

A geometrically exact Cosserat shell model for defective elastic crystals. Justification via Γ -convergence

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We derive the Γ -limit to a three-dimensional Cosserat model as the aspect ratio $h > 0$ of a flat domain tends to zero. The bulk model involves already exact rotations as a second independent field intended to describe the rotations of the lattice in defective elastic crystals. The Γ -limit based on the natural scaling consists of a membrane-like energy and a transverse shear energy both scaling with h , augmented by a curvature energy due to the Cosserat bulk, also scaling with h . A technical difficulty is to establish equi-coercivity of the sequence of functionals as the aspect ratio h tends to zero. Usually, equi-coercivity follows from a local coerciveness assumption. While the three-dimensional problem is well-posed for the Cosserat couple modulus $\mu_c \geq 0$, equi-coercivity needs a strictly positive μ_c . Then the Γ -limit model determines the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$. For the true defective crystal case, however, $\mu_c = 0$ is appropriate. Without equi-coercivity, we first obtain an estimate of the Γ -lim inf and Γ -lim sup which can be strengthened to the Γ -convergence result. The Reissner–Mindlin model is “almost” the linearization of the Γ -limit for $\mu_c = 0$.

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1. Introduction

1.1 *Aspects of shell theory*

The dimensional reduction of a given continuum-mechanical model is already an old subject and has seen many “solutions”. One possible way to proceed is the so called *derivation approach*, i.e., reducing a given three-dimensional model via physically reasonable constitutive assumptions on the kinematics to a two-dimensional model. This is opposed to either the *intrinsic approach* which views the shell from the onset as a two-dimensional surface and invokes concepts from differential geometry, or the *asymptotic methods* which try to establish two-dimensional equations by formal expansion of the three-dimensional solution in power series in terms of a small

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nondimensional thickness parameter, the aspect ratio h . The intrinsic approach is closely related to the *direct approach* which takes the shell to be a two-dimensional medium with additional *extrinsic directors* in the sense of a *restricted Cosserat surface* [17].¹ There, two-dimensional equilibrium in appropriate new resultant stress and strain variables is postulated ab initio more or less independently of three-dimensional considerations (cf. [3, 34, 23, 14–16, 55]).

A comprehensive presentation of the different approaches in classical shell theories can be found in the monograph [42]. A thorough mathematical analysis of linear, infinitesimal-displacement shell theory, based on asymptotic methods, is found in [12] and the extensive references therein (see also [11, 13, 3, 21, 22, 32, 4]). Excellent reviews of the modelling and finite element implementation may be found in [59, 56, 58, 35, 36, 69, 5, 9] and in the series of papers [61–66]. Properly invariant, geometrically exact, elastic plate theories are derived by formal asymptotic methods in [25]. This formal derivation is extended to curvilinear shells in [41, 40]. Apart from the pure bending case [30, 28], which is justified as the Γ -limit of the three-dimensional model as $h \rightarrow 0$ and which can be shown to be intrinsically well-posed, the finite-strain models obtained have not yet been shown to be well-posed. Indeed, the membrane energy contribution is notoriously not Legendre–Hadamard elliptic. The different membrane model formally justified in [39] by Γ -convergence is geometrically exact and automatically quasiconvex/elliptic but unfortunately does not coincide upon linearization with the otherwise well-established infinitesimal-displacement membrane model. Moreover, this model does not describe the detailed geometry of deformation in compression but reduces to a tension-field theory [67]. The quasiconvexifying step in [39] appears since the membrane energy takes then into account the energy reducing effect of possible fine scale oscillations (wrinkles). The development of [39] has been generalized to Young measures in [27]. A hierarchy of limiting theories based on Γ -convergence, distinguished by different scaling exponents of the energy as a function of the aspect ratio h , is developed in [31]. There the different scaling exponents can be controlled by scaling assumptions on the applied forces.

It is possible to include *interfacial energy* (here a second derivative term $\kappa \|D^2\varphi\|^2$ in the bulk energy) in the description of the material. The Γ -limit for constant κ has been investigated in [6] in an application to thin martensitic films. As a result, no quasiconvexification step is necessary (the higher derivative excludes arbitrary fine scale wrinkles) and in the limit one independent “Cosserat director” appears. If simultaneously $\kappa \rightarrow 0$ faster than $h \rightarrow 0$, then the Γ -limit coincides [60, Rem. 5] with that of [39]. In our context (see below), including such an interfacial energy is tantamount to setting $\mu_c = \infty$ in the Cosserat bulk model, i.e. the Cosserat bulk model would degenerate into a *second gradient model*.

There are numerous proposals in the engineering literature for a finite-strain, geometrically exact plate formulation (see e.g. [26, 57–59, 69, 5, 9]). These models are based on the Reissner–Mindlin kinematical assumption which is a variant of the direct approach; usually one independent director vector appears in the model. In many cases the need has been felt to devote attention to rotations $R \in \text{SO}(3)$, since rotations are the dominant deformation mode of a thin flexible structure. This has led to the *drill-rotation formulation*, which means that proper rotations either appear in the formulation as independent fields (leading to a restricted Cosserat surface) or they are an intermediary ingredient in the numerical treatment (constraint Cosserat surface, only continuum rotations matter finally). While the computational merit of this approach is well documented, such models lacked any asymptotic basis.

¹ Restricted, since no material length scale enters the direct approach, only the nondimensional *aspect ratio* h appears in the model. In terminology it is useful to distinguish between a “true” Cosserat model operating on $\text{SO}(3)$ and theories with any number of directors.

1.2 Outline of this contribution

In [45] the first author has proposed a Cosserat shell model for materials with rotational microstructure. In the underlying Cosserat bulk model the Cosserat rotation \bar{R} and the gradients of \bar{R} enter into the measure of deformation of the body. In fact, variation of \bar{R} leads to a balance of substructural interaction [10]. These gradients account therefore for the presence of interfaces between substructural units in a smeared sense. One may think of, e.g., liquid crystals, defective single crystals or metallic foams [47, 52].

Assuming a strict principle of scale separation rules out the possibility of a direct comparison between macroscopic quantities (the usual deformation) and the microscopic ones (for example the lattice vectors in a defective crystal) and makes it reasonable to assume that they behave independently of each other. For definiteness, we may view the Cosserat rotations \bar{R} as averaged lattice rotations, independent of the macroscopic rotation.² It can be shown that the Cosserat rotation follows closely the macroscopic rotation in the bulk model provided that a constitutive parameter, the Cosserat couple modulus μ_c , is strictly positive. Therefore, the interesting case with independent microstructure is represented by $\mu_c = 0$. In this case, the amount of incompatibility of the lattice rotations, measured through $\text{Curl } \bar{R}$, decisively influences the elastic response of the material, and elastic coercivity can only be established for a reasonably smooth distribution of incompatibilities and defects. Every real pure single crystal contains still a massive amount of defects and incompatibilities. Thus, giving up the idealization of a defect free single crystal adds to the physical realism of the model. Let us henceforth refer to $\mu_c = 0$ as defective elastic crystal case.

The above mentioned shell model is shown to be well-posed in [45] for $\mu_c > 0$ and in [50] for $\mu_c = 0$. Apart for technical details, this Cosserat shell model includes the generalized drill-rotation formulations alluded to above. Notably for $\mu_c = 0$, the in-plane drill-energy is absent in conformity with the classical Reissner–Mindlin model.

The formal derivation of the new shell model [45], based on an asymptotic ansatz for a Cosserat bulk model with kinematical and physical assumptions appropriate for thin structures, however, still gives rise to questions as far as the asymptotic correctness and convergence are concerned. In this paper we address this point by showing that the Γ -limit of the Cosserat bulk model for $h \rightarrow 0$ (under natural scaling assumptions) is, after descaling, given by the corresponding formal derivation, provided the energy contributions scaling with h are retained and the coefficient of the transverse shear energy is slightly modified. Given that the information provided by the Γ -limit hinges also on scaling assumptions, we think that this result is a justification of the formal derivation in [45] and the employed kinematical ansatz.

Central to our development is therefore the notion of Γ -convergence, a powerful theory initiated by De Giorgi [19, 20] and especially suited for a variational framework on which in turn the numerical treatment with finite elements is based. This approach has thus far provided the only known convergence theorems for justifying lower dimensional nonlinear, frame-indifferent theories of elastic bodies.

Now, after presenting the notation, we recall in Section 2 the underlying “parent” three-dimensional finite-strain frame-indifferent Cosserat model with rotational substructure embodied by the *Cosserat rotations* \bar{R} , i.e., a *triad of rigid directors* $(\bar{R}_1 | \bar{R}_2 | \bar{R}_3) = \bar{R} \in \text{SO}(3)$, and provide

² Compare with [70], where it is observed that lattice rotations are, in fact, independent of the macroscopic rotations in nano-indent single crystal copper experiments.

the existence results for this bulk model. Then we perform in Section 3 the transformation of the bulk model in physical space to a nondimensional thin domain and introduce the further scaling to a fixed reference domain Ω_1 with constant thickness on which the Γ -convergence procedure is finally based.

In Section 4 we recapitulate some points from Γ -convergence theory and introduce the Γ -limit for the rescaled formulation with respect to the two independent fields (φ, \bar{R}) of deformations and microrotations in Section 5. Two limit cases, $\mu_c = 0$ and $\mu_c = \infty$, deserve additional attention. Next, we provide the proof for the Γ -convergence results: first for the simple case $\mu_c > 0$ in Section 6 similar to the development in [39], and then for the case of defective elastic crystals, $\mu_c = 0$, in Section 7. The case $\mu_c = \infty$ will be dealt with rigorously in a separate contribution. Our geometrically exact results have been first announced in [51, 46, 49]. In the meantime, the geometrically linear case for $\mu_c > 0$ has been treated by others in [2, 1].

1.3 Notation

1.3.1 Notation for bulk material. Let $\Omega \subset \mathbb{R}^3$ always be a bounded open domain with Lipschitz boundary $\partial\Omega$ and let Γ be a smooth subset of $\partial\Omega$ with nonvanishing 2-dimensional Hausdorff measure. For $a, b \in \mathbb{R}^3$ we let $\langle a, b \rangle_{\mathbb{R}^3}$ denote the scalar product on \mathbb{R}^3 with associated norm $\|a\|_{\mathbb{R}^3}^2 = \langle a, a \rangle_{\mathbb{R}^3}$. We denote by $\mathbb{M}^{3 \times 3}$ the set of real 3×3 second order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{M}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{M}^{3 \times 3}} = \text{tr}[XY^T]$, and the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{M}^{3 \times 3}}$. In the following we omit the indices $\mathbb{R}^3, \mathbb{M}^{3 \times 3}$. The identity tensor on $\mathbb{M}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}[X] = \langle X, \mathbb{1} \rangle$ and $\text{tr}[X]^2 = \langle X, \mathbb{1} \rangle^2$. We let Sym and PSym denote the symmetric and positive definite symmetric tensors respectively. We adopt the usual abbreviations of Lie group theory, i.e., $\text{GL}(3) := \{X \in \mathbb{M}^{3 \times 3} \mid \det[X] \neq 0\}$ the general linear group, $\text{O}(3) := \{X \in \text{GL}(3) \mid X^T X = \mathbb{1}\}$, $\text{SO}(3) := \{X \in \text{GL}(3) \mid X^T X = \mathbb{1}, \det[X] = 1\}$ with corresponding Lie algebra $\mathfrak{so}(3) := \{X \in \mathbb{M}^{3 \times 3} \mid X^T = -X\}$ of skew symmetric tensors. We denote by $\text{Adj } X$ the tensor of transposed cofactors $\text{Cof}(X)$ so that $\text{Adj } X = \det[X]X^{-1} = \text{Cof}(X)^T$ if $X \in \text{GL}(3)$. We set $\text{sym}(X) = \frac{1}{2}(X^T + X)$ and $\text{skew}(X) = \frac{1}{2}(X - X^T)$ such that $X = \text{sym}(X) + \text{skew}(X)$. For $\xi, \eta \in \mathbb{R}^n$ we have the tensor product $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. We write the polar decomposition in the form $F = RU = \text{polar}(F)U$ with $R = \text{polar}(F)$ the orthogonal part of F . For a second order tensor X we define the third order tensor $\mathfrak{h} = D_x X(x) = (\nabla(X(x).e_1), \nabla(X(x).e_2), \nabla(X(x).e_3)) = (\mathfrak{h}^1, \mathfrak{h}^2, \mathfrak{h}^3) \in \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \times \mathbb{M}^{3 \times 3} \cong \mathfrak{T}(3)$. For third order tensors $\mathfrak{h} \in \mathfrak{T}(3)$ we set $\|\mathfrak{h}\|^2 = \sum_{i=1}^3 \|\mathfrak{h}^i\|^2$, $\text{sym}(\mathfrak{h}) := (\text{sym } \mathfrak{h}^1, \text{sym } \mathfrak{h}^2, \text{sym } \mathfrak{h}^3)$ and $\text{tr}[\mathfrak{h}] := (\text{tr}[\mathfrak{h}^1], \text{tr}[\mathfrak{h}^2], \text{tr}[\mathfrak{h}^3]) \in \mathbb{R}^3$. Moreover, for any second order tensor X we define $X \cdot \mathfrak{h} := (X\mathfrak{h}^1, X\mathfrak{h}^2, X\mathfrak{h}^3)$ and $\mathfrak{h} \cdot X$, correspondingly. Quantities with a bar, e.g. the micropolar rotation \bar{R} , represent the micropolar replacement of the corresponding classical continuum rotation R . For the deformation $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^3)$ we have the deformation gradient $F = \nabla\varphi \in C(\bar{\Omega}, \mathbb{M}^{3 \times 3})$. $S_1(F) = D_F W(F)$ and $S_2(F) = F^{-1} D_F W(F)$ denote the first and second Piola–Kirchhoff stress tensors. The first and second differential of a scalar-valued function $W(F)$ are written $D_F W(F).H$ and $D_F^2 W(F).(H, H)$. We employ the standard notation for Sobolev spaces, i.e. $L^2(\Omega), H^{1,2}(\Omega), H_o^{1,2}(\Omega), W^{1,q}(\Omega)$, which we use indifferently for scalar-valued functions as well as for vector-valued and tensor-valued functions. The set $W^{1,q}(\Omega, \text{SO}(3))$ denotes orthogonal tensors whose components are in $W^{1,q}(\Omega)$. Moreover, we set $\|X\|_\infty = \sup_{x \in \Omega} \|X(x)\|$. We denote by $C_0^\infty(\Omega)$ the infinitely differentiable functions with compact support in Ω . We use capital letters to denote possibly large positive constants, e.g. C^+, K , and lower case letters to denote possibly small positive constants, e.g. c^+, d^+ .

1.3.2 *Notation for plates and shells.* Let $\omega \subset \mathbb{R}^2$ always be a bounded open domain with Lipschitz boundary $\partial\omega$ and let γ_0 be a smooth subset of $\partial\omega$ with nonvanishing 1-dimensional Hausdorff measure. The aspect ratio of the plate is $h > 0$. We denote by $\mathbb{M}^{m \times n}$ the set of matrices mapping $\mathbb{R}^n \rightarrow \mathbb{R}^m$. For $H \in \mathbb{M}^{3 \times 2}$ and $\xi \in \mathbb{R}^3$ we write $(H|\xi) \in \mathbb{M}^{3 \times 3}$ for the matrix composed of H and the column ξ . Likewise $(v|\xi|\eta)$ is the matrix composed of the columns v, ξ, η . This allows us to write $\nabla\varphi = (\varphi_x|\varphi_y|\varphi_z) = (\partial_x\varphi|\partial_y\varphi|\partial_z\varphi)$ for $\varphi \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. The identity tensor on $\mathbb{M}^{2 \times 2}$ is $\mathbb{1}_2$. The mapping $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the deformation of the midsurface, ∇m is the corresponding deformation gradient and \vec{n}_m is the outer unit normal on m . A matrix $X \in \mathbb{M}^{3 \times 3}$ can now be written as $X = (X.e_2|X.e_2|X.e_3) = (X_1|X_2|X_3)$. We write $v : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ for the displacement of the midsurface such that $m(x, y) = (x, y, 0)^T + v(x, y)$. The standard volume element is $dx \, dy \, dz = dV = d\omega \, dz$.

2. The underlying three-dimensional Cosserat model

2.1 Problem statement in variational form

In [48] a finite-strain, fully frame-indifferent, three-dimensional Cosserat micropolar model is introduced. The two-field problem has been posed in a variational setting. The task is to find a pair $(\varphi, \bar{R}) : \Omega \subset \mathbb{E}^3 \rightarrow \mathbb{E}^3 \times \text{SO}(3)$ of a deformation φ and an independent Cosserat rotation $\bar{R} \in \text{SO}(3)$, defined on the ambient physical space \mathbb{E}^3 , minimizing the energy functional I ,

$$I(\varphi, \bar{R}) = \int_{\Omega} [W_{\text{mp}}(\bar{R}^T \nabla\varphi) + W_{\text{curv}}(\bar{R}^T D_x \bar{R}) - \Pi_f(\varphi) - \Pi_M(\bar{R})] dV - \int_{\Gamma_S} \Pi_N(\varphi) dS - \int_{\Gamma_C} \Pi_{M_c}(\bar{R}) dS \mapsto \min \text{ w.r.t. } (\varphi, \bar{R}), \tag{2.1}$$

together with the Dirichlet boundary condition for the deformation φ on Γ : $\varphi|_{\Gamma} = g_d$ and three possible *alternative* boundary conditions for the microrotations \bar{R} on Γ ,

$$\bar{R}|_{\Gamma} = \begin{cases} \bar{R}_d, & \text{the case of rigid prescription,} \\ \text{polar}(\nabla\varphi), & \text{the case of strong consistent coupling,} \\ \text{no condition for } \bar{R} \text{ on } \Gamma, & \text{induced Neumann-type relations for } \bar{R} \text{ on } \Gamma. \end{cases}$$

The constitutive assumptions on the densities are

$$W_{\text{mp}}(\bar{U}) = \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U})\|^2 + \frac{\lambda}{2} \text{tr}[\text{sym}(\bar{U} - \mathbb{1})]^2, \bar{U} = \bar{R}^T F, F = \nabla\varphi, \\ W_{\text{curv}}(\mathfrak{K}) = \mu \frac{L_c^{1+p}}{12} (1 + \alpha_4 L_c^q \|\mathfrak{K}\|^q) (\alpha_5 \|\text{sym } \mathfrak{K}\|^2 + \alpha_6 \|\text{skew } \mathfrak{K}\|^2 + \alpha_7 \text{tr}[\mathfrak{K}]^2)^{(1+p)/2}, \\ \mathfrak{K} = \bar{R}^T D_x \bar{R} := (\bar{R}^T \nabla(\bar{R}.e_1), \bar{R}^T \nabla(\bar{R}.e_2), \bar{R}^T \nabla(\bar{R}.e_3)), \text{ the third order curvature tensor,}$$

under the minimal requirement $p \geq 1, q \geq 0$. The total elastically stored energy $W = W_{\text{mp}} + W_{\text{curv}}$ is quadratic in the stretch \bar{U} and possibly super-quadratic in the curvature \mathfrak{K} . The strain energy W_{mp} depends on the deformation gradient $F = \nabla\varphi$ and the microrotations $\bar{R} \in \text{SO}(3)$, which do not necessarily coincide with the *continuum rotations* $R = \text{polar}(F)$. The curvature energy W_{curv} depends moreover on the space derivatives $D_x \bar{R}$ which describe the self-interaction of the

microstructure.³ In general, the *micropolar stretch tensor* \bar{U} is *not symmetric* and does not coincide with the *symmetric continuum stretch tensor* $U = R^T F = \sqrt{F^T F}$. By abuse of notation we set $\|\text{sym } \mathfrak{K}\|^2 := \sum_{i=1}^3 \|\text{sym } \mathfrak{K}^i\|^2$ for third order tensors \mathfrak{K} (cf. 1.3.1).

