $q>3$ be the minimal q for these sets. Further, the C^1 property of $\partial\Omega_\circ\cap\Omega$ ensures for every $\mathbf{x}\in\partial\Omega_\circ\cap\Omega$ the existence of a positive number $\alpha_{\mathbf{x}}$, an open neighbourhood $\mathcal{V}_{\mathbf{x}}\subset\Omega$ of $\mathbf{x}$ and a C^1 diffeomorphism $\Phi_{\mathbf{x}} : \mathcal{V}_{\mathbf{x}}\to\alpha_{\mathbf{x}}\mathcal{E}$ such that $\Phi_{\mathbf{x}}(\partial\Omega_\circ\cap\mathcal{V}_{\mathbf{x}})=\alpha_{\mathbf{x}}\mathcal{P}$, $\Phi_{\mathbf{x}}(\mathbf{x})=0$ and the determinant of the corresponding Jacobian is identically 1 (see [56, Ch. I, Thm. 2.5]). The transform of $(-\nabla\cdot\mu\nabla+1)|_{\mathcal{V}_{\mathbf{x}}}$ under $\Phi_{\mathbf{x}}$ (see [7, Ch. 0.8]) is then of the form $-\nabla\cdot\hat{\mu}_{\mathbf{x}}\nabla+1$, where $\hat{\mu}_{\mathbf{x}}$ is uniformly continuous on $\alpha_{\mathbf{x}}\mathcal{E}_-$ and on $\alpha_{\mathbf{x}}\mathcal{E}_+$. We define

$$\sigma_{\mathbf{x}}^- := \lim_{\mathbf{y}\in\mathcal{E}_-,\,\mathbf{y}\to 0}\hat{\mu}_{\mathbf{x}}(\mathbf{y}) \quad\text{and}\quad \sigma_{\mathbf{x}}^+ := \lim_{\mathbf{y}\in\mathcal{E}_+,\,\mathbf{y}\to 0}\hat{\mu}_{\mathbf{x}}(\mathbf{y})$$

Now let $\sigma_{\mathbf{x}}$ be the coefficient function on $\mathbb{R}^d$ given by

$$\sigma_{\mathbf{x}}=\sigma_{\mathbf{x}}^{\pm}\quad\text{on } \mathbb{R}^d_{\pm}.$$

By Theorem 3.11, for all $\mathbf{x}\in\partial\Omega_\circ\cap\Omega$ and all $t\in\,]1,\infty[$ the operator $-\nabla\cdot\sigma_{\mathbf{x}}\nabla+1$ is a topological isomorphism between $W^{1,t}(\mathbb{R}^d)$ and $W^{-1,t}(\mathbb{R}^d)$. Let $\beta_{\mathbf{x}}\in\,]0,\alpha_{\mathbf{x}}]$ be such that

$$\|\sigma_{\mathbf{x}}-\hat{\mu}_{\mathbf{x}}\|_{L^\infty(\beta_{\mathbf{x}}\mathcal{E};\mathcal{B}(\mathbb{C}^d))}\sup_{t\in[2,q]}\|(-\nabla\cdot\sigma_{\mathbf{x}}\nabla+1)^{-1}\|_{\mathcal{B}(W^{-1,t}(\mathbb{R}^d);W^{1,t}(\mathbb{R}^d))}\leqslant 1/2. \tag{3.35}$$

