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Finite element approximation of a Cahn–Hilliard–Navier–Stokes system

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We consider a semi-discrete and a practical fully discrete finite element approximation of a Cahn–Hilliard–Navier–Stokes system. This system arises in the modelling of multiphase fluid systems. We show order h error estimate between the solution of the system and the solution of the semi-discrete approximation. We also show the convergence of the fully discrete approximation. Finally, we present an efficient implementation of the fully discrete scheme together with some numerical simulations.

1. Introduction

The Cahn–Hilliard equation [10, 11] is a phenomenological model of phase transitions (see the survey work of Elliott [15] for details). Coupling the Cahn–Hilliard and Navier–Stokes equations yields a model for the dynamics of multiphase fluids that is used to model phenomena such as hydrodynamic effects during spinodal decomposition and the behaviour of polymer fluids (see for example [27, 26]). The Cahn–Hilliard–Navier–Stokes system that we consider is known as 'Model H' in the nomenclature of Hohenberg and Halperin [22]:

(**R**) Find $\{c(x,t), w(x,t), \mathbf{u}(x,t), p(x,t)\}$ such that

$$\partial_t c - \frac{1}{\text{Pe}}\nabla \cdot (b(c)\nabla w) + \mathbf{u}\cdot\nabla c = 0 \qquad \text{in } \Omega_T := \Omega \times (0,T), \tag{1.1a}$$

$$w = \Phi'(c) - \gamma^2 \Delta c \qquad \text{in } \Omega_T, \tag{1.1b}$$

$$\partial_t \mathbf{u} - \frac{1}{\text{Re}}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p + Kc\nabla w = 0 \qquad \text{in } \Omega_T, \tag{1.1c}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega_T, \tag{1.1d}$$

$$c(x,0) = c_0(x), \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \qquad \forall x \in \Omega, \tag{1.1e}$$

$$\partial_{\mathbf{n}} c = \partial_{\mathbf{n}} w = 0, \quad \mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega \times (0,T), \tag{1.1f}$$

where $\mathbf{g}$ satisfies $\mathbf{g}\cdot\mathbf{n} = 0$ on $\partial\Omega$. Here Ω is a bounded convex polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$ that has outward pointing unit normal $\mathbf{n}$. The concentration order parameter c is such that $c(x,t) \approx 1$ (respectively $c(x,t) \approx -1$) if and only if at time t fluid 1 (respectively fluid 2) is

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present at the point x. The mean velocity field $\mathbf{u}$ is defined to be

$$\mathbf{u} = \frac{1+c}{2}\mathbf{u}_1 + \frac{1-c}{2}\mathbf{u}_2, \tag{1.2}$$

where $\mathbf{u}_i$ is the velocity field of fluid component i. The chemical potential and the pressure in the system are defined by w and p respectively, and the Péclet number Pe, the Reynolds number Re and the capillary number K are given constants. The interface parameter $\gamma > 0$ is also a given constant that is assumed to be small. We take the mobility function $b(\cdot)$ in (1.1a) to be of the form

$$\exists b_1, b_2 > 0 \quad b_1 \leqslant b(x) \leqslant b_2 \quad \forall x \in \Omega \tag{1.3}$$

and we take the free energy $\Phi(\cdot)$ in (1.1b) to be

$$\Phi(c) = \frac{1}{4}(1-c^2)^2. \tag{1.4}$$

By $\partial_t \eta$ we mean $\partial \eta / \partial t$ and similarly for $\partial_{\mathbf{n}} \eta$.

In many fluid flow applications more than one fluid is present, with the components often separated by an interfacial region of partial miscibility, even if the fluids themselves are immiscible. Such multicomponent systems can use either a sharp or a diffuse interface to model these free boundaries, but in the evolution of such problems often breakup or coalescence of interfaces can occur and it is important that the model used can accurately capture this change of topology. Phase field models, such as the Cahn–Hilliard equation, place a diffuse interface between the phases of the system, which allows for a natural description of topological changes in the model without relying on any additional input from the user. The numerical simulation of these models is an area of research currently undergoing intense study (see for example [2, 9, 23, 26, 27]), with the majority of the work so far concentrating on finite difference discretisations. However, despite there being much work on finite element discretizations for both the Cahn–Hilliard (see for example [3, 4, 6]) and the Navier–Stokes equations (see for example [21]) separately, there has been little work on finite element discretizations for Cahn–Hilliard–Navier–Stokes problems.

Adaptive finite element techniques are well suited to phase field modelling since the solution of such models rapidly varies over the interfacial regions while away from the interface, in the bulk regions, it is close to ± 1. This means that it is natural to assume that most of the computational work is needed in the interfacial region. Additionally this region needs to be accurately resolved, as otherwise spurious numerical solutions can occur (see [18]). As a result it is common to couple phase field modelling with adaptive mesh refinement whereby the mesh is locally refined close to and inside of the interfacial region and coarsened elsewhere (see Section 5).

The model (**R**) gives a diffuse interface description of binary incompressible fluid flow (see [1]). An existence/uniqueness result for (**R**), with (1.4) replaced by a more general form for $\Phi(\cdot)$, is given in [12], and in [7] an existence result for the problem with degenerate mobility is presented. The motivation behind this paper is to analyse semi-discrete and fully discrete finite element discretizations of (**R**) and to present some numerical computations. In Section 2 we introduce notation and define a weak formulation, (**P**), of (**R**); then in Section 3 we introduce a continuous in time finite element approximation, $(\mathbf{P}_h)$, of (**P**) and we use the techniques presented in [16] to show order h error estimate between the solutions of (**P**) and $(\mathbf{P}_h)$. In Section 4 we introduce a practical discrete in time and space finite element approximation, $(\mathbf{P}_{h,\tau})$, of (**P**) that we show converges to (**P**) as the spatial and temporal parameters tend to zero. We conclude with Section 5 in which we present an efficient implementation of the practical fully discrete scheme together with some numerical results.

REMARK 1.1 1. Setting $\mathbf{u} \equiv \mathbf{0}$ in (**R**) yields the Cahn–Hilliard equation, while setting $c \equiv 0$ in (**R**) yields the Navier–Stokes equation.

2. The term $c\nabla w$ in the Navier–Stokes equation (1.1c) can be replaced by $w\nabla c$ where the additional term $\nabla(wc)$ is absorbed into the pressure term.

REMARK 1.2 Upon completion of this work we became aware of a related paper by Feng [19]. In this paper a fully discrete finite element approximation for a Cahn–Hilliard–Navier–Stokes system in three space dimensions is developed and analysed. In particular the author shows the convergence of the numerical scheme and establishes the sharp interface limit of the model by utilising a discrete energy law. The main differences between the convergence proof in [19] and the one in Section 4 of this paper is that the finite element discretization in [19] is fully implicit, resulting in a Cahn–Hilliard–Navier–Stokes coupled system. In this paper we analyse a semi-implicit scheme that only requires the solutions of separate Cahn–Hilliard and Navier–Stokes equations. As a result we require different techniques to obtain the stability bounds on the approximate solutions.

2. Notation and auxiliary results

Let $L^p(\Omega)$ denote the space of p-integrable functions with norm denoted by $\|\cdot\|_{0,p}$, where for simplicity of notation we set $\|\cdot\|_{0,2} = \|\cdot\|$. Furthermore let $W^{m,p}(\Omega)$ and $H^m(\Omega)$ be the usual Sobolev spaces with norms $\|\cdot\|_{m,p}$ and $\|\cdot\|_m$ respectively and let $(H^1(\Omega))'$ denote the dual space of $H^1(\Omega)$. Let $H_0^1(\Omega)$ be defined by

$$H_0^1(\Omega) := \{\eta \in H^1(\Omega) : \eta = 0 \text{ on } \partial\Omega\}, \tag{2.1}$$

and let $H^{-1}(\Omega)$ denote the dual space of $H_0^1(\Omega)$ with norm $\|\cdot\|_{-1}$. Let $\mathbf{L}^p(\Omega) = (L^p(\Omega))^2$ and $\mathbf{H}^m(\Omega) := (H^m(\Omega))^2$ for $m = -1, 1$ and 2. We define the following spaces:

$$\begin{aligned} V &= \{v \in H^1(\Omega) : (v, 1) = 0\}, \\ \mathcal{F} &= \{v \in (H^1(\Omega))' : \langle v, 1\rangle = 0\}, \\ \mathbf{W} &= \{\mathbf{v} \in \mathbf{H}^1(\Omega) : (\nabla\cdot\mathbf{v}, \eta) = 0 \ \forall \eta \in L^2(\Omega)\}, \\ \mathbf{W}_0 &= \{\mathbf{v} \in \mathbf{W} : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}, \end{aligned}$$

where $(\cdot,\cdot)$ denotes the L^2 inner product and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.

For $\mathbf{v} \in \mathbf{H}^1(\Omega),\ \eta, \xi \in H^1(\Omega)$ we define

$$\mathbf{a}(\eta, \xi, \mathbf{v}) := \int_\Omega [\eta\nabla\xi\cdot\mathbf{v}]\,\mathrm{d}x, \tag{2.2}$$

and for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ we define

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}]\,\mathrm{d}x - \frac{1}{2}\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}]\,\mathrm{d}x. \tag{2.3}$$

Finally, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$ we define

$$(\nabla\mathbf{v} : \nabla\mathbf{w}) := \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{w}\,\mathrm{d}x$$

where the product $\mathcal{E} : \mathcal{F}$ of two $n \times n$ matrices $\mathcal{E}$ and $\mathcal{F}$ is defined to be $\sum_{i,j=1}^n \mathcal{E}_{ij}\mathcal{F}_{ij}$.

REMARK 2.1 The trilinear form $B(\mathbf{u},\mathbf{v},\mathbf{w})$ in (2.3) that is associated with the nonlinearity in the Navier–Stokes equation has the following property:

$$B(\mathbf{v},\mathbf{u},\mathbf{u}) = 0. \tag{2.4}$$

Next we introduce the Green's operator $\mathcal{G} : \mathcal{F} \to V$ such that

$$(\nabla[\mathcal{G}z], \nabla\eta) = \langle z, \eta\rangle \quad \forall \eta \in H^1(\Omega). \tag{2.5}$$

The well-posedness of $\mathcal{G}$ follows from the generalised Lax–Milgram theorem and the Poincaré inequality

$$\|\eta\| \leqslant C(\|\nabla\eta\| + |(\eta, 1)|) \quad \forall \eta \in H^1(\Omega). \tag{2.6}$$

Let X, Y, Z be Banach spaces with a compact embedding $X \hookrightarrow Y$ and a continuous embedding $Y \hookrightarrow Z$. Then the embeddings

$$\{\eta \in L^2(0,T;X) : \partial_t\eta \in L^2(0,T;Z)\} \hookrightarrow L^2(0,T;Y) \tag{2.7a}$$

and

$$\{\eta \in L^\infty(0,T;X) : \partial_t\eta \in L^2(0,T;Z)\} \hookrightarrow C([0,T];Y) \tag{2.7b}$$

are compact. In the following we often use the well-known result

$$\|\eta\|_{0,r} \leqslant C\|\eta\|_1 \quad \forall \eta \in H^1(\Omega) \text{ and } r \in [2,\infty),$$

in particular when bounding the triple

$$\mathbf{a}(\eta,\chi,\boldsymbol{\xi}) = \left|\int_\Omega \eta\nabla\chi\cdot\boldsymbol{\xi}\,\mathrm{d}x\right| \leqslant \|\eta\boldsymbol{\xi}\|\,\|\nabla\chi\| \leqslant \|\eta\|_{0,4}\|\boldsymbol{\xi}\|_{0,4}\|\nabla\chi\| \leqslant \|\eta\|_1\|\boldsymbol{\xi}\|_1\|\nabla\chi\|. \tag{2.8}$$

Furthermore we note that for $\mu = 1 - 2/r$ and $\sigma = 1/2 - 2/r$ we have

$$\|\eta\|_{0,r} \leqslant C\|\eta\|^{1-\mu}\|\eta\|_1^{\mu} \quad \forall \eta \in H^1(\Omega) \text{ and } r \in [2,\infty) \tag{2.9}$$

and

$$\|\eta\|_{0,r} \leqslant C\|\eta\|_{0,4}^{1-\sigma}\|\eta\|_{1,4}^{\sigma} \quad \forall \eta \in W^{1,4}(\Omega) \text{ and } r \in [4,\infty]. \tag{2.10}$$

Throughout this work C, C_1 and C_2 will denote constants whose value may change from line to line. Furthermore C_1 will denote a constant that can be taken to be arbitrarily small.

2.1 *A weak formulation of the problem*

We now introduce a weak formulation of (1.1a–f):

Find $\{c(x,t), w(x,t), \mathbf{u}(x,t), p(x,t)\}$ such that

$$(\partial_t c, \eta) + \frac{1}{\mathrm{Pe}}(b(c)\nabla w, \nabla\eta) = \mathbf{a}(c,\eta,\mathbf{u}) \quad \forall \eta \in H^1(\Omega), \tag{2.11a}$$

$$(w,\eta) = (\Phi'(c),\eta) + \gamma^2(\nabla c, \nabla\eta) \quad \forall \eta \in H^1(\Omega), \tag{2.11b}$$

$$(\partial_t\mathbf{u},\mathbf{v}) + \frac{1}{\mathrm{Re}}(\nabla\mathbf{u} : \nabla\mathbf{v}) + B(\mathbf{u},\mathbf{u},\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) + K\mathbf{a}(c,w,\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{2.11c}$$

$$(\nabla\cdot\mathbf{u}, \chi) = 0 \quad \forall \chi \in L^2(\Omega), \tag{2.11d}$$

$$c(x,0) = c_0(x) \in H^2(\Omega)\cap V, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathbf{W}_0 \cap \mathbf{H}^2(\Omega) \quad \forall x \in \Omega, \tag{2.11e}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega\times(0,T). \tag{2.11f}$$

In Sections 3 and 4 we analyse the following weak formulation of (2.11a–f) with $\mathbf{g} = \mathbf{0}$ in which for simplicity of presentation we have set $b(\cdot) \equiv 1$ and $\mathrm{Pe} = \mathrm{Re} = K = 1$:

(**P**) Find $\{c(x,t), w(x,t), \mathbf{u}(x,t)\} \in V \times H^1(\Omega) \times \mathbf{W}_0$ such that

$$(\partial_t c, \eta) + (\nabla w, \nabla \eta) = \mathbf{a}(c, \eta, \mathbf{u}) \qquad \forall \eta \in H^1(\Omega), \tag{2.12a}$$

$$(w, \eta) = (\Phi'(c), \eta) + \gamma^2 (\nabla c, \nabla \eta) \qquad \forall \eta \in H^1(\Omega), \tag{2.12b}$$

$$(\partial_t \mathbf{u}, \mathbf{v}) + (\nabla \mathbf{u} : \nabla \mathbf{v}) + B(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \mathbf{a}(c, w, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{W}_0, \tag{2.12c}$$

$$c(x,0) = c_0(x) \in H^2(\Omega) \cap V, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathbf{W}_0 \cap \mathbf{H}^2(\Omega) \qquad \forall x \in \Omega. \tag{2.12d}$$

REMARK 2.2 We note that (2.12c) can also be written as

$$(\partial_t \mathbf{u}, \mathbf{v}) + (\nabla \mathbf{u} : \nabla \mathbf{v}) + B(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + \mathbf{a}(c, w, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega). \tag{2.13}$$

Since $\mathbf{g} = \mathbf{0}$, from [12] and [30] it follows that there exists a unique solution $\{c, w, \mathbf{u}\}$ of (**P**) that satisfies

$$\begin{cases} c \in L^\infty(0,T; H^2(\Omega) \cap V) \cap H^1(0,T; L^2(\Omega)), \\ \mathbf{u} \in L^\infty(0,T; \mathbf{W}_0 \cap \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{L}^2(\Omega)), \\ w \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega)), \\ p \in L^2(0,T; H^1(\Omega)). \end{cases} \tag{2.14}$$

2.2 *Finite element notation*

In Sections 3 and 4 we consider a semi-discrete and a fully discrete finite element approximation of (**P**) under the following assumptions on the mesh.