Here $\Gamma \subset \partial\Omega$ is that part of the boundary where the Dirichlet conditions g_d, \bar{R}_d for the deformations and microrotations or the coupling conditions for the microrotations are prescribed. $\Gamma_S \subset \partial\Omega$ is the part of the boundary where the traction boundary conditions in the form of the potential of applied surface forces Π_N are given with $\Gamma \cap \Gamma_S = \emptyset$. In addition, $\Gamma_C \subset \partial\Omega$ is the part of the boundary where the potential of external surface couples Π_{M_c} is applied with $\Gamma \cap \Gamma_C = \emptyset$. On the free boundary $\partial\Omega \setminus \{\Gamma \cup \Gamma_S \cup \Gamma_C\}$ the corresponding natural boundary conditions for (φ, \bar{R}) apply. The potential of the external applied volume force is Π_f and Π_M takes on the role of the potential of applied external volume couples. For simplicity we assume

$$\Pi_f(\varphi) = \langle f, \varphi \rangle, \quad \Pi_M(\bar{R}) = \langle M, \bar{R} \rangle, \quad \Pi_N(\varphi) = \langle N, \varphi \rangle, \quad \Pi_{M_c}(\bar{R}) = \langle M_c, \bar{R} \rangle, \quad (2.2)$$

for the potentials of applied loads with given functions $f \in L^2(\Omega, \mathbb{R}^3)$, $M \in L^2(\Omega, \mathbb{M}^{3 \times 3})$, $N \in L^2(\Gamma_S, \mathbb{R}^3)$, $M_c \in L^2(\Gamma_C, \mathbb{M}^{3 \times 3})$.

The parameters $\mu, \lambda > 0$ are the Lamé constants of classical isotropic elasticity; the additional parameter $\mu_c \geq 0$ is called the *Cosserat couple modulus*. For $\mu_c > 0$ the elastic strain energy density $W_{\text{mp}}(\bar{U})$ is *uniformly convex* in \bar{U} and satisfies the *standard growth assumption*

$$\begin{aligned} \forall F \in \text{GL}^+(3, \mathbb{R}) : \quad W_{\text{mp}}(\bar{U}) &= W_{\text{mp}}(\bar{R}^T F) \\ &\geq \min(\mu, \mu_c) \|\bar{R}^T F - \mathbb{1}\|^2 \\ &= \min(\mu, \mu_c) \|F - \bar{R}\|^2 \\ &\geq \min(\mu, \mu_c) \inf_{R \in \text{O}(3, \mathbb{R})} \|F - R\|^2 \\ &= \min(\mu, \mu_c) \text{dist}^2(F, \text{O}(3, \mathbb{R})) \\ &= \min(\mu, \mu_c) \text{dist}^2(F, \text{SO}(3)) \\ &= \min(\mu, \mu_c) \|F - \text{polar}(F)\|^2 \\ &= \min(\mu, \mu_c) \|U - \mathbb{1}\|^2. \end{aligned} \quad (2.3)$$

In contrast, for the interesting limit case of defective elastic crystals $\mu_c = 0$, where the Cosserat rotations \bar{R} are viewed as the lattice rotations, the strain energy density is *only convex* with respect to F and does not satisfy (2.3).⁴

The parameter $L_c > 0$ (with dimension of length) introduces an *internal length* which is *characteristic* for the material, e.g., related to the interaction length of the lattices in a defective single crystal. The internal length $L_c > 0$ is responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. We assume throughout that $\alpha_4, \alpha_5, \alpha_6 > 0$, $\alpha_7 \geq 0$. This implies the *coercivity of curvature*

$$\exists c^+ > 0 \forall \mathfrak{K} \in \mathfrak{T}(3) : \quad W_{\text{curv}}(\mathfrak{K}) \geq c^+ \|\mathfrak{K}\|^{1+p+q},$$

³ Observe that $\bar{R}^T \nabla(\bar{R} \cdot e_i) \neq \bar{R}^T \partial_{x_i} \bar{R} \in \mathfrak{so}(3)$.

⁴ The condition $F \in \text{GL}^+(3, \mathbb{R})$ is necessary, otherwise $\|F - \text{polar}(F)\|^2 = \text{dist}^2(F, \text{O}(3, \mathbb{R})) < \text{dist}^2(F, \text{SO}(3))$, as can be easily seen for the reflection $F = \text{diag}(1, -1, 1)$.

which is a basic ingredient of the mathematical analysis. Note that every subsequent result can also be obtained for a true lattice incompatibility measure W_{defect} replacing W_{curv} with

$$W_{\text{defect}} = \mu L_c^{1+p+q} \|\bar{R}^T \text{Curl } \bar{R}\|^{1+p+q}$$

(see [53]). W_{defect} accounts for interfacial energy between adjacent regions of lattice orientations.

The nonstandard boundary condition of *strong consistent coupling* ensures that no unwanted nonclassical, polar effects may occur at the Dirichlet boundary Γ . It implies that the micropolar stretch satisfies $\bar{U}|_{\Gamma} \in \text{Sym}$ and the second Piola–Kirchhoff stress tensor satisfies $S_2 := F^{-1} D_F W_{\text{mp}}(\bar{U}) \in \text{Sym}$ on Γ as in the classical, nonpolar case. We refer to the weaker boundary condition $\bar{U}|_{\Gamma} \in \text{Sym}$ as *weak consistent coupling*.

It is of prime importance to realize that a linearization of this Cosserat bulk model in the case of defective elastic crystals $\mu_c = 0$ for small displacement and small microrotations completely decouples the two fields of deformation φ and Cosserat lattice rotations \bar{R} and leads to the classical linear elasticity problem for the deformation.⁵ For more details on the modelling of the three-dimensional Cosserat model we refer the reader to [48].

2.2 Mathematical results for the Cosserat bulk problem

We recall the results obtained for the case without external loads [47, 44]:

THEOREM 2.1 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c > 0$) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume that the boundary data satisfy $g_d \in H^1(\Omega, \mathbb{R}^3)$ and $\bar{R}_d \in W^{1,1+p}(\Omega, \text{SO}(3))$. Then (2.1) with $\mu_c > 0, \alpha_4 \geq 0, p \geq 1, q \geq 0$ and either free or rigid prescription for \bar{R} on Γ admits at least one minimizing solution pair $(\varphi, \bar{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p}(\Omega, \text{SO}(3))$. \square

In the case of defective elastic crystals a more stringent control of the lattice incompatibility (higher curvature exponent) is necessary. Using the extended Korn inequality [43, 54], the following has been shown in [47]:

THEOREM 2.2 (Existence for 3D-finite-strain elastic Cosserat model with $\mu_c = 0$) Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and assume for the boundary data $g_d \in H^1(\Omega, \mathbb{R}^3)$ and $\bar{R}_d \in W^{1,1+p+q}(\Omega, \text{SO}(3))$. Then (2.1) with $\mu_c = 0, \alpha_4 > 0, p \geq 1, q > 1$ and either free or rigid prescription for \bar{R} on Γ admits at least one minimizing solution pair $(\varphi, \bar{R}) \in H^1(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \text{SO}(3))$. \square

3. Dimensional reduction of the Cosserat bulk model

3.1 The three-dimensional Cosserat problem on a thin domain

The basic task of any shell theory is a consistent reduction of some presumably “exact” 3D-theory to 2D. The three-dimensional problem (2.1) defined on the physical space \mathbb{E}^3 will now be adapted to a

⁵ Thinking in the context of an infinitesimal-displacement Cosserat theory one might believe that $\mu_c > 0$ is necessary also for a “true” finite-strain Cosserat theory.

shell-like theory. Let us therefore assume that the problem is already transformed in *nondimensional* form. This means we are given a three-dimensional (nondimensional) *thin domain* $\Omega_h \subset \mathbb{R}^3$

$$\Omega_h := \omega \times [-h/2, h/2], \quad \omega \subset \mathbb{R}^2,$$

with *transverse boundary* $\partial\Omega_h^{\text{trans}} = \omega \times \{-h/2, h/2\}$ and *lateral boundary* $\partial\Omega_h^{\text{lat}} = \partial\omega \times [-h/2, h/2]$, where ω is a bounded open domain⁶ in \mathbb{R}^2 with smooth boundary $\partial\omega$ and $h > 0$ is the *nondimensional relative characteristic thickness (aspect ratio)*, $h \ll 1$. Moreover, assume we are given a deformation φ and microrotation \bar{R} ,

$$\varphi : \Omega_h \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \bar{R} : \Omega_h \subset \mathbb{R}^3 \rightarrow \text{SO}(3),$$

solving the following two-field minimization problem on the thin domain Ω_h :

$$\begin{aligned} I(\varphi, \bar{R}) &= \int_{\Omega_h} [W_{\text{mp}}(\bar{U}) + W_{\text{curv}}(\mathfrak{K}) - \langle f, \varphi \rangle] \, dV \\ &\quad - \int_{\partial\Omega_h^{\text{trans}} \cup \{\gamma_s \times [-h/2, h/2]\}} \langle N, \varphi \rangle \, dS \mapsto \min \text{ w.r.t. } (\varphi, \bar{R}), \\ \bar{U} &= \bar{R}^T F, \quad \varphi|_{\Gamma_0^h} = g_d(x, y, z), \quad \Gamma_0^h = \gamma_0 \times [-h/2, h/2], \quad \gamma_0 \subset \partial\omega, \quad \gamma_s \cap \gamma_0 = \emptyset, \\ \bar{U}|_{\Gamma_0^h} &= \bar{R}^T \nabla\varphi|_{\Gamma_0^h} \in \text{Sym}(3), \quad \text{weak consistent coupling boundary condition or} \\ \bar{R} &: \text{ free on } \Gamma_0^h, \text{ alternative Neumann-type boundary condition,} \\ W_{\text{mp}}(\bar{U}) &= \mu \|\text{sym}(\bar{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\bar{U})\|^2 + \frac{\lambda}{2} \text{tr}[\text{sym}(\bar{U} - \mathbb{1})]^2, \\ W_{\text{curv}}(\mathfrak{K}) &= \mu \frac{\widehat{L}_c^{1+p}}{12} (1 + \alpha_4 \widehat{L}_c^q \|\mathfrak{K}\|^q) (\alpha_5 \|\text{sym } \mathfrak{K}\|^2 + \alpha_6 \|\text{skew } \mathfrak{K}\|^2 + \alpha_7 \text{tr}[\mathfrak{K}]^2)^{(1+p)/2}, \\ \mathfrak{K} &= \bar{R}^T D_x \bar{R} = (\bar{R}^T \nabla(\bar{R}.e_1), \bar{R}^T \nabla(\bar{R}.e_2), \bar{R}^T \nabla(\bar{R}.e_3)), \end{aligned}$$

where $\widehat{L}_c = L_c/L$ is a nondimensional ratio. Without loss of mathematical generality we assume that $M, M_c \equiv 0$ in (2.2), i.e. that no external volume or surface couples are present in the bulk problem. We want to find a reasonable approximation (φ_s, \bar{R}_s) of (φ, \bar{R}) involving only two-dimensional quantities.

3.2 Transformation on a fixed domain

In order to apply standard techniques of Γ -convergence, we transform the problem onto a *fixed domain* Ω_1 , independent of the aspect ratio $h > 0$. Define therefore

$$\Omega_1 = \omega \times [-1/2, 1/2] \subset \mathbb{R}^3, \quad \omega \subset \mathbb{R}^2.$$

The scaling transformation

$$\begin{aligned} \zeta : \eta \in \Omega_1 \subset \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & \zeta(\eta_1, \eta_2, \eta_3) &:= (\eta_1, \eta_2, h \cdot \eta_3), \\ \zeta^{-1} : \xi \in \Omega_h \subset \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & \zeta^{-1}(\xi_1, \xi_2, \xi_3) &:= (\xi_1, \xi_2, \xi_3/h), \end{aligned}$$

⁶ For definiteness, one can think of $\omega = [0, 1] \times [0, 1]$.

maps Ω_1 into Ω_h and $\zeta(\Omega_1) = \Omega_h$. We consider the correspondingly scaled function (subsequently, scaled functions defined on Ω_1 will be indicated with a superscript \sharp) $\varphi^\sharp : \Omega_1 \rightarrow \mathbb{R}^3$, defined by

$$\begin{aligned} \varphi(\xi_1, \xi_2, \xi_3) &= \varphi^\sharp(\zeta^{-1}(\xi_1, \xi_2, \xi_3)) \quad \forall \xi \in \Omega_h; \quad \varphi^\sharp(\eta) = \varphi(\zeta(\eta)) \quad \forall \eta \in \Omega_1, \\ \nabla\varphi(\xi_1, \xi_2, \xi_3) &= \left(\partial_{\eta_1}\varphi^\sharp(\eta_1, \eta_2, \eta_3) \Big| \partial_{\eta_2}\varphi^\sharp(\eta_1, \eta_2, \eta_3) \Big| \frac{1}{h}\partial_{\eta_3}\varphi^\sharp(\eta_1, \eta_2, \eta_3) \right) =: \nabla_\eta^h\varphi^\sharp = F_h^\sharp. \end{aligned}$$

Similarly, we define a scaled rotation tensor $\bar{R}^\sharp : \Omega_1 \subset \mathbb{R}^3 \rightarrow \text{SO}(3)$ by

$$\begin{aligned} \bar{R}(\xi_1, \xi_2, \xi_3) &= \bar{R}^\sharp(\zeta^{-1}(\xi_1, \xi_2, \xi_3)) \quad \forall \xi \in \Omega_h; \quad \bar{R}^\sharp(\eta) = \bar{R}(\zeta(\eta)) \quad \forall \eta \in \Omega_1, \\ \nabla_\xi[\bar{R}(\xi_1, \xi_2, \xi_3).e_i] &= \left(\partial_{\eta_1}[\bar{R}^\sharp(\eta).e_i] \Big| \partial_{\eta_2}[\bar{R}^\sharp(\eta).e_i] \Big| \frac{1}{h}\partial_{\eta_3}[\bar{R}^\sharp(\eta).e_i] \right) \\ &=: \nabla_\eta^h[\bar{R}^\sharp(\eta).e_i] \in \mathbb{M}^{3 \times 3}, \end{aligned} \tag{3.1}$$

$$D_\eta^h \bar{R}_h^{3d,\sharp}(\eta) := (\nabla_\eta^h[\bar{R}^\sharp(\eta).e_1], \nabla_\eta^h[\bar{R}^\sharp(\eta).e_2], \nabla_\eta^h[\bar{R}^\sharp(\eta).e_3]) \in \mathfrak{T}(3).$$

This allows us to define scaled nonsymmetric stretches $\bar{U}_h^\sharp = \bar{R}^{\sharp,T} F_h^\sharp$ and the scaled third order curvature tensor $\mathfrak{K}_h^\sharp : \Omega_1 \rightarrow \mathfrak{T}(3)$,

$$\begin{aligned} \mathfrak{K}_h^\sharp(\eta) &= \left(\bar{R}^{\sharp,T}(\eta) \left(\partial_{\eta_1}[\bar{R}^\sharp(\eta).e_1] \Big| \partial_{\eta_2}[\bar{R}^\sharp(\eta).e_1] \Big| \frac{1}{h}\partial_{\eta_3}[\bar{R}^\sharp(\eta).e_1] \right), \right. \\ &\quad \bar{R}^{\sharp,T}(\eta) \left(\partial_{\eta_1}[\bar{R}^\sharp(\eta).e_2] \Big| \partial_{\eta_2}[\bar{R}^\sharp(\eta).e_2] \Big| \frac{1}{h}\partial_{\eta_3}[\bar{R}^\sharp(\eta).e_2] \right), \\ &\quad \left. \bar{R}^{\sharp,T}(\eta) \left(\partial_{\eta_1}[\bar{R}^\sharp(\eta).e_3] \Big| \partial_{\eta_2}[\bar{R}^\sharp(\eta).e_3] \Big| \frac{1}{h}\partial_{\eta_3}[\bar{R}^\sharp(\eta).e_3] \right) \right) \\ &= (\bar{R}^{\sharp,T}(\eta)\nabla_\eta^h[\bar{R}^\sharp(\eta).e_1], \bar{R}^{\sharp,T}(\eta)\nabla_\eta^h[\bar{R}^\sharp(\eta).e_2], \bar{R}^{\sharp,T}(\eta)\nabla_\eta^h[\bar{R}^\sharp(\eta).e_3]) \\ &= \bar{R}^{\sharp,T} D_\eta^h \bar{R}^\sharp(\eta). \end{aligned}$$

Moreover, we define similarly scaled functions by setting

$$f^\sharp(\eta) := f(\zeta(\eta)), \quad g_d^\sharp(\eta) = g_d(\zeta(\eta)), \quad N^\sharp(\eta) := N(\zeta(\eta)).$$

