The Allen–Cahn action functional in higher dimensions

Luca Mugnai[†] and Matthias Röger[‡]

Max Planck Institute for Mathematics in the Sciences,
Inselstr. 22, D-04103 Leipzig, Germany

The Allen–Cahn action functional is related to the probability of rare events in the stochastically perturbed Allen–Cahn equation. Formal calculations suggest a *reduced action functional* in the sharp interface limit. We prove the corresponding lower bound in two and three space dimensions. One difficulty is that diffuse interfaces may collapse in the limit. We therefore consider the limit of diffuse surface area measures and introduce a generalized velocity and generalized reduced action functional in a class of evolving measures.

1. Introduction

In this paper we study the (renormalized) *Allen–Cahn action functional*

$$\mathcal{S}_\varepsilon(u) := \int_0^T \int_\Omega \left(\sqrt{\varepsilon}\partial_t u + \frac{1}{\sqrt{\varepsilon}} \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} W'(u) \right) \right)^2 \mathrm{d}x\,\mathrm{d}t. \tag{1.1}$$

This functional arises in the analysis of the stochastically perturbed Allen–Cahn equation [2, 20, 12, 30, 7, 9, 11] and is related to the probability of rare events such as switching between deterministically stable states.

Compared to the purely deterministic setting, stochastic perturbations add new features to the theory of phase separations, and the analysis of action functionals has drawn some attention [7, 12, 17, 18, 26]. Kohn *et al.* [17] considered the *sharp-interface limit* $\varepsilon \to 0$ of $\mathcal{S}_\varepsilon$ and identified a *reduced action functional* that is more easily accessible for a qualitative analysis. The sharp interface limit reveals a connection between minimizers of $\mathcal{S}_\varepsilon$ and mean curvature flow.

The reduced action functional in [17] is defined for phase indicator functions $u : (0, T) \times \Omega \to \{-1, 1\}$ with the additional properties that the measure of the phase $\{u(t, \cdot) = 1\}$ is continuous and the common boundary of the two phases $\{u = 1\}$ and $\{u = -1\}$ is, apart from a countable set of singular times, given as union of smoothly evolving hypersurfaces $\Sigma := \bigcup_{t \in (0,T)} \{t\} \times \Sigma_t$. The

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[†] E-mail: mugnai@mis.mpg.de
[‡] E-mail: roeger@mis.mpg.de

reduced action functional is then defined as

$$\mathcal{S}^0(u) := c_0 \int_0^T \int_{\Sigma_t} |v(t,x) - H(t,x)|^2 \, \mathrm{d}\mathcal{H}^{n-1}(x) \, \mathrm{d}t + 4\mathcal{S}_{\mathrm{nuc}}^0(u), \qquad (1.2)$$

$$\mathcal{S}_{\mathrm{nuc}}^0(u) := 2c_0 \sum_i \mathcal{H}^{n-1}(\Sigma_i), \qquad (1.3)$$

where Σ_i denotes the ith component of Σ at the time of creation, v denotes the normal velocity of the evolution $(\Sigma_t)_{t \in (0,T)}$, $H(t, \cdot)$ denotes the mean curvature vector of Σ_t, and the constant c_0 is determined by W,

$$c_0 := \int_{-1}^1 \sqrt{2W(s)} \, \mathrm{d}s. \qquad (1.4)$$

(See Section 9 for a more rigourous definition of $\mathcal{S}^0$.)

Several arguments suggest that $\mathcal{S}^0$ describes the Gamma-limit of $\mathcal{S}_\varepsilon$:

- The *upper bound* necessary for the Gamma-convergence was formally proved [17] by the construction of good 'recovery sequences'.
- The *lower bound* was proved in [17] for sequences $(u_\varepsilon)_{\varepsilon>0}$ such that the associated 'energy measures' have *equipartitioned energy* and *single multiplicity* as $\varepsilon \to 0$.
- In one space dimension Reznikoff and Tonegawa [26] proved that $\mathcal{S}_\varepsilon$ Gamma-converges to an appropriate relaxation of the one-dimensional version of $\mathcal{S}^0$.

The approach used in [17] is based on the evolution of the phases and is sensible to cancellations of phase boundaries in the sharp interface limit. Therefore in [17] a sharp lower bound is achieved only under a single-multiplicity assumption for the limit of the diffuse interfaces. As a consequence, it could not be excluded that creating multiple interfaces reduces the action.

In the present paper we prove a sharp lower bound of the functional $\mathcal{S}_\varepsilon$ in space dimensions $n = 2, 3$ without any additional restrictions on the approximate sequences.

To circumvent problems with cancellations of interfaces we analyze the evolution of the (diffuse) *surface-area measures*, which makes information available that is lost in the limit of phase fields. With this aim we generalize the functional $\mathcal{S}^0$ to a suitable class of *evolving energy measures* and introduce a generalized formulation of velocity, similar to Brakke's generalization of mean curvature flow [5].

Let us informally describe our approach and main results. Comparing the two functionals $\mathcal{S}_\varepsilon$ and $\mathcal{S}^0$, the first and second term of the sum in the integrand (1.1) describe a 'diffuse velocity' and 'diffuse mean curvature' respectively. We will make this statement precise in (6.13) and (7.1). The mean curvature is given by the first variation of the area functional, and a lower estimate for the square integral of the diffuse mean curvature is available in a time-independent situation [28]. The velocity of the evolution of the phase boundaries is determined by the time derivative of the surface-area measures, and the nucleation term in the functional $\mathcal{S}^0$ in fact describes a singular part of this time derivative.

Our first main result is a compactness result: the diffuse surface-area measures converge to an evolution of measures with a square-integrable generalized mean curvature and a square-integrable generalized velocity. In the class of such evolutions of measures we provide a generalized formulation of the reduced action functional. We prove a lower estimate that counts the propagation cost with the multiplicity of the interface. This shows that it is more expensive to move phase

boundaries with higher multiplicity. Finally we prove two statements on the Gamma-convergence (with respect to $L^1(\Omega_T)$) of the action functional. The first result is for evolutions in the domain of $\mathcal{S}^0$ that have nucleations only at the initial time. This is in particular desirable since minimizers of $\mathcal{S}^0$ are supposed to be in this class. The second result proves the Gamma-convergence in $L^1(\Omega_T)$ under an assumption on the structure of the set of measures arising as sharp interface limits of sequences with uniformly bounded action.

We give a precise statement of our main results in Section 4. In the remainder of this introduction we describe some background and motivation.

1.1 *Deterministic phase field models and sharp interface limits*

Most *diffuse interface models* are based on the *Van der Waals–Cahn–Hilliard* energy

$$E_\varepsilon(u) := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \mathrm{d}x. \tag{1.5}$$

The energy E_ε favors a decomposition of Ω into two regions (phases) where $u \approx -1$ and $u \approx 1$, separated by a transition layer (diffuse interface) of thickness of order ε. Modica and Mortola [22, 21] proved that E_ε Gamma-converges (with respect to L^1-convergence) to a constant multiple of the perimeter functional $\mathcal{P}$, restricted to phase indicator functions,

$$E_\varepsilon \to c_0 \mathcal{P}, \quad \mathcal{P}(u) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \mathrm{d}|\nabla u| & \text{if } u \in BV(\Omega; \{-1, 1\}), \\ \infty & \text{otherwise.} \end{cases}$$

$\mathcal{P}$ measures the surface area of the phase boundary $\partial^*\{u = 1\} \cap \Omega$. In this sense E_ε describes a diffuse approximation of the surface-area functional.

Various tighter connections between the functionals E_ε and $\mathcal{P}$ have been proved. We mention here just two that are important for our analysis. The (accelerated) L^2-gradient flow of E_ε is given by the *Allen–Cahn equation*

$$\varepsilon \partial_t u = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \tag{1.6}$$

for phase fields in the time-space cylinder $(0, T) \times \Omega$. It is proved in different formulations [24, 8, 16] that (1.6) converges to the *mean curvature flow*

$$H(t, \cdot) = v(t, \cdot) \tag{1.7}$$

for the evolution of phase boundaries.

Another connection between the first variations of E_ε and $\mathcal{P}$ is expressed in a (modified) conjecture of De Giorgi [6]: Considering

$$\mathcal{W}_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon} \left(-\varepsilon \Delta u + \frac{1}{\varepsilon} W'(u) \right)^2 \mathrm{d}x \tag{1.8}$$

the sum $E_\varepsilon + \mathcal{W}_\varepsilon$ Gamma-converges up to the constant factor c_0 to the sum of the perimeter functional and the *Willmore functional* $\mathcal{W}$,

$$E_\varepsilon + \mathcal{W}_\varepsilon \to c_0 \mathcal{P} + c_0 \mathcal{W}, \qquad \mathcal{W}(u) = \int_\Gamma H^2 \, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.9}$$

where Γ denotes the phase boundary $\partial^*\{u = 1\} \cap \Omega$. This statement was recently proved by Röger and Schätzle [28] in space dimensions $n = 2, 3$ and is an essential ingredient to obtain the lower bound for the action functional.

1.2 *Stochastic interpretation of the action functional*

Phenomena such as the nucleation of a new phase or switching between two (local) energy minima require an energy barrier crossing and are out of the scope of deterministic models that are energy dissipative. If thermal fluctuations are taken into account such an energy barrier crossing becomes possible. In [17] 'thermally activated switching' was considered for the *stochastically perturbed Allen–Cahn equation*

$$\varepsilon \partial_t u = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) + \sqrt{2\gamma}\, \eta_\lambda. \tag{1.10}$$

Here $\gamma > 0$ is a parameter that represents the temperature of the system, η is a time-space white noise, and η_λ is a spatial regularization with $\eta_\lambda \to \eta$ as $\lambda \to 0$. This regularization is necessary for $n \geqslant 2$ since the white noise is too singular to ensure well-posedness of (1.10) in higher space dimensions.

Large deviation theory and (extensions of) results by Wentzell and Freidlin [14, 13] yield an estimate on the probability distribution of solutions of stochastic ODEs and PDEs in the small-noise limit. This estimate is expressed in terms of a (deterministic) action functional. For instance, thermally activated switching within a time $T > 0$ is described by the set of paths

$$\mathcal{B} := \{u(0, \cdot) = -1, \, \|u(t, \cdot) - 1\|_{L^\infty(\Omega)} \leqslant \delta \text{ for some } t \leqslant T\}, \tag{1.11}$$

where $\delta > 0$ is a fixed constant. The probability of switching for solutions of (1.10) then satisfies

$$\lim_{\gamma \to 0} \gamma \ln \mathsf{Prob}(\mathcal{B}) = - \inf_{u \in \mathcal{B}} \mathcal{S}_\varepsilon^{(\lambda)}(u). \tag{1.12}$$

Here $\mathcal{S}_\varepsilon^{(\lambda)}$ is the action functional associated to (1.10) and it converges (formally) to the action functional $\mathcal{S}_\varepsilon$ as $\lambda \to 0$ [17]. Large deviation theory not only estimates the probability of rare events but also identifies the 'most likely switching path' as the minimizer u in (1.12).

We focus here on the sharp interface limit $\varepsilon \to 0$ of the action functional $\mathcal{S}_\varepsilon$. The small parameter $\varepsilon > 0$ corresponds to a specific diffusive scaling of the time and space domains. This choice was identified [7, 17] as particularly interesting, exhibiting a competition between *nucleation versus propagation* to achieve the optimal switching. Depending on the value of $|\Omega|^{1/d}/\sqrt{T}$ a cascade of more and more complex spatial patterns is observed [7, 17, 18]. The interest in the sharp interface limit is motivated by an interest in applications where the switching time is small compared to the deterministic time scale (see for instance [19]).

1.3 *Organization*

We fix some notation and assumptions in the next section. In Section 3 we introduce the concept of L^2-flows and generalized velocity. Our main results are stated in Section 4 and proved in Sections 5–8. We discuss some implications for the Gamma-convergence of the action functional in Section 9. Finally, in the Appendix we collect some definitions from geometric measure theory.

2. Notation and assumptions

Throughout the paper we will adopt the following notation: Ω is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary; $T > 0$ is a real number and $\Omega_T := (0, T) \times \Omega$; $x \in \Omega$ and $t \in (0, T)$ denote the space and time variables respectively; ∇ and Δ denote the spatial gradient and Laplacian, and ∇' the full gradient in $\mathbb{R} \times \mathbb{R}^n$.

We choose W to be the standard quartic double-well potential

$$W(r) = \frac{1}{4}(1 - r^2)^2.$$

For a family $(\mu^t)_{t \in (0,T)}$ of measures we denote by $\mathcal{L}^1 \otimes \mu^t$ the product measure defined by

$$(\mathcal{L}^1 \otimes \mu^t)(\eta) := \int_0^T \mu^t(\eta(t, \cdot)) \, \mathrm{d}t$$

for any $\eta \in C_c^0(\Omega_T)$.

We next state our main assumptions.

ASSUMPTION 2.1 Let $n = 2, 3$ and let $(u_\varepsilon)_{\varepsilon > 0}$ be a sequence of smooth functions such that for all $\varepsilon > 0$,

$$\mathcal{S}_\varepsilon(u_\varepsilon) \leqslant \Lambda_1, \tag{A1}$$

$$\int_\Omega \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)(0, x) \, \mathrm{d}x \leqslant \Lambda_2, \tag{A2}$$

where the constants Λ_1, Λ_2 are independent of $\varepsilon > 0$. Moreover,

$$\nabla u_\varepsilon \cdot \nu_\Omega = 0 \quad \text{on } [0, T] \times \partial\Omega. \tag{A3}$$

REMARK 2.2 It follows from (A3) that for any $0 \leqslant t_0 \leqslant T$,

$$\int_0^{t_0} \int_\Omega \left(\sqrt{\varepsilon}\partial_t u_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\left(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\right)\right)^2 \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^{t_0} \int_\Omega \varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}\left(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\right)^2 \mathrm{d}x \, \mathrm{d}t$$

$$+ 2\int_\Omega \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)(t_0, x) \, \mathrm{d}x - 2\int_\Omega \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)(0, x) \, \mathrm{d}x.$$

By the uniform bounds (A1), (A2) this implies that

$$\int_{\Omega_T} \varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}\left(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\right)^2 \mathrm{d}x \, \mathrm{d}t \leqslant \Lambda_3, \tag{2.1}$$

$$\max_{0 \leqslant t \leqslant T} \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right)(t, x) \, \mathrm{d}x \leqslant \Lambda_4, \tag{2.2}$$

where

$$\Lambda_3 := \Lambda_1 + 2\Lambda_2, \qquad \Lambda_4 := \frac{1}{2}\Lambda_1 + \Lambda_2.$$

REMARK 2.3 Our arguments would also work for any boundary conditions for which the scalar product $\partial_t u \nabla u \cdot \nu_\Omega$ vanishes on $\partial\Omega$, in particular for time-independent Dirichlet conditions or periodic boundary conditions.

