Interfaces and Free Boundaries **10** (2008), 223–243

On the energy of a flow arising in shape optimization

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[Received 10 July 2006 and in revised form 11 September 2007]

In [8] we have defined a viscosity solution for the gradient flow of the exterior Bernoulli free boundary problem. We prove here that the associated energy is non-increasing along the flow. For this we build a discrete gradient flow in the flavour of Almgren, Taylor and Wang [2].

1. Introduction

In this paper we continue our investigation of a "gradient flow" for the Bernoulli free boundary problem initiated in [8]. The exterior Bernoulli free boundary problem is to minimize the capacity of a set under volume constraints. Using a Lagrange multiplier $\lambda > 0$, this problem can be recast into the minimization with respect to the set Ω of the functional

$$\mathcal{E}_\lambda(\Omega) = \mathrm{cap}_S(\Omega) + \lambda|\Omega|,$$

where $\mathrm{cap}_S(\Omega)$ denotes the capacity of the set Ω with respect to some fixed set S and $|\Omega|$ denotes the volume of Ω. The set Ω is constrained to satisfy the inclusion $S \subset\subset \Omega$. Notice that there is a "competition" between the two terms in the minimization: the capacity is non-increasing with respect to inclusion whereas the volume is non-decreasing.

Such a problem has quite a long history and we refer to the survey paper [12] for references and interpretations in physics. Our study is motivated by several papers in numerical analysis where discrete gradient flows are built via a level-set approach in order to solve free boundary and shape optimization problems: see [1] and the references therein for the recent advances in this area. In this framework, the exterior Bernoulli free boundary problem appears as a model problem in order to better understand this numerical approach. In this work, we prove that the energy $\mathcal{E}_\lambda$ is non-increasing along the generalized flow we built in [8]. This question is certainly essential to better explain the numerical schemes of [1].

Let us now go further into the description of the gradient flow for $\mathcal{E} := \mathcal{E}_1$ (we work here in the case $\lambda = 1$ for simplicity of notation). The energy $\mathcal{E}$ being defined on sets, a gradient flow for $\mathcal{E}$

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is a family $(\Omega(t))_{t\geqslant 0}$ of sets evolving with a normal velocity which "instantaneously decreases the energy the most". For the Bernoulli problem, the corresponding evolution law is given by

$$V_{t,x} = h(x, \Omega(t)) := -1 + \bar{h}(x, \Omega(t)) \quad \text{for all } t \geqslant 0,\ x \in \partial\Omega(t). \tag{1}$$

In the above equation, $V_{t,x}$ is the normal velocity of the set $\Omega(t)$ at the point x at time t and $\bar{h}(x, \Omega)$ is a non-local term of Hele–Shaw type, given, for any set Ω with smooth boundary, by

$$\bar{h}(x, \Omega) = |\nabla u(x)|^2, \tag{2}$$

where $u : \Omega \to \mathbb{R}$ is the capacity potential of Ω with respect to S, i.e., the solution of the partial differential equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus S, \\ u = 1 & \text{on } \partial S, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The set S is a fixed source and we always assume above that S is smooth and $S \subset\subset \Omega(t)$. Let us underline that $h(x, \Omega)$ is well defined as soon as Ω has "smooth" (say $\mathcal{C}^2$) boundary and that $S \subset\subset \Omega$.

The reason why a smooth solution $(\Omega(t))$ of the geometric equation (1) can be considered as a gradient flow of the energy

$$\mathcal{E}(\Omega) = |\Omega| + \mathrm{cap}_S(\Omega) \tag{4}$$

is the following: from the Hadamard formula we have

$$\frac{d}{dt}\mathcal{E}(\Omega(t)) = \int_{\partial\Omega(t)} (1 - |\nabla u|^2) V_{t,x} = -\int_{\partial\Omega(t)} (-1 + |\nabla u|^2)^2 \leqslant 0.$$

Hence the choice of $V_{t,x} = h(x, \Omega(t))$ in (1) appears to be the one which decreases the energy $\mathcal{E}$ the most. In order to minimize the energy $\mathcal{E}$, it is therefore very natural to follow the gradient flow (1). This is precisely what is done numerically in [1].