Let $\mathcal{T}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices $\{\kappa\}$ with $h_\kappa := \mathrm{diam}(\kappa)$ and $h := \max_{\kappa \in \mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}^h} \overline{\kappa}$.

Also we assume that $\mathcal{T}^h$ is weakly acute; that is, for any pair of adjacent triangles the sum of the opposite angles relative to the common side does not exceed π. We define $\mathcal{T}^{h/2}$ to be the mesh obtained from $\mathcal{T}^h$ by refining each simplex κ into four similar triangles by joining the midpoints of each edge of κ.

Associated with $\mathcal{T}^h$ and $\mathcal{T}^{h/2}$ are the finite element spaces

$$S^h := \{v \in C(\overline{\Omega}) : v|_\kappa \text{ is linear } \forall \kappa \in \mathcal{T}^h\} \subset H^1(\Omega),$$
$$S^{h/2} := \{v \in C(\overline{\Omega}) : v|_\kappa \text{ is linear } \forall \kappa \in \mathcal{T}^{h/2}\} \subset H^1(\Omega).$$

Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the piecewise linear Lagrange interpolation operator such that $(\pi^h v)(x_j) = v(x_j)$ for all $j = 1, \dots, J$, where J denotes the number of nodes of $\mathcal{T}^h$. Similarly we define $\boldsymbol{\pi}^h : (C(\overline{\Omega}))^2 \to (S^{h/2})^2$ and we note that for all $q \in (2, \infty]$,

$$\|\nabla(I - \boldsymbol{\pi}^h)\boldsymbol{\eta}\| \leqslant Ch\|\boldsymbol{\eta}\|_2 \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^2(\Omega), \tag{2.15a}$$

$$\|\nabla(I - \boldsymbol{\pi}^h)\boldsymbol{\eta}\|_{0,q} \leqslant C\|\nabla \boldsymbol{\eta}\|_{0,q} \qquad \forall \boldsymbol{\eta} \in \mathbf{H}^{1,q}(\Omega). \tag{2.15b}$$

We define the finite element spaces, analogous to the earlier continuous spaces,

$$
\begin{aligned}
V^h &= \{v \in S^h : (v, 1) = 0\}, \\
\mathbf{W}^h &= \{\mathbf{v} \in (S^{h/2})^2 : (\nabla \cdot \mathbf{v}, \chi) = 0 \; \forall \chi \in S^h\}, \\
\mathbf{W}_0^h &= \{\mathbf{v} \in \mathbf{W}^h : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}.
\end{aligned}
$$

We define a discrete $L^2(\Omega)$ inner product on $C(\overline{\Omega})$ via

$$(u, v)_h := \int_\Omega \pi^h(uv)\,\mathrm{d}x. \tag{2.16}$$

For $\eta, \chi \in C(\overline{\Omega})$ we set

$$I_h(\eta, \chi) := (\eta, \chi)_h - (\eta, \chi) \tag{2.17}$$

and we note that

$$|I_h(\eta, \chi)| = |(\eta, \chi)_h - (\eta, \chi)| \leqslant Ch\|\eta\|\,\|\nabla\chi\| \quad \forall \eta, \chi \in S^h. \tag{2.18}$$

It is well known that the discrete inner product (2.16) induces a norm on $S^h \subset C(\overline{\Omega})$, via

$$\|\chi\|_h^2 := (\chi, \chi)_h \quad \forall \chi \in S^h \tag{2.19}$$

and that there is an equivalence of norms between $\|\cdot\|$ and $\|\cdot\|_h$, i.e.

$$C\|\chi\| \leqslant \|\chi\|_h \leqslant C\|\chi\|, \quad \forall \chi \in S^h. \tag{2.20}$$

The Poincaré inequality (2.6) together with (2.18)–(2.20) yields, for h sufficiently small, the following discrete Poincaré inequality:

$$\|\chi\|_h \leqslant C(\|\nabla\chi\| + |(\chi, 1)_h|) \quad \forall \chi \in S^h, \tag{2.21}$$

so

$$\|\chi\|_1 \leqslant C\|\nabla\chi\| \quad \forall \chi \in V^h. \tag{2.22}$$

We introduce the L^2 projection $Q^h : L^2(\Omega) \to S^h$ defined by

$$(Q^h\eta, \chi)_h = (\eta, \chi) \quad \forall \chi \in S^h \tag{2.23}$$

and we note that

$$\|(I - Q^h)\eta\| + h\|\nabla(I - Q^h)\eta\| \leqslant Ch\|\nabla\eta\| \quad \forall \eta \in H^1(\Omega). \tag{2.24}$$

We also introduce the projection P^h on S^h, with respect to the inner product $(\nabla v, \nabla w)$, so that for $\eta \in H^1(\Omega)$,

$$(\nabla P^h\eta, \nabla\chi) = (\nabla\eta, \nabla\chi) \quad \forall \chi \in S^h \quad \text{and} \quad (P^h\eta, 1) = (\eta, 1) \tag{2.25}$$

and we note that

$$\|P^h\eta - \eta\| + h\|\nabla(P^h\eta - \eta)\| \leqslant Ch^s\|\eta\|_s \quad \forall \eta \in H^s(\Omega),\ s = 1, 2. \tag{2.26}$$

We also define $\mathbf{P}^h : \mathbf{W}_0 \to \mathbf{W}_0^h$ to be the Stokes projection such that for all $\mathbf{v} \in \mathbf{W}_0$,

$$(\nabla \mathbf{P}^h \mathbf{v} : \nabla \boldsymbol{\chi}) = (\nabla \mathbf{v} : \nabla \boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in \mathbf{W}_0^h \tag{2.27}$$

and we note that

$$\|\mathbf{P}^h \mathbf{v} - \mathbf{v}\| + h\|\nabla(\mathbf{P}^h \mathbf{v} - \mathbf{v})\| \leqslant Ch^s \|\mathbf{v}\|_s \quad \forall \mathbf{v} \in \mathbf{H}^s(\Omega) \cap \mathbf{W}_0,\ s = 1, 2. \tag{2.28}$$

We introduce the ‘discrete Laplacian’ operator $\Delta^h : S^h \to V^h$ such that

$$(\Delta^h z_h, \chi) = -(\nabla z_h, \nabla \chi) \quad \forall \chi \in S^h \tag{2.29a}$$

and since we have a quasi-uniform family of partitionings and Ω is convex we have (see [5])

$$\|\nabla z_h\|_{0,s} \leqslant C\|\Delta^h z_h\| \quad \forall s \in (1, \infty). \tag{2.29b}$$

Next we introduce, similar to (2.5), the discrete Green's operators $\mathcal{G}^h : \mathcal{F} \to V^h$ and $\widehat{\mathcal{G}}^h : V^h \to V^h$ such that

$$(\nabla[\mathcal{G}^h \eta], \nabla \chi) = \langle \eta, \chi \rangle \quad \forall \chi \in S^h \tag{2.30a}$$

and

$$(\nabla[\widehat{\mathcal{G}}^h z_h], \nabla \chi) = (z_h, \chi)_h \quad \forall \chi \in S^h. \tag{2.30b}$$

From [17] have

$$\|\nabla \mathcal{G} v\| \leqslant C\|\nabla \mathcal{G}^h v\| \quad \forall v \in S^h. \tag{2.31}$$

Finally we note the well-known inverse inequalities

$$\|\chi\|_{0,4} \leqslant Ch^{-1/2}\|\chi\| \qquad \forall \chi \in S^h, \tag{2.32a}$$

$$\|\chi\|_{0,\infty} \leqslant C\sqrt{\ln(1/h)}\,\|\chi\|_1 \qquad \forall \chi \in S^h. \tag{2.32b}$$

3. Semi-discrete scheme

In this section we consider a continuous in time finite element approximation, $(\mathbf{P}_h)$, of $(\mathbf{P})$ and we use the techniques presented in [16] to prove an error estimate between the solutions of $(\mathbf{P}_h)$ and $(\mathbf{P})$.

We consider the following semi-discrete approximation of $(\mathbf{P})$:

$(\mathbf{P}_h)$ Find $\{c_h, w_h, \mathbf{u}_h\} \in V^h \times S^h \times \mathbf{W}_0^h$ such that

$$(\partial_t c_h, \chi) + (\nabla w_h, \nabla \chi) = \mathbf{a}(c_h, \chi, \mathbf{u}_h) \qquad \forall \chi \in S^h, \tag{3.1a}$$

$$(w_h, \chi) = (\Phi'(c_h), \chi) + \gamma^2(\nabla c_h, \nabla \chi) \qquad \forall \chi \in S^h, \tag{3.1b}$$

$$(\partial_t \mathbf{u}_h, \boldsymbol{\chi}) + (\nabla \mathbf{u}_h : \nabla \boldsymbol{\chi}) + B(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\chi}) + \mathbf{a}(c_h, w_h, \boldsymbol{\chi}) = 0 \qquad \forall \boldsymbol{\chi} \in \mathbf{W}_0^h, \tag{3.1c}$$

$$c_h(x, 0) = Q^h(c_0(x)) \in V^h \cap L^\infty(\Omega), \quad \mathbf{u}_h(x, 0) = \mathbf{P}^h(\mathbf{u}_0(x)) \in \mathbf{W}_0^h \quad \forall x \in \Omega. \tag{3.1d}$$

REMARK 3.1 From (3.1d), (2.24), (2.28) and (2.14) we have

$$(\Phi(c_h(\cdot, 0)), 1) + \|c_h(\cdot, 0)\|_1 + \|\mathbf{u}_h(\cdot, 0)\| \leqslant C. \tag{3.2}$$

LEMMA 3.1 The system $(\mathbf{P}_h)$ satisfies the following stability estimates:

$$(\Phi(c_h(T)),1)+\|c_h(T)\|_1^2+\|\mathbf{u}_h(T)\|^2+\int_0^T(\|\nabla\mathbf{u}_h\|^2+\|\nabla w_h\|^2)\,\mathrm{d}t$$
$$\leqslant C((\Phi(c_h(0)),1)+\|\nabla c_h(0)\|^2+\|\mathbf{u}_h(0)\|^2) \leqslant C, \tag{3.3a}$$

$$\int_0^T \|c_h\|_{0,\infty}^4\,\mathrm{d}t\leqslant C \tag{3.3b}$$

and

$$\frac{1}{2}\|w_h(T)\|^2+\int_0^T\|\partial_t c_h\|^2\,\mathrm{d}t\leqslant C(h) \tag{3.4}$$

where $C(h)$ is a constant that depends on h.

Proof. Setting $\chi = w_h$ in (3.1a), $\chi = \partial_t c_h$ in (3.1b) and $\boldsymbol{\chi}=\mathbf{u}_h$ in (3.1c) and combining the resulting equations gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(c_h),1)+\frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla c_h\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_h\|^2+\|\nabla\mathbf{u}_h\|^2+\|\nabla w_h\|^2+B(\mathbf{u}_h,\mathbf{u}_h,\mathbf{u}_h)=0$$

and (3.3a) follows by noting (2.4), integrating from 0 to T and using (2.22) and (3.2).

To prove (3.3b) we take $\eta=\Delta^h c_h$ in (3.1b) and use (2.29a) to obtain

$$\begin{aligned}\gamma^2\|\Delta^h c_h\|^2&=(\nabla w_h,\nabla c_h)+(\Phi'(c_h),\Delta^h c_h)\\&\leqslant \tfrac12\|\nabla w_h\|^2+\tfrac12\|\nabla c_h\|^2+C\|\Phi'(c_h)\|^2+\tfrac12\gamma^2\|\Delta^h c_h\|^2\\&\leqslant \tfrac12\|\nabla w_h\|^2+\tfrac12\|\nabla c_h\|^2+C(\|c_h\|_{0,6}^6+\|c_h\|_{0,4}^4+\|c_h\|^2)+\tfrac12\gamma^2\|\Delta^h c_h\|^2\\&\leqslant \tfrac12\|\nabla w_h\|^2+\tfrac12\|\nabla c_h\|^2+C\|c_h\|_1^6+\tfrac12\gamma^2\|\Delta^h c_h\|^2.\end{aligned}\tag{3.5}$$

Noting (2.29b) and (2.22), integrating (3.5) from 0 to T and using (3.3a) we obtain

$$\int_0^T\|\nabla c_h\|_{0,4}^2\,\mathrm{d}t\leqslant C. \tag{3.6}$$

From (3.6), (2.10) and (3.3a) we conclude (3.3b).

To prove (3.4) we differentiate (3.1b) with respect to t and then take $\chi = w_h$ in the resulting equation to obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|w_h\|^2=\gamma^2(\partial_t\nabla c_h,\nabla w_h)+(\partial_t(\Phi'(c_h)),w_h). \tag{3.7}$$

Next taking $\chi=\gamma^2\partial_t c_h$ in (3.1a), noting (1.4), (2.2) and combining the resulting equation with (3.7) we obtain

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|w_h\|^2+\gamma^2\|\partial_t c_h\|^2\\&\quad=(\partial_t(\Phi'(c_h)),w_h)+\gamma^2\mathbf{a}(c_h,\partial_t c_h,\mathbf{u}_h)=(\Phi''(c_h)\partial_t c_h,w_h)-\gamma^2(\nabla\cdot(c_h\mathbf{u}_h),\partial_t c_h)\\&\quad\leqslant C\|\Phi''(c_h)\|_{0,\infty}^2\|w_h\|^2+\tfrac12\gamma^2\|\partial_t c_h\|^2+C\|\nabla c_h\|_{0,4}^2\|\mathbf{u}_h\|_{0,4}^2+C\|\nabla\mathbf{u}_h\|^2\|c_h\|_{0,\infty}^2\\&\quad\leqslant C(\|c_h\|_{0,\infty}^4+1)\|w_h\|^2+\tfrac12\gamma^2\|\partial_t c_h\|^2+C\|\nabla c_h\|_{0,4}^2\|\mathbf{u}_h\|_1^2+C\|\nabla\mathbf{u}_h\|^2\|c_h\|_{0,\infty}^2.\end{aligned}$$

Using the above inequality and (2.32a,b) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_h\|^2 + \frac{\gamma^2}{2}\|\partial_t c_h\|^2 \leqslant \frac{C}{h}((\|c_h\|_{0,\infty}^4 + 1)\|w_h\|^2 + \|\nabla c_h\|^2\|\mathbf{u}_h\|_1^2 + \|\nabla \mathbf{u}_h\|^2\|c_h\|_1^2)$$

and (3.4) follows from Gronwall's inequality and (3.3a,b).