Such a $\beta_{\mathbf{x}}$ exists because the second factor is finite by Remark 3.6 and the first factor can be made arbitrarily small by the properties of $\hat{\mu}_{\mathbf{x}}$ and $\sigma_{\mathbf{x}}$ for $\beta_{\mathbf{x}}\to 0$. Define $\mathcal{U}_{\mathbf{x}}$ as the inverse image of $\beta_{\mathbf{x}}\mathcal{E}$ under $\Phi_{\mathbf{x}}$. Finally, for any $\mathbf{x}\in\Omega\setminus\partial\Omega_\circ$ let $B_{\mathbf{x}}\subset\Omega$ be an open ball around $\mathbf{x}$ which does not intersect $\partial\Omega_\circ$. (Clearly, the restriction of the coefficient function to $B_{\mathbf{x}}$ is then uniformly continuous.) The systems $\{\mathcal{O}_{\mathbf{x}_1},\dots,\mathcal{O}_{\mathbf{x}_k}\}$, $\{\mathcal{U}_{\mathbf{x}}\}_{\mathbf{x}\in\Omega\cap\partial\Omega_\circ}$, $\{B_{\mathbf{x}}\}_{\mathbf{x}\in\Omega\setminus\partial\Omega_\circ}$ together form an open covering of $\bar{\Omega}$. Let $\mathcal{O}_{\mathbf{x}_1},\dots,\mathcal{O}_{\mathbf{x}_k},\mathcal{U}_{\mathbf{x}_{k+1}},\dots,\mathcal{U}_{\mathbf{x}_m},B_{\mathbf{x}_{m+1}},\dots,B_{\mathbf{x}_n}$ be an open subcovering and $\eta_1,\dots,\eta_k,\eta_{k+1},\dots,\eta_m,\eta_{m+1},\dots,\eta_n$ be a partition of unity over $\bar{\Omega}$ subordinate to this subcovering. Recalling (3.4), from now on we set $t:=\min(s(2),q)$. Assume $l\in\{1,\dots,k\}$. We put $v_l:=\eta_l u|_{\mathcal{O}_{\mathbf{x}_l}\cap\Omega}$. Then, due to the property $u\in W_0^{1,2}(\Omega)$ and Lemma 3.9, v_l satisfies an equation

$$-\nabla\cdot\mu_l\nabla v_l+v_l=f_l \tag{3.36}$$

where $\mu_l:=\mu|_{\mathcal{O}_{\mathbf{x}_l}\cap\Omega}$ and $f_l\in W^{-1,t}(\mathcal{O}_{\mathbf{x}_l}\cap\Omega)$. Because (3.27) is also a topological isomorphism if q is replaced by t there, we get $v_l\in W_0^{1,t}(\mathcal{O}_{\mathbf{x}_l}\cap\Omega)$, which gives $\eta u_l\in W_0^{1,t}(\Omega)$. Let next l be in $\{k+1,\dots,m\}$. Then the property $u\in W_0^{1,2}(\Omega)$ and Lemma 3.9 imply that $v_l:=\eta_l u|_{\mathcal{U}_{\mathbf{x}_l}}$ satisfies an equation (3.36), where this time $\mu_l:=\mu|_{\mathcal{U}_{\mathbf{x}_l}}$ and $f_l\in W^{-1,t}(\mathcal{U}_{\mathbf{x}_l})$. Moreover, it is clear that both

v_l and f_l have their supports in $\mathcal{U}_{\mathbf{x}_l}$. We transform (3.36) via the C^1-mapping $\Phi_{\mathbf{x}_l}$. This leads to the following equation for the transformed objects

$$-\nabla \cdot \hat{\mu}_{\mathbf{x}_l} \nabla \hat{v}_l + \hat{v}_l = \hat{f}_l \tag{3.37}$$

on $\beta_{\mathbf{x}_l}\mathcal{E}$, where $\hat{f}_l \in W^{-1,t}(\beta_{\mathbf{x}}\mathcal{E})$. Additionally, $\hat{f}_l$ has its support in $\beta_{\mathbf{x}_l}\mathcal{E}$, which is also true for $\hat{v}_l$. Let $\check{\sigma}_l$ be the following coefficient function, defined on $\mathbb{R}^d$:

$$\check{\sigma}_l = \begin{cases} \hat{\mu}_{\mathbf{x}_l} & \text{on } \beta_{\mathbf{x}_l}\mathcal{E}, \\ \sigma_{\mathbf{x}_l} & \text{on } \mathbb{R}^d \setminus \beta_{\mathbf{x}_l}\mathcal{E}. \end{cases}$$