In terms of the introduced scaled deformations and rotations

$$\varphi^\sharp : \Omega_1 \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \bar{R}^\sharp : \Omega_1 \subset \mathbb{R}^3 \rightarrow \text{SO}(3),$$

the scaled problem solves the following two-field minimization problem on the fixed domain Ω_1 :

$$\begin{aligned} I^\sharp(\varphi^\sharp, \nabla_\eta^h\varphi^\sharp, \bar{R}^\sharp, D_\eta^h\bar{R}^\sharp) &= \int_{\eta \in \Omega_1} [W_{\text{mp}}(\bar{U}_h^\sharp) + W_{\text{curv}}(\mathfrak{K}_h^\sharp) - \langle f^\sharp, \varphi^\sharp \rangle] \det[\nabla\zeta(\eta)] \, dV_\eta \\ &\quad - \int_{\partial\Omega_1^{\text{trans}} \cup \{\gamma_s \times [-1/2, 1/2]\}} \langle N^\sharp, \varphi^\sharp \rangle \| \text{Cof } \nabla\zeta(\eta).e_3 \| \, dS_\eta, \\ &= h \int_{\eta \in \Omega_1} [W_{\text{mp}}(\bar{U}_h^\sharp) + W_{\text{curv}}(\mathfrak{K}_h^\sharp) - \langle f^\sharp, \varphi^\sharp \rangle] \, dV_\eta \\ &\quad - \int_{\partial\Omega_1^{\text{trans}}} \langle N^\sharp, \varphi^\sharp \rangle \, dS_\eta \\ &\quad - \int_{\gamma_s \times [-1/2, 1/2]} \langle N^\sharp, \varphi^\sharp \rangle h \, dS_\eta \mapsto \min \text{ w.r.t. } (\varphi^\sharp, \bar{R}^\sharp). \end{aligned}$$

3.3 The rescaled variational Cosserat bulk problem

Since the energy $h^{-1}I^\sharp$ would not be finite for $h \rightarrow 0$ if tractions N^\sharp on the transverse boundary were present, the investigations are in principle restricted to the case of $N^\sharp = 0$ on $\partial\Omega_1^{\text{trans}}$.⁷ For conciseness we investigate the following simplified and rescaled $(N^\sharp, f^\sharp = 0, g_d(\xi_1, \xi_2, \xi_3) := g_d(\xi_1, \xi_2))$ two-field minimization problem on Ω_1 with respect to Γ -convergence (without the factor $h > 0$ now), i.e. we are interested in the limiting behaviour of the energy per unit aspect ratio h :

$$\begin{aligned} I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp) &= \int_{\eta \in \Omega_1} [W_{\text{mp}}(\bar{U}_h^\sharp) + W_{\text{curv}}(\mathfrak{R}_h^\sharp)] dV_\eta \mapsto \min \text{ w.r.t. } (\varphi^\sharp, \bar{R}^\sharp), \\ \bar{U}_h^\sharp &= \bar{R}^{\sharp, T} F_h^\sharp, \quad \varphi_{\Gamma_0^1}^\sharp(\eta) = g_d^\sharp(\eta) = g_d(\zeta(\eta)) = g_d(\eta_1, \eta_2, h \cdot \eta_3) = g_d(\eta_1, \eta_2, 0), \\ \Gamma_0^1 &= \gamma_0 \times [-1/2, 1/2], \quad \gamma_0 \subset \partial\omega, \\ \bar{R}^\sharp &: \text{ free on } \Gamma_0^1, \text{ Neumann-type boundary condition,} \\ \mathfrak{R}_h^\sharp &= \bar{R}^{\sharp, T} D_\eta^h \bar{R}^\sharp(\eta). \end{aligned} \tag{3.2}$$

Here we assume that the boundary condition g_d is already independent of the transverse variable. We restrict attention to the weakest response, the *Neumann boundary conditions* on the Cosserat rotations \bar{R}^\sharp in line with the difficulty to experimentally influence the lattice rotations at the Dirichlet boundary.⁸ Moreover, we assume

$$p \geq 1, \quad q > 1,$$

so that both cases $\mu_c > 0$ and $\mu_c = 0$ can be considered simultaneously. External loads of various sorts can be treated by Remark 4.5.

Within the rescaled formulation (3.2) we want to investigate the possible limit behaviour for $h \rightarrow 0$ and *fixed relative internal length* $\widehat{L}_c > 0$. This amounts to considering sequences of plates with constant physical thickness d , increasing in plane-length L and accordingly increasing curvature strength of the microstructure, similar to letting $\kappa = \text{const}$ in [6].

3.4 On the choice of the scaling

The Γ -limit, if it exists, is unique. The only choice which influences the final form of the Γ -limit is given by the initial scaling assumptions made on the unknowns, in order to relate them to the fixed domain Ω_1 , and the assumption on the scaling of the energies, here the membrane scaling $h^{-1}I^\sharp < \infty$. Our scaling ansatz is consistent with the one proposed in [38, 29], but not with the one taken in [11], which scales transverse components of the displacement differently in order to extract more information from the Γ -limit. Since we deal with a “two-field” model it is not possible to scale the fields differently. The general inadequacy of the scaling of linear elasticity adopted in [11] in a geometrically exact context has been pointed out in [24]. The motivation for our choice is given by the apparent consistency of the results with formal developments and its linearization stability. Here we see that the energy scaling assumptions also introduce an ambiguity in the development.

⁷ The thin plate limit $h \rightarrow 0$ obviously cannot support nonvanishing transverse surface loads.

⁸ We could as well treat the rigid case, i.e. $\bar{R}_{\Gamma_0^1}^\sharp = \bar{R}_d$. The case of weak consistent coupling would need additional provisions, the three-dimensional existence result already needs additional control in order to define the then necessary boundary terms.

For example, starting from classical nonlinear elasticity, considering the present scaling for the unknowns and assuming $h^{-5}I^\sharp < \infty$, a nonlinear von Kármán plate can be rigorously justified by Γ -convergence [29]. These results have been extended to a hierarchy of models in [31].

4. Some facts on Γ -convergence

Let us briefly recapitulate the notions involved by using Γ -convergence. For a detailed treatment we refer to [18, 8]. We start by defining the lower and upper Γ -limits. In the following, X will always denote a metric space such that sequential compactness and compactness coincide. Moreover, we set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We consider a sequence of energy functionals $I_{h_j} : X \rightarrow \overline{\mathbb{R}}, h_j \rightarrow 0$.

DEFINITION 4.1 (Lower and upper Γ -limit) Let X be a metric space and let $I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ with $h_j \rightarrow 0$ be a sequence of functionals. For $x \in X$ we define

$$\begin{aligned} \Gamma\text{-}\liminf_{h_j} I_{h_j} : X \rightarrow \overline{\mathbb{R}}, & \quad \Gamma\text{-}\liminf_{h_j} I_{h_j}(x) := \inf\{\liminf_{h_j} I_{h_j}(x_{h_j}) \mid x_{h_j} \rightarrow x\}, \\ \Gamma\text{-}\limsup_{h_j} I_{h_j} : X \rightarrow \overline{\mathbb{R}}, & \quad \Gamma\text{-}\limsup_{h_j} I_{h_j}(x) := \inf\{\limsup_{h_j} I_{h_j}(x_{h_j}) \mid x_{h_j} \rightarrow x\}. \end{aligned}$$

It is clear that $\Gamma\text{-}\liminf_{h_j} I_{h_j}$ and $\Gamma\text{-}\limsup_{h_j} I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ always exist and are uniquely determined.

DEFINITION 4.2 Let X be a metric space. We say that a sequence of functionals $I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ Γ -converges in X to the limit functional $I_0 : X \rightarrow \overline{\mathbb{R}}$ if for all $x \in X$ we have

$$\begin{aligned} \forall x \in X \ \forall x_{h_j} \rightarrow x : \quad I_0(x) &\leq \liminf_{h_j \rightarrow 0} I_{h_j}(x_{h_j}) \quad (\text{lim inf-inequality}), \\ \forall x \in X \ \exists x_{h_i} \rightarrow x : \quad I_0(x) &\geq \limsup_{h_i \rightarrow 0} I_{h_i}(x_{h_i}) \quad (\text{recovery sequence}). \end{aligned}$$

COROLLARY 4.3 Let X be a metric space. The sequence of functionals $I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ Γ -converges in X to $I_0 : X \rightarrow \overline{\mathbb{R}}$ if and only if

$$\Gamma\text{-}\liminf_{h_j} I_{h_j} = \Gamma\text{-}\limsup_{h_j} I_{h_j} = I_0. \quad \square$$

REMARK 4.4 (Lower semicontinuity of the Γ -limit) The lower and upper Γ -limits are always lower semicontinuous, hence the Γ -limit is a lower semicontinuous functional. Moreover, if the Γ -limit exists, it is unique.

REMARK 4.5 (Stability under continuous perturbations) Assume that $I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ Γ -converges in X to $I_0 : X \rightarrow \overline{\mathbb{R}}$ and let $\Pi : X \rightarrow \mathbb{R}$, independent of h_j , be continuous. Then $I_{h_j} + \Pi$ is Γ -convergent and

$$(\Gamma\text{-}\lim_{h_j} [I_{h_j} + \Pi])(x) = (\Gamma\text{-}\lim_{h_j} I_{h_j})(x) + \Pi(x) = I_0(x) + \Pi(x)$$

(see [18, Prop. 6.21]). Recall that when the functional Π , independent of h_j , is not continuous it can influence whether or not Γ -convergence holds [18, Ex. 6.23].

Let us also recapitulate the important *equi-coerciveness* property. First we recall *coerciveness* of a functional.⁹

DEFINITION 4.6 The functional $I : X \rightarrow \overline{\mathbb{R}}$ is *coercive* with respect to X if for each fixed $C > 0$ the closure of the set $\{x \in X \mid I(x) \leq C\}$ is compact in X , i.e. I has compact sublevels.

Following [18, p. 70] we introduce

DEFINITION 4.7 The sequence of functionals $I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ is *equi-coercive* if for each $C > 0$ there exists a compact set $K_C \subset X$ such that $\{x \in X \mid I_{h_j}(x) \leq C\} \subset K_C$, *independent* of $h_j > 0$.

Hence, if we know that I_{h_j} is equi-coercive over X and that $I_{h_j}(\varphi_j) \leq C$ along a sequence $\varphi_j \in X$, then we can extract a subsequence φ_{j_k} converging in the topology of X to some limit element $\varphi \in X$.

THEOREM 4.8 (Characterization of equi-coerciveness, [18, Prop. 7.7]) The sequence of functionals $I_{h_j} : X \rightarrow \mathbb{R}$ is equi-coercive if and only if there exists a lower semicontinuous coercive function $\Psi : X \rightarrow \overline{\mathbb{R}}$ such that $I_{h_j} \geq \Psi$ on X for every $h_j > 0$. □

The following theorem concerns the convergence of the minimum values of an equi-coercive sequence of functions.

THEOREM 4.9 (Coerciveness of the Γ -limit, [18, Th. 7.8]) Suppose that the sequence of functionals $I_{h_j} : X \rightarrow \mathbb{R}$ is equi-coercive. Then the upper and lower Γ -limits are both coercive and

$$\min_{x \in X} (\Gamma\text{-lim inf}_{h_j} I_{h_j})(x) = \liminf_{h_j} \inf_{x \in X} I_{h_j}(x).$$

If, in addition, the sequence of integral functionals $I_{h_j} : X \rightarrow \overline{\mathbb{R}}$ Γ -converges to a functional $I_0 : X \rightarrow \overline{\mathbb{R}}$, then I_0 itself is coercive and

$$\min_{x \in X} I_0(x) = \liminf_{h_j} \inf_{x \in X} I_{h_j}(x).$$

Note that equi-coercivity is an additional feature in the development of Γ -convergence arguments, which simplifies proofs considerably through compactness arguments. As far as Γ -convergence is concerned, it may be useful to recall [8, p. 19] that *minimizers of the Γ -limit variational problem may not be limits of minimizers, so that Γ -convergence can be interpreted as a choice criterion*. In addition, the Γ -limit of a constant sequence of functionals J , which is not lower semicontinuous, does not coincide with the constant functional J , instead one has $(\Gamma\text{-lim } J)(x) < J(x)$. In this case, $(\Gamma\text{-lim } J)(x) = QJ(x)$, where QJ is the lower semicontinuous envelope of J . In the case of non-lower semicontinuous functionals, the Γ -limit therefore introduces a different physical setting. In this paper we deal with lower semicontinuous functionals.

⁹ Typically, coerciveness is given for $X = L^p(\Omega, \mathbb{R}^3)$, $1 < p < \infty$, with Ω a bounded domain, with smooth boundary and

$$I(\varphi) = \begin{cases} \int_{\Omega} W(\nabla\varphi) \, dV & \text{if } \varphi \in W^{1,p}(\Omega, \mathbb{R}^3), \varphi|_{\partial\Omega} = 0, \\ +\infty & \text{else,} \end{cases}$$

with the *local coercivity assumption* $W(F) \geq c_1^+ \|F\|^p - c_2^+$. Coerciveness follows by Poincaré’s inequality and Rellich’s compact embedding $W^{1,p}(\Omega, \mathbb{R}^3) \subset L^p(\Omega, \mathbb{R}^3)$. Recall that linear elasticity does not satisfy a local coercivity condition. This is the cause for some technical problems of the theory.

5. The “two-field” Cosserat Γ -limit

5.1 *The spaces and admissible sets*

Now let us proceed to the investigation of the Γ -limit for the rescaled problem (3.2). We do not use $I_{h_j}^\sharp$ directly in our investigation of Γ -convergence, since this would imply working with the weak topology of $H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \text{SO}(3))$, which does not give rise to a metric space. Instead, we define suitable “bulk” spaces X, X' and suitable “two-dimensional” spaces X_ω, X'_ω . First, for $p \geq 1, q > 1$ we define the number $r > 1$ by

$$\frac{1}{1+p+q} + \frac{1}{r} = \frac{1}{2}, \quad \text{i.e.} \quad r = \frac{2(1+p+q)}{(1+p+q)-2}, \tag{5.1}$$

such that $L^{1+p+q} \cdot L^r \subset L^2$. Note that for $1+p+q > 3$ we have $r < 6$, which implies the compact embedding $H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^r(\Omega_1, \mathbb{R}^3)$. Now define the spaces

$$\begin{aligned} X &:= L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \text{SO}(3)), \\ X' &:= H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \text{SO}(3)), \\ X_\omega &:= L^r(\omega, \mathbb{R}^3) \times L^{1+p+q}(\omega, \text{SO}(3)), \\ X'_\omega &:= H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \text{SO}(3)), \end{aligned}$$

and the admissible sets

$$\begin{aligned} \mathcal{A}' &:= \{(\varphi, \bar{R}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \text{SO}(3)) \mid \varphi|_{\Gamma_0^1}(\eta) = g_d^\sharp(\eta)\}, \\ \mathcal{A}'_\omega &:= \{(\varphi, \bar{R}) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \text{SO}(3)) \mid \varphi|_{\gamma_0}(\eta_1, \eta_2) = g_d^\sharp(\eta_1, \eta_2, 0)\}, \\ \mathcal{A}'_{\Omega_1, \omega} &:= \{(\varphi, \bar{R}) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \text{SO}(3)) \mid \varphi|_{\Gamma_0^1}(\eta) = g_d^\sharp(\eta)\}. \end{aligned}$$

We note the compact embedding $X' \subset X$ and the natural inclusions $X_\omega \subset X$ and $X'_\omega \subset X'$. Now we extend the rescaled energies to the space X through redefining

$$I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp) = \begin{cases} I_h^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp) & \text{if } (\varphi^\sharp, \bar{R}^\sharp) \in \mathcal{A}', \\ +\infty & \text{else in } X, \end{cases} \tag{5.2}$$

by abuse of notation. This is a classical trick used in applications of Γ -convergence. It has the virtue of incorporating the boundary conditions already in the energy functional. In the following, Γ -convergence results will be shown with respect to the encompassing metric space X .¹⁰

DEFINITION 5.1 (The transverse averaging operator) For $\varphi \in L^2(\Omega_1, \mathbb{R}^3)$ define the averaging operator over the transverse (thickness) variable η_3 by

$$\text{Av} : L^2(\Omega_1, \mathbb{R}^3) \rightarrow L^2(\omega, \mathbb{R}^3), \quad \text{Av}.\varphi(\eta_1, \eta_2) := \int_{-1/2}^{1/2} \varphi(\eta_1, \eta_2, \eta_3) \, d\eta_3.$$

¹⁰ Of course, X, X' as such are not vector spaces, since one cannot add two rotations. Nevertheless, $L^r(\Omega_1, \text{SO}(3)) \subset L^r(\Omega_1, \mathbb{M}^{3 \times 3})$ and the latter is a Banach space.

It is clear that averaging with respect to the transverse variable η_3 commutes with differentiation with respect to the planar variables η_1, η_2 , i.e.

$$[\text{Av.}\nabla_{(\eta_1, \eta_2)}\varphi(\eta_1, \eta_2, \eta_3)](\eta_1, \eta_2) = \nabla_{(\eta_1, \eta_2)}[\text{Av.}\varphi(\eta_1, \eta_2, \eta_3)](\eta_1, \eta_2),$$

for suitable regular functions φ . Note in passing that for a convex function $f : \mathbb{M}^{3 \times 2} \rightarrow \mathbb{R}$ Jensen's inequality implies

$$\begin{aligned} \int_{\omega} f(\nabla_{(\eta_1, \eta_2)}[\text{Av.}\varphi](\eta_1, \eta_2)) \, d\omega &= \int_{\omega} f([\text{Av.}\nabla_{(\eta_1, \eta_2)}\varphi](\eta_1, \eta_2)) \, d\omega \\ &\leq \int_{\omega} \int_{-1/2}^{1/2} f(\nabla_{(\eta_1, \eta_2)}\varphi(\eta_1, \eta_2, \eta_3)) \, d\eta_3 \, d\omega \\ &= \int_{\Omega_1} f(\nabla_{(\eta_1, \eta_2)}\varphi(\eta_1, \eta_2, \eta_3)) \, dV_{\eta}. \end{aligned} \tag{5.3}$$

5.2 The Γ -limit variational “membrane” problem

Our first result is

THEOREM 5.2 (Γ -limit for $\mu_c > 0$) For strictly positive Cosserat couple modulus $\mu_c > 0$ the Γ -limit for problem (3.2) in the setting of (5.2) is given by the limit energy functional $I_0^{\sharp} : X \rightarrow \overline{\mathbb{R}}$,

$$I_0^{\sharp}(\varphi, \overline{R}) := \begin{cases} \int_{\Omega_1} [W_{\text{mp}}^{\text{hom}}(\nabla \text{Av.}\varphi, \overline{R}) + W_{\text{curv}}^{\text{hom}}(\mathcal{R}_s)] \, d\omega - \Pi(\text{Av.}\varphi, \overline{R}_3) & \text{if } (\varphi, \overline{R}) \in \mathcal{A}'_{\Omega_1, \omega}, \\ +\infty & \text{else in } X, \end{cases}$$

with $W_{\text{mp}}^{\text{hom}}$ and $W_{\text{curv}}^{\text{hom}}$ defined below.

The proof of this statement will be given in Section 6.