We set

$$w_\varepsilon := -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \tag{2.3}$$

and define for $\varepsilon > 0$, $t \in (0, T)$ a Radon measure μ_ε^t on $\overline{\Omega}$ by

$$\mu_\varepsilon^t := \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2(t, \cdot) + \frac{1}{\varepsilon} W(u_\varepsilon(t, \cdot)) \right) \mathcal{L}^n, \tag{2.4}$$

and for $\varepsilon > 0$ measures $\mu_\varepsilon, \alpha_\varepsilon$ on $\overline{\Omega}_T$ by

$$\mu_\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \mathcal{L}^{n+1}, \tag{2.5}$$

$$\alpha_\varepsilon := (\varepsilon^{1/2} \partial_t u_\varepsilon + \varepsilon^{-1/2} w_\varepsilon)^2 \mathcal{L}^{n+1}. \tag{2.6}$$

Restricting ourselves to a subsequence $\varepsilon \to 0$ if necessary, we may assume that

$$\mu_\varepsilon \to \mu \quad \text{as Radon measures on } \overline{\Omega}_T, \tag{2.7}$$

$$\alpha_\varepsilon \to \alpha \quad \text{as Radon measures on } \overline{\Omega}_T, \tag{2.8}$$

for two Radon measures μ, α on $\overline{\Omega}_T$, and

$$\alpha(\overline{\Omega}_T) = \liminf_{\varepsilon \to 0} \alpha_\varepsilon(\Omega_T). \tag{2.9}$$

3. L^2-flows

We will show that the uniform bound on the action implies the existence of a square-integrable weak mean curvature and the existence of a square-integrable *generalized velocity*. The formulation of weak mean curvature is standard in geometric measure theory [1, 31]. Our definition of L^2-flow and generalized velocity is similar to Brakke's formulation of mean curvature flow [5].

DEFINITION 3.1 Let $(\mu^t)_{t \in (0,T)}$ be any family of integer rectifiable Radon measures such that $\mu := \mathcal{L}^1 \otimes \mu^t$ defines a Radon measure on Ω_T and such that μ^t has a weak mean curvature $H(t, \cdot) \in L^2(\mu^t)$ for almost all $t \in (0, T)$.

If there exists a positive constant C and a vector field $v \in L^2(\mu, \mathbb{R}^n)$ such that

$$v(t, x) \perp T_x \mu^t \quad \text{for } \mu\text{-almost all } (t, x) \in \Omega_T, \tag{3.1}$$

$$\left| \int_0^T \int_\Omega (\partial_t \eta + \nabla \eta \cdot v) \, d\mu^t \, dt \right| \leqslant C \|\eta\|_{C^0(\Omega_T)} \tag{3.2}$$

for all $\eta \in C_c^1((0, T) \times \overline{\Omega})$, then we call the evolution $(\mu^t)_{t \in (0,T)}$ an L^2-*flow*. A function $v \in L^2(\mu, \mathbb{R}^n)$ satisfying (3.1), (3.2) is called a *generalized velocity vector*.

This definition is based on the observation that for a smooth evolution $(M_t)_{t \in (0,T)}$ with mean curvature $H(t, \cdot)$ and normal velocity vector $V(t, \cdot)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{M_t} \eta(t, x) \, \mathrm{d}\mathcal{H}^{n-1}(x) - \int_{M_t} \partial_t \eta(t, x) \, \mathrm{d}\mathcal{H}^{n-1}(x) - \int_{M_t} \nabla \eta(t, x) \cdot V(t, x) \, \mathrm{d}\mathcal{H}^{n-1}(x)$$
$$= \int_{M_t} H(t, x) \cdot V(t, x) \eta(t, x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Integrating this equality in time implies (3.2) for any evolution with square-integrable velocity and mean curvature.

REMARK 3.2   Choosing $\eta(t, x) = \zeta(t) \psi(x)$ with $\zeta \in C_c^1(0, T)$, $\psi \in C^1(\overline{\Omega})$, we deduce from (3.2) that $t \mapsto \mu^t(\psi)$ belongs to $BV(0, T)$. Choosing a countable dense subset $(\psi_i)_{i \in \mathbb{N}} \subset C^0(\overline{\Omega})$ this implies that there exists a countable set $S \subset (0, T)$ of *singular times* such that any good representative of $t \mapsto \mu^t(\psi)$ is continuous in $(0, T) \setminus S$ for all $\psi \in C^1(\overline{\Omega})$.

Any generalized velocity is (on a set of good points) uniquely determined by the evolution $(\mu_t)_{t \in (0,T)}$.

PROPOSITION 3.3   Let $(\mu^t)_{t \in (0,T)}$ be an L^2-flow and set $\mu := \mathcal{L}^1 \otimes \mu^t$. Let $v \in L^2(\mu)$ be a generalized velocity field in the sense of Definition 3.1. Then

$$\begin{pmatrix} 1 \\ v(t_0, x_0) \end{pmatrix} \in T_{(t_0, x_0)} \mu \tag{3.3}$$

at μ-almost all points $(t_0, x_0) \in \Omega_T$ where the tangential plane of μ exists. The evolution $(\mu^t)_{t \in (0,T)}$ uniquely determines v at all points $(t_0, x_0) \in \Omega_T$ where both tangential planes $T_{(t_0, x_0)} \mu$ and $T_{x_0} \mu^{t_0}$ exist.

We postpone the proof to Section 8.

On the set of points where a tangential plane of μ exists, the generalized velocity field v coincides with the normal velocity introduced in [4].

We now turn to the statement of a lower bound for sequences $(u_\varepsilon)_{\varepsilon > 0}$ satisfying Assumption 2.1. As $\varepsilon \to 0$ we will obtain a phase indicator function u as the limit of the sequence $(u_\varepsilon)_{\varepsilon > 0}$ and an L^2-flow $(\mu^t)_{t \in (0,T)}$ as the limit of the measures $(\mu_\varepsilon)_{\varepsilon > 0}$. We will show that at $\mathcal{H}^n$-almost all points of the phase boundary $\partial^* \{u = 1\} \cap \Omega_T$ the tangential plane of μ exists. This implies the existence of a unique normal velocity field of the phase boundary.

4. Lower bound for the action functional

We state a lower bound for the functionals $\mathcal{S}_\varepsilon$ in several steps. We postpone all proofs to Sections 5–8.

4.1   *Lower estimate for the mean curvature*

We start with an application of the well-known results of Modica and Mortola [22, 21].

PROPOSITION 4.1   There exists $u \in BV(\Omega_T; \{-1, 1\}) \cap L^\infty(0, T; BV(\Omega))$ such that for a subsequence $\varepsilon \to 0$,

$$u_\varepsilon \to u \qquad \text{in } L^1(\Omega_T), \tag{4.1}$$

$$u_\varepsilon(t, \cdot) \to u(t, \cdot) \quad \text{in } L^1(\Omega) \text{ for almost all } t \in (0, T). \tag{4.2}$$

Moreover,

$$\frac{c_0}{2} \int_{\Omega_T} \mathrm{d}|\nabla' u| \leqslant \Lambda_3 + T\Lambda_4, \quad \frac{c_0}{2} \int_\Omega \mathrm{d}|\nabla u(t, \cdot)| \leqslant \Lambda_4 \tag{4.3}$$

where c_0 was defined in (1.4).

The next proposition basically repeats the arguments in [18, Theorem 1.1].

PROPOSITION 4.2   There exists a countable set $S \subset (0, T)$, a subsequence $\varepsilon \to 0$ and Radon measures μ^t, $t \in [0, T] \setminus S$, such that for all $t \in [0, T] \setminus S$,

$$\mu_\varepsilon^t \to \mu^t \quad \text{as Radon measures on } \overline{\Omega}, \tag{4.4}$$

$$\mu = \mathcal{L}^1 \otimes \mu^t, \tag{4.5}$$

and for all $\psi \in C^1(\overline{\Omega})$ the function

$$t \mapsto \mu^t(\psi) \quad \text{is of bounded variation in } (0, T) \tag{4.6}$$

and has no jumps in $(0, T) \setminus S$.

Exploiting the lower bound [28] for the diffuse approximation of the Willmore functional (1.8) we find that the measures μ^t are integer-rectifiable up to a constant with a weak mean curvature satisfying an appropriate lower estimate.

THEOREM 4.3   For almost all $t \in (0, T)$,

- $(1/c_0)\mu^t$ is an integral $(n-1)$-varifold,
- μ^t has weak mean curvature $H(t, \cdot) \in L^2(\mu^t)$,

and

$$\int_{\Omega_T} |H|^2 \, \mathrm{d}\mu \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_T} \frac{1}{\varepsilon} w_\varepsilon^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{4.7}$$

4.2   *Lower estimate for the generalized velocity*

THEOREM 4.4   Let $(\mu^t)_{t \in (0, T)}$ be the limit measures obtained in Proposition 4.2. Then there exists a generalized velocity $v \in L^2(\mu, \mathbb{R}^n)$ of $(\mu^t)_{t \in (0, T)}$. Moreover,

$$\int_{\Omega_T} |v|^2 \, \mathrm{d}\mu \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_T} \varepsilon(\partial_t u_\varepsilon)^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{4.8}$$

In particular, $(\frac{1}{c_0}\mu^t)_{t \in (0, T)}$ is an L^2-flow.

We obtain v as a limit of suitably defined approximate velocities (see Lemma 6.2). On the phase boundary v coincides with the (standard) distributional velocity of the bulk phase $\{u(t, \cdot) = 1\}$. However, our definition extends the velocity also to 'hidden boundaries', which seems necessary in order to prove the Gamma-convergence of the action functional; see the discussion in Section 9.

PROPOSITION 4.5 Define the *generalized normal velocity* V in the direction of the inner normal of $\{u = 1\}$ by

$$V(t, x) := v(t, x) \cdot \frac{\nabla u}{|\nabla u|}(t, x) \quad \text{for } (t, x) \in \partial^*\{u = 1\}.$$

Then $V \in L^1(|\nabla u|)$ and $V|_{\partial^*\{u=1\}}$ is the unique vector field that satisfies, for all $\eta \in C_c^1(\Omega_T)$,

$$\int_0^T \int_\Omega V(t, x) \eta(t, x) \, d|\nabla u(t, \cdot)|(x) \, dt = -\int_{\Omega_T} u \partial_t \eta \, dx \, dt. \tag{4.9}$$

4.3   *Lower estimate of the action functional*

As our main result we obtain the following lower estimate for $\mathcal{S}_\varepsilon$.

THEOREM 4.6 Let Assumption 2.1 hold, and let μ, $(\mu^t)_{t\in[0,T]}$, and S be the measures and the countable set of singular times that we obtained in Proposition 4.2. Define the *nucleation cost* $\mathcal{S}_{\text{nuc}}(\mu)$ by

$$
\begin{aligned}
\mathcal{S}_{\text{nuc}}(\mu) := &\sum_{t_0 \in S} \sup_\psi (\lim_{t \downarrow t_0} \mu^t(\psi) - \lim_{t \uparrow t_0} \mu^t(\psi)) \\
&+ \sup_\psi (\lim_{t \downarrow 0} \mu^t(\psi) - \mu^0(\psi)) + \sup_\psi (\mu^T(\psi) - \lim_{t \uparrow T} \mu^t(\psi)),
\end{aligned}
\tag{4.10}
$$

where the sup is taken over all $\psi \in C^1(\overline{\Omega})$ with $0 \leqslant \psi \leqslant 1$. Then

$$\liminf_{\varepsilon \to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \geqslant \int_{\Omega_T} |v - H|^2 \, d\mu + 4\mathcal{S}_{\text{nuc}}(\mu). \tag{4.11}$$

In the previous definition of nucleation cost we have tacitly chosen good representatives of $\mu^t(\psi)$ (see [3]). With this choice the jump parts in (4.10) are well-defined.

Finally, let us remark that, in view of Theorem 4.3, $\mathcal{S}_{\text{nuc}}$ does indeed measure only $(n-1)$-dimensional jumps.

Theorem 4.6 improves [17] in the higher-multiplicity case. We will discuss our main results in Section 9.

4.4   *Convergence of the Allen–Cahn equation to mean curvature flow*

Let $n = 2, 3$ and consider solutions $(u_\varepsilon)_{\varepsilon>0}$ of the Allen–Cahn equation (1.6) satisfying (A2) and (A3). Then $S_\varepsilon(u_\varepsilon) = 0$ and the results of Sections 4.1–4.3 apply: There exists a subsequence $\varepsilon \to 0$ such that the phase functions u_ε converge to a phase indicator function u, the energy measures μ_ε^t converge to an L^2-flow $(\mu^t)_{t\in(0,T)}$, and

$$H = v \tag{4.12}$$

μ-almost everywhere, where $H(t, \cdot)$ denotes the weak mean curvature of μ^t, and v is the generalized velocity of $(\mu^t)_{t\in(0,T)}$ in the sense of Definition 3.1. Moreover $\mathcal{S}_{\text{nuc}}(\mu) = 0$, which shows that for any nonnegative $\psi \in C^1(\overline{\Omega})$ the function $t \mapsto \mu^t(\psi)$ cannot jump upwards. From (1.6) and (5.3) below one infers that for any $\psi \in C^1(\overline{\Omega})$ and all $\zeta \in C_c^1(0, T)$,

$$-\int_0^T \partial_t \zeta \mu_\varepsilon^t(\psi) \, dt = -\int_{\Omega_T} \zeta(t) \left(\frac{1}{\varepsilon} \psi(x) w_\varepsilon^2(t, x) + \nabla \psi(x) \cdot \nabla u_\varepsilon w_\varepsilon(t, x) \right) dx \, dt. \tag{4.13}$$

We will show that suitably defined 'diffuse mean curvatures' converge as $\varepsilon \to 0$ (see (7.1)). Using this result we can pass to the limit in (4.13) to obtain, for any nonnegative functions $\psi \in C^1(\overline{\Omega})$ and $\zeta \in C_c^1(0, T)$,

$$-\int_0^T \partial_t \zeta \, \mu^t(\psi) \, dt \leqslant -\int_0^T \zeta(t) \int_\Omega (H(t, x)^2 + \nabla \psi(x) \cdot H(t, x)) \, d\mu^t(x) \, dt,$$

which is a time-integrated version of Brakke's inequality.