Because $\hat{f}_l$ and $\hat{v}_l$ have their supports in $\beta_{\mathbf{x}_l}\mathcal{E}$, (3.37) can be extended to an equation on the whole $\mathbb{R}^d$; namely, if V_l is the extension of $\hat{v}_l$ by zero to the whole $\mathbb{R}^d$, then

$$-\nabla \cdot \check{\sigma}_l \nabla V_l + V_l = -\nabla \cdot \sigma_{\mathbf{x}_l} \nabla V_l + V_l + \nabla \cdot (\sigma_{\mathbf{x}_l} - \check{\sigma}_l)\nabla V_l = F_l, \tag{3.38}$$

with $F_l \in W^{-1,t}(\mathbb{R}^d)$. By definition, one has

$$\|\sigma_{\mathbf{x}_l} - \check{\sigma}_l\|_{L^\infty(\mathbb{R}^d;\mathcal{B}(\mathbb{C}^d))} = \|\sigma_{\mathbf{x}_l} - \hat{\mu}_{\mathbf{x}_l}\|_{L^\infty(\beta_{\mathbf{x}_l}\mathcal{E};\mathcal{B}(\mathbb{C}^d))}.$$

This, together with (3.35) and Lemma 3.7, implies that $-\nabla \cdot \check{\sigma}_l \nabla + 1 : W^{1,t}(\mathbb{R}^d) \to W^{-1,t}(\mathbb{R}^d)$ is also a topological isomorphism for our specified t. Consequently, $V_l \in W^{1,t}(\mathbb{R}^d)$, which gives $\hat{v}_l \in W_0^{1,t}(\beta_{\mathbf{x}_l}\mathcal{E})$, and hence $v_l = \eta_l u|_{\mathcal{U}_{\mathbf{x}_l}} \in W_0^{1,t}(\mathcal{U}_{\mathbf{x}_l})$. Since the support of $\eta_l u$ is in $\mathcal{U}_{\mathbf{x}_l}$ we obtain $\eta_l u \in W_0^{1,t}(\Omega)$ for all $l = k+1, \ldots, m$. Lastly, if $l \in \{m+1, \ldots, n\}$, then one also ends up with an equation for $v_l := \eta_l u|_{B_{\mathbf{x}_l}}$ of type (3.36). The corresponding right hand sides are in $W^{-1,t}(B_{\mathbf{x}_l})$ (see Lemma 3.9). By Proposition 3.1 then $\eta_l u|_{B_{\mathbf{x}_l}} \in W_0^{1,t}(B_{\mathbf{x}_l})$, which yields $\eta_l u \in W_0^{1,t}(\Omega)$, and finally $u \in W_0^{1,t}(\Omega)$. Exploiting this and iterating the above considerations one improves the summability of ∇u in the light of Lemma 3.9 step by step and finally one ends up with $u \in W_0^{1,q}(\Omega)$. This proves the assertion for $\lambda = 1$ and one $q > 3$. The q's from $[2, q[$ are obtained by Remark 3.6. For all other λ's we obtain the assertion by the compactness of the resolvent and the fact that no λ with $\Re\lambda \leqslant 0$ can be an eigenvalue. Finally, the case $q < 2$ is obtained by duality.

REMARK 3.15 The proof shows—under our assumption on $\partial\Omega_\circ \cap \Omega$—that the limitation for q comes exclusively from the boundary points.

REMARK 3.16 The reader may possibly ask why in case (ii) we restrict ourselves to $d = 3$. The answer is: the essential aim of this paper is to prove the isomorphism property for a q which is larger than the space dimension d. In this spirit, the two-dimensional case (even under more general assumptions) is covered by [27]. If $d > 3$ an analogue of Proposition 3.3, giving $q > d$, cannot be expected (see [21]). Nevertheless, $d = 3$ as the 'physical' dimension seems to us the most important case.