If we identify the thickness averaged deformation $\text{Av.}\varphi$ with the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, this problem determines in fact a purely two-dimensional minimization problem for the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and the microrotation of the plate (shell) $\overline{R} : \omega \subset \mathbb{R}^2 \rightarrow \text{SO}(3)$ on ω :

$$I_0^{\sharp}(m, \overline{R}) = \int_{\omega} [W_{\text{mp}}^{\text{hom}}(\nabla m, \overline{R}) + W_{\text{curv}}^{\text{hom}}(\mathcal{R}_s)] \, d\omega - \Pi(m, \overline{R}_3) \mapsto \min \text{ w.r.t. } (m, \overline{R}), \tag{5.4}$$

and the boundary conditions for the midsurface deformation m on the Dirichlet part of the lateral boundary $\gamma_0 \subset \partial\omega$,

$$m|_{\gamma_0} = g_d(x, y, 0) = \text{Av.}g_d(x, y, 0), \quad \text{simply supported (fixed, welded).}$$

The boundary conditions for the microrotations \overline{R} are automatically determined in the variational process. The dimensionally homogenized local density is^{11 12}

¹¹ $\|\text{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\|^2 = (\langle \overline{R}_1, m_y \rangle - \langle \overline{R}_2, m_x \rangle)^2$. Note that $\|\text{skew}((\overline{R}_1|\overline{R}_2)^T \nabla m)\| = 0$ does not imply that $\overline{R}_3 = \overline{n}_m$.

¹² In the following, “intrinsic” refers to classical surface geometry, where intrinsic quantities are those which depend only on the first fundamental form $I_m = \nabla m^T \nabla m \in \mathbb{M}^{2 \times 2}$ of the surface. Then “intrinsic” in our terminology are terms which reduce to such a dependence in the continuum limit $\overline{R} = \text{polar}(\nabla m | \overline{n})$. For example $(\overline{R}_1|\overline{R}_2)^T \nabla m = \sqrt{\nabla m^T \nabla m}$, in this case.

$$\begin{aligned}
 W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) := & \underbrace{\mu \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2}_{\text{“intrinsic” shear-stretch energy}} + \underbrace{\mu_c \|\text{skew}((\bar{R}_1 | \bar{R}_2)^T \nabla m)\|^2}_{\text{“intrinsic” first order drill energy}} \\
 & + 2\mu \underbrace{\frac{\mu_c}{\mu + \mu_c} (\langle \bar{R}_3, m_x \rangle^2 + \langle \bar{R}_3, m_y \rangle^2)}_{\text{homogenized transverse shear energy}} + \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\text{tr}[\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2}_{\text{homogenized elongational stretch energy}}.
 \end{aligned}$$

The dimensionally homogenized curvature density is given by

$$\begin{aligned}
 W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_s) := & \inf_{A \in \mathfrak{so}(3)} W_{\text{curv}}^*(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}, A), \\
 \mathfrak{K}_s = & (\bar{R}^T(\nabla(\bar{R}.e_1)|0), \bar{R}^T(\nabla(\bar{R}.e_2)|0), \bar{R}^T(\nabla(\bar{R}.e_3)|0)) = \bar{R}^T(x, y)D_x \bar{R}(x, y), \\
 \mathfrak{K}_s = & (\mathfrak{K}_s^1, \mathfrak{K}_s^2, \mathfrak{K}_s^3) \in \mathfrak{T}(3), \quad \text{the reduced third order curvature tensor,}
 \end{aligned}$$

where W_{curv}^* is an equivalent representation of the bulk curvature energy in terms of skew-symmetric arguments

$$\begin{aligned}
 W_{\text{curv}}(\mathfrak{K}) &= W_{\text{curv}}^*(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}, \bar{R}^T \partial_{\eta_3} \bar{R}), \\
 W_{\text{curv}}^* &: \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3) \rightarrow \mathbb{R}^+,
 \end{aligned}$$

with $\bar{R}^T \partial_{\eta_i} \bar{R} \in \mathfrak{so}(3)$ since $\partial_{\eta_i}[\bar{R}^T \bar{R}] = \partial_{\eta_i} \mathbb{1} = 0$. We note that W_{curv}^* remains a convex function in its argument as is $W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_s)$. Moreover, $W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_s) = W_{\text{curv}}(\mathfrak{K}_s)$ for $W_{\text{curv}}(\mathfrak{K}) = \widehat{W}(\|\mathfrak{K}\|)$.

In (5.4), Π denotes a general external loading functional, continuous in the topology of X (cf. Remark 4.5). It is clear that the limit functional I_0^\sharp is weakly lower semicontinuous in the topology of $X' = H^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega, \text{SO}(3))$ by simple convexity arguments. We note the twofold appearance of the *harmonic mean* \mathcal{H} ,¹³

$$\frac{1}{2} \mathcal{H}\left(\mu, \frac{\lambda}{2}\right) = \frac{\mu\lambda}{2\mu + \lambda}, \quad \mathcal{H}(\mu, \mu_c) = 2\mu \frac{\mu_c}{\mu + \mu_c}.$$

An advantage of this formulation is that the dimensionally homogenized formulation remains *frame-indifferent*. Note that the limit functional I_0^\sharp is consistent with the following *plane stress* requirement (cf. (6.2)):

$$\forall \eta_3 \in [-1/2, 1/2]: \quad S_1(\eta_1, \eta_2, \eta_3).e_3 = 0,$$

i.e. a vanishing normal stress over the entire thickness of the plate, while for any given thickness $h > 0$ from 3D-equilibrium one can only infer *zero normal stress at the upper and lower faces*

$$\langle \bar{R}^T(\eta_1, \eta_2, \pm 1/2) S_1(\eta_1, \eta_2, \pm 1/2).e_3, e_3 \rangle = 0.$$

In this sense, the Cosserat “membrane” Γ -limit underestimates the real stresses, notably the transverse shear stresses, as noted in [31, 9.3] with respect to the membrane scaling.

¹³ For $a, b > 0$ the harmonic, arithmetic and geometric means are defined as $\mathcal{H}(a, b) := 2/(1/a + 1/b)$, $\mathcal{A}(a, b) = (a + b)/2$, $\mathcal{G}(a, b) = \sqrt{ab}$, respectively and one has the chain of inequalities $\mathcal{H}(a, b) \leq \mathcal{G}(a, b) \leq \mathcal{A}(a, b)$.

5.3 The defective elastic crystal limit case $\mu_c = 0$

Since it is not possible to establish equi-coercivity for the defective crystal case $\mu_c = 0$, one cannot infer a Γ -limit result for $\mu_c = 0$ as a consequence of the result for $\mu_c > 0$. However, since the energy functional $I_{h_j}^\sharp$ for $\mu_c > 0$ is strictly bigger than the same functional for $\mu_c = 0$, independent of $h_j > 0$, it is easy to see [18, Prop. 6.7] that on X we have the inequalities

$$\Gamma\text{-lim inf } I_{h_j|_{\mu_c=0}}^\sharp \leq \Gamma\text{-lim sup } I_{h_j|_{\mu_c=0}}^\sharp \leq \lim_{\mu_c \rightarrow 0} (\Gamma\text{-lim } I_{h_j|_{\mu_c>0}}^\sharp) =: I_0^{\sharp,0}, \tag{5.5}$$

where

$$I_0^{\sharp,0}(\varphi, \bar{R}) = \begin{cases} \int_{\omega} [W_{\text{mp}}^{\text{hom},0}(\nabla \text{Av.}\varphi, \bar{R}) + W_{\text{curv}}^{\text{hom}}(\mathcal{R}_s)] \, d\omega - \Pi(\text{Av.}\varphi, \bar{R}_3) & \text{if } (\varphi, \bar{R}) \in \mathcal{A}_0^{\text{mem}}, \\ +\infty & \text{else in } X, \end{cases} \tag{5.6}$$

with $\mathcal{A}_0^{\text{mem}}$ defined as

$$\mathcal{A}_0^{\text{mem}} := \{(\varphi, \bar{R}) \in X \mid \text{sym}(\bar{R}_1|\bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \text{Av.}\varphi \in L^2(\Omega_1, \mathbb{M}^{2 \times 2}), \bar{R} \in W^{1,1+p+q}(\omega, \text{SO}(3)), \\ \varphi|_{\Gamma_0^\sharp}(\eta) = g_d^\sharp(\eta) = g_d(\eta_1, \eta_2, 0)\},$$

and the understanding of $\nabla_{(\eta_1, \eta_2)} \text{Av.}\varphi$ as *distributional derivative* for $\varphi \in L^r(\Omega_1, \mathbb{R}^3)$. The corresponding local energy density in terms of $m = \text{Av.}\varphi$ is

$$W_{\text{mp}}^{\text{hom},0}(\nabla m, \bar{R}) := \underbrace{\mu \|\text{sym}(\bar{R}_1|\bar{R}_2)^T \nabla m - \mathbb{1}_2\|^2}_{\text{“intrinsic” shear-stretch energy}} \\ + \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\text{tr}[\text{sym}((\bar{R}_1|\bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2}_{\text{homogenized elongational stretch energy}}. \tag{5.7}$$

Observe that the upper bound $I_0^{\sharp,0}$ for the Γ -lim sup energy functional is *not coercive* with respect to $H^{1,2}(\omega, \mathbb{R}^3)$ due to the now missing transverse shear contribution, while it retains lower semicontinuity. This degeneration remains true for whatever form the Γ -limit for $\mu_c = 0$ has, should it exist. Our main result is

THEOREM 5.3 (Γ -limit for defective elastic crystals $\mu_c = 0$) The Γ -limit of (3.2) for $\mu_c = 0$ in the setting of (5.2) exists and is given by (5.6). \square

The proof of this statement is deferred to Section 7.

The loss of coercivity for $\mu_c = 0$ is primarily a loss of control for the “transverse” components $\langle m_x, \bar{R}_3 \rangle, \langle m_y, \bar{R}_3 \rangle$, while with respect to the remaining “in-plane” components compactness for minimizing sequences, whose midsurface deformations are supposed to be already bounded in $L^r(\omega)$, can be established (by appropriate use of an extended Korn second inequality, cf. (7.5)). That homogenization may lead to a loss of (strict) rank-one convexity has been observed in [33] for nonlinearly elastic composites, whose constituents are strictly rank-one convex.

For linearization consistency, it is easy to show that the linearization for $\mu_c = 0$ of the frame-indifferent Γ -limit $I_0^{\sharp,0}$ with respect to small midsurface displacement $v : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and

small curvature decouples the fields of infinitesimal midsurface displacement and infinitesimal microrotations: after de-scaling we are left with the classical infinitesimal “membrane” plate problem for $v : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$,

$$\int_{\omega} h \left(\mu \|\text{sym } \nabla(v_1, v_2)\|^2 + \frac{\mu\lambda}{2\mu + \lambda} \text{tr}[\text{sym } \nabla(v_1, v_2)]^2 \right) d\omega - \langle f, \langle v, e_1 \rangle \cdot e_1 + \langle v, e_2 \rangle \cdot e_2 \rangle \mapsto \min \text{ w.r.t. } v, \quad (5.8)$$

$$\langle v, e_i \rangle|_{y_0} = \langle u^d(x, y, 0), e_i \rangle, \quad i = 1, 2, \text{ simply supported (horizontal components only),}$$

which leaves the vertical midsurface displacement v_3 undetermined due to the nonresistance of a linear “membrane” plate to vertical deflections. This problem coincides with a linearization¹⁴ of the nonlinear membrane plate problem proposed in [25, par. 4.3], based on purely formal asymptotic methods applied to the St. Venant–Kirchhoff energy. The variational problem (5.8) is as well the Γ -limit of the classical linear elasticity bulk problem (if corresponding scaling assumptions are made, cf. [4, Th. 4.2], [7] or [11, Th. 1.11.2]). The classical linear bulk model in turn can be obtained as linearization for $\mu_c = 0$ of the Cosserat bulk problem. Hence, *only in the defective elastic crystal case $\mu_c = 0$, do linearization and taking the Γ -limit commute with the Γ -limit of classical linear elasticity.*¹⁵

5.4 The formal limit $\mu_c = \infty$

This case is interesting, because the formal Γ -limit for $\mu_c \rightarrow \infty$ exists and still gives rise to an independent field of microrotations \bar{R} , while the Cosserat bulk problem for $\mu_c = \infty$ degenerates into a constraint theory (a so called indeterminate couple-stress model or second gradient model), where the microrotations R coincide necessarily with the continuum rotations polar(F) from the polar decomposition.

The formal Γ -limit problem is: find the deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and the microrotation of the plate (shell) $\bar{R} : \omega \subset \mathbb{R}^2 \rightarrow \text{SO}(3)$ on ω such that for $I_0^{\sharp, \infty} : X \rightarrow \bar{\mathbb{R}}$ in terms of the averaged deformation $m = \text{Av.}\varphi$,

$$I_0^{\sharp, \infty}(m, \bar{R}) \mapsto \min \text{ w.r.t. } (m, \bar{R}), \quad (5.9)$$

with

$$I_0^{\sharp, \infty}(m, \bar{R}) = \begin{cases} \int_{\omega} [W_{\text{mp}}^{\text{hom}, \infty}(\nabla m, \bar{R}) + W_{\text{curv}}^{\text{hom}}(\mathcal{R}_s)] d\omega - \Pi(m, \bar{R}_3) & \text{if } (m, \bar{R}) \in \mathcal{A}'_{\omega, \infty}, \\ +\infty & \text{else in } X, \end{cases}$$

and the admissible set

$$\mathcal{A}'_{\omega, \infty} := \{(m, \bar{R}) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \text{SO}(3)) \mid m|_{y_0}(\eta_1, \eta_2) = g_d^{\sharp}(\eta_1, \eta_2, 0), \langle \bar{R}_1, m_y \rangle = \langle \bar{R}_2, m_x \rangle\}.$$

¹⁴ Expansion of the first fundamental form I_m of the midsurface m with respect to planar initial configuration yields $I_m - \mathbb{1}_2 = \nabla m^T \nabla m - \mathbb{1}_2 \approx \text{sym } \nabla_{(x,y)}(v_1, v_2) + O(\|\nabla v\|^2)$. Hence control on vertical deflections v_3 is lost during linearization.

¹⁵ As is well known [13, p. 464] this is not the case with the membrane Γ -limit found in [38], based on the nonelliptic St. Venant–Kirchhoff energy.

The formal local energy density reads

$$\begin{aligned}
 W_{\text{mp}}^{\text{hom},\infty}(\nabla m, \bar{R}) := & \underbrace{\mu \|\bar{R}_1 | \bar{R}_2\|^T \nabla m - \mathbb{1}_2\|^2}_{\text{“intrinsic” shear-stretch energy}} + \underbrace{2\mu(\langle \bar{R}_3, m_x \rangle^2 + \langle \bar{R}_3, m_y \rangle^2)}_{\text{homogenized transverse shear energy}} \\
 & + \frac{\mu\lambda}{2\mu + \lambda} \underbrace{\text{tr}[\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2}_{\text{homogenized elongational stretch energy}} .
 \end{aligned}$$

Note that $\mu_c = \infty$ rules out in-plane drill rotations [37, 26], the transverse shear energy is doubled, but transverse shear is still possible since \bar{R}_3 need not coincide with the normal on m . In this sense, the resulting homogenized transverse shear modulus excludes what could be called “transverse shear locking” in accordance with the “Poisson thickness locking” which occurs if the correct homogenized volumetric modulus is not taken.¹⁶ In a future contribution we will discuss whether the formal limit (5.9) is the rigorous Γ -limit of the constraint Cosserat bulk problem. Note that in [6] it has been shown that the Γ -limit of a second gradient bulk model gives rise to one independent “Cosserat” director, which here would correspond to \bar{R}_3 .

6. Proof for positive Cosserat couple modulus $\mu_c > 0$

We continue by proving Theorem 5.2, i.e. the claim on the form of the Γ -limit for strictly positive Cosserat couple modulus μ_c . The proof is split into several steps.

6.1 *Equi-coercivity of $I_{h_j}^\sharp$, compactness and dimensional reduction*

THEOREM 6.1 (Equi-coercivity of $I_{h_j}^\sharp$) For positive Cosserat couple modulus $\mu_c > 0$ the sequence of rescaled energy functionals $I_{h_j}^\sharp$ defined in (3.2) is equi-coercive on the space X .

Proof. It is clear that for given $h > 0$ the problem (3.2) admits a minimizing pair $(\varphi_h^\sharp, \bar{R}_h^\sharp) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \text{SO}(3))$ by the obvious scaling transformation of the minimizing solution of the bulk problem for values of $p \geq 1, q > 1$ and for both $\mu_c > 0$ and $\mu_c = 0$.¹⁷ This is especially true for Neumann boundary conditions on the microrotations, since for exact rotations, $\|\bar{R}\| = \sqrt{3}$. This leads to a control of microrotations in $W^{1,1+p+q}(\Omega_1, \text{SO}(3))$ already without specification of Dirichlet boundary data on the microrotations.

Consider now a sequence $h_j \rightarrow 0$ for $j \rightarrow \infty$. By inspection of the existence proof for the Cosserat bulk problem, it will become clear that for the corresponding sequences $(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) \in H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \text{SO}(3)) = X'$ with $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) < \infty$ bounded independently of h_j (not necessarily minimizers) we obtain a bound on the sequence $(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp)$ in X' , independent of h_j . To see this, note that for $\mu_c > 0$, it is immediate that $\nabla_\eta^h \varphi^\sharp = F_h^\sharp$ is bounded in $L^2(\Omega_1, \mathbb{M}^{3 \times 3})$,

¹⁶ $\lim_{\lambda \rightarrow \infty} \frac{1}{2} \mathcal{H}(\mu, \lambda/2) = \mu < \infty$ but $\lim_{\lambda \rightarrow \infty} \frac{1}{2} \mathcal{A}(\mu, \lambda/2) = \infty$.