5. Proofs of Propositions 4.1, 4.2 and Theorem 4.3

Proof of Proposition 4.1. By (2.1), (2.2) we obtain

$$\int_{\Omega_T} \left(\frac{\varepsilon}{2} |\nabla' u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx \, dt \leqslant \Lambda_3 + T \Lambda_4.$$

This implies by [21] the existence of a subsequence $\varepsilon \to 0$ and a function $u \in BV(\Omega_T; \{-1, 1\})$ such that

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega_T)$$

and

$$\frac{c_0}{2} \int_{\Omega_T} d|\nabla' u| \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_T} \left(\frac{\varepsilon}{2} |\nabla' u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) dx \, dt \leqslant \Lambda_3 + T \Lambda_4.$$

After possibly taking another subsequence, for almost all $t \in (0, T)$,

$$u_\varepsilon(t, \cdot) \to u(t, \cdot) \quad \text{in } L^1(\Omega). \tag{5.1}$$

Using (2.2) and applying [21] for a fixed $t \in (0, T)$ with (5.1) we get

$$\frac{c_0}{2} \int_\Omega d|\nabla u|(t, \cdot) \leqslant \liminf_{\varepsilon \to 0} \mu_\varepsilon^t(\Omega) \leqslant \Lambda_4. \qquad \square$$

Before proving Proposition 4.2 we show that the time derivative of the energy densities μ_ε^t is controlled.

LEMMA 5.1 There exists $C = C(\Lambda_1, \Lambda_3, \Lambda_4)$ such that for all $\psi \in C^1(\overline{\Omega})$,

$$\int_0^T |\partial_t \mu_\varepsilon^t(\psi)| \, dt \leqslant C \|\psi\|_{C^1(\overline{\Omega})}. \tag{5.2}$$

Proof. Using (A3) we compute that

$$2\partial_t \mu_\varepsilon^t(\psi) = \int_\Omega \left(\sqrt{\varepsilon} \partial_t u_\varepsilon + \frac{1}{\sqrt{\varepsilon}} w_\varepsilon \right)^2 (t, x) \psi(x) \, dx - \int_\Omega \left(\varepsilon (\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2 \right)(t, x) \psi(x) \, dx$$

$$- 2 \int_\Omega \varepsilon \nabla \psi(x) \cdot \partial_t u_\varepsilon(t, x) \nabla u_\varepsilon(t, x) \, dx. \tag{5.3}$$

By (2.1), (2.2) we estimate

$$\left| 2\int_{\Omega_T} \varepsilon \nabla\psi \cdot \partial_t u_\varepsilon \nabla u_\varepsilon \, dx \, dt \right| \leqslant \int_{\Omega_T} |\nabla\psi| \big(\varepsilon(\partial_t u_\varepsilon)^2 + \varepsilon|\nabla u_\varepsilon|^2 \big) \, dx \, dt$$

$$\leqslant (\Lambda_3 + T\Lambda_4)\|\nabla\psi\|_{C^0(\overline{\Omega})}, \tag{5.4}$$

and deduce from (A1), (2.1), (5.3) that

$$\int_0^T |\partial_t \mu_\varepsilon^t(\psi)| \, dt \leqslant (\Lambda_1 + \Lambda_3)\|\psi\|_{C^0(\overline{\Omega})} + C(\Lambda_3, T\Lambda_4)\|\nabla\psi\|_{C^0(\overline{\Omega})},$$

which proves (5.2). □

Proof of Proposition 4.2.   By (2.7), $\mu_\varepsilon \to \mu$ as Radon measures on $\overline{\Omega}_T$. Choose now a countable family $(\psi_i)_{i\in\mathbb{N}} \subset C^1(\overline{\Omega})$ which is dense in $C^0(\overline{\Omega})$. By Lemma 5.1 and a diagonal-sequence argument there exists a subsequence $\varepsilon \to 0$ and functions $m_i \in BV(0,T)$, $i \in \mathbb{N}$, such that for all $i \in \mathbb{N}$,

$$\mu_\varepsilon^t(\psi_i) \to m_i(t) \quad \text{for almost all } t \in (0,T), \tag{5.5}$$

$$\partial_t \mu_\varepsilon^t(\psi_i) \to m_i' \quad \text{as Radon measures on } (0,T). \tag{5.6}$$

Let S denote the countable set of times $t \in (0,T)$ where for some $i \in \mathbb{N}$ the measure m_i' has an atomic part in t. We claim that (5.5) holds on $(0,T) \setminus S$. To see this we choose a point $t \in (0,T) \setminus S$ and a sequence of points $(t_j)_{j\in\mathbb{N}}$ in $(0,T) \setminus S$ such that $t_j \nearrow t$ and (5.5) holds for all t_j. We then obtain

$$\lim_{j\to\infty} m_i'([t_j,t]) = 0 \qquad \text{for all } i \in \mathbb{N}, \tag{5.7}$$

$$\lim_{\varepsilon\to 0} \partial_t \mu^\varepsilon(\psi_i)([t_j,t]) = m_i'([t_j,t]) \quad \text{for all } i,j \in \mathbb{N}. \tag{5.8}$$

Moreover,

$$|m_i(t) - \mu_\varepsilon^t(\psi_i)| \leqslant |m_i(t) - m_i(t_j)| + |m_i(t_j) - \mu_\varepsilon^{t_j}(\psi_i)| + |\mu_\varepsilon^{t_j}(\psi_i) - \mu_\varepsilon^t(\psi_i)|$$

$$\leqslant |m_i'([t_j,t])| + |m_i(t_j) - \mu_\varepsilon^{t_j}(\psi_i)| + |\partial_t \mu_\varepsilon^t(\psi_i)([t_j,t])|.$$

Taking first $\varepsilon \to 0$ and then $t_j \nearrow t$ we deduce from (5.7), (5.8) that (5.5) holds for all $i \in \mathbb{N}$ and all $t \in (0,T) \setminus S$.

Take now an arbitrary $t \in (0,T)$ such that (5.5) holds. By (2.2) there exists a subsequence $\varepsilon \to 0$ such that

$$\mu_\varepsilon^t \to \mu^t \quad \text{as Radon measures on } \Omega. \tag{5.9}$$

We deduce that $\mu^t(\psi_i) = m_i(t)$ and since $(\psi_i)_{i\in\mathbb{N}}$ is dense in $C^0(\overline{\Omega})$ we can identify any limits of $(\mu_\varepsilon^t)_{\varepsilon>0}$ and obtain (5.9) for the whole sequence selected in (5.5), (5.6) and for all $t \in (0,T)$ for which (5.5) holds. Moreover, for any $\psi \in C^0(\overline{\Omega})$ the map $t \mapsto \mu_\varepsilon^t(\psi)$ has no jumps in $(0,T) \setminus S$. After possibly taking another subsequence we can also ensure that as $\varepsilon \to 0$,

$$\mu_\varepsilon^0 \to \mu^0, \quad \mu_\varepsilon^T \to \mu^T$$

as Radon measures on $\overline{\Omega}$. This proves (4.4).

By the dominated convergence theorem we conclude that for any $\eta \in C^0(\overline{\Omega}_T)$,

$$\int_{\Omega_T} \eta \, d\mu = \lim_{\varepsilon \to 0} \int_{\Omega_T} \eta \, d\mu_\varepsilon = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \eta(t,x) \, d\mu_\varepsilon^t(x) \, dt = \int_0^T \int_\Omega \eta(t,x) \, d\mu^t(x) \, dt,$$

which implies (4.5).

By (5.2), the $L^1(0,T)$-compactness of sequences that are uniformly bounded in $BV(0,T)$, the lower semicontinuity of the BV-norm under L^1-convergence, and (4.4) we conclude that (4.6) holds. $\qquad\square$

Proof of Theorem 4.3. Fatou's lemma and (2.1), (2.2) imply that for almost all $t \in (0,T)$,

$$\liminf_{\varepsilon \to 0}\left(\mu_\varepsilon^t(\Omega) + \int_\Omega \frac{1}{\varepsilon} w_\varepsilon(t,x)^2 \, dx \right) < \infty. \tag{5.10}$$

Let $S \subset (0,T)$ be as in Proposition 4.2 and fix a $t \in (0,T) \setminus S$ such that (5.10) holds. Then we deduce from [28, Theorems 4.1 and 5.1] and (4.4) that

$$\frac{1}{c_0}\mu^t \text{ is an integral } (n-1)\text{-varifold}, \quad \mu^t \geqslant \frac{c_0}{2}|\nabla u(t,\cdot)|,$$

and μ^t has weak mean curvature $H(t,\cdot)$ satisfying

$$\int_\Omega |H(t,x)|^2 \, d\mu^t(x) \leqslant \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega w_\varepsilon(t,x)^2 \, dx. \tag{5.11}$$

By (5.11) and Fatou's lemma we obtain

$$\int_0^T \int_\Omega |H(t,x)|^2 \, d\mu^t(x) \, dt \leqslant \int_0^T \left(\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega w_\varepsilon(t,x)^2 \, dx \right) dt \leqslant \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_T} w_\varepsilon^2 \, dx \, dt,$$

which proves (4.7).

For later use we also associate general varifolds to μ_ε^t and consider their convergence as $\varepsilon \to 0$. Let $\nu_\varepsilon(t,\cdot) : \Omega \to S_1^{n-1}(0)$ be an extension of $\nabla u_\varepsilon(t,\cdot)/|\nabla u_\varepsilon(t,\cdot)|$ to the set $\{\nabla u_\varepsilon(t,\cdot) = 0\}$. Define the projections $P_\varepsilon(t,x) := \mathrm{Id} - \nu_\varepsilon(t,x) \otimes \nu_\varepsilon(t,x)$ and consider the general varifolds V_ε^t and the integer rectifiable varifold $c_0^{-1}V^t$ defined by

$$V_\varepsilon^t(f) := \int_\Omega f(x, P_\varepsilon(t,x)) \, d\mu_\varepsilon^t(x), \tag{5.12}$$

$$V^t(f) := \int_\Omega f(x, P(t,x)) \, d\mu^t(x) \tag{5.13}$$

for $f \in C_c^0(\Omega \times \mathbb{R}^{n \times n})$, where $P(t,x) \in \mathbb{R}^{n \times n}$ denotes the projection onto the tangential plane $T_x\mu^t$. Then we deduce from the proof of [28, Theorem 4.1] that

$$V_\varepsilon^t \to V^t \quad \text{as } \varepsilon \to 0 \tag{5.14}$$

in the sense of varifolds. $\qquad\square$

6. Proof of Theorem 4.4

6.1 *Equipartition of energy*

We start with a preliminary result, showing the important *equipartition of energy*: the *discrepancy measure*

$$\xi_\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right) \mathcal{L}^{n+1} \tag{6.1}$$

vanishes in the limit $\varepsilon \to 0$.

To prove this we combine results from [28] with a refined version of Lebesgue's dominated convergence theorem [25] (see also [27, Lemma 4.2]).

PROPOSITION 6.1   For a subsequence $\varepsilon \to 0$,

$$|\xi_\varepsilon| \to 0 \quad \text{as Radon measures on } \overline{\Omega}_T. \tag{6.2}$$

Proof. Let us define the measures

$$\xi_\varepsilon^t := \left(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon} W(u_\varepsilon) \right)(t, \cdot) \mathcal{L}^n$$

on Ω. For $\varepsilon > 0$, $k \in \mathbb{N}$, we define the sets

$$\mathcal{B}_{\varepsilon,k} := \left\{ t \in (0, T) : \int_\Omega \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 \, \mathrm{d}x > k \right\}. \tag{6.3}$$

We then deduce from (2.1) that

$$\Lambda_3 \geqslant \int_0^T \int_\Omega \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 \, \mathrm{d}x \, \mathrm{d}t \geqslant |\mathcal{B}_{\varepsilon,k}| k. \tag{6.4}$$

Next we define the (signed) Radon measures $\xi_{\varepsilon,k}^t$ by

$$\xi_{\varepsilon,k}^t := \begin{cases} \xi_\varepsilon^t & \text{for } t \in (0, T) \setminus \mathcal{B}_{\varepsilon,k}, \\ 0 & \text{for } t \in \mathcal{B}_{\varepsilon,k}. \end{cases} \tag{6.5}$$

By [28, Proposition 4.9], we have

$$|\xi_{\varepsilon_j}^t| \to 0 \quad (j \to \infty) \text{ as Radon measures on } \Omega \tag{6.6}$$

for any subsequence $\varepsilon_j \to 0$ $(j \to \infty)$ such that

$$\limsup_{j \to \infty} \int_\Omega \frac{1}{\varepsilon_j} w_{\varepsilon_j}(t, x)^2 \, \mathrm{d}x < \infty.$$

By (2.2), (6.5) we deduce that for any $\eta \in C^0(\overline{\Omega}_T, \mathbb{R}_0^+)$, $k \in \mathbb{N}$, and almost all $t \in (0, T)$,

$$|\xi_{\varepsilon,k}^t|(\eta(t, \cdot)) \to 0 \quad \text{as } \varepsilon \to 0 \tag{6.7}$$

and

$$|\xi_{\varepsilon,k}^t|(\eta(t, \cdot)) = (1 - \mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t))|\xi_\varepsilon^t|(\eta(t, \cdot)) \leqslant \Lambda_4 \|\eta\|_{C^0(\overline{\Omega}_T)}. \tag{6.8}$$

By the dominated convergence theorem, (6.7) and (6.8) imply that

$$\int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,\mathrm{d}t \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.9}$$

Further, we obtain

$$\int_0^T |\xi_\varepsilon^t|(\eta(t,\cdot))\,\mathrm{d}t \leqslant \int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,\mathrm{d}t + \int_{\mathcal{B}_{\varepsilon,k}} |\xi_\varepsilon^t|(\eta(t,\cdot))\,\mathrm{d}t$$

$$\leqslant \int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,\mathrm{d}t + \int_{\mathcal{B}_{\varepsilon,k}} \mu_\varepsilon^t(\eta(t,\cdot))\,\mathrm{d}t. \tag{6.10}$$

For $k \in \mathbb{N}$ fixed we deduce from (2.2), (6.4), (6.10) that

$$\limsup_{\varepsilon \to 0} \int_0^T \int_\Omega |\xi_\varepsilon^t|(\eta(t,\cdot))\,\mathrm{d}t \leqslant \lim_{\varepsilon \to 0} \int_0^T |\xi_{\varepsilon,k}^t|(\eta(t,\cdot))\,\mathrm{d}t + \|\eta\|_{C^0(\Omega_T)}\Lambda_4 \frac{\Lambda_3}{k}. \tag{6.11}$$

By (6.9) and since $k \in \mathbb{N}$ was arbitrary this proves the proposition. $\qquad\square$

6.2 *Convergence of approximate velocities*

In the next step in the proof of Theorem 4.4 we define approximate velocity vectors and show their convergence as $\varepsilon \to 0$.

LEMMA 6.2   Define $v_\varepsilon : \Omega_T \to \mathbb{R}^n$ by

$$v_\varepsilon := \begin{cases} -\dfrac{\partial_t u_\varepsilon}{|\nabla u_\varepsilon|}\dfrac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} & \text{if } |\nabla u_\varepsilon| \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.12}$$

Then there exists a function $v \in L^2(\mu, \mathbb{R}^n)$ such that

$$(\varepsilon|\nabla u_\varepsilon|^2 \mathcal{L}^{n+1}, v_\varepsilon) \to (\mu, v) \quad \text{as } \varepsilon \to 0 \tag{6.13}$$

in the sense of measure-function pair convergence (see Appendix B) and such that (4.8) is satisfied.