REMARK 3.17 If $\Omega_\circ$ does not touch the boundary of Ω, then one can prove the analogous result for the Neumann operator, namely: $-\nabla \cdot \mu\nabla + \lambda$ provides a topological isomorphism between $W^{1,q}(\Omega)$ and $(W^{-1,q'}(\Omega))^*$ for a $q > 3$ and all λ from the open right half-plane. In this case one uses Zanger's result [57] instead of that of Jerison–Kenig.

REMARK 3.18 The result generalizes to the case where finitely many C^1 domains are included in Ω at positive distance to each other and the coefficient function is uniformly continuous on each of them and, of course, on the complement of their union.

REMARK 3.19 The isomorphy property claimed in Theorem 1.1 remains true in the case of real spaces $W_0^{1,q}(\Omega)$, $W^{-1,q}(\Omega)$ and real λ's, because for these λ the operator $-\nabla \cdot \mu\nabla + \lambda$ commutes with complex conjugation.

REMARK 3.20 Let q be any number as in Theorem 1.1 and assume $a_1, \ldots, a_d \in L^r(\Omega)$, $b_1, \ldots, b_d \in L^s(\Omega)$, $c \in L^t(\Omega)$. Then, under suitable conditions on r, s, t, the first order operator

$$W_0^{1,q}(\Omega) \ni u \mapsto \sum_{l=1}^{d} \left(a_l \frac{\partial u}{\partial x_l} + \frac{\partial (b_l u)}{\partial x_l} \right) + cu \in W^{-1,q}(\Omega) \tag{3.39}$$

is relatively compact with respect to $-\nabla \cdot \mu\nabla$. Hence, if $-\nabla \cdot \mu\nabla$ is perturbed by (3.39), it also has $W_0^{1,q}(\Omega)$ as its domain of definition.

4. Nonsmooth interfaces: a counterexample

The reader may have possibly asked himself whether the C^1 property is necessary or may be weakened without changing the result. The following counterexample (see [21]) shows that the situation changes dramatically if the interface has only one corner point. In particular, this shows that piecewise C^1 is (by far) not sufficient for our result. Namely, quite parallel to the classical example of Meyers (see [44]) the integrability exponent for the gradient of the solution of the (planar) homogeneous elliptic equation tends to 2 in dependence on a certain parameter.

The background for the considerations in this section is the well known connection between singularities for the solution of an elliptic equation and the eigenvalues of an associated operator pencil of Sturm–Liouville operators (see [43] or [21]).

We consider the following coefficient function on $\mathbb{R}^2$:

$$\mu(x, y) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} & \text{if } x, y > 0, \\ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & \text{elsewhere on } \mathbb{R}^2, \ t > 0, \end{cases}$$

and, correspondingly, the elliptic problem

$$\nabla \cdot \mu\nabla u = 0. \tag{4.1}$$

Proceeding as in [43] we are looking for solutions $\tilde{u} \in W^{1,2}(]0, 2\pi[)$ of the (generalized) Sturm–Liouville equation

$$-(b_2\tilde{u}')' - \lambda(b_1\tilde{u})' - \lambda b_1\tilde{u}' - \lambda^2 b_0\tilde{u} = 0, \tag{4.2}$$

combined with the compatibility conditions

$$\begin{gathered} w(\pi/2) = v(\pi/2), \qquad w(0) = v(2\pi), \\ (b_2\partial_\theta w + \lambda b_1 w)|_0 = (b_2\partial_\theta v + \lambda b_1 v)|_{2\pi}, \\ (b_2\partial_\theta w + \lambda b_1 w)|_{\pi/2} = (b_2\partial_\theta v + \lambda b_1 v)|_{\pi/2}, \end{gathered} \tag{4.3}$$

if $w = \tilde{u}|_{[0,\pi/2]}$ and $v = \tilde{u}|_{[\pi/2,2\pi]}$.