REMARK 3.1 From the theory of ordinary differential equations, noting (3.3a) and (3.4) it follows that the system $(\mathbf{P}_h)$ has a unique solution $\{c_h, w_h, \mathbf{u}_h\}$ on $[0, T]$ for all $T < \infty$.

Before we present the main result of this section we introduce some useful notation.
We define

$$E_c^A := c - P^h c, \qquad E_c^h := P^h c - c_h, \qquad E_c := c - c_h = E_c^A + E_c^h, \tag{3.8a}$$

$$E_w^A := w - P^h w, \qquad E_w^h := P^h w - w_h, \qquad E_w := w - w_h = E_w^A + E_w^h, \tag{3.8b}$$

$$E_{\mathbf{u}}^A := \mathbf{u} - \mathbf{P}^h \mathbf{u}, \qquad E_{\mathbf{u}}^h := \mathbf{P}^h \mathbf{u} - \mathbf{u}_h, \qquad E_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_h = E_{\mathbf{u}}^A + E_{\mathbf{u}}^h. \tag{3.8c}$$

From (2.26), (2.28), (3.8a–c) and (2.14) we have

$$\|E_c^A\|_1 + \|E_{\mathbf{u}}^A\|_1 \leqslant Ch \quad \text{and} \quad \|E_w^A\|_1 \leqslant Ch\|w\|_2 \tag{3.9a}$$

and similarly we obtain

$$\|\nabla \mathcal{G}^h(\partial_t E_c^A)\| \leqslant Ch\|\partial_t c\| \quad \text{and} \quad \|\partial_t E_{\mathbf{u}}^A\|_{-1} \leqslant Ch\|\partial_t \mathbf{u}\|. \tag{3.9b}$$

Using (3.8a), (3.9a) and (2.22) we have

$$\|E_c\|_1 \leqslant \|E_c^h\|_1 + \|E_c^A\|_1 \leqslant \|E_c^h\|_1 + Ch \leqslant C\|\nabla E_c^h\| + Ch, \tag{3.10a}$$

and similarly from (3.8c) and (3.9a) we have

$$\|E_{\mathbf{u}}\|_1 \leqslant \|E_{\mathbf{u}}^h\|_1 + \|E_{\mathbf{u}}^A\|_1 \leqslant \|E_{\mathbf{u}}^h\|_1 + Ch. \tag{3.10b}$$

We now proceed to bound the errors E_c^h, E_w^h and $E_{\mathbf{u}}^h$. To this end we note the following lemma.

LEMMA 3.2 We have

$$|(\Phi'(c_h) - \Phi'(c), \partial_t E_c^h)| \leqslant \tilde{C}(h^2 + \|\nabla E_c^h\|^2) + C_1\|\nabla \mathcal{G}^h(\partial_t E_c^h)\|^2 \tag{3.11}$$

where $\tilde{C} = C(\|c_h\|_{0,\infty}^2 + 1)^2$.

Proof. Noting (2.5) and (2.31) it follows that

$$\begin{aligned} |(\Phi'(c) - \Phi'(c_h), \partial_t E_c^h)| &= |(\nabla(\Phi'(c) - \Phi'(c_h)), \nabla \mathcal{G}(\partial_t E_c^h))| \\ &\leqslant \|\nabla(\Phi'(c) - \Phi'(c_h))\| \, \|\nabla \mathcal{G}(\partial_t E_c^h)\| \\ &\leqslant C\|\nabla(\Phi'(c) - \Phi'(c_h))\| \, \|\nabla \mathcal{G}^h(\partial_t E_c^h)\|. \end{aligned} \tag{3.12}$$

From (2.14) and the Lipschitz continuity of Φ'' we have

$$\begin{aligned} \|\nabla(\Phi'(c) - \Phi'(c_h))\| &= \|\Phi''(c)\nabla c - \Phi''(c_h)\nabla c_h\| \\ &\leqslant \|\Phi''(c_h)\nabla(c - c_h)\| + \|(\Phi''(c) - \Phi''(c_h))\nabla c\| \\ &\leqslant C(\|c_h\|_{0,\infty}^2 + 1)\|\nabla E_c\| + C(\|c_h\|_{0,\infty} + \|c\|_{0,\infty})\|c - c_h\|_{0,4}\|\nabla c\|_{0,4} \\ &\leqslant C(\|c_h\|_{0,\infty}^2 + 1)(\|\nabla E_c\| + \|E_c\|_1). \end{aligned} \tag{3.13}$$

The desired result follows from (3.12), (3.13), (3.3b) and (3.10a).

REMARK 3.2 We note from (2.28), (2.32a), (2.15b) and (2.14) that

$$\begin{aligned}\|\nabla \mathbf{P}^h\mathbf{u}\|_{0,4} &\leqslant \|\nabla(\boldsymbol{\pi}^h\mathbf{u}-\mathbf{u})\|_{0,4} + \|\nabla(\mathbf{P}^h\mathbf{u}-\boldsymbol{\pi}^h\mathbf{u})\|_{0,4} + \|\nabla\mathbf{u}\|_{0,4}\\ &\leqslant C\|\nabla\mathbf{u}\|_{0,4} + Ch^{-1/2}\|\nabla(\mathbf{P}^h\mathbf{u}-\boldsymbol{\pi}^h\mathbf{u})\| + \|\nabla\mathbf{u}\|_{0,4}\\ &\leqslant C\|\nabla\mathbf{u}\|_{0,4} + Ch^{1/2}\|\mathbf{u}\|_2 \leqslant C. \end{aligned} \tag{3.14}$$

Similarly using (2.32b) we have

$$\begin{aligned}\|\mathbf{P}^h\mathbf{u}\|_{0,\infty} &\leqslant \|\boldsymbol{\pi}^h\mathbf{u}-\mathbf{u}\|_{0,\infty} + \|\mathbf{P}^h\mathbf{u}-\boldsymbol{\pi}^h\mathbf{u}\|_{0,\infty} + \|\mathbf{u}\|_{0,\infty}\\ &\leqslant C\|\mathbf{u}\|_{0,\infty} + C\sqrt{\ln(1/h)}\,\|\mathbf{P}^h\mathbf{u}-\boldsymbol{\pi}^h\mathbf{u}\|_1 + \|\mathbf{u}\|_{0,\infty}\\ &\leqslant C\|\mathbf{u}\|_{0,\infty} + Ch\sqrt{\ln(1/h)} \leqslant C. \end{aligned} \tag{3.15}$$

LEMMA 3.3 For almost every $t \in [0, T]$, we have

$$\begin{aligned}&\frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E_c^h\|^2 + \frac{3}{4}\|\nabla E_w^h\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_{\mathbf{u}}^h\|^2 + \frac{3}{4}\|\nabla E_{\mathbf{u}}^h\|^2\\ &\qquad\qquad \leqslant \widehat{C}(h^2 + \|\nabla E_c^h\|^2 + \|E_{\mathbf{u}}^h\|^2) + C_1\|\nabla\mathcal{G}^h(\partial_t E_c^h)\|^2\end{aligned}$$

where $\widehat{C} = C(1 + \|c_h\|_{0,\infty}^4 + \|w\|_2^2 + \|\partial_t c\|^2 + \|\partial_t \mathbf{u}\|^2)$.

Proof. From (2.12a), (2.25) and (3.8a) we have

$$(\partial_t P^h c, \eta) + (\nabla P^h w, \nabla\eta) = \mathbf{a}(c, \eta, \mathbf{u}) - (\partial_t E_c^A, \eta) \quad \forall \eta \in S^h. \tag{3.16}$$

Setting $\chi = E_w^h$ in (3.1a) and subtracting the resulting equation from (3.16) with $\eta = E_w^h$ we obtain

$$(\partial_t E_c^h, E_w^h) + \|\nabla E_w^h\|^2 = \mathbf{a}(c, E_w^h, \mathbf{u}) - \mathbf{a}(c_h, E_w^h, \mathbf{u}_h) - (\partial_t E_c^A, E_w^h). \tag{3.17}$$

Similarly setting $\eta = \partial_t c_h - \partial_t P^h c = \partial_t E_c^h$ in (2.12b) and noting (2.25) and (3.8b) we obtain

$$(P^h w, \partial_t E_c^h) = (\Phi'(c), \partial_t E_c^h) + \gamma^2(\nabla P^h c, \nabla\partial_t E_c^h) - (E_w^A, \partial_t E_c^h). \tag{3.18}$$

Setting $\chi = \partial_t E_c^h$ in (3.1b) and subtracting the resulting equation from (3.18) we obtain

$$(E_w^h, \partial_t E_c^h) = (\Phi'(c) - \Phi'(c_h), \partial_t E_c^h) + \frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E_c^h\|^2 - (E_w^A, \partial_t E_c^h). \tag{3.19}$$

Combining (3.17) and (3.19) we obtain

$$\begin{aligned}\frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E_c^h\|^2 + \|\nabla E_w^h\|^2 &= \mathbf{a}(c, E_w^h, \mathbf{u}) - \mathbf{a}(c_h, E_w^h, \mathbf{u}_h) - (\partial_t E_c^A, E_w^h)\\ &\qquad + (\Phi'(c_h) - \Phi'(c) + E_w^A, \partial_t E_c^h). \end{aligned} \tag{3.20}$$

Next setting $\mathbf{v} = \mathbf{P}^h\mathbf{u} - \mathbf{u}_h = E^h_{\mathbf{u}}$ in (2.13) and noting (2.27) and (3.8c) gives

$$(\partial_t \mathbf{P}^h\mathbf{u}, E^h_{\mathbf{u}}) + (\nabla \mathbf{P}^h\mathbf{u} : \nabla E^h_{\mathbf{u}}) = (p, \nabla\cdot E^h_{\mathbf{u}}) - \mathbf{a}(c, w, E^h_{\mathbf{u}}) - B(\mathbf{u}, \mathbf{u}, E^h_{\mathbf{u}}) - (\partial_t E^A_{\mathbf{u}}, E^h_{\mathbf{u}}). \quad (3.21)$$

Since $\mathbf{P}^h\mathbf{u} - \mathbf{u}_h \in \mathbf{W}^h_0$ it follows that

$$(\nabla\cdot E^h_{\mathbf{u}}, q_h) = (\nabla\cdot(\mathbf{P}^h\mathbf{u} - \mathbf{u}_h), q_h) = 0 \quad \forall q_h \in S^h. \quad (3.22)$$

Setting $\chi = \mathbf{P}^h\mathbf{u} - \mathbf{u}_h = E^h_{\mathbf{u}}$ in (3.1c), subtracting the resulting equation from (3.21) and using (3.22) with $q_h = Q^h p$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E^h_{\mathbf{u}}\|^2 + \|\nabla E^h_{\mathbf{u}}\|^2 = \mathbf{a}(c_h, w_h, E^h_{\mathbf{u}}) - \mathbf{a}(c, w, E^h_{\mathbf{u}}) + B(\mathbf{u}_h, \mathbf{u}_h, E^h_{\mathbf{u}}) - B(\mathbf{u}, \mathbf{u}, E^h_{\mathbf{u}})$$
$$-(\partial_t E^A_{\mathbf{u}}, E^h_{\mathbf{u}}) + (\nabla\cdot E^h_{\mathbf{u}}, p - Q^h p). \quad (3.23)$$

Next we note from (3.8a–c) that

$$\begin{aligned}\mathbf{a}(c, E^h_w, \mathbf{u}) - \mathbf{a}(c_h, E^h_w, \mathbf{u}_h) + \mathbf{a}(c_h, w_h, E^h_{\mathbf{u}}) - \mathbf{a}(c, w, E^h_{\mathbf{u}})&\\ = \mathbf{a}(E_c, E^h_w, \mathbf{u}) + \mathbf{a}(c_h, E^h_w, E_{\mathbf{u}}) - \mathbf{a}(c_h, E_w, E^h_{\mathbf{u}}) - \mathbf{a}(E_c, w, E^h_{\mathbf{u}})&\\ = \mathbf{a}(E_c, E^h_w, \mathbf{u}) - \mathbf{a}(E_c, w, E^h_{\mathbf{u}}) - \mathbf{a}(c_h, E^A_w, E^h_{\mathbf{u}}) + \mathbf{a}(c_h, E^h_w, E^A_{\mathbf{u}}).& \quad (3.24)\end{aligned}$$

Combining (3.20), (3.23) and (3.24) yields

$$\begin{aligned}&\frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E^h_c\|^2 + \|\nabla E^h_w\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E^h_{\mathbf{u}}\|^2 + \|\nabla E^h_{\mathbf{u}}\|^2\\ &\quad = \mathbf{a}(E_c, E^h_w, \mathbf{u}) - \mathbf{a}(E_c, w, E^h_{\mathbf{u}}) - \mathbf{a}(c_h, E^A_w, E^h_{\mathbf{u}})\\ &\qquad + \mathbf{a}(c_h, E^h_w, E^A_{\mathbf{u}}) - (\partial_t E^A_c, E^h_w) + (E^A_w, \partial_t E^h_c) + (\nabla\cdot E^h_{\mathbf{u}}, p - Q^h p)\\ &\qquad + B(\mathbf{u}_h, \mathbf{u}_h, E^h_{\mathbf{u}}) - B(\mathbf{u}, \mathbf{u}, E^h_{\mathbf{u}}) - (\partial_t E^A_{\mathbf{u}}, E^h_{\mathbf{u}}) + (\Phi'(c_h) - \Phi'(c), \partial_t E^h_c). \quad (3.25)\end{aligned}$$

We now bound the terms on the right hand side of (3.25). From (2.8), (2.14), (3.3a), (3.9a) and (3.10a) we obtain

$$\begin{aligned}&|\mathbf{a}(E_c, E^h_w, \mathbf{u})| + |\mathbf{a}(E_c, w, E^h_{\mathbf{u}})| + |\mathbf{a}(c_h, E^A_w, E^h_{\mathbf{u}})| + |\mathbf{a}(c_h, E^h_w, E^A_{\mathbf{u}})|\\ &\quad \leqslant \|E_c\|_1(\|\mathbf{u}\|_1\|\nabla E^h_w\| + \|\nabla w\|\,\|E^h_{\mathbf{u}}\|_1) + \|c_h\|_1(\|E^h_{\mathbf{u}}\|_1\|\nabla E^A_w\| + \|E^A_{\mathbf{u}}\|_1\|\nabla E^h_w\|)\\ &\quad \leqslant \widehat{C}\|E_c\|_1^2 + C_1\|\nabla E^h_w\|^2 + C_1\|E^h_{\mathbf{u}}\|_1^2 + C\|\nabla E^A_w\|^2 + C\|E^A_{\mathbf{u}}\|_1^2\\ &\quad \leqslant \widehat{C}(h^2 + \|\nabla E^h_c\|^2) + C_1(\|E^h_{\mathbf{u}}\|_1^2 + \|\nabla E^h_w\|^2). \quad (3.26)\end{aligned}$$