Proof. We define Radon measures

$$\tilde{\mu}_\varepsilon := \varepsilon|\nabla u_\varepsilon|^2 \mathcal{L}^{n+1} = \mu_\varepsilon + \xi_\varepsilon. \tag{6.14}$$

From (2.7), (6.2) we deduce that

$$\tilde{\mu}_\varepsilon \to \mu \quad \text{as Radon measures on } \overline{\Omega}_T. \tag{6.15}$$

Next we observe that $(\tilde{\mu}_\varepsilon, v_\varepsilon)$ is a measure-function pair in the sense of [15] (see also Definition B.1 in Appendix B) and that by (2.1),

$$\int_{\Omega_T} |v_\varepsilon|^2 \,\mathrm{d}\tilde{\mu}_\varepsilon \leqslant \int_{\Omega_T} \varepsilon(\partial_t u_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \Lambda_3. \tag{6.16}$$

By Theorem B.3 we therefore deduce that there exists a subsequence $\varepsilon \to 0$ and a function $v \in L^2(\mu, \mathbb{R}^n)$ such that (6.13) and (4.8) hold. $\qquad\square$

LEMMA 6.3 For μ-almost all $(t, x) \in \Omega_T$,

$$v(t, x) \perp T_x \mu^t. \tag{6.17}$$

Proof. We follow [23, Proposition 3.2]. Let $\nu_\varepsilon : \Omega_T \to S_1^{n-1}(0)$ be an extension of $\nabla u_\varepsilon / |\nabla u_\varepsilon|$ to the set $\{\nabla u_\varepsilon = 0\}$ and define projection-valued maps $P_\varepsilon : \Omega_T \to \mathbb{R}^{n \times n}$ by

$$P_\varepsilon := \mathrm{Id} - \nu_\varepsilon \otimes \nu_\varepsilon.$$

Consider next the general varifolds $\tilde{V}_\varepsilon$, V defined by

$$\tilde{V}_\varepsilon(f) := \int_{\Omega_T} f(t, x, P_\varepsilon(t, x)) \, \mathrm{d}\tilde{\mu}_\varepsilon(t, x), \tag{6.18}$$

$$V(f) := \int_{\Omega_T} f(t, x, P(t, x)) \, \mathrm{d}\mu^t(x) \tag{6.19}$$

for $f \in C_c^0(\Omega_T \times \mathbb{R}^{n \times n})$, where $P(t, x) \in \mathbb{R}^{n \times n}$ denotes the projection onto the tangential plane $T_x \mu^t$.

From (5.14), Proposition 6.1, and Lebesgue's dominated convergence theorem we deduce that

$$\lim_{\varepsilon \to 0} \tilde{V}_\varepsilon = V \tag{6.20}$$

as Radon measures on $\Omega_T \times \mathbb{R}^{n \times n}$.

Next we define functions $\hat{v}_\varepsilon$ on $\Omega_T \times \mathbb{R}^{n \times n}$ by

$$\hat{v}_\varepsilon(t, x, Y) = v_\varepsilon(t, x) \quad \text{for all } (t, x) \in \Omega_T, Y \in \mathbb{R}^{n \times n}.$$

We then observe that

$$\int_{\Omega_T \times \mathbb{R}^{n \times n}} \hat{v}_\varepsilon^2 \, \mathrm{d}V_\varepsilon = \int_{\Omega_T} v_\varepsilon^2 \, \mathrm{d}\tilde{\mu}_\varepsilon \leqslant \Lambda_3$$

and deduce from (6.20) and Theorem B.3 the existence of $\hat{v} \in L^2(V, \mathbb{R}^n)$ such that $(V_\varepsilon, \hat{v}_\varepsilon)$ converges to $(V, \hat{v})$ as measure-function pairs on $\Omega_T \times \mathbb{R}^{n \times n}$ with values in $\mathbb{R}^n$.

We now consider $h \in C_c^0(\mathbb{R}^{n \times n})$ such that $h(Y) = 1$ for all projections Y. We deduce that for any $\eta \in C_c^0(\Omega_T, \mathbb{R}^n)$,

$$\int_{\Omega_T} \eta \cdot v \, \mathrm{d}\mu = \lim_{\varepsilon \to 0} \int_{\Omega_T \times \mathbb{R}^{n \times n}} \eta(t, x) \cdot h(Y) \hat{v}_\varepsilon(t, x, Y) \, \mathrm{d}V_\varepsilon(t, x, Y)$$

$$= \int_{\Omega_T} \eta(t, x) \cdot \hat{v}(t, x, P(t, x)) \, \mathrm{d}\mu(t, x),$$

which shows that for μ-almost all $(t, x) \in \Omega_T$,

$$\hat{v}(t, x, P(t, x)) = v(t, x). \tag{6.21}$$

Finally, we observe that for h, η as above,

$$\int_{\Omega_T} \eta(t, x) \cdot P(t, x) v(t, x) \, \mathrm{d}\mu(t, x) = \int_{\Omega_T \times \mathbb{R}^{n \times n}} \eta(t, x) h(Y) \cdot Y \hat{v}(t, x, Y) \, \mathrm{d}V(t, x, Y)$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega_T \times \mathbb{R}^{n \times n}} \eta(t, x) h(Y) \cdot Y \hat{v}_\varepsilon(t, x, Y) \, \mathrm{d}V_\varepsilon(t, x, Y)$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega_T} \eta(t, x) \cdot P_\varepsilon(t, x) v_\varepsilon(t, x) \, \mathrm{d}\tilde{\mu}_\varepsilon(t, x) = 0$$

since $P_\varepsilon v_\varepsilon = 0$. This shows that $P(t, x) v(t, x) = 0$ for μ-almost all $(t, x) \in \Omega_T$. $\qquad\square$

Proof of Theorem 4.4. By (2.1) there exists a subsequence $\varepsilon \to 0$ and a Radon measure β on $\overline{\Omega}_T$ such that

$$\left(\varepsilon (\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2 \right) \mathcal{L}^{n+1} \to \beta, \qquad \beta(\overline{\Omega}_T) \leqslant \Lambda_3. \tag{6.22}$$

Using (A3) we compute that for any $\eta \in C_c^1((0,T) \times \overline{\Omega})$,

$$\begin{aligned}
\int_{\Omega_T} \eta \, \mathrm{d}\alpha_\varepsilon &= \int_{\Omega_T} \eta \left(\varepsilon (\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon} w_\varepsilon^2 \right) \mathrm{d}x \, \mathrm{d}t - 2 \int_{\Omega_T} \partial_t \eta \, \mathrm{d}\mu_\varepsilon \\
&\quad + 2 \int_{\Omega_T} \varepsilon \nabla \eta \cdot \partial_t u_\varepsilon \nabla u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{6.23}$$

As ε tends to zero the term on the left-hand side and the first two terms on the right-hand side converge by (2.7), (2.8) and (6.22). For the third term on the right-hand side of (6.23) we find from (6.13) that

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \nabla \eta \cdot \varepsilon \partial_t u_\varepsilon \nabla u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = - \lim_{\varepsilon \to 0} \int_{\Omega_T} \nabla \eta \cdot v_\varepsilon \varepsilon |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t = - \int_{\Omega_T} \nabla \eta \cdot v \, \mathrm{d}\mu.$$

Therefore, taking $\varepsilon \to 0$ in (6.23) we deduce that

$$\int_{\Omega_T} \eta \, \mathrm{d}\alpha = \int_{\Omega_T} \eta \, \mathrm{d}\beta - 2 \int_{\Omega_T} \partial_t \eta \, \mathrm{d}\mu - 2 \int_{\Omega_T} \nabla \eta \cdot v \, \mathrm{d}\mu$$

for all $\eta \in C_c^1((0,T) \times \overline{\Omega})$. This yields

$$\left| \int_{\Omega_T} (\partial_t \eta + \nabla \eta \cdot v) \, \mathrm{d}\mu \right| \leqslant \|\eta\|_{C^0(\overline{\Omega}_T)} \frac{1}{2} (\alpha(\overline{\Omega}_T) + \beta(\overline{\Omega}_T)),$$

which together with (6.17) shows that v is a generalized velocity vector for $(\mu^t)_{t \in (0,T)}$ in the sense of Definition 3.1. The estimate (4.8) was already proved in Lemma 6.2. □

7. Proof of Theorem 4.6

We start with the convergence of a 'diffuse mean curvature term'.

LEMMA 7.1   Define

$$H_\varepsilon := \frac{1}{\varepsilon} w_\varepsilon \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|^2},$$

let $\tilde{\mu}_\varepsilon = \varepsilon |\nabla u_\varepsilon|^2 \mathcal{L}^{n+1}$, and let v_ε, v be as in (6.12), (6.13). Then

$$(\tilde{\mu}_\varepsilon, H_\varepsilon) \to (\mu, H), \tag{7.1}$$

$$(\tilde{\mu}_\varepsilon, v_\varepsilon - H_\varepsilon) \to (\mu, v - H) \tag{7.2}$$

as $\varepsilon \to 0$ in the sense of measure-function pair convergence. In particular

$$\int_{\Omega_T} \eta |v - H|^2 \, \mathrm{d}\mu \leqslant \alpha(\eta) \tag{7.3}$$

for all $\eta \in C^0(\overline{\Omega}_T, \mathbb{R}_0^+)$.

Proof. We use arguments similar to the proof of Proposition 6.1. For $\varepsilon > 0$, $k \in \mathbb{N}$, we define

$$\mathcal{B}_{\varepsilon,k} := \left\{ t \in (0, T) : \int_{\Omega} \frac{1}{\varepsilon} w_{\varepsilon}(t, x)^2 \, \mathrm{d}x > k \right\}. \tag{7.4}$$

We then deduce from (2.1) that

$$\Lambda_3 \geqslant \int_{\Omega_T} \frac{1}{\varepsilon} w_{\varepsilon}^2 \, \mathrm{d}x \, \mathrm{d}t \geqslant |\mathcal{B}_{\varepsilon,k}| k. \tag{7.5}$$

Next we define functionals $T_{\varepsilon,k}^t \in C_c^0(\Omega, \mathbb{R}^n)^*$ by

$$T_{\varepsilon,k}^t(\psi) := \begin{cases} \displaystyle\int_{\Omega} \psi(x) \cdot w_{\varepsilon}(t, x) \nabla u_{\varepsilon}(t, x) \, \mathrm{d}x & \text{for } t \in (0, T) \setminus \mathcal{B}_{\varepsilon,k}, \\ \displaystyle\int_{\Omega} \psi(x) \cdot H(t, x) \, \mathrm{d}\mu^t(x) & \text{for } t \in \mathcal{B}_{\varepsilon,k}. \end{cases} \tag{7.6}$$

Considering the general $(n-1)$-varifolds V_{ε}^t, V^t defined in (5.12), (5.13) we infer from [28, Proposition 4.10] and (5.14) that

$$\lim_{j \to \infty} \int_{\Omega} \psi \cdot w_{\varepsilon_j}(t, x) \nabla u_{\varepsilon_j}(t, x) \, \mathrm{d}x = -\lim_{j \to \infty} \delta V_{\varepsilon_j}^t(\psi) = -\delta \mu^t(\psi) = \int_{\Omega} \psi \cdot H(t, x) \, \mathrm{d}\mu^t(x) \tag{7.7}$$

for any subsequence $\varepsilon_j \to 0$ $(j \to \infty)$ such that

$$\limsup_{j \to \infty} \int_{\Omega} \frac{1}{\varepsilon_j} w_{\varepsilon_j}^2 \, \mathrm{d}x \, \mathrm{d}t < \infty.$$

Therefore we deduce from (7.6), (7.7) that for all $\eta \in C_c^0(\Omega_T, \mathbb{R}^n)$, $k \in \mathbb{N}$, and almost all $t \in (0, T)$,

$$T_{\varepsilon,k}^t(\eta(t, \cdot)) \to \int_{\Omega} \eta(t, x) \cdot H(t, x) \, \mathrm{d}\mu^t(x) \quad \text{as } \varepsilon \to 0 \tag{7.8}$$

and

$$\begin{aligned} |T_{\varepsilon,k}^t(\eta(t, \cdot))| &\leqslant (1 - \mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t)) \left| \int_{\Omega} \eta(t, x) \cdot w_{\varepsilon}(t, x) \nabla u_{\varepsilon}(t, x) \, \mathrm{d}x \right| \\ &\quad + \mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t) \left| \int_{\Omega} \eta(t, x) \cdot H(t, x) \, \mathrm{d}\mu^t(x) \right| \\ &\leqslant \|\eta\|_{C^0(\Omega_T)} (1 - \mathcal{X}_{\mathcal{B}_{\varepsilon,k}}(t)) \left(\int_{\Omega} \frac{1}{2\varepsilon} w_{\varepsilon}(t, x)^2 \, \mathrm{d}x \right)^{1/2} \left(\int_{\Omega} \frac{\varepsilon}{2} |\nabla u_{\varepsilon}(t, x)|^2 \, \mathrm{d}x \right)^{1/2} \\ &\quad + \int_{\Omega} |\eta(t, x)| \, |H(t, x)| \, \mathrm{d}\mu^t(x) \\ &\leqslant \|\eta\|_{C^0(\Omega_T)} \sqrt{\frac{k}{2}} \sqrt{\Lambda_4} + \int_{\Omega} |\eta(t, x)| \, |H(t, x)| \, \mathrm{d}\mu^t(x), \end{aligned} \tag{7.9}$$

where the right-hand side is bounded in $L^1(0, T)$, uniformly with respect to $\varepsilon > 0$.