The coefficient functions b_0, b_1, b_2 are defined as follows:

$$
\begin{aligned}
b_0(\theta) &:= \begin{cases} \cos^2\theta + t^2\sin^2\theta & \text{if } \theta \in [0, \pi/2[, \\ t & \text{if } \theta \in [\pi/2, 2\pi[, \end{cases} \\
b_2(\theta) &:= \begin{cases} \sin^2\theta + t^2\cos^2\theta & \text{if } \theta \in [0, \pi/2[, \\ t & \text{if } \theta \in [\pi/2, 2\pi[, \end{cases} \\
b_1(\theta) &:= \begin{cases} (t^2-1)\sin\theta\cos\theta & \text{if } \theta \in [0, \pi/2[, \\ 0 & \text{if } \theta \in [\pi/2, 2\pi[. \end{cases}
\end{aligned} \tag{4.4}
$$

In order to determine the λ with the smallest possible (positive) real part, we use the ansatz functions (see [17])

$$
\begin{aligned}
w(\theta) &:= c_+(t\cos\theta + i\sin\theta)^\lambda + c_-(t\cos\theta - i\sin\theta)^\lambda, \\
v(\theta) &:= d_+\cos\lambda\theta + d_-\sin\lambda\theta
\end{aligned}
$$

with unknown coefficients $c_\pm$ and $d_\pm$. Using (4.3) and (4.4), we can eliminate $c_\pm$ to get the equations

$$
\begin{aligned}
d_+(t^\lambda - \cos 2\pi\lambda) - d_-\sin 2\pi\lambda &= 0, \\
d_+\sin 2\pi\lambda + d_-(t^\lambda - \cos 2\pi\lambda) &= 0.
\end{aligned} \tag{4.5}
$$

Obviously, the system (4.5) is nontrivially solvable in d_+, d_- iff

$$(t^\lambda - \cos 2\pi\lambda)^2 + \sin^2 2\pi\lambda = 0,$$

or, what is the same,

$$\cos 2\pi\lambda = \frac{t^\lambda + t^{-\lambda}}{2} = \cosh(\lambda\ln t). \tag{4.6}$$

Writing $\cosh(\lambda\ln t) = \cos(i\lambda\ln t)$ and taking into account the identity

$$\cos\theta - \cos\tau = -2\sin\frac{\theta+\tau}{2}\sin\frac{\theta-\tau}{2}$$

shows that (4.6) is equivalent to

$$\sin\left(\frac{\lambda}{2}(2\pi + i\ln t)\right)\sin\left(\frac{\lambda}{2}(2\pi - i\ln t)\right) = 0.$$

This is the case iff

$$\frac{\lambda}{2}(2\pi \pm i\ln t) = 2k\pi, \quad k \in \mathbb{Z}.$$

Thus, the λ with the smallest (positive) real part is

$$\lambda = \frac{8\pi^2}{4\pi^2 + \ln^2 t} \pm i\frac{4\pi\ln t}{4\pi^2 + \ln^2 t}.$$

One easily notices that as $t \to \infty$, the real parts of these λ's converge to zero. Assume that λ with $\Re\lambda \in (0, 1)$ is a complex number and $\tilde{u}_\lambda \in W^{1,2}(0, 2\pi)$ a corresponding function which satisfies (4.2) together with the compatibility conditions (4.3). Then the function

$$u(x_1, x_2) := (x_1^2 + x_2^2)^{\lambda/2}\tilde{u}_\lambda(\arg(x_1 + ix_2)) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$$

is a solution of equation (4.1) in the distributional sense. Moreover, $\tilde{u}_\lambda$ does not vanish identically, and hence its absolute value has a strictly positive lower bound at least on a (nontrivial) subinterval of $(0, 2\pi)$. Thus, $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2)$ for $q \in [2, 2/(1 - \Re\lambda))$, but not for $q = 2/(1 - \Re\lambda)$. If we let t tend to ∞, these solutions lack any common (local) integrability exponent larger than 2 for their first order derivatives.