¹⁷ In contrast to Γ -convergence arguments based on the finite-strain St. Venant–Kirchhoff energy [38], which might not admit minimizers for any given $h > 0$.

independently of $\bar{R}_{h_j}^\sharp$ on account of the *local coercivity* condition

$$\begin{aligned} W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} F_{h_j}^\sharp) &\geq \min(\mu_c, \mu) \|\bar{R}_{h_j}^{\sharp,T} F_{h_j}^\sharp - \mathbb{1}\|^2 = \min(\mu_c, \mu) (\|F_{h_j}^\sharp\|^2 - 2\langle \bar{R}_{h_j}^{\sharp,T} F_{h_j}^\sharp, \mathbb{1} \rangle + 3) \\ &\geq \min(\mu_c, \mu) (\|F_{h_j}^\sharp\|^2 - 2\sqrt{3}\|F_{h_j}^\sharp\| + 3), \end{aligned}$$

and after integration

$$\begin{aligned} \infty &> I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) > \int_{\Omega_1} [W_{\text{mp}}(\bar{U}_{h_j}^\sharp) + W_{\text{curv}}(\mathfrak{R}_{h_j}^\sharp)] \, dV_\eta \geq \int_{\Omega_1} W_{\text{mp}}(\bar{U}_{h_j}^\sharp) \, dV_\eta \\ &\geq \int_{\Omega_1} \min(\mu_c, \mu) (\|F_{h_j}^\sharp\|^2 - 2\sqrt{3}\|F_{h_j}^\sharp\| + 3) \, dV_\eta \\ &\geq \min(\mu_c, \mu) \int_{\Omega_1} \left(\left[\|\partial_{\eta_1} \varphi^\sharp\|^2 + \|\partial_{\eta_2} \varphi^\sharp\|^2 + \frac{1}{h_j^2} \|\partial_{\eta_3} \varphi^\sharp\|^2 \right] \right. \\ &\quad \left. - 2\sqrt{3} \left[\|\partial_{\eta_1} \varphi^\sharp\| + \|\partial_{\eta_2} \varphi^\sharp\| + \frac{1}{h_j} \|\partial_{\eta_3} \varphi^\sharp\| \right] + 3 \right) \, dV_\eta. \end{aligned} \tag{6.1}$$

This implies a bound, independent of h_j , for the gradient $\nabla \varphi_{h_j}^\sharp$ in $L^2(\Omega_1, \mathbb{R}^3)$. The Dirichlet boundary conditions for $\varphi_{h_j}^\sharp$ together with Poincaré’s inequality yield the boundedness of $\varphi_{h_j}^\sharp$ in $H^{1,2}(\Omega_1, \mathbb{R}^3)$.¹⁸ With a similar argument, based on the local coercivity of curvature, the bound on $\bar{R}_{h_j}^\sharp$ can be obtained: we only need to observe that for a constant $c^+ > 0$, depending on the positivity of $\alpha_4, \alpha_5, \alpha_6, \alpha_7$, but independent of h_j ,

$$\begin{aligned} \infty &> I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) > \int_{\Omega_1} [W_{\text{mp}}(\bar{U}_{h_j}^\sharp) + W_{\text{curv}}(\mathfrak{R}_{h_j}^\sharp)] \, dV_\eta \geq \int_{\Omega_1} W_{\text{curv}}(\mathfrak{R}_{h_j}^\sharp) \, dV_\eta \\ &\geq \int_{\Omega_1} c^+ \|\mathfrak{R}_{h_j}^\sharp\|^{1+p+q} \, dV_\eta = c^+ \int_{\Omega_1} \|\bar{R}_{h_j}^{\sharp,T} D_\eta^{h_j} \bar{R}_{h_j}^\sharp\|^{1+p+q} \, dV_\eta = c^+ \int_{\Omega_1} \|D_\eta^{h_j} \bar{R}_{h_j}^\sharp\|^{1+p+q} \, dV_\eta, \end{aligned}$$

which establishes a bound on the gradient of rotations $\nabla_\eta^{h_j} [\bar{R}_{h_j}^\sharp(\eta) \cdot e_i], i = 1, 2, 3$, independent of h_j . Moreover, $\|\bar{R}_{h_j}^\sharp\| = \sqrt{3}$, establishing the $W^{1,1+p+q}(\Omega_1, \text{SO}(3))$ bound on $\bar{R}_{h_j}^\sharp$. Thus we may obtain a subsequence, not relabelled, such that

$$\varphi_{h_j}^\sharp \rightharpoonup \varphi_0^\sharp \text{ in } H^{1,2}(\Omega_1, \mathbb{R}^3), \quad \bar{R}_{h_j}^\sharp \rightharpoonup \bar{R}_0^\sharp \text{ in } W^{1,1+p+q}(\Omega_1, \text{SO}(3)).$$

Both weak limits $(\varphi_0^\sharp, \bar{R}_0^\sharp)$ must be independent of the transverse coordinate η_3 , otherwise the energy $I_{h_j}^\sharp$ could not remain finite for $h_j \rightarrow 0$ (see (6.1) and compare with the definition of $D_\eta^{h_j}$ in (3.1)). Hence the solution must be found in terms of functions defined on the two-dimensional domain ω . In this sense the domain of the limit problem is two-dimensional and the corresponding space is X_ω . Since the embedding $X' \subset X$ is compact, it is shown that the sequence of energy functionals $I_{h_j}^\sharp$ is equi-coercive with respect to X . \square

¹⁸ This argument fails for the limit case $\mu_c = 0$ since local coercivity does not hold, which is realistic for defective elastic crystals.

6.2 Lower bound—the lim inf-condition

If I_0^\sharp is the Γ -limit of the sequence of energy functionals $I_{h_j}^\sharp$ then we must have (lim inf-inequality) that

$$I_0^\sharp(\varphi_0, \bar{R}_0) \leq \liminf_{h_j} I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp)$$

whenever

$$\varphi_{h_j}^\sharp \rightarrow \varphi_0^\sharp \quad \text{in } L^r(\Omega_1, \mathbb{R}^3), \quad \bar{R}_{h_j}^\sharp \rightarrow \bar{R}_0^\sharp \quad \text{in } L^{1+p+q}(\Omega_1, \text{SO}(3)),$$

for arbitrary $(\varphi_0^\sharp, \bar{R}_0^\sharp) \in X$. Observe that we can restrict attention to sequences $(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) \in X$ such that $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) < \infty$ since otherwise the statement is true anyway. Sequences with $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) < \infty$ are uniformly bounded in the space X' , as seen previously. This implies weak convergence of a subsequence in X' . But we already know that the original sequences converge strongly in X to the limit $(\varphi_0^\sharp, \bar{R}_0^\sharp) \in X$. Hence we must have as well weak convergence to $\varphi_0^\sharp \in H^{1,2}(\omega, \mathbb{R}^3)$ and $\bar{R}_0^\sharp \in W^{1,1+p+q}(\omega, \text{SO}(3))$, independent of the transverse variable η_3 .

In a first step we consider now the *local energy contribution*: along sequences $(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) \in X$ with finite energy $I_{h_j}^\sharp$, the third column of the deformation gradient $\nabla_\eta^{h_j} \varphi_{h_j}^\sharp$ remains bounded but otherwise undetermined. Therefore, a trivial lower bound is obtained by minimizing the effect of the derivative in this direction in the local energy W_{mp} . To continue our development, we need some calculations: For smooth $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\bar{R} : \omega \subset \mathbb{R}^2 \rightarrow \text{SO}(3)$ define the “director” vector $b^* \in \mathbb{R}^3$ formally through

$$W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) = W_{\text{mp}}(\bar{R}^T(\nabla m|b^*)) := \inf_{b \in \mathbb{R}^3} W_{\text{mp}}(\bar{R}^T(\nabla m|b)).$$

The vector b^* , which realizes this infimum, can be explicitly determined. Set $\tilde{F} := (\nabla m|b^*)$. The corresponding local optimality condition reads

$$\begin{aligned} \forall \delta b^* \in \mathbb{R}^3 : \quad & \langle DW_{\text{mp}}(\bar{R}^T(\nabla m|b^*)), \bar{R}^T(0|0|\delta b^*) \rangle = 0 \Rightarrow \\ & \langle \bar{R}DW_{\text{mp}}(\bar{R}^T(\nabla m|b^*)), (0|0|\delta b^*) \rangle = 0 \Rightarrow \\ & \bar{R}DW_{\text{mp}}(\bar{R}^T(\nabla m|b^*)).e_3 = 0 \Rightarrow D_{\tilde{F}}W_{\text{mp}}(\bar{R}^T(\nabla m|b^*)).e_3 = 0 \Rightarrow \\ & S_1((\nabla m|b^*), \bar{R}).e_3 = 0. \end{aligned} \tag{6.2}$$

Since

$$S_1(F, \bar{R}) = \bar{R}(\mu(F^T \bar{R} + \bar{R}^T F - 2\mathbb{1}) + 2\mu_c \text{skew}(\bar{R}^T F) + \lambda \text{tr}[\bar{R}^T F - \mathbb{1}]\mathbb{1})$$

and

$$\begin{aligned} \bar{R}^T \tilde{F} &= \begin{pmatrix} \langle \bar{R}_1, m_x \rangle & \langle \bar{R}_1, m_y \rangle & \langle \bar{R}_1, b^* \rangle \\ \langle \bar{R}_2, m_x \rangle & \langle \bar{R}_2, m_y \rangle & \langle \bar{R}_2, b^* \rangle \\ \langle \bar{R}_3, m_x \rangle & \langle \bar{R}_3, m_y \rangle & \langle \bar{R}_3, b^* \rangle \end{pmatrix}, \\ \tilde{F}^T \bar{R} + \bar{R}^T \tilde{F} - 2\mathbb{1} &= \begin{pmatrix} 2[\langle \bar{R}_1, m_x \rangle - 1] & \langle \bar{R}_1, m_y \rangle + \langle \bar{R}_2, m_x \rangle & \langle \bar{R}_1, b^* \rangle + \langle \bar{R}_3, m_x \rangle \\ \langle \bar{R}_2, m_x \rangle + \langle \bar{R}_1, m_y \rangle & 2[\langle \bar{R}_2, m_y \rangle - 1] & \langle \bar{R}_2, b^* \rangle + \langle \bar{R}_3, m_y \rangle \\ \langle \bar{R}_3, m_x \rangle + \langle \bar{R}_1, b^* \rangle & \langle \bar{R}_3, m_y \rangle + \langle \bar{R}_2, b^* \rangle & 2[\langle \bar{R}_3, b^* \rangle - 1] \end{pmatrix}, \end{aligned}$$

$$\text{skew}(\bar{R}^T \tilde{F}) = \begin{pmatrix} 0 & \frac{1}{2}(\langle \bar{R}_1, m_y \rangle - \langle \bar{R}_2, m_x \rangle) & \frac{1}{2}(\langle \bar{R}_1, b^* \rangle - \langle \bar{R}_3, m_x \rangle) \\ * & 0 & \frac{1}{2}(\langle \bar{R}_2, b^* \rangle - \langle \bar{R}_3, m_y \rangle) \\ * & * & 0 \end{pmatrix},$$

the (plane-stress) requirement $S_1 \cdot e_3 = 0$ (6.2) implies

$$\begin{aligned} \mu \begin{pmatrix} \langle \bar{R}_1, b^* \rangle + \langle \bar{R}_3, m_x \rangle \\ \langle \bar{R}_2, b^* \rangle + \langle \bar{R}_3, m_y \rangle \\ 2[\langle \bar{R}_3, b^* \rangle - 1] \end{pmatrix} + \mu_c \begin{pmatrix} \langle \bar{R}_1, b^* \rangle - \langle \bar{R}_3, m_x \rangle \\ \langle \bar{R}_2, b^* \rangle - \langle \bar{R}_3, m_y \rangle \\ 0 \end{pmatrix} \\ + \lambda(\langle \bar{R}_1, m_x \rangle + \langle \bar{R}_2, m_y \rangle + \langle \bar{R}_3, b^* \rangle - 3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The solution of the last system can conveniently be expressed in the orthonormal triad $(\bar{R}_1, \bar{R}_2, \bar{R}_3)$ as

$$\begin{aligned} b^* &= \frac{\mu_c - \mu}{\mu + \mu_c} \langle \bar{R}_3, m_x \rangle \bar{R}_1 + \frac{\mu_c - \mu}{\mu + \mu_c} \langle \bar{R}_3, m_y \rangle \bar{R}_2 + \varrho_m^* \bar{R}_3, \\ \varrho_m^* &= 1 - \frac{\lambda}{2\mu + \lambda} [\langle \nabla m | 0 \rangle, \bar{R}] - 2]. \end{aligned}$$

Note that for $\bar{R} \in SO(3)$ and $\nabla m \in L^2(\Omega_1, \mathbb{R}^3)$ it follows that $b^* \in L^2(\Omega_1, \mathbb{R}^3)$. Reinserting the solution b^* we have

$$\begin{aligned} \bar{R}^T \tilde{F} &= \begin{pmatrix} \langle \bar{R}_1, m_x \rangle & \langle \bar{R}_1, m_y \rangle & \frac{\mu_c - \mu}{\mu + \mu_c} \langle \bar{R}_3, m_x \rangle \\ \langle \bar{R}_2, m_x \rangle & \langle \bar{R}_2, m_y \rangle & \frac{\mu_c - \mu}{\mu + \mu_c} \langle \bar{R}_3, m_y \rangle \\ \langle \bar{R}_3, m_x \rangle & \langle \bar{R}_3, m_y \rangle & \varrho_m^* \end{pmatrix}, \\ \tilde{F}^T \bar{R} + \bar{R}^T \tilde{F} - 2\mathbb{1} &= \begin{pmatrix} 2[\langle \bar{R}_1, m_x \rangle - 1] & \langle \bar{R}_1, m_y \rangle + \langle \bar{R}_2, m_x \rangle & (1 + \frac{\mu_c - \mu}{\mu + \mu_c}) \langle \bar{R}_3, m_x \rangle \\ \langle \bar{R}_2, m_x \rangle + \langle \bar{R}_1, m_y \rangle & 2[\langle \bar{R}_2, m_y \rangle - 1] & (1 + \frac{\mu_c - \mu}{\mu + \mu_c}) \langle \bar{R}_3, m_y \rangle \\ (1 + \frac{\mu_c - \mu}{\mu + \mu_c}) \langle \bar{R}_3, m_x \rangle & (1 + \frac{\mu_c - \mu}{\mu + \mu_c}) \langle \bar{R}_3, m_y \rangle & 2[\varrho_m^* - 1] \end{pmatrix}, \\ \text{skew}(\bar{R}^T \tilde{F}) &= \begin{pmatrix} 0 & \frac{1}{2}(\langle \bar{R}_1, m_y \rangle - \langle \bar{R}_2, m_x \rangle) & \frac{1}{2}((\frac{\mu_c - \mu}{\mu + \mu_c} - 1) \langle \bar{R}_3, m_x \rangle) \\ * & 0 & \frac{1}{2}((\frac{\mu_c - \mu}{\mu + \mu_c} - 1) \langle \bar{R}_3, m_y \rangle) \\ * & * & 0 \end{pmatrix}, \\ 1 + \frac{\mu_c - \mu}{\mu + \mu_c} &= \frac{2\mu_c}{\mu + \mu_c}, \quad \frac{\mu_c - \mu}{\mu + \mu_c} - 1 = \frac{-2\mu}{\mu + \mu_c}. \end{aligned}$$

Finally for $W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) := W_{\text{mp}}(\bar{R}^T(\nabla m|b^*))$ with $\tilde{U} = \bar{R}^T(\nabla m|b^*) = \bar{R}^T \tilde{F}$, after a lengthy but otherwise straightforward computation we obtain

$$\begin{aligned} W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) &:= W_{\text{mp}}(\tilde{U}) = \mu \|\text{sym}(\tilde{U} - \mathbb{1})\|^2 + \mu_c \|\text{skew}(\tilde{U})\|^2 + \frac{\lambda}{2} \text{tr}[\text{sym}(\tilde{U} - \mathbb{1})]^2 \\ &= \mu \|\text{sym}((\bar{R}_1|\bar{R}_2)^T \nabla m - \mathbb{1}_2)\|^2 + \mu_c \|\text{skew}((\bar{R}_1|\bar{R}_2)^T \nabla m)\|^2 \end{aligned}$$

$$\begin{aligned}
 &+ 2\mu \frac{\mu_c}{\mu + \mu_c} (\langle \bar{R}_3, m_x \rangle^2 + \langle \bar{R}_3, m_y \rangle^2) \\
 &+ \frac{\mu\lambda}{2\mu + \lambda} \operatorname{tr}[\operatorname{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla m - \mathbb{1}_2)]^2.
 \end{aligned}$$

Along the sequence $(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp)$ we have, by construction,

$$W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^\sharp) = W_{\text{mp}}\left(\bar{R}_{h_j}^{\sharp,T} \left(\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\sharp \Big| \frac{1}{h_j} \partial_{\eta_3} \varphi_{h_j}^\sharp\right)\right) \geq W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp).$$

Hence, integrating and taking the \liminf we also have

$$\liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^\sharp) \, dV_{\eta} \geq \liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) \, dV_{\eta}. \tag{6.3}$$

Now we use weak convergence of $\varphi_{h_j}^\sharp$ and strong convergence of $\bar{R}_{h_j}^\sharp$, together with the convexity with respect to ∇m and continuity with respect to \bar{R} of $\int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla m, \bar{R}) \, dV_{\eta}$ to get lower semicontinuity of the right hand side in (6.3) and to obtain altogether

$$\liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^\sharp) \, dV_{\eta} \geq \int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \varphi_0^\sharp, \bar{R}_0^\sharp) \, dV_{\eta}. \tag{6.4}$$

Next we are concerned with the *curvature contribution*: it is always possible to uniquely rewrite the curvature energy expression in terms of skew-symmetric quantities

$$W_{\text{curv}}^* : \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3) \rightarrow \mathbb{R}^+, \quad W_{\text{curv}}^*(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}, \bar{R}^T \partial_{\eta_3} \bar{R}) := W_{\text{curv}}(\mathfrak{K}),$$

where $\bar{R}^T \partial_{\eta_i} \bar{R} \in \mathfrak{so}(3)$ since $\partial_{\eta_i}[\bar{R}^T \bar{R}] = 0$. We note that W_{curv}^* remains a convex function in its argument since $\mathfrak{K} \in \mathfrak{T}(3)$ can be obtained by a linear mapping from $(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}, \bar{R}^T \partial_{\eta_3} \bar{R}) \in \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3)$. We define the ‘‘homogenized’’ (relaxed) curvature energy through

$$\begin{aligned}
 W_{\text{curv}}^{*,\text{hom}}(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}) &:= W_{\text{curv}}^*(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}, A^*) \\
 &= \inf_{A \in \mathfrak{so}(3)} W_{\text{curv}}^*(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}, A),
 \end{aligned} \tag{6.5}$$

and set accordingly

$$\begin{aligned}
 W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_s) &:= W_{\text{curv}}^{*,\text{hom}}(\bar{R}^T \partial_{\eta_1} \bar{R}, \bar{R}^T \partial_{\eta_2} \bar{R}), \\
 \mathfrak{K}_s &= (\bar{R}^T(\nabla(\bar{R}.e_1)|0), \bar{R}^T(\nabla(\bar{R}.e_2)|0), \bar{R}^T(\nabla(\bar{R}.e_3)|0)),
 \end{aligned}$$

in terms of the reduced curvature tensor $\mathfrak{K}_s \in \mathfrak{T}(3)$.