Futhermore noting (2.30a) and (3.9a,b) we have

$$\begin{aligned}&|(\partial_t E^A_c, E^h_w)| + |(E^A_w, \partial_t E^h_c)| + |(\partial_t E^A_{\mathbf{u}}, E^h_{\mathbf{u}})|\\ &\quad \leqslant \|\nabla\mathcal{G}^h(\partial_t E^A_c)\|\,\|\nabla E^h_w\| + \|\nabla E^A_w\|\,\|\nabla\mathcal{G}^h(\partial_t E^h_c)\| + \|\partial_t E^A_{\mathbf{u}}\|_{-1}\|E^h_{\mathbf{u}}\|_1\\ &\quad \leqslant C\|\nabla\mathcal{G}^h(\partial_t E^A_c)\|^2 + C_1\|\nabla E^h_w\|^2 + C\|\nabla E^A_w\|^2\\ &\qquad + C_1\|\nabla\mathcal{G}^h(\partial_t E^h_c)\|^2 + C\|\partial_t E^A_{\mathbf{u}}\|_{-1}^2 + C_1\|E^h_{\mathbf{u}}\|_1^2\\ &\quad \leqslant \widehat{C}h^2 + C_1(\|\nabla E^h_w\|^2 + \|\nabla\mathcal{G}^h(\partial_t E^h_c)\|^2 + \|E^h_{\mathbf{u}}\|_1^2). \quad (3.27)\end{aligned}$$

From Remark 3.2, (2.4), (2.14), (3.9a) and (3.10b) we have

$$\begin{aligned}|B(\mathbf{u},\mathbf{u},E^h_{\mathbf{u}})-B(\mathbf{u}_h,\mathbf{u}_h,E^h_{\mathbf{u}})| &= |B(\mathbf{u},E^A_{\mathbf{u}},E^h_{\mathbf{u}})+B(E_{\mathbf{u}},\mathbf{P}^h\mathbf{u},E^h_{\mathbf{u}})+B(\mathbf{u}_h,E^h_{\mathbf{u}},E^h_{\mathbf{u}})|\\ &= |B(\mathbf{u},E^A_{\mathbf{u}},E^h_{\mathbf{u}})+B(E_{\mathbf{u}},\mathbf{P}^h\mathbf{u},E^h_{\mathbf{u}})|\\ &\leqslant \|\mathbf{u}\|_{0,\infty}(\|\nabla E^A_{\mathbf{u}}\|\,\|E^h_{\mathbf{u}}\|+\|\nabla E^h_{\mathbf{u}}\|\,\|E^A_{\mathbf{u}}\|)\\ &\quad+\|E_{\mathbf{u}}\|\,\|\nabla\mathbf{P}^h\mathbf{u}\|_{0,4}\|E^h_{\mathbf{u}}\|_{0,4}+\|\mathbf{P}^h\mathbf{u}\|_{0,\infty}\|\nabla E^h_{\mathbf{u}}\|\,\|E_{\mathbf{u}}\|\\ &\leqslant C_1\|E^h_{\mathbf{u}}\|_1^2+C\|E^A_{\mathbf{u}}\|_1^2+C\|E_{\mathbf{u}}\|^2\\ &\leqslant Ch^2+C_1\|E^h_{\mathbf{u}}\|_1^2+C\|E^h_{\mathbf{u}}\|^2. \end{aligned} \tag{3.28}$$

Finally, from (2.24) and (2.14) we have

$$|(\nabla\cdot E^h_{\mathbf{u}},Q^hp-p)|\leqslant\|\nabla E^h_{\mathbf{u}}\|\,\|Q^hp-p\|\leqslant Ch^2+C_1\|\nabla E^h_{\mathbf{u}}\|^2. \tag{3.29}$$

Combining (3.25)–(3.29) and noting (3.11) we obtain

$$\begin{aligned}&\frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E^h_c\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E^h_{\mathbf{u}}\|^2+\|\nabla E^h_{\mathbf{u}}\|^2+\|\nabla E^h_w\|^2\\ &\qquad\leqslant C_1(\|\nabla E^h_{\mathbf{u}}\|^2+\|\nabla E^h_w\|^2+\|\nabla\mathcal{G}^h(\partial_tE^h_c)\|^2)\\ &\qquad\quad+\widehat{C}(h^2+\|\nabla E^h_c\|^2+\|E^h_{\mathbf{u}}\|^2).\end{aligned} \tag{3.30}$$

Taking $C_1=1/4$ in (3.30) gives the desired result. □

LEMMA 3.4 For all $t\in[0,T]$ we have

$$\|\nabla\mathcal{G}^h(\partial_tE^h_c)\|^2\leqslant\widehat{C}h^2+C_2(\|\nabla E^h_w\|^2+\|E^h_{\mathbf{u}}\|_1^2+\|\nabla E^h_c\|^2)$$

where $\widehat{C}=C(1+\|c_h\|^4_{0,\infty}+\|w\|_2^2+\|\partial_tc\|^2+\|\partial_t\mathbf{u}\|^2)$.

Proof. Replacing η with χ in (3.16), combining the resulting equation with (3.1a) and noting (2.30a) and (3.8a) we obtain

$$(\nabla\mathcal{G}^h(\partial_tE^h_c),\nabla\chi)=(\partial_tE^h_c,\chi)=-(\nabla E^h_w,\nabla\chi)+\mathbf{a}(c,\chi,\mathbf{u})-\mathbf{a}(c_h,\chi,\mathbf{u}_h)-(\partial_tE^A_c,\chi),$$

so

$$(\nabla\mathcal{G}^h(\partial_tE^h_c),\nabla\chi)=-(\nabla E^h_w,\nabla\chi)+\mathbf{a}(E_c,\chi,\mathbf{u})-\mathbf{a}(c_h,\chi,E_{\mathbf{u}})-(\partial_tE^A_c,\chi). \tag{3.31}$$

From (3.31), (2.8), (3.9b), (2.14) and (3.3a) we have

$$\begin{aligned}(\nabla\mathcal{G}^h(\partial_tE^h_c),\nabla\chi)&\leqslant\|\nabla E^h_w\|\,\|\nabla\chi\|+\|\nabla\chi\|(\|\mathbf{u}\|_{0,\infty}\|E_c\|+\|c_h\|_1\|E_{\mathbf{u}}\|_1)+\|\nabla\mathcal{G}^h(\partial_tE^A_c)\|\,\|\nabla\chi\|\\ &\leqslant\widehat{C}h+C(\|\nabla E^h_w\|+\|E_{\mathbf{u}}\|_1+\|E_c\|)\|\nabla\chi\|.\end{aligned} \tag{3.32}$$

From (3.10a,b) we obtain the desired result. □

From Lemmas 3.3 and 3.4 with C_1 chosen such that $4C_2C_1\leqslant1$ we obtain the following result.

LEMMA 3.5 For almost every $t\in[0,T]$, we have

$$\frac{\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E^h_c\|^2+\frac{1}{2}\|\nabla E^h_w\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E^h_{\mathbf{u}}\|^2+\frac{1}{2}\|\nabla E^h_{\mathbf{u}}\|^2\leqslant\widehat{C}(h^2+\|\nabla E^h_c\|^2+\|E^h_{\mathbf{u}}\|^2). \tag{3.33}$$

LEMMA 3.6 For all $t \in [0, T]$ we have

$$\|E_c^h(t)\|_1^2 + \|E_{\mathbf{u}}^h(t)\|^2 + \int_0^T (\|\nabla E_w^h\|^2 + \|\nabla E_{\mathbf{u}}^h\|^2)\,\mathrm{d}t \leqslant C(h^2 + \|\nabla E_c^h(0)\|^2 + \|E_{\mathbf{u}}^h(0)\|^2). \quad (3.34)$$

Proof. Applying Gronwall's inequality to (3.33) and noting (2.14), (2.22) and (3.3b) we obtain the required result. □

THEOREM 3.2 If

$$\|c(0) - c_h(0)\|_1 + \|\mathbf{u}(0) - \mathbf{u}_h(0)\| \leqslant Ch \quad (3.35)$$

then

$$\|c - c_h\|_{L^\infty(H^1)} + \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\mathbf{L}^2)} + \|\nabla(w - w_h)\|_{L^2(L^2)} + \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathbf{H}^1)} \leqslant Ch. \quad (3.36)$$

Proof. The desired result is a direct consequence of Lemma 3.6, (2.26) and (2.28).

REMARK 3.3 We note that in the above error bound γ enters in the form $\exp(\gamma^{-2})$. Since γ is required to be a small parameter, ideally one would like to obtain an estimate that depends polynomially on γ^{-1}. Such an estimate has been proved in [20] in which a spectrum estimate result for the linearized Cahn–Hilliard operator is used; however, as is mentioned in [19], we know of no such spectrum estimate for the linearized operator associated with the coupled system (1.1a–d).

4. Fully discrete scheme

In this section we prove the convergence of a fully discrete finite element approximation $(\mathbf{P}_{h,\tau})$, of $(\mathbf{P})$.

We define $\mathcal{T}^h$ to be as in Section 2.2 and in addition we let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a partitioning of $[0, T]$ into possibly variable time steps $\tau_n := t_n - t_{n-1}$, for $n = 1 \to N$. We set $\tau := \max_{n=1\to N} \tau_n$.

Finally, for simplicity of notation we set

$$\delta_\tau c_h^n := \frac{c_h^n - c_h^{n-1}}{\tau_n} \quad \text{and} \quad \delta_\tau \mathbf{u}_h^n := \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau_n}.$$

We now define a fully discrete finite element approximation of $(\mathbf{P})$:

$(\mathbf{P}_{h,\tau})$ For $n \geqslant 1$ find $\{c_h^n, w_h^n, \mathbf{u}_h^n\} \in V^h \times S^h \times \mathbf{W}_0^h$ such that

$$(\delta_\tau c_h^n, \chi)_h + (\nabla w_h^n, \nabla\chi) = \mathbf{a}(c_h^{n-1}, \chi, \mathbf{u}_h^{n-1}) \quad \forall \chi \in S^h, \quad (4.1a)$$

$$(w_h^n, \chi)_h = \gamma^2(\nabla c_h^n, \nabla\chi) + (\phi(c_h^n) - c_h^{n-1}, \chi)_h \quad \forall \chi \in S^h, \quad (4.1b)$$

$$(\delta_\tau \mathbf{u}_h^n, \boldsymbol{\chi}) + (\nabla \mathbf{u}_h^n : \nabla\boldsymbol{\chi}) + B(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\chi}) + \mathbf{a}(c_h^{n-1}, w_h^n, \boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\chi} \in \mathbf{W}_0^h, \quad (4.1c)$$

$$c_h^0(x) = Q^h(c_0(x)) \in V^h \cap L^\infty(\Omega), \quad \mathbf{u}_h^0(x) = \mathbf{P}^h(\mathbf{u}_0(x)) \in \mathbf{W}_0^h \quad \forall x \in \Omega, \quad (4.1d)$$

where $\phi(s) = s^3$.

REMARK 4.1 From (4.1d), (2.24), (2.28) and (2.14) we have

$$(\Phi(c_h^0(\cdot)), 1)_h + \|c_h^0(\cdot)\|_1 + \|\mathbf{u}_h^0(\cdot)\| \leqslant C. \tag{4.2}$$

LEMMA 4.1 For all $h, \tau_n > 0$ there exists a unique solution $\{\mathbf{u}_h^n, c_h^n, w_h^n\}$ to the n-th step of $(\mathbf{P}_{h,\tau})$.

Proof. In order to prove existence of a unique solution $\{c_h^n, w_h^n\} \in V^h \times S^h$ to (4.1a,b) first we set $\chi = \widehat{\mathcal{G}}^h \chi$ in (4.1a) and note (2.2) to obtain

$$\begin{aligned}(\delta_\tau c_h^n, \widehat{\mathcal{G}}^h \chi)_h + (\nabla w_h^n, \nabla \widehat{\mathcal{G}}^h \chi) &= (c_h^{n-1}, \nabla(\widehat{\mathcal{G}}^h \chi) \cdot \mathbf{u}_h^{n-1}) \\ &= -(\nabla \cdot (c_h^{n-1} \mathbf{u}_h^{n-1}), \widehat{\mathcal{G}}^h \chi) \quad \forall \chi \in V^h.\end{aligned} \tag{4.3}$$

Using (4.3) and (2.30a,b) we obtain

$$(w_h^n, \chi)_h = (\nabla w_h^n, \nabla \widehat{\mathcal{G}}^h \chi) = -(\widehat{\mathcal{G}}^h[\delta_\tau c_h^n], \chi)_h - (\mathcal{G}^h[\nabla \cdot (c_h^{n-1} \mathbf{u}_h^{n-1})], \chi)_h \quad \forall \chi \in V^h. \tag{4.4}$$

Hence (4.1a,b) can be restated as: Find $c_h^n \in V^h$ such that for all $\chi \in V^h$

$$\gamma^2 (\nabla c_h^n, \nabla \chi) + \left(\phi(c_h^n) + \frac{1}{\tau_n} \widehat{\mathcal{G}}^h[\delta_\tau c_h^n], \chi \right)_h = (c_h^{n-1} - \mathcal{G}^h[\nabla \cdot (c_h^{n-1} \mathbf{u}_h^{n-1})], \chi)_h. \tag{4.5}$$

There exists $c_h^n \in V^h$ solving (4.5) since, on noting (2.30b), this is the Euler–Lagrange equation of the convex minimization problem

$$\min_{z^h \in V^h} \left\{ \frac{\gamma^2}{2} \|\nabla z^h\|^2 + \frac{1}{4}((z^h)^4, 1)_h + \frac{1}{2\tau_n} \|\nabla \widehat{\mathcal{G}}^h (z^h - c_h^{n-1})\|^2 - (c_h^{n-1} - \mathcal{G}^h[\nabla \cdot (c_h^{n-1} \mathbf{u}_h^{n-1})], z^h)_h \right\}.$$

Therefore on noting (4.4), we have existence of a unique solution $\{c_h^n, w_h^n\} \in V^h \times S^h$ to (4.1a,b). The existence of a unique solution $\mathbf{u}_h^n \in \mathbf{W}_0^h$ to (4.1c) follows from standard results.