REMARK 4.1 The above example is not restricted to two dimensions. One can add arbitrarily many dimensions by extending the solution constantly in these directions—at least in a neighbourhood of zero.

5. Parabolic operators

Very often elliptic operators in divergence form occur as the elliptic part of parabolic operators (see [5] or [29]). In this section we will deduce functional-analytic properties for the corresponding parabolic operators from our elliptic regularity result. If X is a complex Banach space, then we denote by $W^{1,r}(]0, T[; X)$ the set of elements from $L^r(]0, T[; X)$ whose distributional derivatives with respect to time also belong to $L^r(]0, T[; X)$. The main result reads as follows:

THEOREM 5.1 Let Λ be a bounded domain with Lipschitz boundary and ρ a measurable, essentially bounded, elliptic coefficient function which takes its values in the set of real, symmetric $d \times d$ matrices. Assume that $q \in]1, \infty[$ is such that

$$-\nabla \cdot \rho\nabla : W^{1,q}_0(\Lambda) \to W^{-1,q}(\Lambda)$$

is a topological isomorphism. Then $\partial/\partial t - \nabla \cdot \rho\nabla$ has maximal parabolic regularity on $W^{-1,q}(\Lambda)$, precisely: If $r \in]1, \infty[$ is fixed, then for any $f \in L^r(]0, T[; W^{-1,q}(\Lambda))$ there is exactly one function $w \in L^r(]0, T[; W^{1,q}(\Lambda)) \cap W^{1,r}(]0, T[; W^{-1,q}(\Lambda))$ such that

$$\frac{\partial w}{\partial t} - \nabla \cdot \rho\nabla w = f \quad \text{and} \quad w(0) = 0. \tag{5.1}$$

COROLLARY 5.2 Under the above assumptions $-\nabla \cdot \rho\nabla$ generates an analytic semigroup on $W^{-1,q}(\Lambda)$.

In order to prove Theorem 5.1 we first establish some auxiliary results:

THEOREM 5.3 Let Λ be a Lipschitz domain and ρ as in the previous theorem. Assume $q \in]1, \infty[$ and let A_q be the $L^q(\Lambda)$ realization of $\nabla \cdot \rho\nabla$, with domain D_q. Then $\partial/\partial t - A_q$ hes maximal regularity over $L^q(\Lambda)$, in other words: If $r \in]1, \infty[$ is fixed, then for any $f \in L^r(]0, T[; L^q(\Lambda))$ there is exactly one $w \in L^r(]0, T[; D_q) \cap W^{1,r}(]0, T[; L^q(\Lambda))$ such that (5.1) is satisfied.

Proof. The semigroup generated by A_2 on $L^2(\Lambda)$ admits upper Gaussian estimates (see [8] or [6]), which implies maximal parabolic regularity on L^p spaces [30] (see also [18]). □

THEOREM 5.4 Under the assumptions of Theorem 5.1, $(-\nabla \cdot \rho\nabla)^{1/2}$ provides a topological isomorphism between $W^{1,s}_0(\Lambda)$ and $L^s(\Lambda)$ and between $L^s(\Lambda)$ and $W^{-1,s}(\Lambda)$ for all $s \in [q', q]$.

Proof. First, interpolation (see Theorem 3.5) and duality show that $-\nabla \cdot \rho\nabla$ is a topological isomorphism between $W^{1,s}_0(\Lambda)$ and $W^{-1,s}(\Lambda)$ for all $s \in [q', q]$. A deep result of [8, Thm. 4] yields the continuity of the map

$$(-\nabla \cdot \rho\nabla)^{1/2} : W^{1,s}_0(\Lambda) \to L^s(\Lambda) \tag{5.2}$$

for all $s \in]1, \infty[$. By duality one obtains the continuity of

$$(-\nabla \cdot \rho\nabla)^{1/2} : L^s(\Lambda) \to W^{-1,s}(\Lambda) \tag{5.3}$$