Similarly to (6.2) the infinitesimal rotation $A^* \in \mathfrak{so}(3)$, which realizes the infimum in (6.5), can be explicitly determined. We refrain from giving the explicit result. Suffice it to note that $W_{\text{curv}}^{\text{hom}}$ is uniquely defined, remains convex in its argument and has the same growth as W_{curv} . Then

$$\begin{aligned}
 W_{\text{curv}}(\bar{R}_{h_j}^{\sharp,T} \mathbb{D}_{\eta}^{h_j} \bar{R}_{h_j}^\sharp) &= W_{\text{curv}}^*\left(\bar{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \bar{R}_{h_j}^\sharp, \bar{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \bar{R}_{h_j}^\sharp, \frac{1}{h_j} \bar{R}_{h_j}^{\sharp,T} \partial_{\eta_3} \bar{R}_{h_j}^\sharp\right) \\
 &\geq W_{\text{curv}}^{*,\text{hom}}(\bar{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \bar{R}_{h_j}^\sharp, \bar{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \bar{R}_{h_j}^\sharp).
 \end{aligned}$$

Integrating the last inequality, taking the \liminf on both sides and using fact that $W_{\text{curv}}^{*,\text{hom}}$ is convex in its argument, together with weak convergence of the two in-plane components of the curvature tensor, i.e.

$$(\bar{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \bar{R}_{h_j}^{\sharp}, \bar{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \bar{R}_{h_j}^{\sharp}, 0) \rightharpoonup (\bar{R}_0^{\sharp,T} \partial_{\eta_1} \bar{R}_0^{\sharp}, \bar{R}_0^{\sharp,T} \partial_{\eta_2} \bar{R}_0^{\sharp}, 0) \quad \text{in } L^{1+p+q}(\Omega_1, \mathfrak{T}(3)),$$

we obtain

$$\begin{aligned} \liminf_{h_j} \int_{\Omega_1} W_{\text{curv}}(\bar{R}_{h_j}^{\sharp,T} D_{\eta}^{h_j} \bar{R}_{h_j}^{\sharp}) dV_{\eta} &\geq \int_{\Omega_1} W_{\text{curv}}^{*,\text{hom}}(\bar{R}_{h_j}^{\sharp,T} \partial_{\eta_1} \bar{R}_{h_j}^{\sharp}, \bar{R}_{h_j}^{\sharp,T} \partial_{\eta_2} \bar{R}_{h_j}^{\sharp}) dV_{\eta} \\ &= \int_{\Omega_1} W_{\text{curv}}^{\text{hom}}(\bar{R}_0^{\sharp,T} D \bar{R}_0^{\sharp}) dV_{\eta}. \end{aligned} \quad (6.6)$$

Then, because $W_{\text{curv}}, W_{\text{mp}} \geq 0$,

$$\begin{aligned} \liminf_{h_j} \int_{\Omega_1} [W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^{\sharp}) + W_{\text{curv}}(\bar{R}_{h_j}^{\sharp,T} D_{\eta}^{h_j} \bar{R}_{h_j}^{\sharp})] dV_{\eta} \\ \geq \liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} \nabla_{\eta}^{h_j} \varphi_{h_j}^{\sharp}) dV_{\eta} + \liminf_{h_j} \int_{\Omega_1} W_{\text{curv}}(\bar{R}_{h_j}^{\sharp,T} D_{\eta}^{h_j} \bar{R}_{h_j}^{\sharp}) dV_{\eta} \\ \geq \int_{\Omega_1} [W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \varphi_0^{\sharp}, \bar{R}_0^{\sharp}) + W_{\text{curv}}^{\text{hom}}(\bar{R}_0^{\sharp,T} D \bar{R}_0^{\sharp})] dV_{\eta}, \end{aligned}$$

where we have used (6.4) and (6.6). Now we use the fact that φ_0^{\sharp} is independent of the transverse variable η_3 , which allows us to insert the averaging operator without any change to see that

$$\int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \varphi_0^{\sharp}, \bar{R}_0^{\sharp}) = \int_{\Omega_1} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \text{Av.} \varphi_0^{\sharp}, \bar{R}_0^{\sharp}) dV_{\eta} = \int_{\omega} W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \text{Av.} \varphi_0^{\sharp}, \bar{R}_0^{\sharp}) d\omega,$$

since \bar{R}_0^{\sharp} is also independent of the transverse variable. Hence we obtain altogether the desired \liminf -inequality

$$I_0^{\sharp}(\varphi_0^{\sharp}, \bar{R}_0^{\sharp}) \leq \liminf_{h_j} I_{h_j}^{\sharp}(\varphi_{h_j}^{\sharp}, \bar{R}_{h_j}^{\sharp})$$

for

$$\begin{aligned} I_0^{\sharp}(\varphi_0, \bar{R}_0) &:= \int_{\Omega_1} [W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \text{Av.} \varphi_0, \bar{R}_0) + W_{\text{curv}}^{\text{hom}}(\bar{R}_0^T D \bar{R}_0)] dV_{\eta} \\ &= \int_{\omega} [W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \text{Av.} \varphi_0, \bar{R}_0) + W_{\text{curv}}^{\text{hom}}(\bar{R}_0^T D \bar{R}_0)] d\omega. \quad \square \end{aligned}$$

6.3 Upper bound—the recovery sequence

Now we show that the lower bound will actually be reached. A sufficient requirement for the recovery sequence is that

$$\begin{aligned} \forall (\varphi_0, \bar{R}_0) \in X = L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \text{SO}(3)) \\ \exists \varphi_{h_j}^{\sharp} \rightarrow \varphi_0 \quad \text{in } L^r(\Omega_1, \mathbb{R}^3), \quad \bar{R}_{h_j}^{\sharp} \rightarrow \bar{R}_0 \quad \text{in } L^{1+p+q}(\Omega_1, \text{SO}(3)) : \\ \limsup I_{h_j}^{\sharp}(\varphi_{h_j}^{\sharp}, \bar{R}_{h_j}^{\sharp}) \leq I_0^{\sharp}(\varphi_0, \bar{R}_0). \end{aligned}$$

Observe that this is now only a condition if $I_0^\sharp(\varphi_0, \bar{R}_0) < \infty$. In this case the uniform coercivity of $I_{h_j}^\sharp(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp)$ over $X' = H^{1,2}(\Omega_1, \mathbb{R}^3) \times W^{1,1+p+q}(\Omega_1, \text{SO}(3))$ implies that we can restrict attention to sequences $(\varphi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp)$ converging weakly to some $(\varphi_0, \bar{R}_0) \in H^{1,2}(\omega, \mathbb{R}^3) \times W^{1,1+p+q}(\omega, \text{SO}(3)) = X'_\omega$, defined over the two-dimensional domain ω only. Note, however, that finally it is strong convergence in X that matters.

The natural candidate for the recovery sequence for the bulk deformation is given by the “reconstruction”

$$\varphi_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) := m(\eta_1, \eta_2) + h_j \eta_3 b^*(\eta_1, \eta_2) = \varphi_0(\eta_1, \eta_2) + h_j \eta_3 b^*(\eta_1, \eta_2),$$

where, with the abbreviation $m = \varphi_0 = \text{Av.}\varphi_0$ at places,

$$b^*(\eta_1, \eta_2) := \frac{\mu_c - \mu}{\mu + \mu_c} \langle \bar{R}_{0,3}, m_x \rangle \bar{R}_{0,1} + \frac{\mu_c - \mu}{\mu + \mu_c} \langle \bar{R}_{0,3}, m_y \rangle \bar{R}_{0,2} + \varrho_m^* \bar{R}_{0,3},$$

$$\varrho_m^* = 1 - \frac{\lambda}{2\mu + \lambda} [\langle (\nabla m|_0), \bar{R}_0 \rangle - 2].$$

Observe that $b^* \in L^2(\omega, \mathbb{R}^3)$. Convergence of $\varphi_{h_j}^\sharp$ in $L^r(\Omega_1, \mathbb{R}^3)$ to the limit φ_0 as $h_j \rightarrow 0$ is obvious.

The reconstruction for the rotation \bar{R}_0 is, however, not obvious since on the one hand we have to maintain the rotation constraint along the sequence and on the other hand we must approach the lower bound, which excludes the simple reconstruction $\bar{R}_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) = \bar{R}_0(\eta_1, \eta_2)$. In order to meet both requirements we consider therefore

$$\bar{R}_{h_j}^\sharp(\eta_1, \eta_2, \eta_3) := \bar{R}_0(\eta_1, \eta_2) \cdot \exp(h_j \eta_3 A^*(\eta_1, \eta_2)),$$

where $A^* \in \mathfrak{so}(3)$ is the term obtained in (6.5), depending on the given \bar{R}_0 , and we note that $A^* \in L^{1+p+q}(\omega, \mathfrak{so}(3))$ by the coercivity of W_{curv}^* . It is clear that $\bar{R}_{h_j}^\sharp \in \text{SO}(3)$, since $\exp : \mathfrak{so}(3) \rightarrow \text{SO}(3)$ and we have the convergence $\bar{R}_{h_j}^\sharp \rightarrow \bar{R}_0$ in $L^{1+p+q}(\Omega_1, \text{SO}(3))$ for $h_j \rightarrow 0$.

Since neither b^* nor A^* need be differentiable, we have to consider slightly modified recovery sequences, however. With fixed $\varepsilon > 0$ choose $b_\varepsilon \in W^{1,2}(\omega, \mathbb{R}^3)$ such that

$$\|b_\varepsilon - b^*\|_{L^2(\omega, \mathbb{R}^3)} < \varepsilon$$

and similarly for A^* choose $A_\varepsilon \in W^{1,1+p+q}(\omega, \mathfrak{so}(3))$ such that

$$\|A_\varepsilon - A^*\|_{L^{1+p+q}(\omega, \mathfrak{so}(3))} < \varepsilon.$$

This allows us to present finally our *recovery sequence*

$$\varphi_{h_j, \varepsilon}^\sharp(\eta_1, \eta_2, \eta_3) := \varphi_0(\eta_1, \eta_2) + h_j \eta_3 b_\varepsilon(\eta_1, \eta_2),$$

$$\bar{R}_{h_j, \varepsilon}^\sharp(\eta_1, \eta_2, \eta_3) := \bar{R}_0(\eta_1, \eta_2) \cdot \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2)).$$

This definition implies

$$\begin{aligned}
 \nabla\varphi_{h_j,\varepsilon}^\sharp(\eta_1, \eta_2, \eta_3) &= (\nabla\varphi_0(\eta_1, \eta_2)|h_j b_\varepsilon(\eta_1, \eta_2)) + h_j \eta_3 (\nabla b_\varepsilon(\eta_1, \eta_2)|0), \\
 \overline{R}_{h_j,\varepsilon}^{\sharp,T} \partial_{\eta_1} \overline{R}_{h_j,\varepsilon}^\sharp &= \exp(h_j \eta_3 A_\varepsilon)^T \overline{R}_0^T [\partial_{\eta_1} \overline{R}_0 \exp(h_j \eta_3 A_\varepsilon) \\
 &\quad + \overline{R}_0 D \exp(h_j \eta_3 A_\varepsilon) \cdot [h_j \eta_3 \partial_{\eta_1} A_\varepsilon]], \\
 \overline{R}_{h_j,\varepsilon}^{\sharp,T} \partial_{\eta_2} \overline{R}_{h_j,\varepsilon}^\sharp &= \exp(h_j \eta_3 A_\varepsilon)^T \overline{R}_0^T [\partial_{\eta_2} \overline{R}_0 \exp(h_j \eta_3 A_\varepsilon) \\
 &\quad + \overline{R}_0 D \exp(h_j \eta_3 A_\varepsilon) \cdot [h_j \eta_3 \partial_{\eta_2} A_\varepsilon]], \\
 \overline{R}_{h_j,\varepsilon}^{\sharp,T} \partial_{\eta_3} \overline{R}_{h_j,\varepsilon}^\sharp &= \exp(h_j \eta_3 A_\varepsilon)^T \overline{R}_0^T [\partial_{\eta_3} \overline{R}_0 \exp(h_j \eta_3 A_\varepsilon) \\
 &\quad + \overline{R}_0 D \exp(h_j \eta_3 A_\varepsilon) \cdot [h_j A_\varepsilon]] \\
 &= h_j \exp(h_j \eta_3 A_\varepsilon)^T D \exp(h_j \eta_3 A_\varepsilon) \cdot [A_\varepsilon],
 \end{aligned} \tag{6.7}$$

with $\partial_{\eta_i} A_\varepsilon \in \mathfrak{so}(3)$. In view of the prominent appearance of the exponential in these expressions it is useful to briefly recapitulate the basic features of the matrix exponential \exp acting on $\mathfrak{so}(3)$. We note

$$\begin{aligned}
 \exp : \mathfrak{so}(3) &\rightarrow \text{SO}(3) \quad \text{is infinitely differentiable,} \\
 \forall A \in \mathfrak{so}(3) : \quad &\|\exp(A)\| = \sqrt{3}, \quad \text{hence} \\
 \exp : L^{1+p+q}(\Omega_1, \mathfrak{so}(3)) &\rightarrow L^{1+p+q}(\Omega_1, \text{SO}(3)) \quad \text{is continuous,} \\
 D \exp : \mathfrak{so}(3) &\rightarrow \text{Lin}(\mathfrak{so}(3), \mathbb{M}^{3 \times 3}) \quad \text{is locally continuous,} \\
 \forall H \in \mathfrak{so}(3) : \quad &D \exp(0) \cdot H = H, \\
 \forall A, H \in \mathfrak{so}(3) : \quad &\exp(A)^T \cdot D \exp(A) \cdot H \in \mathfrak{so}(3).
 \end{aligned} \tag{6.8}$$

Note that by appropriately choosing $h_j, \varepsilon > 0$ we can arrange that strong convergence of all terms in (6.7) to the correct limit still obtains by using (6.8)₃. Now abbreviate

$$\begin{aligned}
 \tilde{U} &:= \overline{R}_0^T (\nabla\varphi_0(\eta_1, \eta_2)|b^*) \in \mathbb{M}^{3 \times 3}, \\
 \tilde{V}_{h_j}^\varepsilon &:= \overline{R}_{h_j,\varepsilon}^{\sharp,T} [(\nabla\varphi_0(\eta_1, \eta_2)|b_\varepsilon(\eta_1, \eta_2)) + h_j \eta_3 (\nabla b_\varepsilon(\eta_1, \eta_2)|0)] \in \mathbb{M}^{3 \times 3}, \\
 \tilde{V}_0^\varepsilon &:= \overline{R}_0^T (\nabla\varphi_0(\eta_1, \eta_2)|b_\varepsilon(\eta_1, \eta_2)) \in \mathbb{M}^{3 \times 3}, \\
 \mathfrak{k}_{h_j,\varepsilon}^{\sharp,i} &:= \overline{R}_{h_j,\varepsilon}^{\sharp,T} \partial_{\eta_i} \overline{R}_{h_j,\varepsilon}^\sharp \in \mathfrak{so}(3), \quad i = 1, 2, 3, \\
 \mathfrak{k}_0^i &:= \overline{R}_0^T \partial_{\eta_i} \overline{R}_0 \in \mathfrak{so}(3), \quad i = 1, 2, \\
 \tilde{A}_{h_j,\varepsilon} &:= \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2))^T D \exp(h_j \eta_3 A_\varepsilon(\eta_1, \eta_2)) \cdot [A_\varepsilon] \in \mathfrak{so}(3), \\
 \mathfrak{R}_{h_j,\varepsilon}^\sharp &:= \overline{R}_{h_j,\varepsilon}^{\sharp,T} D_{\eta_j}^{h_j} \overline{R}_{h_j,\varepsilon}^\sharp(\eta_1, \eta_2, \eta_3) \in \mathfrak{T}(3), \\
 \mathfrak{R}_0(\eta_1, \eta_2) &= \overline{R}_0^T D \overline{R}_0(\eta_1, \eta_2) \in \mathfrak{T}(3).
 \end{aligned} \tag{6.9}$$

We note that by the smoothness of $A_\varepsilon \in W^{1,1+p+q}(\omega_1, \mathfrak{so}(3))$,

$$\begin{aligned} \|\tilde{A}_{h_j,\varepsilon} - A_\varepsilon\|_{L^{1+p+q}(\Omega_1, \mathfrak{so}(3))} &\rightarrow 0 && \text{as } h_j \rightarrow 0, \\ \|\mathfrak{k}_{h_j,\varepsilon}^{\sharp,i} - \mathfrak{k}_0^i\|_{L^{1+p+q}(\Omega_1, \mathfrak{so}(3))} &\rightarrow 0 && \text{as } h_j \rightarrow 0, \\ \|\tilde{V}_{h_j}^\varepsilon - \tilde{V}_0^\varepsilon\|_{L^2(\Omega_1, \mathbb{M}^{3 \times 3})} &\rightarrow 0 && \text{as } h_j \rightarrow 0, \\ \|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\|_{L^2(\Omega_1, \mathbb{M}^{3 \times 3})} &\rightarrow 0 && \text{as } h_j, \varepsilon \rightarrow 0. \end{aligned} \tag{6.10}$$

The abbreviations in (6.9) imply

$$\begin{aligned} I_{h_j}^\sharp(\varphi_{h_j,\varepsilon}^\sharp, \bar{R}_{h_j,\varepsilon}^\sharp) &= \int_{\Omega_1} \left(W_{\text{mp}}(\tilde{V}_{h_j}^\varepsilon) + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \frac{1}{h_j} \bar{R}_{h_j,\varepsilon}^{\sharp,T} \partial_{\eta_3} \bar{R}_{h_j,\varepsilon}^\sharp) \right) dV_\eta \\ &= \int_{\Omega_1} [W_{\text{mp}}(\tilde{V}_{h_j}^\varepsilon) + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta, \end{aligned}$$

where we have used the fact that $h_j \cdot b_\varepsilon$ in the definition of the recovery deformation gradient (6.7)₁ is cancelled by the factor $1/h_j$ in the definition of $I_{h_j}^\sharp$. Hence, adding and subtracting $W_{\text{mp}}(\tilde{U})$,

$$\begin{aligned} I_{h_j}^\sharp(\varphi_{h_j,\varepsilon}^\sharp, \bar{R}_{h_j,\varepsilon}^\sharp) &= \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{mp}}(\tilde{V}_{h_j}^\varepsilon) - W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta \\ &= \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{mp}}(\tilde{U} + \tilde{V}_{h_j}^\varepsilon - \tilde{U}) - W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}(\mathfrak{K}_{h_j})] dV_\eta \\ &\text{since } W_{\text{mp}} \text{ and } W_{\text{curv}} \text{ are both positive, we get from the triangle inequality} \\ &\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + |W_{\text{mp}}(\tilde{U} + \tilde{V}_{h_j}^\varepsilon - \tilde{U}) - W_{\text{mp}}(\tilde{U})| + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta \\ &\text{expanding the quadratic energy } W_{\text{mp}} \text{ we obtain} \\ &= \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + |W_{\text{mp}}(\tilde{U}) + \langle DW_{\text{mp}}(\tilde{U}), \tilde{V}_{h_j}^\varepsilon - \tilde{U} \rangle \\ &\quad + D^2 W_{\text{mp}}(\tilde{U}) \cdot (\tilde{V}_{h_j}^\varepsilon - \tilde{U}, \tilde{V}_{h_j}^\varepsilon - \tilde{U})| + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta \\ &\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + \|DW_{\text{mp}}(\tilde{U})\| \|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\| + C \|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\|^2 + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta \end{aligned}$$

for $\|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\| \leq 1$ we have

$$\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + (C + \|DW_{\text{mp}}(\tilde{U})\|) \|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\| + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta$$

since $\|DW_{\text{mp}}(\tilde{U})\| \leq C_2 \|\tilde{U}\|$ we obtain

$$\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + (C + \|\tilde{U}\|) \|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\| + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta$$

and by Hölder's inequality we get

$$\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})] dV_\eta + (C + \|\tilde{U}\|_{L^2(\Omega_1)}) \|\tilde{V}_{h_j}^\varepsilon - \tilde{U}\|_{L^2(\Omega_1)}.$$