In general the geometric flow (1) does not have classical solutions. In order to define the flow after the onset of singularities, we have introduced in [8] a notion of generalized (viscosity) solution and investigated its existence as well as uniqueness. In order to prove that the energy is non-increasing along the generalized flow, we face a main difficulty: energy estimates are hard to derive from the notion of viscosity solutions. Indeed, this latter notion is defined through a comparison principle, which has very little to do with the energy associated to the flow. To the best of our knowledge, such a question has only been settled for the mean curvature motion (MCM for short), which corresponds (at least formally) to the gradient flow of the perimeter. There are two proofs of the fact that the perimeter of the viscosity solution to the mean curvature flow decreases: the first one is due to Evans and Spruck in their seminal papers [10, 11]; it is based on a regularized version of the level set formulation for the flow and is probably specific to local evolution equations. The other proof is due to Chambolle [9]. Its starting point is the fundamental construction of Almgren, Taylor and Wang [2] who built generalized solutions of the MCM in a variational way as limits of a "discrete gradient flow" for the perimeter (the so-called minimizing movements; see also Ambrosio [5]). The key argument of Chambolle's paper [9] is that Almgren, Taylor and Wang's generalized solutions coincide with the viscosity solutions, at least for a large class of initial sets. Hence the energy estimate available from [2]—which allows comparing the energy of the evolving set with the energy of the initial position—can also be applied to the viscosity solution. Since the viscosity solution enjoys a semigroup property, one can conclude that the energy is decreasing along the flow.

To prove that the energy $\mathcal{E}$ is decreasing along our viscosity solutions of (1), we borrow several ideas from Almgren, Taylor and Wang [2] and Chambolle [9]. As in [2] for the MCM, we start with the construction of a discrete gradient flow (Ω_n^h) for the energy $\mathcal{E}$: namely Ω_{n+1}^h is obtained from Ω_n^h as a minimizer of a functional $J_h(\Omega_n^h, \cdot)$ which is equal to $\mathcal{E}$ plus a penalizing term. The penalizing term—which depends on the time step h—prevents the minimizing set Ω_{n+1}^h from being too far from Ω_n^h. Then, as in Chambolle [9], we prove that the limits of these discrete gradient flows converge to the viscosity solution of our equation (1) as the time step h goes to 0. In [9], this convergence is proved by using the convexity of the equivalent of our functional $J_h(\Omega_h^n, \cdot)$ for the MCM. Here we use instead directly a weak form of the Euler equation for minimizers of $J_h(\Omega_h^n, \cdot)$ as described by Alt and Caffarelli [3] for the Bernoulli problem. We then conclude that the energy of the flow is non-increasing.

The paper is organized in the following way. In Section 2 we recall the construction of [8] for the viscosity solutions of (1). Section 3 is devoted to suitable generalizations of the capacity and capacity potential needed for our estimates. In Section 4 we introduce the functional J_h and build the discrete motions, the limits of which are discussed in Section 5. The fact that the energy is non-increasing along the flow is finally proved in Section 6.

2. Definitions and notations for the generalized flow

Let us first fix some basic notations: if A, B are subsets of $\mathbb{R}^N$, then $A \subset\subset B$ means that the closure $\overline{A}$ of A is a compact subset which satisfies $\overline{A} \subset \text{int}(B)$, where $\text{int}(B)$ is the interior of B. We set

$$\mathcal{D} = \{K \subset\subset \mathbb{R}^N : S \subset\subset K\}.$$

Throughout the paper $|\cdot|$ denotes the euclidean norm (in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, depending on the context) and $B(x, R)$ denotes the open ball centered at x and of radius R. If E is a measurable subset of $\mathbb{R}^N$, we also denote by $|E|$ the Lebesgue measure of E. If K is a subset of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$, then $d_K(x)$ denotes the usual distance from x to K: $d_K(x) = \inf_{y\in K} |y - x|$. The *signed distance* d_K^s to K is defined by

$$d_K^s(x) = \begin{cases} d_K(x) & \text{if } x \notin K, \\ -d_{\partial K}(x) & \text{if } x \in K, \end{cases} \tag{5}$$

where $\partial K = \overline{K} \setminus \text{int}(K)$ is the boundary of K.