By the dominated convergence theorem, (7.8) and (7.9) imply that

$$\int_0^T T_{\varepsilon,k}^t(\eta(t,\cdot))\,\mathrm{d}t \to \int_{\Omega_T} \eta \cdot H\,\mathrm{d}\mu \quad \text{as } \varepsilon \to 0. \tag{7.10}$$

Further, we obtain

$$\left| \int_{\Omega_T} \eta \cdot w_\varepsilon \nabla u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \eta \cdot H\,\mathrm{d}\mu \right|$$

$$\leqslant \left| \int_0^T T_{\varepsilon,k}^t(\eta(t,\cdot))\,\mathrm{d}t - \int_{\Omega_T} \eta \cdot H\,\mathrm{d}\mu \right|$$

$$+ \left| \int_{\mathcal{B}_{\varepsilon,k}} \int_\Omega \eta(t,x) \cdot H(t,x)\,\mathrm{d}\mu^t(x)\,\mathrm{d}t \right| + \left| \int_{\mathcal{B}_{\varepsilon,k}} \int_\Omega \eta \cdot w_\varepsilon \nabla u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \right|. \tag{7.11}$$

The last term on the right-hand side is estimated by

$$\left| \int_{\mathcal{B}_{\varepsilon,k}} \int_\Omega \eta(t,x) \cdot w_\varepsilon(t,x) \nabla u_\varepsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t \right| \leqslant \|\eta\|_{C^0(\Omega_T)} \left(\int_{\Omega_T} \frac{1}{2\varepsilon} w_\varepsilon^2\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} |\mathcal{B}_{\varepsilon,k}|^{1/2} \sqrt{\Lambda_4}$$

$$\leqslant \|\eta\|_{C^0(\Omega_T)} \Lambda_3 \frac{1}{\sqrt{k}} \sqrt{\Lambda_4}, \tag{7.12}$$

where we have used (2.2) and (7.5). For the second term on the right-hand side of (7.11) we obtain

$$\left| \int_{\mathcal{B}_{\varepsilon,k}} \int_\Omega \eta(t,x) \cdot H(t,x)\,\mathrm{d}\mu^t(x)\,\mathrm{d}t \right| \leqslant |\mathcal{B}_{\varepsilon,k}|^{1/2} \|\eta\|_{C^0(\Omega_T)}^{1/2} \left(\int_{\operatorname{supp}(\eta)} H^2\,\mathrm{d}\mu \right)^{1/2}$$

$$\leqslant \frac{\sqrt{\Lambda_3}}{\sqrt{k}} \|\eta\|_{C^0(\Omega_T)}^{1/2} \sqrt{\Lambda_3}, \tag{7.13}$$

where we have used (4.7) and (2.1). Finally, for $k \in \mathbb{N}$ fixed, by (7.10) we deduce that

$$\lim_{\varepsilon \to 0} \left| \int_0^T T_{\varepsilon,k}^t(\eta(t,\cdot))\,\mathrm{d}t - \int_{\Omega_T} \eta \cdot H\,\mathrm{d}\mu \right| = 0. \tag{7.14}$$

Taking $\varepsilon \to 0$ in (7.11) we find by (7.12)–(7.14) that

$$\lim_{\varepsilon \to 0} \left| \int_{\Omega_T} \eta \cdot w_\varepsilon \nabla u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} \eta \cdot H\,\mathrm{d}\mu \right| \leqslant \frac{\Lambda_3}{\sqrt{k}} \|\eta\|_{C^0(\Omega_T)} \sqrt{\Lambda_4} + \frac{1}{\sqrt{k}} \Lambda_3 \tag{7.15}$$

for any $k \in \mathbb{N}$, which proves (7.1). Together with (6.13) this implies (7.2). Finally, we fix an arbitrary nonnegative $\eta \in C^0(\overline{\Omega}_T)$ and deduce that the measure-function pair $(\tilde{\mu}_\varepsilon, \sqrt{\eta}(v_\varepsilon - H_\varepsilon))$ converges to $(\mu, \sqrt{\eta}(v - H))$. The estimate (7.3) then follows from Theorem B.3. $\qquad\square$

Let $\Pi : [0,T] \times \overline{\Omega} \to [0,T]$ denote the projection onto the first component and $\Pi_\#$ the pushforward of measures by Π. For $\psi \in C^0(\overline{\Omega})$ we consider the measures

$$\alpha_\psi := \Pi_\#(\psi\alpha),$$

on $[0, T]$, that is,

$$\alpha_\psi(\zeta) := \int_{\Omega_T} \zeta(t)\psi(x)\, d\alpha(t, x) \quad \text{for } \zeta \in C^0([0, T]),$$

and set

$$\alpha_\Omega := \Pi_\# \alpha.$$

We can then estimate the atomic part of α_Ω in terms of the nucleation cost.

LEMMA 7.2   Let $\mathcal{S}_{\text{nuc}}(\mu)$ be the nucleation cost defined in (4.10). Then

$$(\alpha_\Omega)_{\text{atomic}}[0, T] \geqslant 4\mathcal{S}_{\text{nuc}}(\mu). \tag{7.16}$$

Proof.   Let $\eta \in C^1(\overline{\Omega}_T, \mathbb{R}_0^+)$ be nonnegative. We compute that

$$
\begin{aligned}
\int_{\Omega_T} \eta\, d\alpha_\varepsilon &= \int_{\Omega_T} \eta\left(\varepsilon(\partial_t u_\varepsilon)^2 + \frac{1}{\varepsilon}w_\varepsilon^2 + 2\partial_t u_\varepsilon w_\varepsilon\right) dx\, dt \\
&\geqslant 4\int_{\Omega_T} \eta\partial_t u_\varepsilon w_\varepsilon\, dx\, dt \\
&= -4\int_{\Omega_T} \partial_t \eta\, d\mu_\varepsilon + 4\int_{\Omega_T} \nabla\eta \cdot \varepsilon\partial_t u_\varepsilon \nabla u_\varepsilon\, dx\, dt \\
&\quad + 4\mu_\varepsilon^T(\eta(T, \cdot)) - 4\mu_\varepsilon^0(\eta(0, \cdot)).
\end{aligned}
\tag{7.17}
$$

Passing to the limit $\varepsilon \to 0$ we infer from (2.7), (4.4), (6.13) that

$$\int_{\Omega_T} \eta\, d\alpha \geqslant -4\int_{\Omega_T} \partial_t \eta\, d\mu - 4\int_{\Omega_T} \nabla\eta \cdot v\, d\mu + 4\mu^T(\eta(T, \cdot)) - 4\mu^0(\eta(0, \cdot)). \tag{7.18}$$

We now choose $\eta(t, x) = \zeta(t)\psi(x)$ where $\zeta \in C^1([0, T], \mathbb{R}_0^+)$, $\psi \in C^1(\overline{\Omega}, \mathbb{R}_0^+)$ in (7.18) and deduce that

$$
\begin{aligned}
\int_0^T \zeta\, d\alpha_\psi &\geqslant -4\int_0^T \partial_t\zeta\, \mu^t(\psi)\, dt + 4\int_0^T \zeta\int_\Omega \nabla\psi \cdot v(t, x)\, d\mu^t(x)\, dt \\
&\quad + 4\zeta(T)\mu^T(\psi) - 4\zeta(0)\mu^0(\psi).
\end{aligned}
\tag{7.19}
$$

This shows that

$$
\begin{aligned}
\alpha_\psi &\geqslant 4\partial_t(\mu^t(\psi)) + 4\left(\int_\Omega \nabla\psi(x) \cdot v(t, x)\, d\mu^t(x)\right)\mathcal{L}^1 \\
&\quad + 4(\mu^T(\psi) - \lim_{t\uparrow T}\mu^t(\psi))\delta_T + 4(\lim_{t\downarrow 0}\mu^t(\psi) - \mu^0(\psi))\delta_0.
\end{aligned}
\tag{7.20}
$$

Evaluating the atomic parts we see that for any $0 < t_0 < T$,

$$\alpha_\psi(\{t_0\}) \geqslant 4\partial_t(\mu^t(\psi))(\{t_0\}),$$

which implies that

$$\alpha_\Omega(\{t_0\}) \geqslant 4\sup_\psi \partial_t(\mu^t(\psi))(\{t_0\}), \tag{7.21}$$

where the supremum is taken over all $\psi \in C^1(\overline{\Omega})$ with $0 \leqslant \psi \leqslant 1$.

Moreover we deduce from (7.20) that

$$\alpha_\Omega(\{0\}) \geqslant 4 \sup_\psi (\lim_{t \downarrow 0} \mu^t(\psi) - \mu^0(\psi)), \tag{7.22}$$

$$\alpha_\Omega(\{T\}) \geqslant 4 \sup_\psi (\mu^T(\psi) - \lim_{t \uparrow T} \mu^t(\psi)), \tag{7.23}$$

where the supremum is taken over $\psi \in C(\overline{\Omega})$ with $0 \leqslant \psi \leqslant 1$. By (7.21)–(7.23) we conclude that (7.16) holds. □

Proof of Theorem 4.6. By (7.3) we know that $\alpha \geqslant |v - H|^2 \mu$. Since $\mu = \mathcal{L}^1 \otimes \mu^t$ we deduce from the Radon–Nikodym theorem that

$$(\alpha_\Omega)_{\mathrm{ac}}[0, T] \geqslant \int_{\Omega_T} |v - H|^2 \, \mathrm{d}\mu, \tag{7.24}$$

and from (7.16) that

$$(\alpha_\Omega)_{\mathrm{atomic}}[0, T] \geqslant 4 \mathcal{S}_{\mathrm{nuc}}(\mu), \tag{7.25}$$

where $(\alpha_\Omega)_{\mathrm{ac}}$ and $(\alpha_\Omega)_{\mathrm{atomic}}$ denote the absolutely continuous and atomic parts of the measure α_Ω with respect to $\mathcal{L}^1$. Adding the two estimates and recalling (2.9) we obtain (4.11). □

8. Proofs of Propositions 3.3 and 4.5

For $r > 0$, $(t_0, x_0) \in \Omega_T$ define the cylinders

$$Q_r(t_0, x_0) := (t_0 - r, t_0 + r) \times B_r^n(x_0).$$

Proof of Proposition 3.3. Define

$$\Sigma_n(\mu) := \{(t, x) \in \Omega_T : \text{the tangential plane of } \mu \text{ at } (t, x) \text{ exists}\} \tag{8.1}$$

and choose $(t_0, x_0) \in \Sigma_n(\mu)$ such that

$$v \text{ is approximately continuous with respect to } \mu \text{ at } (t_0, x_0). \tag{8.2}$$

Since $v \in L^2(\mu)$ we deduce from [10, Theorem 2.9.13] that (8.2) holds μ-almost everywhere. Let

$$P_0 := T_{(t_0,x_0)}\mu, \quad \theta_0 > 0, \tag{8.3}$$

denote the tangential plane and multiplicity at (t_0, x_0) respectively, and define for any $\varphi \in C_c^0(Q_1(0))$ the scaled functions $\varphi_\varrho \in C_c^0(Q_\varrho(t_0, x_0))$ by

$$\varphi_\varrho(t, x) := \varrho^{-n} \varphi(\varrho^{-1}(t - t_0), \varrho^{-1}(x - x_0)).$$

Then (8.3) shows that

$$\int_{\Omega_T} \varphi_\varrho \, \mathrm{d}\mu \to \theta_0 \int_{P_0} \varphi \, \mathrm{d}\mathcal{H}^n \quad \text{as } \varrho \searrow 0. \tag{8.4}$$

From (3.2), the Hahn–Banach theorem, and the Riesz theorem we deduce that the functional

$$\vartheta \in C_c^1(\Omega_T)^*, \qquad \vartheta(\eta) := \int_{\Omega_T} \nabla'\eta \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} \, \mathrm{d}\mu, \tag{8.5}$$

can be extended to a (signed) Radon measure on Ω_T. Since by the Radon–Nikodym theorem, $D_\mu|\vartheta|$ exists and is finite μ-almost everywhere, we may assume without loss of generality that

$$D_\mu|\vartheta|(t_0, x_0) < \infty. \tag{8.6}$$

We next fix $\eta \in C_c^1(Q_1(0))$ and compute

$$\vartheta(\varrho\eta_\varrho) = \int_{\Omega_T} \left(\nabla'\eta\right)_\varrho \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} \, \mathrm{d}\mu. \tag{8.7}$$

From (8.2), (8.4) we deduce that the right-hand side converges as $\varrho \to 0$,

$$\lim_{\varrho\to 0} \int_{\Omega_T} (\nabla'\eta)_\varrho \cdot \begin{pmatrix} 1 \\ v \end{pmatrix} \, \mathrm{d}\mu = \theta_0 \begin{pmatrix} 1 \\ v(t_0, x_0) \end{pmatrix} \cdot \int_{P_0} \nabla'\eta \, \mathrm{d}\mu. \tag{8.8}$$

For the left-hand side of (8.7) we deduce that

$$\liminf_{\varrho\searrow 0} |\vartheta(\varrho\eta_\varrho)| \leqslant \|\eta\|_{C_c^0(Q_1(0))} \liminf_{\varrho\searrow 0} \varrho^{-n+1}|\vartheta|(Q_\varrho(t_0, x_0)) \tag{8.9}$$

and observe that (8.6) implies

$$\infty > \lim_{\varrho\searrow 0} \frac{|\vartheta|(Q_\varrho(t_0, x_0))}{\mu(Q_\varrho(t_0, x_0))} \geqslant \liminf_{\varrho\searrow 0} \varrho^{-n}|\vartheta|(Q_\varrho(t_0, x_0)) \cdot (\limsup_{\varrho\searrow 0} \varrho^{-n}\mu(Q_\varrho(t_0, x_0)))^{-1}$$
$$\geqslant c \liminf_{\varrho\searrow 0} \varrho^{-n}|\vartheta|(Q_\varrho(t_0, x_0)), \tag{8.10}$$

since by (8.4) for any $\varphi \in C_c^0(Q_2(0), \mathbb{R}_0^+)$ with $\varphi \geqslant 1$ on $Q_1(0)$,

$$\limsup_{\varrho\searrow 0} \varrho^{-n}\mu(Q_\varrho(t_0, x_0)) \leqslant \limsup_{\varrho\searrow 0} \int_{\Omega_T} \varphi_\varrho \, \mathrm{d}\mu \leqslant C(\varphi).$$

Therefore (8.7)–(8.10) yield

$$\theta_0 \begin{pmatrix} 1 \\ v(t_0, x_0) \end{pmatrix} \cdot \int_{P_0} \nabla'\eta \, \mathrm{d}\mu = 0. \tag{8.11}$$

Now we observe that the integral over the projection of $\nabla'\eta$ onto P_0 vanishes. This shows that

$$\int_{P_0} \nabla'\eta \, \mathrm{d}\mathcal{H}^n \in P_0^\perp. \tag{8.12}$$

Since η can be chosen such that the integral in (8.12) takes an arbitrary direction normal to P_0 we see from (8.11) that $v(t_0, x_0)$ satisfies (3.3). If $T_{x_0}\mu^{t_0}$ exists then

$$T_{(t_0, x_0)}\mu = (\{0\} \times T_{x_0}\mu^{t_0}) \oplus \mathrm{span} \begin{pmatrix} 1 \\ v(x_0) \end{pmatrix}$$

and we conclude that v is uniquely determined. $\qquad\square$

To prepare the proof of Proposition 4.5 we first show that μ is absolutely continuous with respect to $\mathcal{H}^n$.