for all $s \in]1, \infty[$. Hence, for $s \in [q', q]$ we can estimate

$$\|(-\nabla \cdot \rho\nabla)^{-1/2}\|_{\mathcal{B}(L^s(\Lambda); W_0^{1,s}(\Lambda))} \leqslant \|(-\nabla \cdot \rho\nabla)^{1/2}\|_{\mathcal{B}(L^s(\Lambda); W^{-1,s}(\Lambda))} \|(-\nabla \cdot \rho\nabla)^{-1}\|_{\mathcal{B}(W^{-1,s}(\Lambda); W_0^{1,s}(\Lambda))}.$$

This proves that (5.2) is in fact a topological isomorphism if $s \in [q', q]$. Being an isomorphism between $L^s(\Lambda)$ and $W^{-1,s}(\Lambda)$ follows from this by duality. □

COROLLARY 5.5 Let D_q denote the domain of the $L^q(\Lambda)$ realization of $-\nabla \cdot \rho\nabla$. Then $(-\nabla \cdot \rho\nabla)^{1/2}$ provides a topological isomorphism between D_q and $W_0^{1,q}(\Lambda)$.

Proof. $-\nabla \cdot \rho\nabla$ is a topological isomorphism between D_q and $L^q(\Lambda)$ while $(-\nabla \cdot \rho\nabla)^{1/2}$ is a topological isomorphism between $W_0^{1,q}(\Lambda)$ and $L^q(\Lambda)$. □

We now give the proof of Theorem 5.1. It is clear that the established isomorphisms for $(-\nabla \cdot \rho\nabla)^{1/2}$ induce the following isomorphisms:

$$(-\nabla \cdot \rho\nabla)^{-1/2} : L^r(]0, T[; W^{-1,q}(\Lambda)) \to L^r(]0, T[; L^q(\Lambda)), \tag{5.4}$$

$$(-\nabla \cdot \rho\nabla)^{1/2} : L^r(]0, T[; D_q) \to L^r(]0, T[; W_0^{1,q}(\Lambda)), \tag{5.5}$$

$$(-\nabla \cdot \rho\nabla)^{1/2} : W^{1,r}(]0, T[; L^q(\Lambda)) \to W^{1,r}(]0, T[; W^{-1,q}(\Lambda)). \tag{5.6}$$

Further, it is well known that the solution w of (5.1) is obtained as

$$w(t) = \int_0^t e^{(t-s)\nabla\cdot\rho\nabla} f(s)\, \mathrm{d}s.$$

Hence, the parabolic solution operator commutes with $(-\nabla \cdot \rho\nabla)^{1/2}$. Consequently, the maximal regularity property on $L^q(\Lambda)$ transfers via the isomorphisms (5.4)–(5.6) to the space $W^{-1,q}(\Lambda)$. Corollary 5.2 follows from the well known fact that maximal parabolic regularity implies the generation property of an analytic semigroup.

REMARK 5.6 The authors are convinced that the results on the parabolic operators are adequate instruments for the treatment of (even nonautonomous) semilinear and quasilinear parabolic problems. The key point concerning quasilinear equations of the type, say,

$$\frac{\partial w}{\partial t} - \nabla \cdot G(w)\mu\nabla w = H(t, w, \nabla w)$$

is that for three-dimensional domains and $q > 3$ suitable interpolation spaces between $W_0^{1,q}$ and $W^{-1,q}$ embed continuously into Hölder spaces. Thus, if G is a strictly positive C^1 function, then the coefficient functions $G(w)\mu$ are of the same quality as μ (in the spirit of Theorem 1.1). Hence, the domains of the operators $\nabla \cdot G(w)\mu\nabla$ do not depend on u if u runs through a suitable interpolation space (see [46])—which is often required in quasilinear parabolic theory. We will study these matters in detail elsewhere.

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