Here and throughout the paper, we assume that

S is the closure of an open, nonempty, bounded subset of $\mathbb{R}^N$ with a $\mathcal{C}^2$ boundary. (6)

The *generalized solution* of the front propagation problem (1) is defined through its graph: if $(\Omega(t))_{t\geqslant 0}$ is family of evolving sets, then its graph is the subset of $\mathbb{R}^+ \times \mathbb{R}^N$ defined by

$$\mathcal{K} = \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N : x \in \Omega(t)\}.$$

We denote by (t, x) an element of such a set, where $t \in \mathbb{R}^+$ denotes the time and $x \in \mathbb{R}^N$ is the space coordinate. We set

$$\mathcal{K}(t) = \{x \in \mathbb{R}^N : (t, x) \in \mathcal{K}\}.$$

The closure of the set $\mathcal{K}$ in $\mathbb{R}^{N+1}$ is denoted by $\overline{\mathcal{K}}$. The closure of the complement of $\mathcal{K}$ is denoted $\widehat{\mathcal{K}}$:

$$\widehat{\mathcal{K}} = \overline{(\mathbb{R}^+ \times \mathbb{R}^N) \setminus \mathcal{K}},$$

and we set

$$\widehat{\mathcal{K}}(t) = \{x \in \mathbb{R}^N \mid (t, x) \in \widehat{\mathcal{K}}\}.$$

We use here repeatedly the terminology of [6, 7, 8]:

- A *tube* $\mathcal{K}$ is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$ such that $\overline{\mathcal{K}} \cap ([0, t] \times \mathbb{R}^N)$ is a compact subset of $\mathbb{R}^{N+1}$ for any $t \geqslant 0$.
- A tube $\mathcal{K}$ is *left lower semicontinuous* if

$$\forall t > 0,\ \forall x \in \mathcal{K}(t),\ \forall t_n \to t^-,\ \exists x_n \in \mathcal{K}(t_n), \quad x_n \to x.$$

- If $s = 1, 2$ or $(1, 1)$, then a $\mathcal{C}^s$ *tube* $\mathcal{K}$ on some open interval I of $\mathbb{R}^+$ is a tube such that the relative boundary of $\mathcal{K}$ in $I \times \mathbb{R}^N$ is at least $\mathcal{C}^s$ regular.
- A *regular tube* $\mathcal{K}_r$ on some open interval I of $\mathbb{R}^+$ is a tube with non-empty interior such that the relative boundary of $\mathcal{K}_r$ in $I \times \mathbb{R}^N$ is at least $\mathcal{C}^1$ regular, and at any point (t, x) of this boundary, the outward normal (ν_t, ν_x) to $\mathcal{K}_r$ at (t, x) satisfies $\nu_x \neq 0$. In this case, the *normal velocity* $V_{(t,x)}^{\mathcal{K}_r}$ at $(t, x) \in \partial\mathcal{K}_r$ is defined by

$$V_{(t,x)}^{\mathcal{K}_r} = -\frac{\nu_t}{|\nu_x|},$$

  where (ν_t, ν_x) is the outward normal to $\mathcal{K}_r$ at (t, x).
- A $\mathcal{C}^1$ regular tube $\mathcal{K}_r$ on some open interval I of $\mathbb{R}^+$ is *externally tangent* to a tube $\mathcal{K}$ at $(t, x) \in \mathcal{K}$ if $t \in I$ and

$$(I \times \mathbb{R}^N) \cap \mathcal{K} \subset \mathcal{K}_r \quad \text{and} \quad (t, x) \in \partial\mathcal{K}_r.$$

  It is *internally tangent* to $\mathcal{K}$ at $(t, x) \in \widehat{\mathcal{K}}$ if $t \in I$ and

$$(I \times \mathbb{R}^N) \cap \mathcal{K}_r \subset \mathcal{K} \quad \text{and} \quad (t, x) \in \partial\mathcal{K}_r.$$

- We say that a sequence of $\mathcal{C}^{1,1}$ tubes $(\mathcal{K}_n)$ converges to some $\mathcal{C}^{1,1}$ tube $\mathcal{K}$ in some open interval I *in the* $\mathcal{C}^{1,\mathrm{b}}$ *sense* if $(\mathcal{K}_n)$ converges to $\mathcal{K}$ and $(\partial\mathcal{K}_n)$ converges to $\partial\mathcal{K}$ in the Hausdorff distance, and if there is an open neighborhood $\mathcal{O}$ of the relative boundary of $\mathcal{K}$ in $I \times \mathbb{R}^N$ such that, if $d^s_{\mathcal{K}}$ (respectively $d^s_{\mathcal{K}_n}$) is the signed distance to $\mathcal{K}$ (respectively to $\mathcal{K}_n$), then $(d^s_{\mathcal{K}_n})$ and $(\nabla d^s_{\mathcal{K}_n})$ converge uniformly to $d^s_{\mathcal{K}}$ and $D\mathbf{d}_{\mathcal{K}}$ on $\mathcal{O}$ and $\|D^2 d^s_{\mathcal{K}_n}\|_\infty$ is uniformly bounded on $\mathcal{O}$.