Continuing the estimate with regard to $W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon})$ and adding and subtracting \tilde{V}_0^ε we obtain

$$\begin{aligned} I_{h_j}^\sharp(\varphi_{h_j,\varepsilon}^\sharp, \bar{R}_{h_j,\varepsilon}^\sharp) &\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*) + W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon}) \\ &\quad - W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)] dV_\eta \\ &\quad + (C + \|\tilde{U}\|_{L^2(\Omega_1)}) \|\tilde{V}_{h_j}^\varepsilon - \tilde{V}_0^\varepsilon + \tilde{V}_0^\varepsilon - \tilde{U}\|_{L^2(\Omega_1)} \\ &\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)] dV_\eta \\ &\quad + \|W_{\text{curv}}^*(\mathfrak{k}_{h_j,\varepsilon}^{\sharp,1}, \mathfrak{k}_{h_j,\varepsilon}^{\sharp,2}, \tilde{A}_{h_j,\varepsilon}) - W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A_\varepsilon)\|_{L^1(\Omega_1)} \\ &\quad + \|W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A_\varepsilon) - W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)\|_{L^1(\Omega_1)} \\ &\quad + (C + \|\tilde{U}\|_{L^2(\Omega_1)}) (\|\tilde{V}_{h_j}^\varepsilon - \tilde{V}_0^\varepsilon\|_{L^2(\Omega_1)} + \|\tilde{V}_0^\varepsilon - \tilde{U}\|_{L^2(\Omega_1)}). \end{aligned}$$

Now take $h_j \rightarrow 0$ to obtain, by the continuity of W_{curv}^* in its first two arguments and (6.10)₃,

$$\begin{aligned} \limsup_{h_j \rightarrow 0} I_{h_j}^\sharp(\varphi_{h_j,\varepsilon}^\sharp, \bar{R}_{h_j,\varepsilon}^\sharp) &\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)] dV_\eta \\ &\quad + \|W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A_\varepsilon) - W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)\|_{L^1(\Omega_1)} \\ &\quad + (C + \|\tilde{U}\|_{L^2(\Omega_1)}) \|\tilde{V}_0^\varepsilon - \tilde{U}\|_{L^2(\Omega_1)}. \end{aligned}$$

Since

$$\begin{aligned} \|\tilde{V}_0^\varepsilon - \tilde{U}\|^2 &= \|\bar{R}_0^T(\nabla\varphi_0(\eta_1, \eta_2)|b_\varepsilon) - \bar{R}_0^T(\nabla\varphi_0(\eta_1, \eta_2)|b^*)\|^2 \\ &= \|\bar{R}_0^T((\nabla\varphi_0(\eta_1, \eta_2)|b_\varepsilon) - (\nabla\varphi_0(\eta_1, \eta_2)|b^*))\|^2 \\ &= \|(\nabla\varphi_0(\eta_1, \eta_2)|b_\varepsilon) - (\nabla\varphi_0(\eta_1, \eta_2)|b^*)\|^2 = \|b_\varepsilon - b^*\|^2, \end{aligned}$$

we get, by letting $\varepsilon \rightarrow 0$ and using now the continuity of W_{curv}^* in its last argument together with $\|A_\varepsilon - A^*\|_{L^{1+p+q}(\omega, \mathfrak{so}(3))} < \varepsilon$, the bound

$$\begin{aligned} \limsup_{h_j \rightarrow 0} I_{h_j}^\sharp(\varphi_{h_j,\varepsilon}^\sharp, \bar{R}_{h_j,\varepsilon}^\sharp) &\leq \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^*(\mathfrak{k}_0^1, \mathfrak{k}_0^2, A^*)] dV_\eta \\ &= \int_{\Omega_1} [W_{\text{mp}}(\tilde{U}) + W_{\text{curv}}^{\text{hom}}(\mathfrak{k}_0^1, \mathfrak{k}_0^2)] dV_\eta \\ &= \int_{\Omega_1} [W_{\text{mp}}^{\text{hom}}(\nabla\varphi_0, \bar{R}_0) + W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_0)] dV_\eta. \end{aligned}$$

Since φ_0, \bar{R}_0 are two-dimensional (independent of the transverse variable), we may write as well

$$\begin{aligned} \limsup_{h_j \rightarrow 0} I_{h_j}^\sharp(\varphi_{h_j,\varepsilon}^\sharp, \bar{R}_{h_j,\varepsilon}^\sharp) &\leq \int_{\Omega_1} [W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \text{Av}.\varphi_0, \bar{R}_0) + W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_0)] dV_\eta \\ &= \int_\omega [W_{\text{mp}}^{\text{hom}}(\nabla_{(\eta_1, \eta_2)} \text{Av}.\varphi_0, \bar{R}_0) + W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_0)] d\omega = I_0^\sharp(\varphi_0, \bar{R}_0), \end{aligned}$$

which shows the desired upper bound. Note that the appearance of the averaging operator Av is not strictly necessary since the limit problem for $\mu_c > 0$ is independent of the transverse variable anyhow. This finishes the proof of Theorem 5.2. \square

7. Proof for zero Cosserat couple modulus $\mu_c = 0$

Now we supply the proof for Theorem 5.3, i.e. we show that the formal limit as $\mu_c \rightarrow 0$ of the Γ -limit for $\mu_c > 0$ is in fact the Γ -limit for $\mu_c = 0$. This result cannot be inferred from the case with $\mu_c > 0$ since equi-coercivity is lost.

REMARK 7.1 (Loss of equi-coercivity) If we consider Γ -convergence in the weak topology of $W^{1,2}(\Omega, \mathbb{R}^3)$ for the deformations φ instead of working with the strong topology of $L^r(\Omega, \mathbb{R}^3)$, i.e. assuming for minimizing sequences a priori that $\|\nabla\varphi_{h_j}\|_{L^2(\Omega)}$ is bounded, then the problem related to a loss of equi-coercivity does not appear and the Γ -limit result for $\mu_c = 0$ is an easy consequence of the case for $\mu_c > 0$.

For $\mu_c > 0$ equi-coercivity is enough to provide the uniform bound on the deformation gradients in the minimization process. The crucial question is whether we obtain a uniform bound on the deformation gradients in the minimization process also for $\mu_c = 0$. For thickness $h \rightarrow 0$ the deformations of the thin structure might develop high oscillations (wrinkles) which exclude such a bound on the gradients but the sequence of deformations could still converge strongly in $L^r(\Omega, \mathbb{R}^3)$. Therefore, the strong topology of $L^r(\Omega, \mathbb{R}^3)$ is the convenient framework for Γ -convergence results.

In order to circumvent the loss of equi-coercivity we investigate first a lower bound of the rescaled three-dimensional formulation for the limit case $\mu_c = 0$.

7.1 The “membrane” lower bound for $\mu_c = 0$

We introduce a new family of functionals $I_h^{\sharp, \text{mem}} : X' \rightarrow \overline{\mathbb{R}}$, where all transverse shear terms have been omitted, more precisely

$$I_h^{\sharp, \text{mem}}(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \overline{R}^\sharp, D_\eta^h \overline{R}^\sharp) = \int_{\eta \in \Omega_1} [W_{\text{mp}}(\overline{U}_h^{\sharp, \text{mem}}) + W_{\text{curv}}(\mathfrak{K}_h^\sharp)] dV_\eta \mapsto \min \text{ w.r.t. } (\varphi^\sharp, \overline{R}^\sharp),$$

$$\overline{U}_h^\sharp = \overline{R}^{\sharp, T} F_h^\sharp, \quad \varphi_{\Gamma_0^1}^\sharp(\eta) = g_d^\sharp(\eta) = g_d(\zeta(\eta)) = g_d(\eta_1, \eta_2, h \cdot \eta_3) = g_d(\eta_1, \eta_2, 0),$$

$$\begin{aligned} \overline{U}_h^{\sharp, \text{mem}} &= \begin{pmatrix} \overline{U}_{h,11}^\sharp & \overline{U}_{h,12}^\sharp & 0 \\ \overline{U}_{h,21}^\sharp & \overline{U}_{h,22}^\sharp & 0 \\ 0 & 0 & \overline{U}_{h,33}^\sharp \end{pmatrix} \\ &= \begin{pmatrix} \langle \overline{R}_1^{3d, \sharp}, \partial_{\eta_1} \varphi^\sharp \rangle & \langle \overline{R}_1^{3d, \sharp}, \partial_{\eta_2} \varphi^\sharp \rangle & 0 \\ \langle \overline{R}_2^{3d, \sharp}, \partial_{\eta_1} \varphi^\sharp \rangle & \langle \overline{R}_2^{3d, \sharp}, \partial_{\eta_2} \varphi^\sharp \rangle & 0 \\ 0 & 0 & \frac{1}{h} \langle \overline{R}_3^{3d, \sharp}, \partial_{\eta_3} \varphi^\sharp \rangle \end{pmatrix}, \end{aligned}$$

$$\Gamma_0^1 = \gamma_0 \times [-1/2, 1/2], \quad \gamma_0 \subset \partial\omega,$$

$$\overline{R}^\sharp : \text{ free on } \Gamma_0^1, \text{ Neumann-type boundary condition,}$$

$$W_{\text{mp}}(\bar{U}_h^{\sharp, \text{mem}}) = \mu \|\text{sym}(\bar{U}_h^{\sharp, \text{mem}} - \mathbb{1})\|^2 + \frac{\lambda}{2} \text{tr}[\text{sym}(\bar{U}_h^{\sharp, \text{mem}} - \mathbb{1})]^2 =: W_{\text{mp}}^{\text{mem}}(\nabla \varphi_h^\sharp, \bar{R}^\sharp),$$

$$W_{\text{curv}}(\mathfrak{K}_h^\sharp) = \mu \frac{\widehat{L}_c^{1+p}}{12} (1 + \alpha_4 \widehat{L}_c^q \|\mathfrak{K}_h^\sharp\|^q) (\alpha_5 \|\text{sym} \mathfrak{K}_h^\sharp\|^2 + \alpha_6 \|\text{skew} \mathfrak{K}_h^\sharp\|^2 + \alpha_7 \text{tr}[\mathfrak{K}_h^\sharp]^2)^{(1+p)/2},$$

$$\mathfrak{K}_h^\sharp = \bar{R}^{\sharp, T} D_\eta^h \bar{R}^\sharp(\eta).$$

Note that for $(\varphi^\sharp, \bar{R}^\sharp) \in X$ the product \bar{U}_h^\sharp does not have a classical meaning if $\nabla \varphi^\sharp \notin L^2(\Omega_1, \mathbb{M}^{3 \times 3})$. However, the product \bar{U}_h^\sharp does already have a distributional meaning because $\bar{R}^\sharp \in W^{1, 1+p+q}(\Omega_1, \text{SO}(3))$ and $\nabla \varphi^\sharp \in W^{-1, r}(\Omega_1, \mathbb{M}^{3 \times 3})$. Accordingly, we define the admissible set

$$\mathcal{A}_h^{\text{mem}} := \{(\varphi, \bar{R}) \in X \mid \text{sym} \bar{U}_h^{\sharp, \text{mem}} \in L^2(\Omega_1, \mathbb{M}^{3 \times 3}), \bar{R} \in W^{1, 1+p+q}(\Omega_1, \text{SO}(3)), \\ \varphi|_{\Gamma_0^1}(\eta) = g_d^\sharp(\eta) = g_d(\eta_1, \eta_2, 0)\},$$

where the distribution $\bar{U}_h^{\sharp, \text{mem}}$ is regular and belongs to $L^2(\Omega_1, \mathbb{M}^{3 \times 3})$. As in (5.2) we extend the rescaled energies to the larger space X through redefining

$$I_h^{\sharp, \text{mem}}(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp) = \begin{cases} I_h^{\sharp, \text{mem}}(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp) & \text{if } (\varphi^\sharp, \bar{R}^\sharp) \in \mathcal{A}_h^{\text{mem}}, \\ +\infty & \text{else in } X. \end{cases}$$

Observe that

$$\forall h > 0: \quad I_{h|_{\mu_c=0}}^\sharp(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp) \geq I_h^{\sharp, \text{mem}}(\varphi^\sharp, \nabla_\eta^h \varphi^\sharp, \bar{R}^\sharp, D_\eta^h \bar{R}^\sharp),$$

which implies [18, Prop. 6.7] that

$$\Gamma\text{-lim inf}_h I_{h|_{\mu_c=0}}^\sharp \geq \Gamma\text{-lim inf}_h I_h^{\sharp, \text{mem}}. \quad (7.1)$$

Hence $\Gamma\text{-lim inf}_h I_h^{\sharp, \text{mem}}$ provides a lower bound for $\Gamma\text{-lim inf}_h I_{h|_{\mu_c=0}}^\sharp$. Putting inequalities (5.5) and (7.1) together, we obtain the natural chain of inequalities on X ,

$$\Gamma\text{-lim inf}_h I_h^{\sharp, \text{mem}} \leq \Gamma\text{-lim inf}_h I_{h|_{\mu_c=0}}^\sharp \\ \leq \Gamma\text{-lim sup}_h I_{h|_{\mu_c=0}}^\sharp \leq \lim_{\mu_c \rightarrow 0} (\Gamma\text{-lim}_h I_{h|_{\mu_c > 0}}^\sharp) =: I_0^{\sharp, 0}. \quad (7.2)$$

7.2 A lower bound for the “membrane” lower bound

Let us consider the following energy functional $I_0^{\sharp, \text{mem}} : X \rightarrow \bar{\mathbb{R}}$:

$$I_0^{\sharp, \text{mem}}(\varphi, \bar{R}) := \begin{cases} \int_\omega [W_{\text{mp}}^{\text{hom}, 0}(\nabla_{(\eta_1, \eta_2)} \text{Av} \cdot \varphi(\eta_1, \eta_2, \eta_3), \bar{R}) + W_{\text{curv}}^{\text{hom}}(\mathfrak{K}_s)] d\omega & \text{if } (\varphi, \bar{R}) \in \mathcal{A}_0^{\text{mem}}, \\ +\infty & \text{else in } X, \end{cases}$$

where $W_{\text{mp}}^{\text{hom}, 0}$ is defined in (5.7) and the admissible set is now

$$\mathcal{A}_0^{\text{mem}} := \{(\varphi, \bar{R}) \in X \mid \text{sym}(\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \text{Av.} \varphi \in L^2(\Omega_1, \mathbb{M}^{2 \times 2}), \\ \bar{R} \in W^{1, 1+p+q}(\omega, \text{SO}(3)), \varphi|_{r_0^1}^\#(\eta) = g_d^\#(\eta) = g_d(\eta_1, \eta_2, 0)\},$$

with a distributional meaning for $(\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \text{Av.} \varphi$. Note that $I_0^{\#, \text{mem}} = I_0^{\#, 0}$. We next show

LEMMA 7.2 (Membrane lower bound) For arbitrary $(\varphi_0^\#, \bar{R}_0^\#) \in X$,

$$I_0^{\#, \text{mem}}(\varphi_0, \bar{R}_0) \leq \liminf_{h_j} I_{h_j}^{\#, \text{mem}}(\varphi_{h_j}^\#, \bar{R}_{h_j}^\#)$$

whenever

$$\varphi_{h_j}^\# \rightarrow \varphi_0^\# \text{ in } L^r(\Omega_1, \mathbb{R}^3), \quad \bar{R}_{h_j}^\# \rightarrow \bar{R}_0^\# \text{ in } L^{1+p+q}(\Omega_1, \text{SO}(3)).$$

Proof. Observe that we can restrict attention to sequences $(\varphi_{h_j}^\#, \bar{R}_{h_j}^\#) \in X$ such that $I_{h_j}^{\#, \text{mem}}(\varphi_{h_j}^\#, \bar{R}_{h_j}^\#) < \infty$ since otherwise the statement is true anyway. If $I_{h_j}^{\#, \text{mem}}(\varphi_{h_j}^\#, \bar{R}_{h_j}^\#) < \infty$, then equi-coercivity with respect to rotations remains untouched by a change from W_{mp} to $W_{\text{mp}}^{\text{mem}}$ in the local energy. Hence, as usual by now, we can restrict attention to sequences of rotations $\bar{R}_{h_j}^\#$ converging weakly to some $\bar{R}_0 \in W^{1, 1+p+q}(\omega, \text{SO}(3))$, defined over the two-dimensional domain ω only. However, we cannot conclude that φ_0 is independent of the transverse variable, in contrast to the case with $\mu_c > 0$.