PROPOSITION 8.1   For any $D \subset\subset \Omega$ there exists $C(D) > 0$ such that for all $x_0 \in D$ and almost all $t_0 \in (0, T)$,

$$\limsup_{r \searrow 0} r^{-n} \mu(Q_r(t_0, x_0)) \leqslant C(D)\Lambda_4 + \liminf_{\varepsilon \to 0} \int_D \frac{1}{\varepsilon} w_\varepsilon(t_0, x)^2 \, \mathrm{d}x. \qquad (8.13)$$

In particular,

$$\limsup_{\rho \to 0} \frac{\mu(B_\rho(t_0, x_0))}{\rho^n} < \infty \quad \text{for } \mu\text{-almost every } (t_0, x_0) \qquad (8.14)$$

and μ is absolutely continuous with respect to $\mathcal{H}^n$,

$$\mu \ll \mathcal{H}^n. \qquad (8.15)$$

Proof. Let

$$r_0 := \min \left\{ 1, \frac{1}{2} \operatorname{dist}(D, \partial\Omega), |t_0|, |T - t_0| \right\}.$$

Then for all $r < r_0$, $x_0 \in D$, from (6.2) and [28, Proposition 4.5] we obtain

$$\frac{1}{r} \int_{t_0-r}^{t_0+r} r^{1-n} \mu^t(B_r^n(x_0)) \, \mathrm{d}t$$

$$\leqslant \frac{1}{r} \int_{t_0-r}^{t_0+r} r_0^{1-n} \mu^t(B_{r_0}^n(x_0)) \, \mathrm{d}t + \frac{1}{4(n-1)^2} \frac{1}{r} \int_{t_0-r}^{t_0+r} \left(\liminf_{\varepsilon \to 0} \int_D \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 \, \mathrm{d}x \right) \mathrm{d}t. \qquad (8.16)$$

By Fatou's lemma and (2.1),

$$t \mapsto \liminf_{\varepsilon \to 0} \int_D \frac{1}{\varepsilon} w_\varepsilon(t, x)^2 \, \mathrm{d}x \quad \text{is in } L^1(0, T), \qquad (8.17)$$

and by (2.2) we deduce that for almost all $t_0 \in (0, T)$,

$$\limsup_{r \searrow 0} \frac{1}{r} \int_{t_0-r}^{t_0+r} r^{1-n} \mu^t(B_r^n(x_0)) \, \mathrm{d}t \leqslant 2r_0^{1-n} \Lambda_4 + \frac{1}{2(n-1)^2} \liminf_{\varepsilon \to 0} \int_D \frac{1}{\varepsilon} w_\varepsilon(t_0, x)^2 \, \mathrm{d}x.$$

Since r_0 depends only on D and Ω, the inequality (8.13) follows.

By (8.17) the right-hand side in (8.13) is finite for $\mathcal{L}^1$-almost all $t_0 \in (0, T)$, and $\theta^{*n}(\mu, (t, x))$ is bounded for almost all $t \in (0, T)$ and all $x \in \Omega$. By (2.2) we deduce that for any $I \subset (0, T)$ with $|I| = 0$,

$$\mu(I \times \Omega) \leqslant \Lambda_4 |I| = 0,$$

which implies (8.14).

To prove the final statement let $B \subset \Omega_T$ be given with

$$\mathcal{H}^n(B) = 0. \qquad (8.18)$$

Consider the family of sets $(D_k)_{k \in \mathbb{N}}$, given by

$$D_k := \{z \in \Omega_T : \theta^{*n}(\mu, z) \leqslant k\}.$$

By (8.14), [31, Theorem 3.2], and (8.18) we find that for all $k \in \mathbb{N}$,

$$\mu(B \cap D_k) \leqslant 2^n k \mathcal{H}^n(B \cap D_k) = 0. \tag{8.19}$$

Moreover,

$$\mu\Big(B \setminus \bigcup_{k \in \mathbb{N}} D_k\Big) = 0 \tag{8.20}$$

by (8.14). By (8.19), (8.20) we conclude that $\mu(B) = 0$, which proves (8.15). $\qquad\square$

To prove Proposition 4.5 we need that $\mathcal{H}^n$-almost everywhere on $\partial^*\{u = 1\}$ the generalized tangent plane of μ exists. We first obtain the following relation between the measures μ and $|\nabla' u|$.

PROPOSITION 8.2   There exists a nonnegative function $g \in L^2(\mu, \mathbb{R}_0^+)$ such that

$$g\mu \geqslant \frac{c_0}{2} |\nabla' u|. \tag{8.21}$$

In particular, $|\nabla' u|$ is absolutely continuous with respect to μ,

$$|\nabla' u| \ll \mu. \tag{8.22}$$

Proof. Let

$$G(r) = \int_0^r \sqrt{2W(s)}\,\mathrm{d}s. \tag{8.23}$$

On the set $\{|\nabla u_\varepsilon| \neq 0\}$ we have

$$
\begin{aligned}
|\nabla G(u_\varepsilon)| &= \frac{|\nabla G(u_\varepsilon)|}{|\nabla' G(u_\varepsilon)|} |\nabla' G(u_\varepsilon)| = \frac{|\nabla G(u_\varepsilon)|}{\sqrt{\partial_t G(u_\varepsilon)^2 + |\nabla G(u_\varepsilon)|^2}} |\nabla' G(u_\varepsilon)| \\
&= \frac{1}{\sqrt{1 + |v_\varepsilon|^2}} |\nabla' G(u_\varepsilon)|.
\end{aligned}
\tag{8.24}
$$

Letting $\tilde{\mu}_\varepsilon$ be as in (6.14) we get from (2.2), (6.16), and Theorem B.3 the existence of a function $g \in L^2(\mu)$ such that (up to a subsequence)

$$\lim_{\varepsilon \to 0} \big(\tilde{\mu}_\varepsilon, \sqrt{1 + |v_\varepsilon|^2}\big) = (\mu, g) \tag{8.25}$$

as measure-function pairs on Ω_T with values in $\mathbb{R}$.

Let $\eta \in C_c^0(\Omega_T)$. Then

$$
\begin{aligned}
&\left| \int_{\Omega_T} \eta \sqrt{1 + |v_\varepsilon|^2}\,|\nabla G(u_\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t - \int \eta \sqrt{1 + |v_\varepsilon|^2}\,\mathrm{d}\tilde{\mu}_\varepsilon \right| \\
&\quad = \left| \int_{\Omega_T} \eta \sqrt{1 + |v_\varepsilon|^2}\left(\sqrt{\frac{2W(u_\varepsilon)}{\varepsilon}} - \sqrt{\varepsilon}\,|\nabla u_\varepsilon| \right)\sqrt{\varepsilon}\,|\nabla u_\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad \leqslant \left(\int_{\Omega_T} \eta^2 (1 + |v_\varepsilon|^2)\varepsilon |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} \left\| \sqrt{\frac{2W(u_\varepsilon)}{\varepsilon}} - \sqrt{\varepsilon}\,|\nabla u_\varepsilon| \right\|_{L^2(\Omega_T)} \\
&\quad \leqslant \|\eta\|_{L^\infty} (2T \Lambda_4 + \Lambda_3)^{1/2} (2|\xi_\varepsilon|(\Omega_T))^{1/2}.
\end{aligned}
\tag{8.26}
$$

Thanks to (8.25), (8.26) and (6.2) we conclude that

$$\lim_{\varepsilon \to 0} \left(|\nabla G(u_\varepsilon)| \mathcal{L}^{n+1}, \sqrt{1 + |v_\varepsilon|^2} \right) = (\mu, g) \tag{8.27}$$

as measure-function pairs on Ω_T with values in $\mathbb{R}$.

Again by (2.1) we have

$$\int_{\{0 = |\nabla u_\varepsilon| < W(u_\varepsilon)\}} |\nabla' G(u_\varepsilon)| \, dx \, dt = \int_{\{0 = |\nabla u_\varepsilon| < W(u_\varepsilon)\}} |\partial_t u_\varepsilon| \sqrt{2W(u_\varepsilon)} \, dx \, dt$$

$$\leqslant \sqrt{2} \left(\int_{\Omega_T} \varepsilon (\partial_t u_\varepsilon)^2 \, dx \, dt \right)^{1/2} \left(\int_{\{0 = |\nabla u_\varepsilon| < W(u_\varepsilon)\}} \frac{W(u_\varepsilon)}{\varepsilon} \, dx \, dt \right)^{1/2} \leqslant \sqrt{2\Lambda_3} (|\xi_\varepsilon|(\Omega_T))^{1/2},$$

which vanishes as $\varepsilon \to 0$ by (6.2). This implies together with (8.24) and (8.27) that

$$\int \eta g \, d\mu = \lim_{\varepsilon \to 0} \int \eta \sqrt{1 + |v_\varepsilon|^2} \, |\nabla G(u_\varepsilon)| \, dx \, dt = \lim_{\varepsilon \to 0} \int_{\Omega_T} \eta |\nabla' G(u_\varepsilon)| \, dx \, dt \geqslant \frac{c_0}{2} \int_{\Omega_T} \eta \, d|\nabla' u|,$$

where in the last inequality we have used the fact that

$$\frac{c_0}{2} \int_{\Omega_T} \eta \, d|\nabla' u| = \int_{\Omega_T} \eta \, d|\nabla' G(u)| \leqslant \liminf_{\varepsilon \to 0} \int_{\Omega_T} \eta |\nabla' G(u_\varepsilon)| \, dx \, dt.$$

Considering now a set $B \subset \partial^*\{u = 1\}$ with $\mu(B) = 0$ we conclude that

$$|\nabla' u|(B) \leqslant \frac{2}{c_0} \int_B g \, d\mu = 0,$$

since $g \in L^2(\mu)$. $\square$

PROPOSITION 8.3   At $\mathcal{H}^n$-almost all points in $\partial^*\{u = 1\}$ the tangential plane of μ exists.

Proof. The Radon–Nikodym theorem shows that the derivative

$$f(z) := D_{|\nabla' u|} \mu(z) := \lim_{r \searrow 0} \frac{\mu(B_r^{n+1}(z))}{|\nabla' u|(B_r^{n+1}(z))} \tag{8.28}$$

exists for $|\nabla' u|$-almost all $z \in \Omega_T$, and $f \in L^1(|\nabla' u|)$. By (8.15) we deduce that

$$\mu \lfloor \partial^*\{u = 1\} = f |\nabla' u|. \tag{8.29}$$

Similarly

$$\frac{1}{f(z)} = D_\mu |\nabla' u|(z)$$

is finite for μ-almost all $z \in \partial^*\{u = 1\}$. By (8.22) this implies that

$$f > 0 \quad |\nabla' u|\text{-almost everywhere in } \Omega_T. \tag{8.30}$$

Since $|\nabla' u|$ is rectifiable and f measurable with respect to $|\nabla' u|$ we find from (8.29), (8.30) and [31, Remark 11.5] that

$$\mu \lfloor \partial^*\{u = 1\} \text{ is rectifiable.} \tag{8.31}$$

Moreover, $\mathcal{H}^n$-almost all $z \in \partial^*\{u = 1\}$ satisfy

$$\lim_{r \searrow 0} \frac{\mu(B_r^{n+1}(z) \setminus \partial^*\{u = 1\})}{\mu(B_r^{n+1}(z))} = 0, \tag{8.32}$$

$$\limsup_{r \searrow 0} \frac{\mu(B_r^{n+1}(z))}{\omega_n r^n} < \infty. \tag{8.33}$$

In fact, (8.32) follows from [10, Theorem 2.9.11] and (8.22), and (8.33) from Proposition 8.1 and (8.22). Let now $z_0 \in \partial^*\{u = 1\}$ satisfy (8.32), (8.33). For an arbitrary $\eta \in C_c^0(B_1^{n+1}(0))$ we then deduce that

$$\limsup_{r \to 0} \left| \int_{\Omega_T \setminus \partial^*\{u=1\}} \eta(r^{-1}(z - z_0)) r^{-n} \, \mathrm{d}\mu(z) \right|$$

$$\leqslant \|\eta\|_{C_c^0(B_1^{n+1}(0))} \limsup_{r \to 0} \frac{\mu(B_r^{n+1}(z_0) \setminus \partial^*\{u = 1\})}{\mu(B_r^{n+1}(z_0))} \limsup_{r \to 0} \frac{\mu(B_r^{n+1}(z_0))}{r^n} = 0$$

by (8.32), (8.33). Therefore

$$\lim_{r \to 0} \int_{\Omega_T} \eta(r^{-1}(z - z_0)) r^{-n} \, \mathrm{d}\mu(z) = \lim_{r \to 0} \int_{\partial^*\{u=1\}} \eta(r^{-1}(z - z_0)) r^{-n} \, \mathrm{d}\mu(z)$$

if the latter limit exists. By (8.31) we therefore conclude that at $\mathcal{H}^n$-almost all points of $\partial^*\{u = 1\}$ the tangential plane of μ exists and coincides with the tangential plane of $\mu \lfloor \partial^*\{u = 1\}$. $\qquad \square$

Proof of Proposition 4.5. Since $u \in BV(\Omega_T)$ and $u(t, \cdot) \in BV(\Omega)$ for almost all $t \in (0, T)$, we know that $\partial_t u, \nabla u$ are Radon measures on Ω_T and that $\nabla u(t, \cdot)$ is a Radon measure on Ω for almost all $t \in (0, T)$. Moreover we observe that $v \in L^1(|\nabla u|)$ since

$$\int_{\Omega_T} |v| \, \mathrm{d}|\nabla u| \leqslant \int_{\Omega_T} |v| \, \mathrm{d}|\nabla' u| \leqslant \frac{2}{c_0} \int_{\Omega_T} g|v| \, \mathrm{d}\mu \leqslant \frac{2}{c_0} \|g\|_{L^2(\mu)} \|v\|_{L^2(\mu)} < \infty$$

by Theorem 4.4 and Proposition 8.2. From (3.3) and Proposition 8.3 we deduce that for any $\eta \in C_c^1(\Omega_T)$,

$$-\int_{\Omega_T} \eta \, \mathrm{d}\partial_t u = \int_{\Omega_T} \eta v \, \mathrm{d}\nabla u = \int_{\Omega_T} \eta v \cdot \frac{\nabla u}{|\nabla u|} \, \mathrm{d}|\nabla u| = \int_0^T \int_\Omega \eta V \, \mathrm{d}|\nabla u(t, \cdot)| \, \mathrm{d}t,$$

which proves (4.9). $\qquad \square$

9. Conclusions

Theorem 4.6 suggests defining a generalized action functional $\mathcal{S}$ in the class of L^2-flows by

$$\mathcal{S}(\mu) := \inf_v \int_{\Omega_T} |v - H|^2 \, \mathrm{d}\mu + 4\mathcal{S}_{\mathrm{nuc}}(\mu), \tag{9.1}$$

where the infimum is taken over all generalized velocities v for the evolution $(\mu^t)_{t \in (0,T)}$. In the class of n-rectifiable L^2-flows we have

$$\mathcal{S}(\mu) = \int_{\Omega_T} |v - H|^2 \, \mathrm{d}\mu + 4\mathcal{S}_{\mathrm{nuc}}(\mu), \tag{9.2}$$

where v is the unique normal velocity of $(\mu^t)_{t \in (0,T)}$ (see Proposition 3.3).

In the present section we compare the functional $\mathcal{S}$ with the functional $\mathcal{S}^0$ defined in [17] (see (1.2)) and discuss the implications of Theorem 4.6 on a full Gamma-convergence result for the action functional. For ease of exposition we focus in this section on the switching scenario.