LEMMA 4.2 Let $c_h^{n-1} \in V^h$. Then for $\tau_n \leqslant Ch/(4\|c_h^{n-1}\|_{0,4}^2)$ we have

$$\begin{aligned}\mathcal{E}(c_h^n, \mathbf{u}_h^n) + \tfrac{1}{2}\gamma^2 \|\nabla(c_h^n - c_h^{n-1})\|^2 + \tfrac{1}{4}\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \tfrac{1}{2}\tau_n \|\nabla w_h^n\|^2 + \tau_n \|\nabla \mathbf{u}_h^n\|^2 \\ \leqslant \mathcal{E}(c_h^{n-1}, \mathbf{u}_h^{n-1})\end{aligned} \tag{4.6}$$

where

$$\mathcal{E}(c_h^n, \mathbf{u}_h^n) := \tfrac{1}{2}\gamma^2 \|\nabla c_h^n\|^2 + (\Phi(c_h^n), 1)_h + \tfrac{1}{2}\|\mathbf{u}_h^n\|^2. \tag{4.7}$$

Proof. Choosing $\chi \equiv w_h^n$ in (4.1a) and $\chi \equiv c_h^n - c_h^{n-1}$ in (4.1b) yields respectively

$$(c_h^n - c_h^{n-1}, w_h^n)_h + \tau_n (\nabla w_h^n, \nabla w_h^n) = \tau_n \mathbf{a}(c_h^{n-1}, w_h^n, \mathbf{u}_h^{n-1}) \tag{4.8}$$

and

$$\gamma^2 (\nabla c_h^n, \nabla [c_h^n - c_h^{n-1}]) = (w_h^n - \phi(c_h^n) + c_h^{n-1}, c_h^n - c_h^{n-1})_h. \tag{4.9}$$

Combining (4.8) and (4.9), using the elementary identity

$$2r(r-s) = (r^2 - s^2) + (r-s)^2 \quad \forall r, s \in \mathbb{R}, \tag{4.10}$$

and noting the following result from [18]:

$$(\phi(r) - s)(r - s) \geqslant \Phi(r) - \Phi(s) \tag{4.11}$$

we obtain

$$\tfrac{1}{2}\gamma^2\|\nabla(c_h^n - c_h^{n-1})\|^2 + \tfrac{1}{2}\gamma^2\|\nabla c_h^n\|^2 + (\Phi(c_h^n), 1)_h + \tau_n\|\nabla w_h^n\|^2 \\ \leqslant (\Phi(c_h^{n-1}), 1)_h + \tfrac{1}{2}\gamma^2\|\nabla c_h^{n-1}\|^2 + \tau_n \mathbf{a}(c_h^{n-1}, w_h^n, \mathbf{u}_h^{n-1}). \tag{4.12}$$

Choosing $\chi \equiv \mathbf{u}_h^n$ in (4.1c) and noting (4.10) and (2.4) we obtain

$$\tfrac{1}{2}\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \tfrac{1}{2}\|\mathbf{u}_h^n\|^2 - \tfrac{1}{2}\|\mathbf{u}_h^{n-1}\|^2 + \tau_n\|\nabla \mathbf{u}_h^n\|^2 + \tau_n \mathbf{a}(c_h^{n-1}, w_h^n, \mathbf{u}_h^n) = 0. \tag{4.13}$$

Combining (4.12), (4.13) and noting (4.7) we have

$$\mathcal{E}(c_h^n, \mathbf{u}_h^n) + \tfrac{1}{2}\gamma^2\|\nabla(c_h^n - c_h^{n-1})\|^2 + \tfrac{1}{2}\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \tau_n\|\nabla w_h^n\|^2 + \tau_n\|\nabla \mathbf{u}_h^n\|^2 \\ \leqslant \mathcal{E}(c_h^{n-1}, \mathbf{u}_h^{n-1}) + \tau_n \mathbf{a}(c_h^{n-1}, w_h^n, \mathbf{u}_h^{n-1} - \mathbf{u}_h^n). \tag{4.14}$$

Noting (2.32a) and choosing τ_n such that $\tau_n \leqslant Ch/(4\|c_h^{n-1}\|_{0,4}^2)$ we have

$$\begin{aligned} \tau_n|\mathbf{a}(c_h^{n-1}, w_h^n, \mathbf{u}_h^{n-1} - \mathbf{u}_h^n)| &\leqslant \tau_n\|c_h^{n-1}\|_{0,4}\|\nabla w_h^n\|\,\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{0,4} \\ &\leqslant \tfrac{1}{2}\tau_n\|c_h^{n-1}\|_{0,4}^2\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{0,4}^2 + \tfrac{1}{2}\tau_n\|\nabla w_h^n\|^2 \\ &\leqslant C\tau_n h^{-1}\|c_h^{n-1}\|_{0,4}^2\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \tfrac{1}{2}\tau_n\|\nabla w_h^n\|^2 \\ &\leqslant \tfrac{1}{4}\|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2 + \tfrac{1}{2}\tau_n\|\nabla w_h^n\|^2, \end{aligned} \tag{4.15}$$

and hence from (4.14) and (4.15) we conclude (4.6).

THEOREM 4.1 For all $h > 0$ and for all time partitions $\{\tau_n\}_{n=1}^N$ such that $\tau_n \leqslant Ch/(4\|c_h^{n-1}\|_{0,4}^2)$, the solution $\{c_h^n, w_h^n, \mathbf{u}_h^n\}_{n=1}^N$ to $(\mathbf{P}_{h,\tau})$ is such that

$$\max_{n=1\to N}[\gamma^2\|c_h^n\|_1^2 + \|\mathbf{u}_h^n\|^2 + (\Phi(c_h^n), 1)_h] + \sum_{n=1}^N[\gamma^2\|\nabla(c_h^n - c_h^{n-1})\|^2 + \|\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|^2] \\ + \sum_{n=1}^N \tau_n[\|\nabla w_h^n\|^2 + \|\nabla \mathbf{u}_h^n\|^2] \leqslant C(\|\nabla c_h^0\|^2 + \|\mathbf{u}_h^0\|^2 + (\Phi(c_h^0), 1)_h) \leqslant C. \tag{4.16a}$$

In addition

$$\sum_{n=1}^N \tau_n\|\nabla\mathcal{G}[\delta_\tau c_h^n]\|^2 \leqslant C. \tag{4.16b}$$

Proof. The result (4.16a) follows by summing (4.6) from $n = 1 \to N$ and noting (2.22) and (4.2). From (2.5), (2.23), (4.1a), (2.8) and (2.24) we have for any $\eta \in H^1(\Omega)$,

$$\begin{aligned} (\nabla\mathcal{G}[\delta_\tau c_h^n], \nabla\eta) = (\delta_\tau c_h^n, \eta) = (\delta_\tau c_h^n, Q^h\eta)_h &= -(\nabla w_h^n, \nabla[Q^h\eta]) + \mathbf{a}(c_h^{n-1}, Q^h\eta, \mathbf{u}_h^{n-1}) \\ &\leqslant [\|\nabla w_h^n\| + \|\mathbf{u}_h^{n-1}\|_1\|c_h^{n-1}\|_1]\|\nabla Q^h\eta\| \\ &\leqslant C[\|\nabla w_h^n\| + \|\mathbf{u}_h^{n-1}\|_1\|c_h^{n-1}\|_1]\|\nabla\eta\|. \end{aligned} \tag{4.17}$$

The bound in (4.16b) then follows from (4.17) and (4.16a).

REMARK 4.2 We note that the bound $\tau_n \leqslant Ch/(4\|c_h^{n-1}\|_{0,4}^2)$ in Lemma 4.2 and Theorem 4.1 is attainable since from (4.16a) and (2.9) we have $\|c_h^{n-1}\|_{0,4} \leqslant C\|c_h^{n-1}\|_1 \leqslant C$.

We now consider the convergence of $(\mathbf{P}_{h,\tau})$ as h and τ tend to zero. To this end we let

$$c_{h,\tau}(t) := \frac{t - t_{n-1}}{\tau_n} c_h^n + \frac{t_n - t}{\tau_n} c_h^{n-1}, \quad t \in [t_{n-1}, t_n],\ n \geqslant 1, \tag{4.18a}$$

$$c_{h,\tau}^+(t) := c_h^n, \quad c_{h,\tau}^-(t) := c_h^{n-1}, \qquad t \in (t_{n-1}, t_n],\ n \geqslant 1. \tag{4.18b}$$

We note for future reference that

$$c_{h,\tau} - c_{h,\tau}^{\pm} = (t - t_n^{\pm})\partial_t c_{h,\tau}, \quad t \in (t_{n-1}, t_n),\ n \geqslant 1, \tag{4.19}$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$. We also introduce

$$\bar{\tau}(t) := \tau_n, \quad t \in (t_{n-1}, t_n],\ n \geqslant 1. \tag{4.20}$$

Using the above notation, and introducing analogous notation for $w_{h,\tau}$ and $\mathbf{u}_{h,\tau}$, we can restate $(\mathbf{P}_{h,\tau})$ as:

Find $\{c_{h,\tau}, w_{h,\tau}, \mathbf{u}_{h,\tau}\} \in L^2(0,T;V^h) \times L^2(0,T;S^h) \times L^2(0,T;\mathbf{W}_0^h)$ such that

$$\int_0^T ((\partial_t c_{h,\tau}, \chi)_h + (\nabla w_{h,\tau}^+, \nabla\chi) - \mathbf{a}(c_{h,\tau}^-, \chi, \mathbf{u}_{h,\tau}^-))\,\mathrm{d}t = 0 \quad \forall \chi \in L^2(0,T;S^h), \tag{4.21a}$$

$$\int_0^T (w_{h,\tau}^+, \chi)_h\,\mathrm{d}t = \int_0^T (\gamma^2(\nabla c_{h,\tau}^+, \nabla\chi) + (\phi(c_{h,\tau}^+) - c_{h,\tau}^-, \chi)_h)\,\mathrm{d}t \quad \forall \chi \in L^2(0,T;S^h), \tag{4.21b}$$

$$\int_0^T ((\partial_t \mathbf{u}_h, \boldsymbol{\chi}) + (\nabla\mathbf{u}_{h,\tau}^+ : \nabla\boldsymbol{\chi}) + B(\mathbf{u}_{h,\tau}^-, \mathbf{u}_{h,\tau}^+, \boldsymbol{\chi}) + \mathbf{a}(c_{h,\tau}^-, w_{h,\tau}^+, \boldsymbol{\chi}))\,\mathrm{d}t = 0 \quad \forall \boldsymbol{\chi} \in L^2(0,T;\mathbf{W}_0^h). \tag{4.21c}$$

LEMMA 4.3 There exists a subsequence of $\{c_{h,\tau}, w_{h,\tau}, \mathbf{u}_{h,\tau}\}_h$, where $\{c_{h,\tau}, w_{h,\tau}, \mathbf{u}_{h,\tau}\}$ solve $(\mathbf{P}_{h,\tau})$, and functions

$$\begin{cases} c \in L^\infty(0,T;V) \cap H^1(0,T;(H^1(\Omega))'), & w \in L^2(0,T;H^1(\Omega)), \\ \mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbf{W}_0) \end{cases} \tag{4.22}$$

with $c(\cdot,0) = c_0(\cdot)$ in $L^2(\Omega)$ and $\mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot)$ in $\mathbf{L}^2(\Omega)$ are such that as $h \to 0$,

$$c_{h,\tau},\ c_{h,\tau}^{\pm} \to c \quad \text{weak-}* \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{4.23a}$$

$$c_{h,\tau},\ c_{h,\tau}^{\pm} \to c \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{4.23b}$$

$$\mathcal{G}(\partial_t c_{h,\tau}) \to \mathcal{G}(\partial_t c) \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{4.23c}$$

$$w_{h,\tau},\ w_{h,\tau}^{\pm} \to w \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{4.23d}$$

$$\mathbf{u}_{h,\tau}, \mathbf{u}_{h,\tau}^{\pm} \to \mathbf{u} \quad \text{weakly in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{4.23e}$$

Proof. Noting the definitions (4.18a,b), (4.20), the bounds in Theorem 4.1, (4.16b) together with (2.6) and (4.2) imply that

$$\|c_{h,\tau}^{(\pm)}\|_{L^\infty(H^1)}^2 + \|\mathbf{u}_{h,\tau}^{(\pm)}\|_{L^\infty(\mathbf{L}^2)}^2 + \|\bar\tau^{1/2}\partial_t c_{h,\tau}\|_{L^2(H^1)}^2 + \|\bar\tau^{1/2}\partial_t \mathbf{u}_h\|_{L^2(\mathbf{L}^2)}^2 + \|\nabla w_{h,\tau}^+\|_{L^2(L^2)}^2 + \|\mathbf{u}_{h,\tau}^+\|_{L^2(\mathbf{H}^1)}^2 + \|\mathcal{G}(\partial_t c_{h,\tau})\|_{L^2(H^1)}^2 \leqslant C. \qquad (4.24)$$

Furthermore, we deduce from (4.19) and (4.24) that

$$\|c_{h,\tau} - c_{h,\tau}^\pm\|_{L^2(H^1)}^2 \leqslant \|\bar\tau\partial_t c_{h,\tau}\|_{L^2(H^1)}^2 \leqslant C\tau \qquad (4.25a)$$

and

$$\|\mathbf{u}_{h,\tau} - \mathbf{u}_{h,\tau}^\pm\|_{L^2(\mathbf{L}^2)}^2 \leqslant \|\bar\tau\partial_t \mathbf{u}_{h,\tau}\|_{L^2(\mathbf{L}^2)}^2 \leqslant C\tau. \qquad (4.25b)$$

Hence on noting (4.24), (4.25a,b) and (2.7a) we can choose a subsequence $\{c_{h,\tau}, w_{h,\tau}, \mathbf{u}_{h,\tau}\}_h$ such that the convergence results (4.22) and (4.23a–f) hold. Then (4.22) yields on noting (2.7b), (2.24) and (2.28) that the subsequences satisfy the additional initial conditions.

We follow the arguments used in [28] to prove the following lemma.