We are now ready to define the generalized solutions of (1):

DEFINITION 2.1 Let $\mathcal{K}$ be a tube and $K_0 \in \mathcal{D}$ be an initial set.

1. $\mathcal{K}$ is a *viscosity subsolution* to the front propagation problem (1) if $\mathcal{K}$ is left lower semicontinuous and $\mathcal{K}(t) \in \mathcal{D}$ for any t, and, for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ externally tangent to $\mathcal{K}$ at some point (t, x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have

$$V_{(t,x)}^{\mathcal{K}_r} \leqslant h(x, \mathcal{K}_r(t)),$$

   where $V_{(t,x)}^{\mathcal{K}_r}$ is the normal velocity of $\mathcal{K}_r$ at (t, x).

We say that $\mathcal{K}$ is a subsolution to the front propagation problem with initial position K_0 if $\mathcal{K}$ is a subsolution and $\overline{\mathcal{K}}(0) \subset \overline{K_0}$.

2. $\mathcal{K}$ is a *viscosity supersolution* to the front propagation problem if $\widehat{\mathcal{K}}$ left lower semicontinuous and $\mathcal{K}(t) \subset \mathcal{D}$ for any t, and, for any $\mathcal{C}^2$ regular tube $\mathcal{K}_r$ internally tangent to $\mathcal{K}$ at some point (t,x), with $\mathcal{K}_r(t) \in \mathcal{D}$ and $t > 0$, we have

$$V_{(t,x)}^{\mathcal{K}_r} \geqslant h(x, \mathcal{K}_r(t)).$$

We say that $\mathcal{K}$ is a supersolution to the front propagation problem with initial position K_0 if $\mathcal{K}$ is a supersolution and $\widehat{\mathcal{K}}(0) \subset \overline{\mathbb{R}^N \setminus K_0}$.

3. Finally, we say that a tube $\mathcal{K}$ is a *viscosity solution* to the front propagation problem (with initial position K_0) if $\mathcal{K}$ is a sub- and a supersolution to the front propagation problem (with initial position K_0).

In [8] we have proved that for any initial position there is a maximal solution, with a closed graph, which contains any subsolution of the problem, as well as a minimal solution, which has an open graph, and is contained in any supersolution of the problem.

3. Capacity and capacity potential

Let Ω be an open bounded subset of $\mathbb{R}^N$. We denote by $\mathcal{C}_c^\infty(\Omega)$ the set of smooth functions with compact support in Ω, and by $H_0^1(\Omega)$ its closure in the H^1 norm. By convention, if $u \in H_0^1(\Omega)$, then we extend u by setting $u = 0$ on $\mathbb{R}^N \setminus \Omega$. Let S be as in (6). For an open bounded subset Ω of $\mathbb{R}^N$ such that $S \subset\subset \Omega$, the *capacity* of Ω with respect to S is defined by

$$\mathrm{cap}_S(\Omega) = \inf\left\{\int_{\Omega\setminus S} |\nabla\phi|^2 : \phi \in \mathcal{C}_c^\infty(\Omega),\ \phi = 1 \text{ on } S\right\}.$$

Since S is a fixed set in what follows, we will write $\mathrm{cap}(\Omega)$ instead of $\mathrm{cap}_S(\Omega)$.

Obviously $\mathrm{cap}(\Omega)$ is non-increasing with respect to Ω (for inclusion). Note that

$$\mathrm{cap}(\Omega) = \inf\left\{\int_{\Omega\setminus S} |\nabla v|^2 : v \in H_0^1(\Omega),\ v = 1 \text{ on } S\right\} \tag{7}$$

and the infimum is achieved for a unique $u \in H_0^1(\Omega)$, called the *capacity potential* of Ω with respect to S, such that $u = 1$ on S, and u is harmonic in $\Omega \setminus S$. If Ω has a $\mathcal{C}^{1,1}$ boundary, then it is known that the infimum is achieved by some $u \in \mathcal{C}^2(\Omega \setminus S) \cap \mathcal{C}^1(\overline{\Omega \setminus S})$ which is a classical solution to (3).