ASSUMPTION 9.1 Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth functions $u_\varepsilon : \Omega_T \to \mathbb{R}$ with uniformly bounded action (A1), zero Neumann boundary data (A3), and assume that for all $\varepsilon > 0$,

$$u_\varepsilon(0, \cdot) = -1, \quad u_\varepsilon(T, \cdot) = 1 \quad \text{in } \Omega. \tag{9.3}$$

Following [17] we define the reduced action functional on the set $\mathcal{M} \subset BV(\Omega_T; \{-1, 1\}) \cap L^\infty(0, T; BV(\Omega))$ such that

- for every $\psi \in C_c^0(\Omega)$ the function

$$t \mapsto \int_\Omega u(t, \cdot)\psi \, \mathrm{d}x$$

  is absolutely continuous on $[0, T]$;
- $(\partial^*\{u(t, \cdot) = 1\})_{t \in (0,T)}$ is up to countably many times given as a smooth evolution of hypersurfaces.

By Assumption 9.1 the functional $\mathcal{S}_{\mathrm{nuc}}^0$ can be rewritten as

$$\mathcal{S}^0(u) := c_0 \int_0^T \int_{\Sigma_t} |v(t, x) - H(t, x)|^2 \, \mathrm{d}\mathcal{H}^{n-1}(x) \, \mathrm{d}t + 4\mathcal{S}_{\mathrm{nuc}}^0(u), \tag{9.4}$$

$$\mathcal{S}_{\mathrm{nuc}}^0(u) := \sum_{t_0 \in S} \sup_\psi \left(\lim_{t \downarrow t_0} \frac{c_0}{2} |\nabla u(t, \cdot)|(\psi) - \lim_{t \uparrow t_0} \frac{c_0}{2} |\nabla u(t, \cdot)|(\psi) \right)$$
$$+ \sup_\psi \lim_{t \downarrow 0} \frac{c_0}{2} |\nabla u(t, \cdot)|(\psi), \tag{9.5}$$

where the sup is taken over all $\psi \in C^1(\overline{\Omega})$ with $0 \leqslant \psi \leqslant 1$.

In [17, Proposition 2.2] a (formal) proof of the limsup estimate was given for a subclass of 'nice' functions in $\mathcal{M}$. Following the ideas of that proof, using the one-dimensional construction [17, Proposition 3.1], and a density argument we expect that the limsup estimate can be extended to the whole set $\mathcal{M}$. We do not give a rigorous proof here but assume the limsup estimate in the following.

ASSUMPTION 9.2 For all $u \in \mathcal{M}$ there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ that satisfies Assumption 9.1 such that

$$u = \lim_{\varepsilon \to 0} u_\varepsilon, \quad \mathcal{S}^0(u) \geqslant \limsup_{\varepsilon \to 0} \mathcal{S}_\varepsilon(u_\varepsilon). \tag{9.6}$$

The natural candidate for the Gamma-limit of $\mathcal{S}_\varepsilon$ with respect to $L^1(\Omega_T)$ is the $L^1(\Omega_T)$-lower semicontinuous envelope of $\mathcal{S}^0$,

$$\overline{\mathcal{S}}(u) := \inf\{\liminf_{k \to \infty} \mathcal{S}^0(u_k) : (u_k)_{k \in \mathbb{N}} \subset \mathcal{M}, u_k \to u \text{ in } L^1(\Omega_T)\}. \tag{9.7}$$

9.1   *Comparison of $\mathcal{S}$ and $\mathcal{S}^0$*

If we associate with a function $u \in \mathcal{M}$ the measure $|\nabla u|$ on Ω_T we can compare $\mathcal{S}^0(u)$ and $\mathcal{S}(\frac{c_0}{2}|\nabla u|)$.

PROPOSITION 9.3   Let $u \in \mathcal{M}$ and let $\mu = \mathcal{L}^1 \otimes \mu^t$ be an L^2-flow of measures. Assume that for almost all $t \in (0, T)$,

$$\mu^t \geqslant \frac{c_0}{2} |\nabla u(t, \cdot)|, \tag{9.8}$$

and the nucleation cost $\mathcal{S}^0_{\mathrm{nuc}}(u)$ is not larger than the nucleation cost $\mathcal{S}_{\mathrm{nuc}}(\mu)$. Then

$$\mathcal{S}^0(u) \leqslant \mathcal{S}(\mu). \tag{9.9}$$

For $\mu = \frac{c_0}{2}|\nabla u|$ we obtain

$$\mathcal{S}^0(u) = \mathcal{S}\left(\frac{c_0}{2}|\nabla u|\right). \tag{9.10}$$

Proof. The locality of the mean curvature [29] shows that the weak mean curvature of μ^t and the (classical) mean curvature coincide on $\partial\{u(t, \cdot) = 1\}$. By Proposition 4.5 any generalized velocity v and the (classical) normal velocity V are equal on the phase boundary. This shows that the integral part of $\mathcal{S}^0(u)$ is not larger than the integral part of $\mathcal{S}(\mu)$, with equality if $\mu^t = \frac{c_0}{2}|\nabla u(t, \cdot)|$ for almost all $t \in (0, T)$. This proves (9.9). For the measure $\frac{c_0}{2}|\nabla u|$ we observe that the nucleation cost $\mathcal{S}_{\mathrm{nuc}}(\frac{c_0}{2}\mu)$ equals the nucleation cost $\mathcal{S}^0_{\mathrm{nuc}}(u)$ and we obtain (9.10). $\qquad\square$
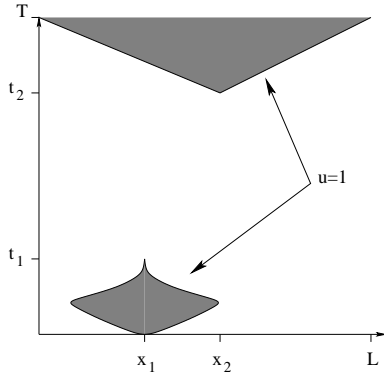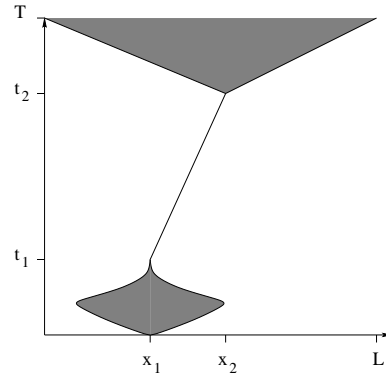

FIG. 1. The phases $\{u = 1\}$.   FIG. 2. The measure μ.

If higher multiplicities occur for the measure μ, the nucleation costs of μ and u may differ and the value of $\mathcal{S}^0(u)$ might be larger than $\mathcal{S}(\mu)$, as the following example shows. Let $\Omega = (0, L)$, let $\{u = 1\}$ be the shaded regions in Figure 1, and let μ be the measure supported on the phase boundary and with double density on a hidden boundary connecting the upper and lower parts of the phase $\{u = 1\}$ (see Figure 2). At time t_2 a new phase is nucleated, but this time is not singular with respect to the evolution $(\mu^t)_{t \in (0, T)}$. On the other hand, no propagation cost occurs for the evolution $(u(t, \cdot))_{t \in (t_1, t_0)}$ whereas there is a propagation cost for $(\mu^t)_{t \in (t_1, t_2)}$. The difference in action is given by

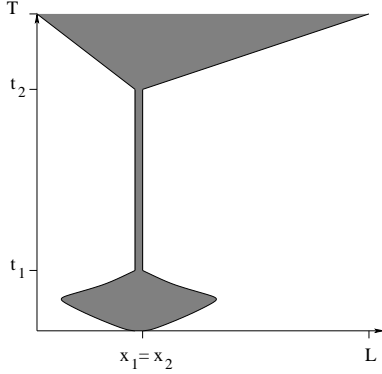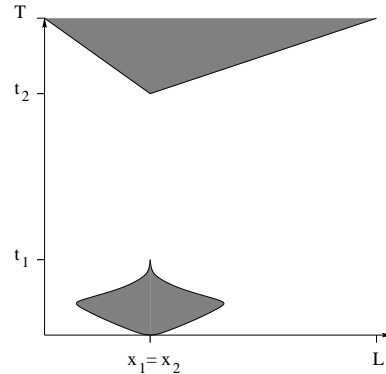$$\mathcal{S}^0(u) - \mathcal{S}(\mu) = 8c_0 - 2c_0 \frac{(x_2 - x_1)^2}{t_2 - t_1},$$

FIG. 3. Phases $\{u_k = 1\}$.

FIG. 4. The limit.

where x_1 is the annihilation point at time t_1 and x_2 the nucleation point at time t_2 (see Figure 1). This shows that as soon as $x_2 - x_1 < 4\sqrt{t_2 - t_1}$ we have

$$\mathcal{S}(\mu) < \mathcal{S}^0(u).$$

The same example with $x_2 = x_1$ shows that $\mathcal{S}^0$ is not lower semicontinuous and that a relaxation is necessary in order to obtain the Gamma-limit of $\mathcal{S}_\varepsilon$. In fact, consider a sequence $(u_k)_{k \in \mathbb{N}}$ with phases $\{u_k = 1\}$ given by the shaded region in Figure 3. Assume that the neck connecting the upper and lower parts of the shaded region disappears as $k \to \infty$ and that u_k converges to the phase indicator function u with phase $\{u = 1\}$ indicated by the shaded regions in Figure 4. Then a nucleation cost at time t_2 appears for u. For the approximations u_k however there is no nucleation cost for $t > 0$ and the approximation can be made such that the propagation cost in (t_1, t_2) is arbitrarily small, which shows that

$$\mathcal{S}^0(u) > \liminf_{k \to \infty} \mathcal{S}^0(u_k).$$

The situation in higher space dimensions is even more involved than in the one-dimensional examples discussed above. For instance one could create a circle with double density (no new phase is created) at a time t_1 and let this double-density circle grow until a time $t_2 > t_1$ where the circle splits and two circles evolve in different directions, one of them shrinking and the other growing. In this way a new phase is created at time t_2. In this example $\mathcal{S}$ counts the creation of a double-density circle at time t_1 and the cost of propagating that circle between the times t_1, t_2. In contrast, $\mathcal{S}^0$ counts the nucleation cost of the new phase at time t_2, which is larger than the nucleation cost $\mathcal{S}_{\text{nuc}}$ at time t_1, but no propagation cost between the times t_1, t_2.

The analysis in [17] suggests that minimizers of the action functional exhibit nucleation and annihilation of phases only at the initial and final times. This class is therefore particularly interesting.

THEOREM 9.4 Let $(u_\varepsilon)_{\varepsilon > 0}$ satisfy Assumption 9.1 and suppose that Assumption 9.2 holds. Suppose that $u_\varepsilon \to u$ in $L^1(\Omega_T)$, $u \in \mathcal{M}$, and u exhibits nucleation and annihilation of phases only at the final and initial times. Then

$$\overline{\mathcal{S}}(u) = \mathcal{S}^0(u) \leqslant \liminf_{\varepsilon \to 0} \mathcal{S}_\varepsilon(u_\varepsilon). \tag{9.11}$$

In particular, $\mathcal{S}_\varepsilon$ Gamma-converges to $\mathcal{S}^0$ for those evolutions in $\mathcal{M}$ that have nucleations only at the initial time.

Proof. From the definition of the functional $\overline{\mathcal{S}}$ we deduce that

$$\overline{\mathcal{S}}(u) \leqslant \mathcal{S}^0(u) \tag{9.12}$$

and there exists a sequence $(u_k)_{k \in \mathbb{N}} \subset \mathcal{M}$ such that

$$u = \lim_{k \to \infty} u_k, \quad \overline{\mathcal{S}}(u) = \lim_{k \to \infty} \mathcal{S}^0(u_k).$$

Assumption 9.2 implies that for all $k \in \mathbb{N}$ there exists a sequence $(u_{\varepsilon,k})_{\varepsilon > 0}$ such that

$$u_k = \lim_{\varepsilon \to 0} u_{\varepsilon,k}, \quad \mathcal{S}^0(u_k) \geqslant \limsup_{\varepsilon \to 0} \mathcal{S}_\varepsilon(u_{\varepsilon,k}). \tag{9.13}$$

Therefore we can choose a diagonal sequence $(u_{\varepsilon(k),k})_{k \in \mathbb{N}}$ such that

$$\overline{\mathcal{S}}(u) \geqslant \limsup_{k \to \infty} \mathcal{S}_{\varepsilon(k)}(u_{\varepsilon(k),k}). \tag{9.14}$$

By Propositions 4.1 and 4.2 there exists a subsequence $k \to \infty$ such that

$$u_{\varepsilon(k),k} \to u, \quad \mu_{\varepsilon(k),k} \to \mu, \quad \mu \geqslant \frac{c_0}{2}|\nabla u|, \tag{9.15}$$

where the last inequality follows from

$$\frac{c_0}{2} \int_\Omega \eta \, \mathrm{d}|\nabla u(t, \cdot)| \leqslant \liminf_{\varepsilon \to 0} \int_\Omega \eta|\nabla G(u_\varepsilon)| \, \mathrm{d}x \leqslant \liminf_{\varepsilon \to 0} \int_\Omega \eta \, \mathrm{d}\mu_\varepsilon^t = \int_\Omega \eta \, \mathrm{d}\mu^t,$$

with G as in (8.23). By Theorem 4.6 we further deduce that

$$\liminf_{k \to \infty} \mathcal{S}_{\varepsilon(k)}(u_{\varepsilon(k),k}) \geqslant \mathcal{S}(\mu).$$

This implies by (9.14) that

$$\overline{\mathcal{S}}(u) \geqslant \mathcal{S}(\mu). \tag{9.16}$$

Since $\mu^0 = 0$ and $\mu^t \geqslant \frac{c_0}{2}|\nabla u(t, \cdot)|$ the nucleation cost of μ at $t = 0$ is not lower than the nucleation cost for u. Since by assumption there are no more nucleation times we can apply Proposition 9.3 to obtain $\mathcal{S}^0(u) \leqslant \mathcal{S}(\mu)$. By (9.12), (9.16) we conclude that $\mathcal{S}^0(u) = \overline{\mathcal{S}}(u) = \mathcal{S}(\mu)$.

Applying Proposition 4.1 and Theorem 4.6 to the sequence $(u_\varepsilon)_{\varepsilon > 0}$ we deduce that there exists a subsequence $\varepsilon \to 0$ such that

$$\mu_\varepsilon \to \tilde{\mu}, \quad \tilde{\mu} \geqslant \frac{c_0}{2}|\nabla u| \tag{9.17}$$

and

$$\liminf_{\varepsilon \to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \geqslant \mathcal{S}(\tilde{\mu}).$$

Repeating the arguments above we deduce from Proposition 9.3 that $\mathcal{S}^0(u) \leqslant \mathcal{S}(\tilde{\mu})$ and

$$\mathcal{S}^0(u) \leqslant \liminf_{\varepsilon \to 0} \mathcal{S}_\varepsilon(u_\varepsilon).$$

Combining the upper bound (9.6) with (9.11) proves the Gamma-convergence of $\mathcal{S}_\varepsilon$ in u. $\qquad \square$

9.2   *Gamma-convergence under an additional assumption*

Using Theorem 4.6 we can prove the Gamma-convergence of $\mathcal{S}_\varepsilon$ under an additional assumption on the structure of the set of those measures that arise as limits of sequences with uniformly bounded action.