LEMMA 4.4 We have

$$\mathbf{u}_{h,\tau}, \mathbf{u}_{h,\tau}^\pm \to \mathbf{u} \quad \text{strongly in } L^2(0,T;\mathbf{L}^2(\Omega)). \qquad (4.26)$$

Proof. Since $\{\mathbf{u}_{h,\tau}\}_h$ is bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega))$, a result of J.-L. Lions [28, 29] states that compactness of the sequence $\{\mathbf{u}_{h,\tau}^\pm\}_h$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ will follow if

$$\int_\delta^T \|\mathbf{u}_{h,\tau}^+(t) - \mathbf{u}_{h,\tau}^+(t-\delta)\|_{\mathbf{L}^2(\Omega)}^2 \leqslant C\delta^\alpha \qquad (4.27)$$

for $0 \leqslant \delta \leqslant T$ and some $\alpha > 0$. Since $\{\mathbf{u}_{h,\tau}^+\}_h$ are piecewise constant in time it suffices to take δ to be a multiple of the time step τ_n. Writing $(t-\delta, t) = (t^m, t^n)$ and setting $\mathbf{v}(t) := \mathbf{u}_{h,\tau}^+(t^n) - \mathbf{u}_{h,\tau}^+(t^m) = \mathbf{v}^{mn}$ in (4.1c) yields

$$(\mathbf{u}_{h,\tau}^+(t^n) - \mathbf{u}_{h,\tau}^+(t^m), \mathbf{v}^{mn}) = -\int_{t^m}^{t^n} [(\nabla\mathbf{u}_{h,\tau}^+ : \nabla\mathbf{v}^{mn}) + B(\mathbf{u}_{h,\tau}^+(\cdot - \tau_n), \mathbf{u}_{h,\tau}^+, \mathbf{v}^{mn})]\,\mathrm{d}s - \int_{t^m}^{t^n} \mathbf{a}(c_{h,\tau}(\cdot - \tau_n), w_{h,\tau}^+, \mathbf{v}^{mn})\,\mathrm{d}s. \qquad (4.28)$$

Using (2.8), (4.24), Hölder's inequality and (4.16a) we have

$$\begin{aligned}
&\left|\int_{t^m}^{t^n} ((\nabla\mathbf{u}_{h,\tau}^+ : \nabla\mathbf{v}^{mn}) - \mathbf{a}(c_{h,\tau}(\cdot - \tau_n), w_{h,\tau}^+, \mathbf{v}^{mn}))\,\mathrm{d}s\right| \\
&\qquad \leqslant \int_{t^m}^{t^n} (\|\nabla\mathbf{u}_{h,\tau}^+\|\,\|\nabla\mathbf{v}^{mn}\| + \|c_{h,\tau}(\cdot - \tau_n)\|_1 \|\nabla w_{h,\tau}^+\|\,\|\mathbf{v}^{mn}\|_1)\,\mathrm{d}s \\
&\qquad \leqslant C\|\mathbf{v}^{mn}\|_1 \int_{t^m}^{t^n} (\|\nabla\mathbf{u}_{h,\tau}^+\| + \|\nabla w_{h,\tau}^+\|) \\
&\qquad \leqslant C\|\mathbf{v}^{mn}\|_1 (t^n - t^m)^{1/2} (\|\nabla\mathbf{u}_{h,\tau}^+\|_{L^2(\mathbf{L}^2)} + \|\nabla w_{h,\tau}^+\|_{L(\mathbf{L}^2)})
\end{aligned} \qquad (4.29)$$

and similarly noting (2.9) and (4.24) we obtain

$$\begin{aligned}
&\left|\int_{t^m}^{t^n} B(\mathbf{u}_{h,\tau}^+(\cdot-\tau_n), \mathbf{u}_{h,\tau}^+, \mathbf{v}^{mn})\,\mathrm{d}s\right| \\
&\quad\leqslant \int_{t^m}^{t^n} \|\mathbf{u}_{h,\tau}^+(\cdot-\tau_n)\|_{0,4}(\|\nabla\mathbf{u}_{h,\tau}^+\|\,\|\mathbf{v}^{mn}\|_{0,4} + \|\nabla\mathbf{v}^{mn}\|\,\|\mathbf{u}_{h,\tau}^+\|_{0,4})\,\mathrm{d}s \\
&\quad\leqslant C\int_{t^m}^{t^n} \|\mathbf{u}_{h,\tau}^+(\cdot-\tau_n)\|^{1/2}\|\mathbf{u}_{h,\tau}^+(\cdot-\tau_n)\|_1^{1/2} \\
&\qquad\times(\|\nabla\mathbf{u}_{h,\tau}^+\|\,\|\mathbf{v}^{mn}\|^{1/2}\|\mathbf{v}^{mn}\|_1^{1/2} + \|\nabla\mathbf{v}^{mn}\|\,\|\mathbf{u}_{h,\tau}^+\|^{1/2}\|\mathbf{u}_{h,\tau}^+\|_1^{1/2})\,\mathrm{d}s \\
&\quad\leqslant C\int_{t^m}^{t^n} \|\mathbf{u}_{h,\tau}^+(\cdot-\tau_n)\|_1^{1/2}(\|\nabla\mathbf{u}_{h,\tau}^+\|\,\|\mathbf{v}^{mn}\|_1^{1/2} + \|\nabla\mathbf{v}^{mn}\|\,\|\mathbf{u}_{h,\tau}^+\|_1^{1/2})\,\mathrm{d}s \\
&\quad\leqslant C(t^n-t^m)^{1/4}\|\mathbf{v}^{mn}\|_1^{1/2}\|\mathbf{u}_{h,\tau}^+\|_{L^2(\mathbf{H}^1)}^{3/2} + C(t^n-t^m)^{1/2}\|\nabla\mathbf{v}^{mn}\|\,\|\mathbf{u}_{h,\tau}^+\|_{L^2(\mathbf{H}^1)}. \qquad (4.30)
\end{aligned}$$

Integrating (4.28) with respect to $t^n \in (\delta, T)$, using (4.29) and (4.30) we obtain (4.27), and noting (4.25b) gives the required result.

LEMMA 4.5 Let $\mathbf{v}_h,\ \eta_h$ and ξ_h be such that

$$\|\xi_h\nabla\eta_h\|_{L^2(\mathbf{L}^2)} \leqslant C, \quad \|\mathbf{v}_h\xi_h\|_{L^2(\mathbf{L}^2)} \leqslant C, \quad \|\mathbf{v}_h\cdot\nabla\eta_h\|_{L^2(L^2)} \leqslant C \qquad (4.31)$$

and

$$\begin{aligned}
\mathbf{v}_h &\to \mathbf{v} && \text{strongly in } L^2(0,T;\mathbf{L}^2(\Omega)),\\
\xi_h &\to \xi && \text{strongly in } L^2(0,T;L^2(\Omega)),\\
\eta_h &\to \eta && \text{weakly in } L^2(0,T;H^1(\Omega)).
\end{aligned} \qquad (4.32)$$

Then

$$\int_0^T \mathbf{a}(\xi_h,\eta_h,\mathbf{v}_h)\,\mathrm{d}t \to \int_0^T \mathbf{a}(\xi,\eta,\mathbf{v})\,\mathrm{d}t.$$

Proof. We have

$$\begin{aligned}
&\left|\int_0^T \mathbf{a}(\xi_h,\eta_h,\mathbf{v}_h) - \mathbf{a}(\xi,\eta,\mathbf{v})\,\mathrm{d}t\right| \\
&\quad\leqslant \left|\int_0^T \mathbf{a}(\xi_h,\eta_h,\mathbf{v}_h-\mathbf{v})\,\mathrm{d}t\right| + \left|\int_0^T \mathbf{a}(\xi_h-\xi,\eta_h,\mathbf{v})\,\mathrm{d}t\right| + \left|\int_0^T \mathbf{a}(\xi,\eta_h-\eta,\mathbf{v})\,\mathrm{d}t\right| \\
&\quad\leqslant \|\mathbf{v}_h-\mathbf{v}\|_{L^2(\mathbf{L}^2)}\|\xi_h\nabla\eta_h\|_{L^2(\mathbf{L}^2)} + \|\mathbf{v}\cdot\nabla\eta_h\|_{L^2(L^2)}\|\xi_h-\xi\|_{L^2(L^2)} + \left|\int_0^T \mathbf{a}(\xi,\eta_h-\eta,\mathbf{v})\,\mathrm{d}t\right|
\end{aligned}$$

and the desired result follows from (4.31) and (4.32).

THEOREM 4.2 There exists a subsequence of $\{c_{h,\tau}, w_{h,\tau}, \mathbf{u}_{h,\tau}\}_h$, where $\{c_{h,\tau}, w_{h,\tau}, \mathbf{u}_{h,\tau}\}$ solve $(\mathbf{P}_{h,\tau})$, and functions $\{c, w, \mathbf{u}\}$ satisfying (4.22). In addition, as $h \to 0$, (4.23a–e) hold. Furthermore, $\{c, w, \mathbf{u}\}$ fulfil $c(\cdot,0) = c_0(\cdot)$ in $L^2(\Omega)$, $\mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot)$ in $\mathbf{L}^2(\Omega)$ and satisfy for all $\eta \in$

$L^2(0, T; H^1(\Omega))$ and all $\mathbf{v} \in L^2(0, T; \mathbf{W}_0) \cap H^1(0, T; \mathbf{L}^2(\Omega))$ with $\mathbf{v}(\cdot, T) = \mathbf{0}$,

$$\int_0^T \langle \partial_t c, \eta \rangle \, \mathrm{d}t + \int_0^T (\nabla w, \nabla \eta) \, \mathrm{d}t = \int_0^T \mathbf{a}(c, \eta, \mathbf{u}) \, \mathrm{d}t, \tag{4.33a}$$

$$\int_0^T (w, \eta) \, \mathrm{d}t = \gamma^2 \int_0^T (\nabla c, \nabla \eta) \, \mathrm{d}t + \int_0^T (\Phi'(c), \eta) \, \mathrm{d}t, \tag{4.33b}$$

$$- \int_0^T (\mathbf{u}, \partial_t \mathbf{v}) \, \mathrm{d}t + \int_0^T (\nabla \mathbf{u} : \nabla \mathbf{v}) \, \mathrm{d}t + \int_0^T B(\mathbf{u}, \mathbf{u}, \mathbf{v}) \, \mathrm{d}t + \int_0^T \mathbf{a}(c, w, \mathbf{v}) \, \mathrm{d}t = (\mathbf{u}_0(\cdot), \mathbf{v}(\cdot, 0)). \tag{4.33c}$$

Proof. For any $\eta \in L^2(0, T; W^{1,4}(\Omega)) \cap H^1(0, T; L^2(\Omega))$ there exists a subsequence $\eta_{h,\tau}(t) \in S^h$ such that

$$\eta_{h,\tau} \to \eta \quad \text{strongly in } L^2(0, T; H^1(\Omega)). \tag{4.34}$$

We choose $\chi \equiv \eta_{h,\tau}$ in (4.21a) and now analyse the subsequent terms. Firstly, (2.18) and (4.24) imply that as $h \to 0$,

$$\begin{aligned} &\left| \int_0^T [(\partial_t c_{h,\tau}, \eta_{h,\tau})_h - (\partial_t c_{h,\tau}, \eta_{h,\tau})] \, \mathrm{d}t \right| \\ &\quad = \bigg| - \int_0^T (c_{h,\tau}, \partial_t \eta_{h,\tau})_h \, \mathrm{d}t + (c_{h,\tau}(\cdot, T), \eta_{h,\tau}(\cdot, T))_h - (c_{h,\tau}(\cdot, 0), \eta_{h,\tau}(\cdot, 0))_h \\ &\qquad + \int_0^T (c_{h,\tau}, \partial_t \eta_{h,\tau}) \, \mathrm{d}t - (c_{h,\tau}(\cdot, T), \eta_{h,\tau}(\cdot, T)) + (c_{h,\tau}(\cdot, 0), \eta_{h,\tau}(\cdot, 0)) \bigg| \to 0. \end{aligned} \tag{4.35}$$

Furthermore, it follows from (2.5), (4.23c) and (4.34) that

$$\int_0^T (\partial_t c_{h,\tau}, \eta_{h,\tau}) \, \mathrm{d}t \to \int_0^T \langle \partial_t c, \eta \rangle \, \mathrm{d}t \quad \text{as } h \to 0. \tag{4.36}$$

Combining (4.35) and (4.36) shows that

$$\int_0^T (\partial_t c_{h,\tau}, \eta_{h,\tau})_h \, \mathrm{d}t \to \int_0^T \langle \partial_t c, \eta \rangle \, \mathrm{d}t \quad \text{as } h \to 0. \tag{4.37}$$

In view of (4.34) and (4.23d) it follows that as $h \to 0$,

$$\int_0^T (\nabla w_{h,\tau}^+, \nabla \eta_{h,\tau}) \, \mathrm{d}t \to \int_0^T (\nabla w, \nabla \eta) \, \mathrm{d}t. \tag{4.38}$$

From Lemma 4.5, (4.23b), (4.26) and (4.34) we conclude that as $h \to 0$,

$$\int_0^T \mathbf{a}(c_{h,\tau}^-, \eta_{h,\tau}, \mathbf{u}_{h,\tau}^-) \, \mathrm{d}t \to \int_0^T \mathbf{a}(c, \eta, \mathbf{u}) \, \mathrm{d}t. \tag{4.39}$$

Combining (4.37), (4.38) and (4.39) yields (4.33a), on recalling (4.22).

From (2.18), (4.24), (4.23a), (4.23d) and (4.34) we infer that as $h \to 0$,

$$\int_0^T [\gamma^2(\nabla c_{h,\tau}^+, \nabla \eta_{h,\tau}) + (c_{h,\tau}^- - w_{h,\tau}^+, \eta_{h,\tau})_h]\, \mathrm{d}t \to \int_0^T [\gamma^2(\nabla c, \nabla \eta) + (c - w, \eta)]\, \mathrm{d}t. \quad (4.40)$$

Furthermore, setting $F(c, c_{h,\tau}) = c_{h,\tau}^2 + c^2 + c_{h,\tau} c$ and recalling (4.22) we have

$$\begin{aligned}
&\left| \int_0^T [(\phi(c_{h,\tau}^+), \eta_{h,\tau}) - (\phi(c), \eta)]\, \mathrm{d}t \right| \\
&\quad \leqslant \left| \int_0^T (\phi(c_{h,\tau}^+) - \phi(c), \eta)\, \mathrm{d}t \right| + \left| \int_0^T (\phi(c_{h,\tau}^+), \eta_{h,\tau} - \eta)\, \mathrm{d}t \right| \\
&\quad \leqslant \int_0^T \|\eta\|_{0,4} \|F(c, c_{h,\tau})\|_{0,4} \|c_{h,\tau}^+ - c\|\, \mathrm{d}t + \int_0^T \|\phi(c_{h,\tau}^+)\|\, \|\eta_{h,\tau} - \eta\|\, \mathrm{d}t \\
&\quad \leqslant C \int_0^T \|\eta\|_{0,4} \|c_{h,\tau}\|_{0,8}^4 \|c_{h,\tau}^+ - c\|\, \mathrm{d}t + C \int_0^T \|c_{h,\tau}^+\|_{0,6}^3 \|\eta_{h,\tau} - \eta\|\, \mathrm{d}t \\
&\quad \leqslant C \int_0^T \|\eta\|_1 \|c_{h,\tau}\|_1^4 \|c_{h,\tau}^+ - c\|\, \mathrm{d}t + C \int_0^T \|c_{h,\tau}^+\|_1^3 \|\eta_{h,\tau} - \eta\|\, \mathrm{d}t. \qquad (4.41)
\end{aligned}$$

From (2.18), (4.24), (4.41), (4.23b) and (4.34) we have

$$\int_0^T (\phi(c_{h,\tau}^+), \eta_{h,\tau})_h\, \mathrm{d}t \to \int_0^T (\phi(c), \eta)\, \mathrm{d}t \quad \text{as } h \to 0. \quad (4.42)$$

Combining (4.40) and (4.42) yields (4.33b), on recalling (4.22).