For any set E (not necessarily open) such that $S \subset\subset E$, we define a generalized capacity by

$$\mathrm{cap}(\overline{E}) = \sup\{\mathrm{cap}(\Omega) : E \subset\subset \Omega,\ \Omega \text{ open and bounded}\}.$$

With this definition, $\mathrm{cap}(\overline{E})$ is non-increasing with respect to E. Notice that this notion of capacity does not take into account "thin closed sets" in the sense that, if $\overline{F} = \overline{E}$, then $\mathrm{cap}(\overline{E}) = \mathrm{cap}(\overline{F})$ even when $|E \setminus F| \neq 0$. By construction, if E is open, then

$$\mathrm{cap}(\overline{E}) \leqslant \mathrm{cap}(E),$$

but equality does not hold in general. Nevertheless, there is equality if the boundary of the set is regular enough:

LEMMA 3.1 If Ω is an open bounded subset of $\mathbb{R}^N$, with $S \subset\subset \Omega$ and with a $\mathcal{C}^{1,1}$ boundary, then $\mathrm{cap}(\Omega) = \mathrm{cap}(\overline{\Omega})$.

Proof. We have to prove that $\mathrm{cap}(\overline{\Omega}) \geqslant \mathrm{cap}(\Omega)$. It is enough to show that, if

$$\Omega_n = \{y \in \mathbb{R}^N : d_\Omega(y) < 1/n\},$$

then $\mathrm{cap}(\Omega_n) \to \mathrm{cap}(\Omega)$ as $n \to +\infty$. Indeed, for n large enough, Ω_n also has a $\mathcal{C}^{1,1}$ boundary. Then from classical regularity arguments, the harmonic potential u_n to Ω_n converges to the capacity potential u of Ω in the $\mathcal{C}^{1,\alpha}$ norm, where $\alpha \in (0, 1)$, whence the result. □

LEMMA 3.2 Let E_n be a bounded sequence of subsets of $\mathbb{R}^N$, for which there exists some $r > 0$ with $S_r \subset E_n$ for any n, where

$$S_r = \{y \in \mathbb{R}^N : d_S(y) \leqslant r\}. \tag{8}$$

Denote by K the Kuratowski upper limit of the (E_n), that is,

$$K = \{x \in \mathbb{R}^N : \liminf_n d_{E_n}(x) = 0\}.$$

Then

$$\liminf_n \mathrm{cap}(\overline{E}_n) \geqslant \mathrm{cap}(\overline{K}).$$

Proof. Let Ω be any open bounded set such that $K \subset\subset \Omega$. Since (E_n) is bounded and has K as upper limit, the inclusion $E_n \subset \Omega$ holds for n large enough. Hence $\mathrm{cap}(\overline{E}_n) \geqslant \mathrm{cap}(\Omega)$ for every n. Therefore

$$\liminf_n \mathrm{cap}(\overline{E}_n) \geqslant \mathrm{cap}(\Omega).$$

The open set Ω being arbitrary, the desired conclusion holds. □

Let Ω be an open bounded subset of $\mathbb{R}^N$ with $S \subset\subset \Omega$. We denote by $H_0^1(\overline{\Omega})$ the intersection of the spaces $H_0^1(\Omega_n)$ where (Ω_n) is a decreasing sequence of open bounded sets such that $\Omega \subset\subset \Omega_n$ and $\overline{\Omega} = \bigcap_n \Omega_n$. One easily checks that $H_0^1(\overline{\Omega})$ does not depend on the sequence (Ω_n).

LEMMA 3.3 Assume that $|\partial\Omega| = 0$. Then

$$\mathrm{cap}(\overline{\Omega}) = \inf\left\{\int_{\Omega\setminus S} |\nabla v|^2 : v \in H_0^1(\overline{\Omega}),\ v = 1 \text{ on } S\right\},$$

and there is a unique $u \in H_0^1(\overline{\Omega})$ such that

$$u = 1 \text{ on } S \quad \text{and} \quad \int_{\mathbb{R}^N\setminus S} |\nabla u|^2 = \int_{\Omega\setminus S} |\nabla u|^2 = \mathrm{cap}(\overline{\Omega}).$$

Moreover, u is harmonic in $\Omega \setminus S$ and $|\{u > 0\} \setminus \Omega| = 0$.

DEFINITION 3.4 Such a function u is called the *capacity potential* of $\overline{\Omega}$ with respect to S.