ASSUMPTION 9.5 Consider any sequence $(u_\varepsilon)_{\varepsilon>0}$ with $u_\varepsilon \to u$ in $L^1(\Omega_T)$ that satisfies Assumption 9.1. Define the energy measures μ_ε according to (2.5) and let μ be any Radon measure such that for a subsequence $\varepsilon \to 0$,

$$\mu = \lim_{\varepsilon \to 0} \mu_\varepsilon. \tag{9.18}$$

Then we assume that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that

$$u = \lim_{k\to\infty} u_k, \quad \mathcal{S}(\mu) \geqslant \lim_{k\to\infty} \mathcal{S}^0(u_k). \tag{9.19}$$

For any $u \in \mathcal{M}$ that exhibits nucleation and annihilation only at initial and final times Assumption 9.5 is always satisfied: The proof of Theorem 9.4 and our results in Section 4 show that for any limit μ as in (9.18) we can apply Proposition 9.3. Therefore $\mathcal{S}^0(u) \leqslant \mathcal{S}(\mu)$ and the constant sequence u satisfies (9.19). However, a characterization of those $u \in \mathcal{M}$ such that Assumption 9.5 holds is open.

THEOREM 9.6 Suppose that Assumptions 9.1, 9.2, and 9.5 hold. Then

$$\mathcal{S}_\varepsilon \to \overline{\mathcal{S}} \quad \text{as } \varepsilon \to 0 \tag{9.20}$$

in the sense of Gamma-convergence with respect to $L^1(\Omega_T)$.

Proof. We first prove the limsup estimate for $\mathcal{S}_\varepsilon, \overline{\mathcal{S}}$. In fact, fix an arbitrary $u \in L^1(\Omega_T; \{-1, 1\})$ with $\overline{\mathcal{S}}(u) < \infty$. We deduce that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ as in (9.7) such that

$$\overline{\mathcal{S}}(u) = \lim_{k\to\infty} \mathcal{S}^0(u_k). \tag{9.21}$$

By (9.6) for all $k \in \mathbb{N}$ there exists a sequence $(u_{\varepsilon,k})_{\varepsilon>0}$ such that

$$\lim_{\varepsilon\to 0} u_{\varepsilon,k} = u_k, \quad \mathcal{S}^0(u_k) \geqslant \limsup_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_{\varepsilon,k}).$$

Choosing a suitable diagonal sequence $u_{\varepsilon(k),k}$ we deduce that

$$\overline{\mathcal{S}}(u) \geqslant \lim_{k\to\infty} \mathcal{S}_{\varepsilon(k)}(u_{\varepsilon(k),k}), \tag{9.22}$$

which proves the limsup estimate.

We next prove the liminf estimate. Consider an arbitrary sequence $(u_\varepsilon)_{\varepsilon>0}$ that satisfies Assumption 9.1. By Theorem 4.6 there exists $u \in BV(\Omega_T; \{-1, 1\})$ and a measure μ on Ω_T such that

$$u_\varepsilon \to u \quad \text{in } L^1(\Omega_T), \quad \mu_\varepsilon \to \mu \tag{9.23}$$

for a subsequence $\varepsilon \to 0$, and

$$\liminf_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \geqslant \mathcal{S}(\mu). \tag{9.24}$$

By Assumption 9.5 there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that (9.19) holds. By (9.24) and the definition of $\overline{\mathcal{S}}$ this yields

$$\liminf_{\varepsilon\to 0} \mathcal{S}_\varepsilon(u_\varepsilon) \geqslant \mathcal{S}(\mu) \geqslant \lim_{k\to\infty} \mathcal{S}^0(u_k) \geqslant \overline{\mathcal{S}}(u) \tag{9.25}$$

and proves the liminf estimate. $\qquad\square$

Appendix A. Rectifiable measures and weak mean curvature

We briefly summarize some definitions from geometric measure theory. We always restrict ourselves to the hypersurface case, that is, 'tangential plane' and 'rectifiability' of a measure in $\mathbb{R}^d$ means '$(d-1)$-dimensional tangential plane' and '$(d-1)$-rectifiable'.

DEFINITION A.1   Let μ be a Radon measure in $\mathbb{R}^d$, $d \in \mathbb{N}$.

1. We say that μ has a (*generalized*) *tangential plane* at $z \in \mathbb{R}^d$ if there exist a number $\Theta > 0$ and a $(d-1)$-dimensional linear subspace $T \subset \mathbb{R}^d$ such that

$$\lim_{r\searrow 0} r^{-d+1} \int \eta\left(\frac{y-z}{r}\right) \mathrm{d}\mu(y) = \Theta \int_T \eta \, \mathrm{d}\mathcal{H}^{d-1} \quad \text{for every } \eta \in C_c^0(\mathbb{R}^d). \tag{A1}$$

    We then set $T_z\mu := T$ and call Θ the *multiplicity* of μ in z.
2. If for μ-almost all $z \in \mathbb{R}^d$ a tangential plane exists then we call μ *rectifiable*. If in addition the multiplicity is integer-valued μ-almost everywhere we say that μ is *integer-rectifiable*.
3. The *first variation* $\delta\mu : C_c^1(\mathbb{R}^d, \mathbb{R}^d)$ of a rectifiable Radon-measure μ on $\mathbb{R}^d$ is defined by

$$\delta\mu(\eta) := \int \mathrm{div}_{T_z\mu}\, \eta \, \mathrm{d}\mu.$$

If there exists a function $H \in L^1_{\mathrm{loc}}(\mu)$ such that

$$\delta\mu(\eta) = -\int H \cdot \eta \, \mathrm{d}\mu$$

we call H the *weak mean-curvature vector* of μ.

Appendix B. Measure-function pairs

We recall some basic facts about the notion of *measure-function pairs* introduced by Hutchinson in [15].

DEFINITION B.1   Let $E \subset \mathbb{R}^d$ be an open subset. Let μ be a positive Radon measure on E. Suppose $f : E \to \mathbb{R}^m$ is well defined μ-almost everywhere, and $f \in L^1(\mu, \mathbb{R}^m)$. Then we say (μ, f) is a *measure-function pair* over E (with values in $\mathbb{R}^m$).

Next we define two notions of convergence for a sequence of measure-function pairs on E with values in $\mathbb{R}^m$.

DEFINITION B.2   Suppose $\{(\mu_k, f_k)\}_k$ and (μ, f) are measure-function pairs over E with values in $\mathbb{R}^m$. Suppose

$$\lim_{k \to \infty} \mu_k = \mu$$

as Radon measures on E. Then we say (μ_k, f_k) *converges to* (μ, f) *in the weak sense* (in E) and write

$$(\mu_k, f_k) \to (\mu, f),$$

if $\mu_k \lfloor f_k \to \mu \lfloor f$ in the sense of vector-valued measures, that is,

$$\lim_{k \to \infty} \int f_k \cdot \eta \, \mathrm{d}\mu_k = \int f \cdot \eta \, \mathrm{d}\mu$$

for all $\eta \in C_c^0(E, \mathbb{R}^m)$.

The following result is a slightly less general version of [15, Theorem 4.4.2], sufficient for our aims.

THEOREM B.3   Let $F : \mathbb{R}^m \to [0, \infty)$ be a continuous, convex function with superlinear growth at infinity, that is,

$$\lim_{|y| \to \infty} \frac{F(y)}{|y|} = \infty.$$

Let $\{(\mu_k, f_k)\}_k$ be measure-function pairs over $E \subset \mathbb{R}^d$ with values in $\mathbb{R}^m$. Suppose μ is a Radon measure on E and $\mu_k \to \mu$ as $k \to \infty$. Then the following are true:

(1) If

$$\sup_k \int \int F(f_k) \, \mathrm{d}\mu_k < \infty, \tag{B1}$$

then some subsequence of $\{(\mu_k, f_k)\}$ converges in the weak sense to some measure-function pair (μ, f) for some f.

(2) If (B1) holds and $(\mu_k, f_k) \to (\mu, f)$ then

$$\liminf_{k \to \infty} \int F(f_k) \, \mathrm{d}\mu_k \geqslant \int F(f) \, \mathrm{d}\mu. \tag{B2}$$

Acknowledgments

REFERENCES

1. ALLARD, W. K.   On the first variation of a varifold.   *Ann. of Math. (2)* **95** (1972), 417–491. Zbl 0252.49028 MR 0307015

2. ALLOUBA, H., & LANGA, J. A. Semimartingale attractors for Allen–Cahn SPDEs driven by space-time white noise. I. Existence and finite dimensional asymptotic behavior. *Stoch. Dyn.* **4** (2004), 223–244. Zbl 1078.60046 MR 2069690

3. AMBROSIO, L., FUSCO, N., & PALLARA, D. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Math. Monogr., Clarendon Press Oxford Univ. Press, New York (2000). Zbl 0957.49001 MR 1857292

4. BELLETTINI, G., & MUGNAI, L. Remarks on the variational nature of the heat equation and of mean curvature flow. Preprint (2007).

5. BRAKKE, K. A. *The Motion of a Surface by its Mean Curvature*. Math. Notes. 20, Princeton Univ. Press, Princeton, NJ (1978). Zbl 0386.53047 MR 0485012

6. DE GIORGI, E. Some remarks on Γ-convergence and least squares method. *Composite Media and Homogenization Theory* (Trieste, 1990), Progr. Nonlinear Differential Equations Appl. 5, Birkhäuser Boston, Boston, MA (1991). 135–142. Zbl 0747.49008 MR 1145748

7. E, W., REN, W., & VANDEN-EIJNDEN, E. Minimum action method for the study of rare events. *Comm. Pure Appl. Math.* **57** (2004), 637–656. Zbl 1050.60068 MR 2032916

8. EVANS, L. C., SONER, H. M., & SOUGANIDIS, P. E. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* **45** (1992), 1097–1123. Zbl 0801.35045 MR 1177477

9. FARIS, W. G., & JONA-LASINIO, G. Large fluctuations for a nonlinear heat equation with noise. *J. Phys. A* **15** (1982), 3025–3055. Zbl 0496.60060 MR 0684578

10. FEDERER, H. *Geometric Measure Theory*. Grundlehren Math. Wiss. 153, Springer, New York (1969). Zbl 0176.00801 MR 0257325

11. FENG, J. Large deviation for diffusions and Hamilton-Jacobi equation in Hilbert spaces. *Ann. Probab.* **34** (2006), 321–385. Zbl 1091.60002 MR 2206350

12. FOGEDBY, H. C., HERTZ, J., & SVANE, A. Domain wall propagation and nucleation in a metastable two-level system. Preprint (2004).

13. FREIDLIN, M. I., & WENTZELL, A. D. *Fluctuations in Dynamical Systems under Small Random Perturbations*. 2nd ed., Nauka, Moscow (1979)(in Russian).

14. FREIDLIN, M. I., & WENTZELL, A. D. *Random Perturbations of Dynamical Systems*. Grundlehren Math. Wiss. 260 Springer, New York (1998). Zbl 0922.60006 MR 1652127

15. HUTCHINSON, J. E. Second fundamental form for varifolds and the existence of surfaces minimising curvature. *Indiana Univ. Math. J.* **35** (1986), 45–71. Zbl 0561.53008 MR 0825628

16. ILMANEN, T. Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* **38** (1993), 417–461. Zbl 0784.53035 MR 1237490

17. KOHN, R., OTTO, F., REZNIKOFF, M. G., & VANDEN-EIJNDEN, E. Action minimization and sharp-interface limits for the stochastic Allen–Cahn equation. *Comm. Pure Appl. Math.* **60** (2007), 393–438. Zbl pre05155934 MR 2284215

18. KOHN, R. V., REZNIKOFF, M. G., & TONEGAWA, Y. Sharp-interface limit of the Allen–Cahn action functional in one space dimension. *Calc. Var. Partial Differential Equations* **25** (2006), 503–534. Zbl 1094.60044 MR 2214622

19. KOHN, R. V., REZNIKOFF, M. G., & VANDEN-EIJNDEN, E. Magnetic elements at finite temperature and large deviation theory. *J. Nonlinear Sci.* **15** (2005), 223–253. Zbl 1107.82068 MR 2173980

20. KOSSIORIS, G. T., KATSOULAKIS, M. A., & LAKKIS, O. Noise regularization and computations for the 1-dimensional stochastic Allen–Cahn problem. *Interfaces Free Bound.* **9** (2007), 1–30. Zbl pre05174077 MR 2317297

21. MODICA, L. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.* **98** (1987), 123–142. Zbl 0616.76004 MR 0866718

22. MODICA, L., & MORTOLA, S. Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B (5)* **14** (1977), 285–299. Zbl 0356.49008 MR 0445362

23. MOSER, R. A generalization of Rellich's theorem and regularity of varifolds minimizing curvature. Preprint 72, Max Planck Institute for Mathematics in the Sciences, Leipzig (2001).

24. DE MOTTONI, P., & SCHATZMAN, M. Development of interfaces in $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A* **116** (1990), 207–220. Zbl 0725.35009 MR 1084732

25. PLOTNIKOV, P. I., & STAROVOĬTOV, V. N. The Stefan problem with surface tension as a limit of the phase field model. *Differentsial′nye Uravneniya* **29** (1993), 461–471, 550 (in Russian). Zbl 0802.35165 MR 1236334

26. REZNIKOFF, M. G., & TONEGAWA, Y. Higher multiplicity in the one-dimensional Allen–Cahn action functional. *Indiana Univ. Math. J.* **56** (2007), 2935–2989.

27. RÖGER, M. Existence of weak solutions for the Mullins–Sekerka flow. *SIAM J. Math. Anal.* **37** (2005), 291–301. Zbl 1088.49031 MR 2176933

28. RÖGER, M., & SCHÄTZLE, R. On a modified conjecture of De Giorgi. *Math. Z.* **254** (2006), 675–714. Zbl pre05204059 MR 2253464

29. SCHÄTZLE, R. Lower semicontinuity of the Willmore functional for currents. Preprint (2004).

30. SHARDLOW, T. Stochastic perturbations of the Allen–Cahn equation. *Electron. J. Differential Equations* **2000**, No. 47, 19 pp. Zbl 0959.60047 MR 1764701

31. SIMON, L. *Lectures on Geometric Measure Theory*. Proc. Centre Math. Anal. Austral. Nat. Univ. 3, Austral. National Univ., Canberra (1983). Zbl 0546.49019 MR 0756417