For $\mathbf{v} \in L^2(0, T; \mathbf{W}_0) \cap H^1(0, T; \mathbf{L}^2(\Omega))$ there exists $\mathbf{v}_{h,\tau}(t) \in \mathbf{W}_0^h$ such that

$$\mathbf{v}_{h,\tau} \to \mathbf{v} \quad \text{strongly in } L^2(0, T; \mathbf{H}^1(\Omega)), \quad (4.43)$$

$$\partial_t \mathbf{v}_{h,\tau} \to \partial_t \mathbf{v} \quad \text{weakly in } L^2(0, T; \mathbf{L}^2(\Omega)). \quad (4.44)$$

We choose $\chi \equiv \mathbf{v}_{h,\tau}$ in (4.21c) and now analyse the subsequent terms. Since $\mathbf{v}_{h,\tau}(\cdot, T) = \mathbf{0}$ we have

$$\int_0^T (\partial_t \mathbf{u}_{h,\tau}, \mathbf{v}_{h,\tau})\, \mathrm{d}t = -\int_0^T (\mathbf{u}_{h,\tau}, \partial_t \mathbf{v}_{h,\tau})\, \mathrm{d}t - (\mathbf{u}_{h,\tau}(\cdot, 0), \mathbf{v}_{h,\tau}(\cdot, 0)) \quad (4.45)$$

and from (4.45), (4.26) and (4.44) we have

$$\int_0^T (\partial_t \mathbf{u}_{h,\tau}, \mathbf{v}_{h,\tau})\, \mathrm{d}t \to -\int_0^T (\mathbf{u}, \partial_t \mathbf{v})\, \mathrm{d}t - (\mathbf{u}_0(\cdot), \mathbf{v}(\cdot, 0)) \quad \text{as } h \to 0. \quad (4.46)$$

Using arguments similar to the proof of Lemma 4.5 and noting (4.23e), (4.26) and (4.43) it follows that as $h \to 0$,

$$\int_0^T B(\mathbf{u}_{h,\tau}^-, \mathbf{u}_{h,\tau}^+, \mathbf{v}_{h,\tau})\, \mathrm{d}t \to \int_0^T B(\mathbf{u}, \mathbf{u}, \mathbf{v})\, \mathrm{d}t, \quad (4.47)$$

and from (4.23e) and (4.43) we find that as $h \to 0$,

$$\int_0^T (\nabla \mathbf{u}_{h,\tau}^+ : \nabla \mathbf{v}_{h,\tau})\, \mathrm{d}t \to \int_0^T (\nabla \mathbf{u} : \nabla \mathbf{v})\, \mathrm{d}t. \tag{4.48}$$

From Lemma 4.5, (4.23b), (4.23d) and (4.43) it follows that as $h \to 0$,

$$\int_0^T \mathbf{a}(c_{h,\tau}^-, w_{h,\tau}^+, \mathbf{v})\, \mathrm{d}t \to \int_0^T \mathbf{a}(c, w, \mathbf{v})\, \mathrm{d}t. \tag{4.49}$$

Combining (4.46)–(4.49) yields (4.33c) on recalling (4.22).

REMARK 4.3 We note that there is an inconsistency of notation between this section and Section 3, since in this section $\{c, w, \mathbf{u}\}$ is used to denote the solution of (4.33a–c), whereas in Section 3, $\{c, w, \mathbf{u}\}$ is used to denote the solution of the smoother problem (**P**).

5. Numerical results

In practice the concentration, c, rapidly varies over the interfacial regions between the two fluids, while away from the interface, in the bulk regions, it is close to ± 1. This interfacial region is of thickness $O(\gamma \ln \gamma)$ and to effectively model it we require an adaptive mesh that is locally refined close to and inside of the interfacial region and coarsened elsewhere. We now define our mesh refinement strategy:

Given an initial mesh $\mathcal{T}^0$, for time step n an element, κ, with vertices $\mathbf{x}_{\kappa,i}$, $i = 1, 2, 3$, is refined if

$$\max_i |c_h^{n-1}(\mathbf{x}_{\kappa,i})| < 1 - \delta$$

and $h_\kappa > \gamma/2$ where $\delta = O(\gamma)$; in practice this guarantees that there are at least eight grid points in the interfacial region. This element is repeatedly refined, using uniform refinement, until $h_\kappa < \gamma/2$. Elements are coarsened if κ and its neighbouring elements $\Omega_\kappa = \{K \in \mathcal{T}^{n-1} : \overline{\kappa} \cap \overline{K} \neq \emptyset\}$, with the set of vertices ω_κ, are such that

$$\max_{\mathbf{x} \in \omega_\kappa} |c_h^{n-1}(\mathbf{x})| > 1 - \delta.$$

We call this grid $\mathcal{T}^n$ and use it for the chemical potential, w, as well. The space of continuous piecewise linear functions associated with this space is denoted by $S^{n,c}$ and we denote the dimension of $S^{n,c}$ by $N^{n,c}$. We use standard continuous piecewise linear interpolation to transfer functions in $S^{n-1,c}$ to $S^{n,c}$.

For the velocity the divergence free space, $\mathbf{W}_0^h$, is not constructed and the pressure function, p, remains in the system. We use a fixed uniform mesh, $\mathcal{T}^h$, for this problem and the mixed finite element $(S^{h/2})^2 \times S^h$; this space, (P1 iso P2 - P1), is known to be inf-sup stable (see [13]). Note that the space $S^{h/2}$ need not be related to the concentration mesh and in general is not. We denote the dimensions of $S^{h/2}$ and S^h by $N^{h,u}$ and $N^{h,p}$, respectively.

We obtain the following fully discrete approximation of (2.11a–f):

$(\tilde{\mathbf{P}}_{h,\tau})$ For $n \geqslant 1$ find $\{c_h^n, w_h^n, \mathbf{u}_h^n, p_h^n\} \in S^{n,c} \times S^{n,c} \times (S^{h/2})^2 \times S^h$ such that

$$(\delta_\tau c_h^n, \chi)_h + \frac{1}{\mathrm{Pe}}(b(c_h^{n-1})\nabla w_h^n, \nabla\chi) = \mathbf{a}(c_h^{n-1}, \chi, \mathbf{u}_h^{n-1}) \qquad \forall \chi \in S^{n,c}, \tag{5.1a}$$

$$\gamma^2(\nabla c_h^n, \nabla\chi) = (w_h^n - \phi(c_h^n) + c_h^{n-1}, \chi)_h \qquad \forall \chi \in S^{n,c}, \tag{5.1b}$$

$$(\delta_\tau \mathbf{u}_h^n, \boldsymbol{\chi}) + \frac{1}{\mathrm{Re}}(\nabla\mathbf{u}_h^n : \nabla\boldsymbol{\chi}) + B(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \boldsymbol{\chi}) + K\mathbf{a}(c_h^{n-1}, w_h^n, \boldsymbol{\chi}) = (p_h^n, \nabla\cdot\boldsymbol{\chi}) \qquad \forall \boldsymbol{\chi} \in (S^{h/2})^2, \tag{5.1c}$$

$$(\nabla\cdot\mathbf{u}_h^n, q_h) = 0 \qquad \forall q_h \in S^h, \tag{5.1d}$$

$$c_h^0(x) = Q^h(c_0(x)), \quad \mathbf{u}_h^0(x) = \mathbf{P}^h(\mathbf{u}_0(x)) \qquad \forall x \in \Omega, \tag{5.1e}$$

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega \times (0, T), \tag{5.1f}$$

where $b(c) = \exp(-c^2)$ and $\mathbf{g}\cdot\mathbf{n} = 0$ on $\partial\Omega$.

We now discuss algorithms for solving the resulting system of algebraic equations for $\{c_h^n, w_h^n, \mathbf{u}_h^n, p_h^n\}$ arising at each time level from the approximation $(\tilde{\mathbf{P}}_{h,\tau})$. We note that with this approximation at each time level we are required to solve a Cahn–Hilliard equation followed by a Navier–Stokes equation.

Let $\mathbf{C}^n = \{c_1^n, c_2^n, \ldots, c_{N^{n,c}}^n\}$ be the vector of coefficients for c_h^n with respect to the standard basis $S^{n,c} = \langle\psi_i^{n,c}\rangle_1^{N^{n,c}}$ and similarly for $\mathbf{W}^n$, $\mathbf{U}^n$ and $\mathbf{P}^n$. Then the system (5.1a–f) can be rewritten as:

Find $\mathbf{C}^n, \mathbf{W}^n \in \mathbb{R}^{N^{n,c}}$, $\mathbf{U}^n \in (\mathbb{R}^{N^{h,u}})^2$ and $\mathbf{P}^n \in \mathbb{R}^{N^{h,p}}$ such that

$$\widehat{M}^{n,c}\mathbf{C}^n + \frac{\tau_n}{\mathrm{Pe}}D^{n,c}\mathbf{W}^n = \widehat{M}^{n,c}\mathbf{C}^{n-1} + \tau_n\mathbf{F}(c_h^{n-1}, \mathbf{u}_h^{n-1}),$$
$$-\widehat{M}^{n,c}\mathbf{W}^n + \widehat{M}^{n,c}\phi(\mathbf{C}^n) + \gamma^2A^{n,c}\mathbf{C}^n = \widehat{M}^{n,c}\mathbf{C}^{n-1},$$
$$\left(\mathbf{M}^{h,u} + \frac{\tau_n}{\mathrm{Re}}\mathbf{A}^{h,u} + \tau_n\mathbf{S}^{n-1,h,u}\right)\mathbf{U}^n - \tau_n(B^h)^T\mathbf{P}^n = \mathbf{M}^{h,u}\mathbf{U}^{n-1} - \tau_nK\mathbf{G}(c_h^{n-1}, w_h^n),$$
$$B^h\mathbf{U}^n = \mathbf{0},$$

where the Cahn–Hilliard matrices and right hand side are given by

$$A^{n,c}(i,j) := \int_\Omega \nabla\psi_i^{n,c}\cdot\nabla\psi_j^{n,c}\,\mathrm{d}x, \qquad D^{n,c}(i,j) := \int_\Omega b(\mathbf{c}_h^{n-1})\nabla\psi_i^{n,c}\cdot\nabla\psi_j^{n,c}\,\mathrm{d}x,$$
$$\widehat{M}^{n,c}(i,i) := \int_\Omega \pi^h[(\psi_i^{n,c})^2]\,\mathrm{d}x, \qquad \mathbf{F}_i(c_h^{n-1}, \mathbf{u}_h^{n-1}) := \int_\Omega c_h^{n-1}\nabla\psi_i^{n,c}\cdot\mathbf{u}_h^{n-1}\,\mathrm{d}x,$$

and the Navier–Stokes matrices are given by

$$\mathbf{A}^{h,u} = \begin{bmatrix} A^{h,u} & 0 \\ 0 & A^{h,u} \end{bmatrix}, \quad \mathbf{M}^{h,u} = \begin{bmatrix} M^{h,u} & 0 \\ 0 & M^{h,u} \end{bmatrix}, \quad \mathbf{S}^{n-1,h,u} = \begin{bmatrix} S^{n-1,h,u} & 0 \\ 0 & S^{n-1,h,u} \end{bmatrix},$$
$$B^h = [B_1^h,\ B_2^h], \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}^1(c_h^{n-1}, w_h^n) \\ \mathbf{G}^2(c_h^{n-1}, w_h^n) \end{bmatrix}$$

with

$$A^{h,u}(i,j) := \int_\Omega \nabla\psi_i^{h,u}\cdot\nabla\psi_j^{h,u}\,dx, \qquad M^{h,u}(i,j) := \int_\Omega \psi_i^{h,u}\psi_j^{h,u}\,\mathrm{d}x,$$

$$S^{n-1,h,u} := \frac{1}{2}\int_\Omega [\mathbf{u}^{h,n-1}\cdot\nabla\psi_i^{h,u}]\cdot\psi_j^{h,u}\;\mathrm{d}x - \frac{1}{2}\int_\Omega [\mathbf{u}^{h,n-1}\cdot\nabla\psi_j^{h,u}]\cdot\psi_i^{h,u}\,\mathrm{d}x,$$

$$B_1^{h,u}(i,j) := \int_\Omega \frac{\partial\psi_i^{h,u}}{\partial x_1}\psi_j^{h,p}\,\mathrm{d}x, \qquad B_2^{h,u}(i,j) := \int_\Omega \frac{\partial\psi_i^{h,u}}{\partial x_2}\psi_j^{h,p}\,\mathrm{d}x,$$

$$\mathbf{G}_i^1(c_h^{n-1},w_h^n) := \int_\Omega c_h^{n-1}\frac{\partial w_h^n}{\partial x_1}\psi_i^{h,u}\,\mathrm{d}x, \qquad \mathbf{G}_i^2(c_h^{n-1},w_h^n) := \int_\Omega c_h^{n-1}\frac{\partial w_h^n}{\partial x_2}w_h^n\psi_i^{h,u}\,\mathrm{d}x.$$

Here $\langle\psi_i^{n,c}\rangle_1^{N^{n,c}}$, $\langle\psi_i^{h,u}\rangle_1^{N^{h,u}}$ and $\langle\psi_i^{h,p}\rangle_1^{N^{h,p}}$ are the standard basis functions for $S^{n,c}$, $S^{h/2}$ and S^h respectively.

To solve the Cahn–Hilliard equation we use the multigrid solver given in [14] which produces an efficient and reliable method with optimal convergence. For the Navier–Stokes equation, we have an Oseen type equation, which is solved using preconditioned GMRES (see [31]), where the F_p preconditioner is used (see [24]). This solver is known to be optimal with respect to mesh refinement and has mild dependence on the Reynolds number. For further details and numerical examples showing the efficiency of this solution method see [25].

We display two numerical experiments; the first shows the evolution of a cross-shaped interface to a circle and the second is similar to the lid driven cavity experiment seen in [8].

5.1 *Rotational boundary condition: cross to circle*

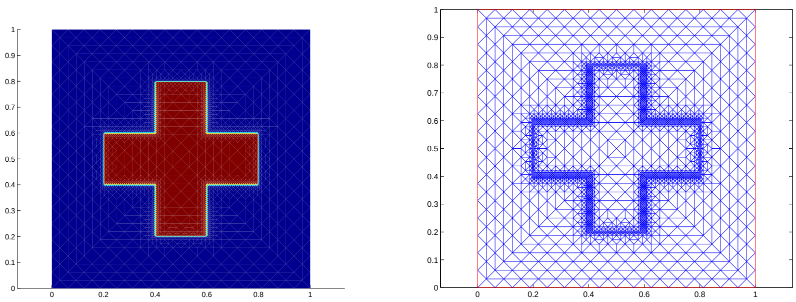

FIG. 1. Initial data for the concentration $c_{h,\tau}$ and the initial mesh.

In this experiment we take $\Omega = (0,1)^2$ and as initial data for the concentration we take a cross set inside Ω (see Figure 1); for the velocity we take $\mathbf{u}_0 = [x_2, -x_1]$ and $\mathbf{g} = [x_2, -x_1]$. We set $\nu = 1/100$, Pe $= 1000$, $K = 0.001$ and $\gamma = 1/120$ and we use a fixed time step with $\tau = 0.05$. For the velocity space we use a uniform mesh of size $h_{\mathbf{u}} \approx 1/90$.

Figures 2–4 show the evolution of the concentration for this problem together with the corresponding meshes.

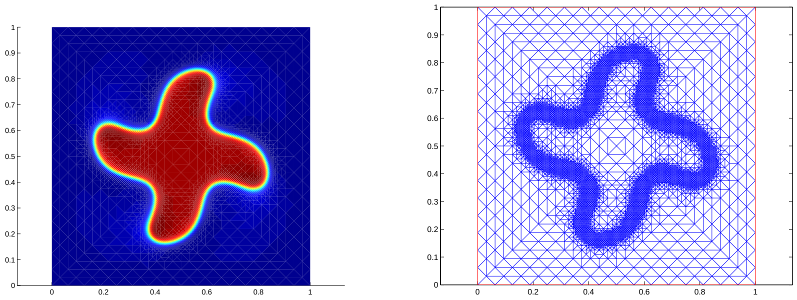

FIG. 2. Concentration $c_{h,\tau}$ and mesh at $t = 3$.