Along sequences $(\varphi_{h_j}^\#, \bar{R}_{h_j}^\#) \in X$ with finite energy the product $(1/h_j) \langle \bar{R}_{h_j, 3}, \partial_{\eta_3} \varphi_{h_j}^\# \rangle$ remains bounded but otherwise indeterminate. Therefore, a trivial lower bound is obtained by minimizing the effect in the 33-component in the local energy $W_{\text{mp}}^{\text{mem}}$. To do this, we need some calculations: for smooth $\varphi : \Omega_1 \rightarrow \mathbb{R}^3$, $\bar{R} : \omega \subset \mathbb{R}^2 \rightarrow \text{SO}(3)$ define the “director” vector $b^* = (0, 0, \varrho^*)^T \in \mathbb{R}^3$ with $b(\varrho) = (0, 0, \varrho)^T \in \mathbb{R}^3$ formally through

$$W_{\text{mp}}^{\text{hom}, 0}(\nabla_{(\eta_1, \eta_2)} \varphi, \bar{R}) = W_{\text{mp}}^{\text{mem}}(\bar{R}^T (\nabla_{(\eta_1, \eta_2)} \varphi | b^*)) := \inf_{\varrho \in \mathbb{R}} W_{\text{mp}}^{\text{mem}}(\bar{R}^T (\nabla_{(\eta_1, \eta_2)} \varphi | b(\varrho))).$$

The real number ϱ^* which realizes this infimum can be explicitly determined. Without giving the calculation, which follows as in (6.2), we obtain

$$\varrho^* = 1 - \frac{\lambda}{2\mu + \lambda} [(\langle \nabla_{(\eta_1, \eta_2)} \varphi | 0 \rangle, \bar{R}) - 2] = 1 - \frac{\lambda}{2\mu + \lambda} \text{tr}[\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \varphi - \mathbb{1}_2)].$$

Note that if $\bar{R} \in \text{SO}(3)$ and $\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \varphi - \mathbb{1}_2) \in L^2(\Omega_1, \mathbb{R}^3)$ one has $\varrho^* \in L^2(\Omega_1, \mathbb{R}^3)$. For $W_{\text{mp}}^{\text{hom}, 0}(\nabla_{(\eta_1, \eta_2)} \varphi, \bar{R}) := W_{\text{mp}}(\bar{R}^T (\nabla_{(\eta_1, \eta_2)} \varphi | b^*))$ after a lengthy but straightforward computation we obtain

$$W_{\text{mp}}^{\text{hom}, 0}(\nabla_{(\eta_1, \eta_2)} \varphi, \bar{R}) := \mu \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \varphi - \mathbb{1}_2)\|^2 \\ + \frac{\mu\lambda}{2\mu + \lambda} \text{tr}[\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \varphi - \mathbb{1}_2)]^2.$$

Along the sequence $(\varphi_{h_j}^\#, \bar{R}_{h_j}^\#)$ we have therefore, by construction,

$$W_{\text{mp}}^{\text{mem}}(\bar{R}_{h_j}^{\#, T} \nabla_{\eta_j}^{h_j} \varphi_{h_j}^\#) = W_{\text{mp}}^{\text{mem}}\left(\bar{R}_{h_j}^{\#, T} \left(\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\# \Big| \frac{1}{h_j} \partial_{\eta_3} \varphi_{h_j}^\#\right)\right) \geq W_{\text{mp}}^{\text{hom}, 0}(\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^\#, \bar{R}_{h_j}^\#).$$

Hence, integrating and taking the lim inf we also have

$$\liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}(\bar{R}_{h_j}^{\sharp,T} \nabla_{\eta_j}^{h_j} \varphi_{h_j}^{\sharp}) \, dV_{\eta} \geq \liminf_{h_j} \int_{\Omega_1} W_{\text{mp}}^{\text{hom},0}(\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^{\sharp}, \bar{R}_{h_j}^{\sharp}) \, dV_{\eta}. \quad (7.3)$$

As in (6.4) (and subsequently) the proof of statement 7.2 would be finished if we could show weak convergence of $\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^{\sharp}$ in $L^2(\Omega_1, \mathbb{M}^{3 \times 3})$ whenever $\varphi_{h_j}^{\sharp} \rightarrow \varphi_0^{\sharp}$ strongly in $L^r(\Omega_1, \mathbb{R}^3)$ and $I_{h_j}^{\sharp, \text{mem}}(\varphi_{h_j}^{\sharp}, \bar{R}_{h_j}^{\sharp}) < \infty$. Boundedness and weak convergence of the sequence $\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^{\sharp}$ in $L^2(\Omega_1, \mathbb{M}^{3 \times 3})$ is, however, not clear at all, since we now basically control only the “symmetric intrinsic” term $\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \varphi - \mathbb{1}_2)\|^2$ in the integrand. Instead, we will prove a weaker statement, namely that

$$(\bar{R}_{1, h_j}^{\sharp} | \bar{R}_{2, h_j}^{\sharp})^T \nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^{\sharp} \rightharpoonup (\bar{R}_{1, 0}^{\sharp} | \bar{R}_{2, 0}^{\sharp})^T \nabla_{(\eta_1, \eta_2)} \varphi_0^{\sharp} \in L^2(\Omega_1, \mathbb{M}^{2 \times 2}), \quad (7.4)$$

after showing that the above expressions have a well-defined distributional meaning along the sequence, since $\nabla_{(\eta_1, \eta_2)} \varphi_{h_j}^{\sharp}$ has no classical meaning if we only know that $\varphi_{h_j}^{\sharp} \in L^r(\Omega_1, \mathbb{R}^3)$.

In order to give a precise distributional meaning to the expression in (7.4) along the sequence we first define, for smooth $\phi \in C^\infty(\bar{\Omega}_1, \mathbb{R}^3)$ and $\bar{R} \in W^{1, 1+p+q}(\Omega_1, \text{SO}(3))$, an intermediate function Ψ ,

$$\Psi : \Omega_1 \rightarrow \mathbb{R}^2, \quad \Psi(\eta_1, \eta_2, \eta_3) := \begin{pmatrix} \langle \bar{R}_1, \phi \rangle \\ \langle \bar{R}_2, \phi \rangle \end{pmatrix}.$$

This implies that $\Psi \in W^{1, 1+p+q}(\Omega_1, \mathbb{R}^2)$. We have

$$\begin{aligned} (\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi &= \begin{pmatrix} \langle \bar{R}_1, \partial_{\eta_1} \phi \rangle & \langle \bar{R}_1, \partial_{\eta_2} \phi \rangle \\ \langle \bar{R}_2, \partial_{\eta_1} \phi \rangle & \langle \bar{R}_2, \partial_{\eta_2} \phi \rangle \end{pmatrix}, \quad D(\bar{R}_1 | \bar{R}_2) \cdot \phi := \begin{pmatrix} \langle \partial_{\eta_1} \bar{R}_1, \phi \rangle & \langle \partial_{\eta_2} \bar{R}_1, \phi \rangle \\ \langle \partial_{\eta_1} \bar{R}_2, \phi \rangle & \langle \partial_{\eta_2} \bar{R}_2, \phi \rangle \end{pmatrix}, \\ \nabla \Psi &= \begin{pmatrix} \partial_{\eta_1} \langle \bar{R}_1, \phi \rangle & \partial_{\eta_2} \langle \bar{R}_1, \phi \rangle \\ \partial_{\eta_1} \langle \bar{R}_2, \phi \rangle & \partial_{\eta_2} \langle \bar{R}_2, \phi \rangle \end{pmatrix} = (\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi + D(\bar{R}_1 | \bar{R}_2) \cdot \phi. \end{aligned}$$

The last equality shows

$$(\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi := \nabla_{(\eta_1, \eta_2)} \Psi - D(\bar{R}_1 | \bar{R}_2) \cdot \phi. \quad (7.5)$$

We note the local estimate

$$\begin{aligned} \|\text{sym} \nabla_{(\eta_1, \eta_2)} \Psi\|^2 &= \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi) + \text{sym}(D(\bar{R}_1 | \bar{R}_2) \cdot \phi)\|^2 \\ &\leq 2\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi)\|^2 + 2\|\text{sym}(D(\bar{R}_1 | \bar{R}_2) \cdot \phi)\|^2 \\ &\leq 2\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi)\|^2 + 2\|D(\bar{R}_1 | \bar{R}_2) \cdot \phi\|^2 \\ &\leq 2\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi)\|^2 + 2\|D(\bar{R}_1 | \bar{R}_2)\|^2 \cdot \|\phi\|^2. \end{aligned}$$

The last inequality implies after integration and Hölder’s inequality (reminder: $r = \frac{2(1+p+q)}{(1+p+q)-2}$, cf. (5.1))

$$\begin{aligned} \int_{\Omega_1} \|\text{sym} \nabla_{(\eta_1, \eta_2)} \Psi\|^2 \, dV_{\eta} \\ \leq 2 \int_{\Omega_1} \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi)\|^2 \, dV_{\eta} + 2\|\bar{R}\|_{W^{1, 1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1, \mathbb{R}^3)}^2. \end{aligned}$$

Moreover,

$$\int_{\Omega_1} (\|\text{sym } \nabla_{(\eta_1, \eta_2)} \Psi\|^2 + \|\Psi\|^2) \, dV_\eta \leq 2 \int_{\Omega_1} \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi)\|^2 \, dV_\eta + 2\|\bar{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1, \mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1, \mathbb{R}^3)}^2,$$

since $\|\Psi\|^2 = \langle \bar{R}_1, \phi \rangle^2 + \langle \bar{R}_2, \phi \rangle^2 \leq \|\bar{R}_1\|^2 \|\phi\|^2 + \|\bar{R}_2\|^2 \|\phi\|^2 = 2\|\phi\|^2$. Furthermore, adding and subtracting $\mathbb{1}_2$ yields

$$\begin{aligned} & \int_{\Omega_1} (\|\text{sym } \nabla_{(\eta_1, \eta_2)} \Psi\|^2 + \|\Psi\|^2) \, dV_\eta \\ & \leq 2 \int_{\Omega_1} \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi)\|^2 \, dV_\eta + 2\|\bar{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1, \mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1, \mathbb{R}^3)}^2 \\ & = 2 \int_{\Omega_1} \|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi - \mathbb{1}_2 + \mathbb{1}_2)\|^2 \, dV_\eta \\ & \quad + 2\|\bar{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1, \mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1, \mathbb{R}^3)}^2 \\ & \leq \int_{\Omega_1} (4\|\text{sym}((\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi - \mathbb{1}_2)\|^2 + 4\|\mathbb{1}_2\|^2) \, dV_\eta \\ & \quad + 2\|\bar{R}\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi\|_{L^r(\Omega_1, \mathbb{R}^3)}^2 + 2\|\phi\|_{L^2(\Omega_1, \mathbb{R}^3)}^2. \end{aligned}$$

Hence, considering $\phi_{h_j}^\sharp$ instead of ϕ , along the sequence $(\phi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) \in X$ with

$$I_{h_j}^{\sharp, \text{mem}}(\phi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) < \infty,$$

and with the distributional meaning of the gradient on $\phi_{h_j}^\sharp$, we obtain the additional uniform bound

$$\begin{aligned} \int_{\Omega_1} (\|\text{sym } \nabla_{(\eta_1, \eta_2)} \Psi\|^2 + \|\Psi\|^2) \, dV_\eta & \leq \frac{4}{\mu} I_{h_j}^{\sharp, \text{mem}}(\phi_{h_j}^\sharp, \bar{R}_{h_j}^\sharp) + \int_{\Omega_1} 4\|\mathbb{1}_2\|^2 \, dV_\eta \\ & \quad + 2\|\bar{R}_{h_j}^\sharp\|_{W^{1,1+p+q}(\Omega_1)}^2 \|\phi_{h_j}^\sharp\|_{L^r(\Omega_1, \mathbb{R}^3)}^2 + 2\|\phi_{h_j}^\sharp\|_{L^2(\Omega_1, \mathbb{R}^3)}^2 \\ & < \infty. \end{aligned}$$

The classical Korn second inequality without boundary conditions on a Lipschitz domain [68, Prop. 1.1] implies therefore that

$$\begin{aligned} \infty & > \int_{\Omega_1} (\|\text{sym } \nabla_{(\eta_1, \eta_2)} \Psi_{h_j}\|^2 + \|\Psi_{h_j}\|^2) \, dV_\eta \\ & = \int_{-1/2}^{1/2} \left[\int_\omega (\|\text{sym } \nabla_{(\eta_1, \eta_2)} \Psi_{h_j}(\eta_1, \eta_2, \eta_3)\|^2 + \|\Psi_{h_j}(\eta_1, \eta_2, \eta_3)\|^2) \, d\omega \right] \, d\eta_3 \\ & \geq \int_{-1/2}^{1/2} \left[c_K^+ \int_\omega (\|\nabla_{(\eta_1, \eta_2)} \Psi_{h_j}(\eta_1, \eta_2, \eta_3)\|^2 + \|\Psi_{h_j}(\eta_1, \eta_2, \eta_3)\|^2) \, d\omega \right] \, d\eta_3, \end{aligned}$$

which yields the boundedness of $\nabla_{(\eta_1, \eta_2)} \Psi_{h_j}$ in $L^2(\Omega_1, \mathbb{R}^2)$ and weak convergence of this sequence of gradients to a limit. By construction we already know that $\Psi_{h_j} \rightharpoonup \Psi_0 \in L^2(\Omega_1, \mathbb{R}^2)$ (by the

assumed strong convergence of \bar{R}_{h_j} and $\phi_{h_j}^\sharp$). Hence $\nabla_{(\eta_1, \eta_2)} \Psi_{h_j}$ converges weakly to $\nabla_{(\eta_1, \eta_2)} \Psi_0$. Since we know as well that $\partial_{\eta_i} \bar{R}_{h_j}^\sharp \rightharpoonup \partial_{\eta_i} \bar{R}_0^\sharp$ in $L^{1+p+q}(\Omega_1, \mathbb{M}^{3 \times 3})$, $i = 1, 2$, and $\phi_{h_j}^\sharp \rightarrow \phi_0^\sharp$ in $L^r(\Omega_1, \mathbb{R}^3)$ we obtain

$$D(\bar{R}_{1,h_j}^\sharp | \bar{R}_{2,h_j}^\sharp) \cdot \phi_{h_j}^\sharp \rightharpoonup D(\bar{R}_{1,0}^\sharp | \bar{R}_{2,0}^\sharp) \cdot \phi_0^\sharp \in L^2(\Omega_1, \mathbb{M}^{2 \times 2}).$$

Looking now back at (7.5) shows that

$$(\bar{R}_{1,h_j}^\sharp | \bar{R}_{2,h_j}^\sharp)^T \nabla_{(\eta_1, \eta_2)} \phi_{h_j}^\sharp \in L^2(\Omega_1, \mathbb{M}^{2 \times 2}) \tag{7.6}$$

is a well defined expression with distributional meaning of $\nabla_{(\eta_1, \eta_2)} \phi_{h_j}^\sharp$ for which (7.4) holds. Due to the convexity of $W_{mp}^{\text{hom},0}$ in the argument $\text{sym}(\bar{R}_1 | \bar{R}_2)^T \nabla_{(\eta_1, \eta_2)} \phi$, we may pass to the limit in (7.3) to obtain

$$\liminf_{h_j} \int_{\Omega_1} W_{mp}(\bar{R}_{h_j}^{\sharp, T} \nabla_{\eta}^{h_j} \phi_{h_j}^\sharp) dV_{\eta} \geq \int_{\Omega_1} W_{mp}^{\text{hom},0}(\nabla_{(\eta_1, \eta_2)} \phi_0^\sharp, \bar{R}_0^\sharp) dV_{\eta}. \tag{7.7}$$

The convexity of $W_{mp}^{\text{hom},0}$ and Jensen’s inequality (5.3) show then

$$\begin{aligned} \int_{\omega} W_{mp}^{\text{hom},0}(\nabla_{(\eta_1, \eta_2)} \text{Av} \cdot \varphi(\eta_1, \eta_2), \bar{R}) d\omega &\leq \int_{\omega} \int_{-1/2}^{1/2} W_{mp}^{\text{hom},0}(\nabla_{(\eta_1, \eta_2)} \varphi(\eta_1, \eta_2, \eta_3), \bar{R}) d\eta_3 d\omega \\ &= \int_{\Omega_1} W_{mp}^{\text{hom},0}(\nabla_{(\eta_1, \eta_2)} \varphi(\eta_1, \eta_2, \eta_3), \bar{R}) dV_{\eta}. \end{aligned} \tag{7.8}$$

Combining (7.8) with (7.7) gives

$$\liminf_{h_j} \int_{\Omega_1} W_{mp}(\bar{R}_{h_j}^{\sharp, T} \nabla_{\eta}^{h_j} \phi_{h_j}^\sharp) dV_{\eta} \geq \int_{\omega} W_{mp}^{\text{hom},0}(\nabla_{(\eta_1, \eta_2)} \text{Av} \cdot \varphi(\eta_1, \eta_2), \bar{R}) d\omega.$$

The proof of Lemma 7.2 is finished along the lines of (6.4). Note that (7.6) definitely does not yield control of $\nabla_{(\eta_1, \eta_2)} \phi_{h_j}^\sharp$ in $L^2(\Omega_1, \mathbb{M}^{3 \times 2})$. \square

Proof of Theorem 5.3. To finish the proof of Γ -convergence for zero Cosserat couple modulus (Theorem 5.3) we observe first that Lemma 7.2 implies that

$$I_0^{\sharp, \text{mem}} \leq \Gamma\text{-}\liminf_{h_j} I_{h_j}^{\sharp, \text{mem}},$$

which is “almost” a \liminf result for $I_h^{\sharp, \text{mem}}$ since $I_0^{\sharp, \text{mem}}$ could be strictly smaller. We combine this result with the chain of inequalities (7.2) to conclude that on $X = L^r(\Omega_1, \mathbb{R}^3) \times L^{1+p+q}(\Omega_1, \text{SO}(3))$,

$$\begin{aligned} I_0^{\sharp, \text{mem}} &\leq \Gamma\text{-}\liminf I_h^{\sharp, \text{mem}} \leq \Gamma\text{-}\liminf I_{h|\mu_c=0}^\sharp \\ &\leq \Gamma\text{-}\limsup I_{h|\mu_c=0}^\sharp \leq \lim_{\mu_c \rightarrow 0} \Gamma\text{-}\lim I_{h|\mu_c>0}^\sharp =: I_0^{\sharp, 0}. \end{aligned} \tag{7.9}$$

Since, however, $I_0^{\sharp, \text{mem}} = I_0^{\sharp, 0}$, the last inequality is in fact an equality, which shows that

$$\Gamma\text{-}\lim_{h_j|_{\mu_c=0}} I_{h_j}^{\sharp} = I_0^{\sharp, 0}.$$

This gives us complete information on the behaviour of sequences of minimizing problems for $\mu_c = 0$, should such sequences exist and converge to a limit in the encompassing space X . \square

8. Conclusion

We have justified the dimensional reduction of a geometrically exact Cosserat bulk model to its two-dimensional counterpart by use of Γ -convergence arguments. The underlying Cosserat bulk model features already independent rotations which may be identified with the averaged lattice rotations in defective elastic crystals if $\mu_c = 0$. Thus the appearance of an independent director field R_3 is natural and not primarily due to the dimensional reduction/relaxation step. The argument is given for plates (flat reference configuration) only, but it is straightforward to extend the result to genuine shells with curvilinear reference configuration, and it should be noted that the extension to shells is independent of geometrical features of this curvilinear reference configuration: the inclusion of transverse shear effects makes the distinction between elliptic, parabolic and hyperbolic surfaces in a certain sense irrelevant. A welcome feature of the resulting Γ -limit for the defective crystal case $\mu_c = 0$ is its linearization consistency.

Apart for bending terms, the resulting Γ -limit is similar to the previously given formal development in [45] and constitutes therefore a rigorous mathematical justification of Reissner–Mindlin type models. Future work will discuss the engineering implications of our results as far as the numerical value of the Cosserat couple modulus μ_c and its relation to the transverse shear modulus in classical Reissner–Mindlin type theories is concerned.

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