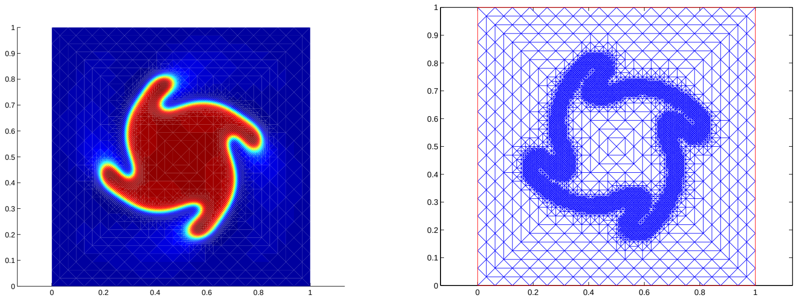

FIG. 3. Concentration $c_{h,\tau}$ and mesh at $t = 7$.

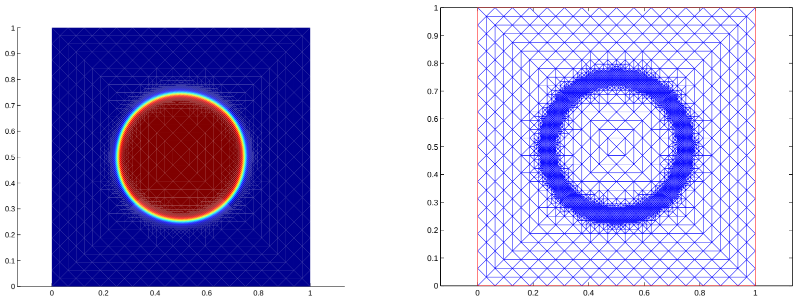

FIG. 4. Concentration $c_{h,\tau}$ and mesh at $t = 20$.

5.2 *Lid driven cavity boundary condition*

In this experiment we take $\Omega = (0, 1)^2$ and as initial data for the concentration we take the horizontal line $x_2 = 0.5$. For the velocity we take the initial data to be the Stokes solution to the lid driven cavity problem and for boundary data we take $\mathbf{g} = [x_1^2(1 - x_1)^2, 0]$ for $x_2 = 1$, and zero otherwise. We set $\nu = 1/500$, Pe $= 200$, $K = 1/5000$ and $\gamma = 1/120$ and we use a fixed time step with $\tau = 0.025$. The velocity space is a uniform mesh of size $h_{\mathbf{u}} \approx 1/90$.

The evolution of the concentration for this problem is shown through Figures 5–8 (see [8] for a similar problem). The corresponding adapted meshes are also given. Figures 9–10 show contour plots of the concentration and the velocity field. From the magnified images we can see the velocity at the front of the fluid is almost perpendicular to the interface and thus forces it to move, while behind this front the velocity is almost tangential and hence in this region the movement of the interface is small.

We conclude with Figure 11 in which we see the flow at time $t = 20$ together with a contour plot of the concentration.

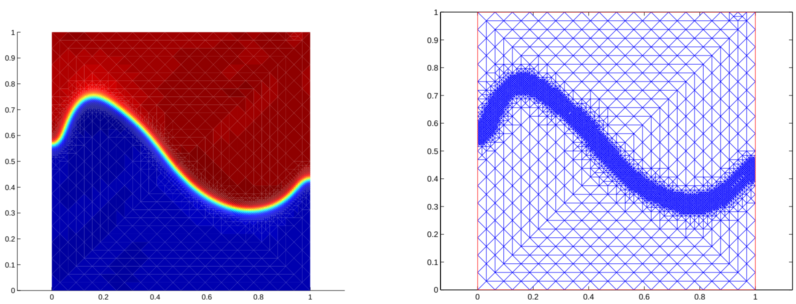

FIG. 5. Concentration $c_{h,\tau}$ and mesh at $t = 3.4$.

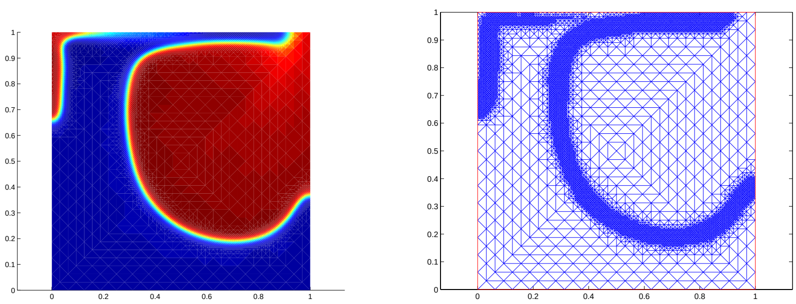

FIG. 6. Concentration $c_{h,\tau}$ and mesh at $t = 6.5$.

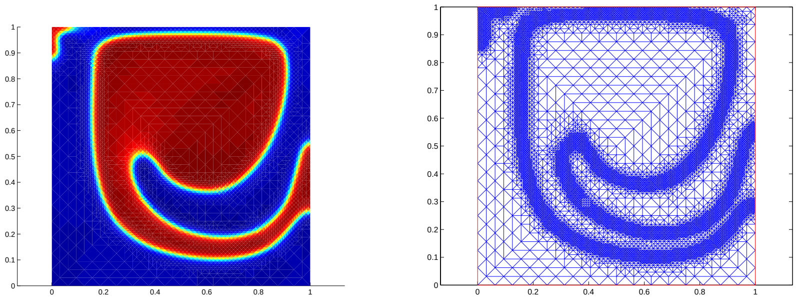

FIG. 7. Concentration $c_{h,\tau}$ and mesh at $t = 12.5$.

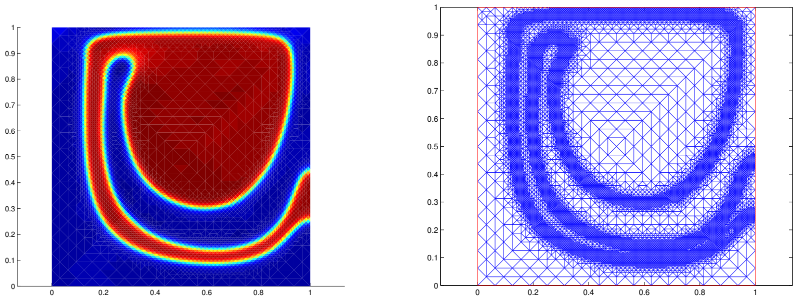

FIG. 8. Concentration $c_{h,\tau}$ and mesh at $t = 20$.

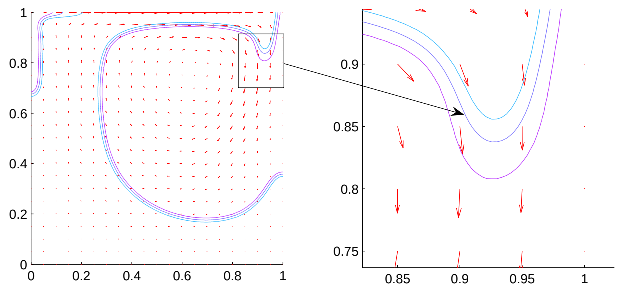

FIG. 9. Flow and concentration at $t = 7$.

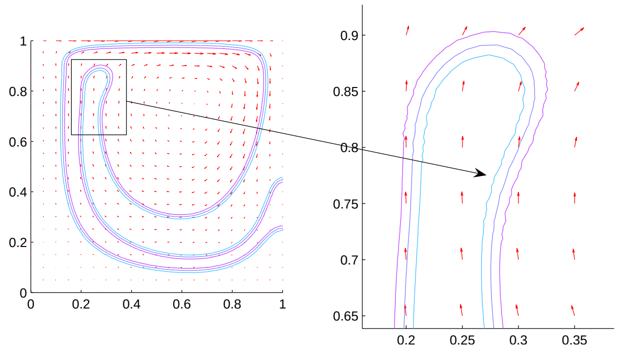

FIG. 10. Flow and concentration at $t = 16.5$.

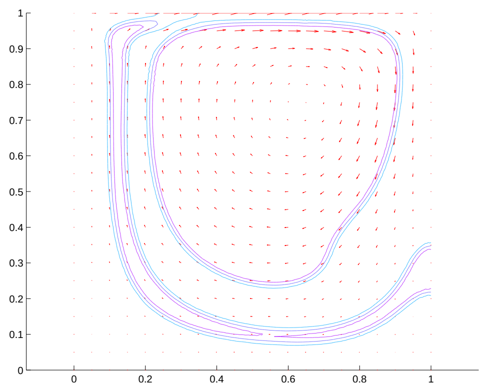

FIG. 11. Flow and concentration at $t = 20$.

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REFERENCES

1. ANDERSON, D. M., MCFADDEN, G. B., & WHEELER, A. A. Diffuse interface methods in fluid mechanics. *Ann. Rev. Fluid Mech.* **30** (1998), 139–165. MR 1609626
2. BADALASSI, V. E., CENICEROS, H. D., & BANERJEE, S. Computation of multiphase systems with phase field models. *J. Comput. Phys.* **190** (2003), 371–397. Zbl 1076.76517 MR 2013023
3. BARRETT, J. W., & BLOWEY, J. F. An error bound for the finite element approximation of the Cahn–Hilliard equation with logarithmic free energy. *Numer. Math.* **72** (1995), 1–20. Zbl 0851.65070 MR 1359705

4. BARRETT, J. W., & BLOWEY, J. F. Finite element approximation of the Cahn–Hilliard equation with concentration mobility. *Math. Comp.* **68** (1999), 487–517. Zbl pre01267989 MR 1609678
5. BARRETT, J. W., LANGDON, S., & NUERNBERG, R. Finite element approximation of a sixth order nonlinear degenerate parabolic equation. *Numer. Math.* **96** (2004), 401–434. Zbl 1041.65076 MR 2028722
6. BLOWEY, J. F., COPETTI, M. I. M., & ELLIOTT, C. M. Numerical analysis of a model for phase separation of a multicomponent alloy. *IMA J. Numer. Anal.* **16** (1996), 111–139. Zbl 0857.65137 MR 1367400
7. BOYER, F. Mathematical study of multiphase flow under shear through order parameter formulation. *Asymptotic Anal.* **20** (1999), 175–212. Zbl 0937.35123 MR 1700669
8. BOYER, F. A theoretical and numerical model for the study of incompressible mixture flows. *Comput. Fluids* **31** (2002), 41–68. Zbl 1057.76060
9. BOYER, F., CHUPIN, L., & FRANCK, B. A. Numerical study of viscoelastic mixtures through a Cahn–Hilliard fluid. *Eur. J. Mech. B Fluids* **23** (2004), 759–780. Zbl 1058.76531 MR 2077449
10. CAHN, J. W. On spinodal decomposition. *Acta Metall. Mater.* **9** (1961), 795–801.
11. CAHN, J. W., & HILLIARD, J. E. Free energy of a nonuniform system, I, interfacial free energy. *J. Chem. Phys.* **28** (1958), 258–267.
12. CHUPIN, L. Existence result for a mixture of non-newtonian flows with stress diffusion using the Cahn–Hilliard formulation. *Discrete Contin. Dynam. Systems Ser. B* **3** (2003), 45–68. Zbl pre01894896 MR 1951567
13. CIARLET, P., JR., & GIRAULT, V. Condition inf-sup pour l'élément fini de Taylor–Hood P_2-iso-P_1, 3-D; application aux équations de Maxwell. *C. R. Math. Acad. Sci. Paris* **335** (2002), 827–832. Zbl 1021.78009 MR 1947708
14. KAY, D., & WELFORD, R. A multigrid finite element solver for the Cahn–Hilliard equation. *J. Comput. Phys.* **212** (2006), 288–304. Zbl 1081.65091 MR 2183612
15. ELLIOTT, C. M. The Cahn–Hilliard model for the kinetics of phase separation. *Mathematical Model for Phase Change Problems* (Óbidos, 1988), Int. Ser. Numer. Math. 88, Birkhäuser, Basel (1989), 35–73. Zbl 0692.73003 MR 1038064
16. ELLIOTT, C. M., FRENCH, D. A., & MILNER, F. A. A second order splitting method for the Cahn–Hilliard equation. *Numer. Math.* **54** (1989), 575–590. Zbl 0668.65097 MR 0978609
17. ELLIOTT, C. M., KAY, D., & STYLES, V. k Finite element analysis of a current density-electric field formulation of bean's model for superconductivity. *IMA J. Numer. Anal.* **25** (2005), 182–204. Zbl 1100.78018 MR 2110240
18. ELLIOTT, C. M., & STUART, A. M. The global dynamics of discrete semilinear parabolic equations. *SIAM J. Numer. Anal.* **30** (1993), 1622–1663. Zbl 0792.65066 MR 1249036
19. FENG, X. Fully discrete finite element approximation of the Navier–Stokes–Cahn–Hilliard diffuse interface model for two-phase flows. *SIAM J. Numer. Anal.* **44** (2006), 1049–1072. Zbl pre05167765 MR 2231855
20. FENG, X., & PROHL, A. Numerical analysis of the Cahn–Hilliard equation and approximation of the Hele–Shaw problem. *Interfaces Free Bound.* **7** (2005), 1–28. Zbl 1072.35150 MR 2126141
21. GIRAULT, V., & RAVIART, P. A. *Finite Element Methods for the Navier–Stokes Equations: Theory and Algorithms*. Springer Ser. Comput. Math., Springer, New York, 1986. Zbl 0585.65077
22. HOHENBERG, P. C., & HALPERIN, B. I. Theory of dynamic critical phenomena. *Rev. Mod. Phys.* **49** (1977), 435–479.
23. JACQMIN, D. Calculations of two-phase Navier–Stokes flows using phase-field modelling. *J. Comput. Phys.* **155** (1999), 96–127. Zbl 0966.76060 MR 1716497
24. KAY, D., LOGHIN, D., & WATHEN, A. A preconditioner for the steady-state Navier–Stokes equations. *SIAM J. Sci. Comput.* **24** (2002), 237–256. Zbl 1013.65039 MR 1924423

25. KAY, D., & WELFORD, R. Efficient numerical solution of Cahn–Hilliard–Navier–Stokes fluids in 2D. *SIAM J. Sci. Comput.*, to appear.

26. KIM, J. A diffuse-interface model for axisymmetric immiscible two-phase flow. *Appl. Math. Comput.* **160** (2005), 589–606. Zbl pre02132586 MR 2102831

27. KIM, J., KANG, K., & LOWENGRUB, J. Conservative multigrid methods for Cahn–Hilliard fluids. *J. Comput. Phys.* **193** (2004), 511–543. Zbl 1109.76348 MR 2030475

28. LIONS, J.-L. On some questions in boundary value problems of mathematical physics. *Contemporary Development in Continuum Mechanics and Partial Differential Equations* (Rio de Janeiro, 1977), North-Holland Math. Stud. 30, North-Holland (1978), 284–346. Zbl 0404.35002 MR 0519648

29. LIONS, J.-L., & MAGENES, E. *Non-Homogeneous Boundary Value Problems and Applications*. Volume I, Springer, Berlin (1972). Zbl 0223.35039 MR 0350177

30. RODRIGUES, J.-F. *Obstacle Problems in Mathematical Physics*. North-Holland Math. Stud. 134, North-Holland, Amsterdam (1987). Zbl 0606.73017 MR 0880369

31. SAAD, Y., & SCHULTZ, M. H. GMRES: A generalized minimum residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.* **7** (1986), 856–869. Zbl 0599.65018 MR 0848568