REMARK 3.1 1. If $\partial\Omega$ is $\mathcal{C}^{1,1}$, then the capacity potential u of $\overline{\Omega}$ with respect to S is the (classical) solution of (3) and is equal to the (classical) capacity potential (7) of Ω.

2. In what follows, we study the energy of subsets $\Omega \supset\supset S$ which is defined as the sum of the capacity and the volume of Ω with respect to S (see (4)). This energy is well-defined for bounded sets $\Omega \supset\supset S$. That is why we assumed all the sets to be bounded. But let us mention that all classical results of this section hold upon replacing Ω, S bounded by $\Omega \setminus S$ bounded. We need this generalization in the proof of Lemma 4.5.

Proof of Lemma 3.3. The proof is easily obtained by approximation. By construction of $\text{cap}(\overline{\Omega})$, we can find a decreasing sequence of open bounded sets Ω_n such that

$$\Omega \subset\subset \Omega_n, \quad \bigcap_n \Omega_n = \overline{\Omega} \quad \text{and} \quad \text{cap}(\overline{\Omega}) = \lim_n \text{cap}(\Omega_n).$$

Let u_n be the (classical) capacity potential of Ω_n. From the maximum principle, the sequence (u_n) is decreasing, and converges to some u which is non-negative with support in $\overline{\Omega}$ and equals 1 on S. In particular, $\{u > 0\} \subset \Omega$ a.e. since $|\partial\Omega| = 0$. Furthermore, by a classical stability result, u is harmonic in Ω because so are the u_n. Since we can find a smooth function ϕ with compact support in Ω such that $\phi = 1$ on S, we have

$$\int_{\Omega_n \setminus S} |\nabla u_n|^2 \leqslant \int_{\Omega_n \setminus S} |\nabla \phi|^2 = \int_{\Omega \setminus S} |\nabla \phi|^2,$$

which proves that (u_n) is bounded in $H^1(\mathbb{R}^N)$. Thus the limit u belongs to $H^1(\mathbb{R}^N)$. Since $u_n \in H_0^1(\Omega_n)$ with $H_0^1(\Omega_{n+1}) \subset H_0^1(\Omega_n)$, u belongs to $H_0^1(\Omega_n)$ for any n. Therefore $u \in H_0^1(\overline{\Omega})$. In particular, the support of u lies in $\overline{\Omega} = \Omega$ a.e. So we have

$$\begin{aligned} \text{cap}(\overline{\Omega}) &= \lim_n \text{cap}(\Omega_n) = \lim_n \int_{\Omega_n \setminus S} |\nabla u_n|^2 \\ &= \liminf_n \int_{\mathbb{R}^N \setminus S} |\nabla u_n|^2 \geqslant \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 = \int_{\Omega \setminus S} |\nabla u|^2. \end{aligned} \tag{9}$$

For every n,

$$\begin{aligned} \text{cap}(\Omega_n) = \int_{\Omega_n \setminus S} |\nabla u_n|^2 &= \inf\left\{ \int_{\Omega_n \setminus S} |\nabla v|^2 : v \in H_0^1(\Omega_n),\ v = 1 \text{ on } S \right\} \\ &\leqslant \inf\left\{ \int_{\Omega \setminus S} |\nabla v|^2 : v \in H_0^1(\overline{\Omega}),\ v = 1 \text{ on } S \right\}, \end{aligned}$$

since $H_0^1(\overline{\Omega}) \subset H_0^1(\Omega_n)$. Letting n go to infinity, we obtain

$$\text{cap}(\overline{\Omega}) \leqslant \inf\left\{ \int_{\Omega \setminus S} |\nabla v|^2 : v \in H_0^1(\overline{\Omega}),\ v = 1 \text{ on } S \right\}.$$

From (9), we get equality in the above inequality and the fact that u is optimal. □

4. The discrete motions

Let us fix $h > 0$ which has to be understood as a time step. Let us recall that S is the closure of an open bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary. We introduce the function space

$$E(S) := \{u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : u = 1 \text{ on } S \text{ and } u \text{ has compact support}\}.$$

If S and S' are two compact subsets of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary such that $S \subset S'$, then we note that $E(S') \subset E(S)$.

For any bounded open subset Ω of $\mathbb{R}^N$ with $S \subset\subset \Omega$ we define the functional $J_h : E(S) \to \mathbb{R}$ by setting

$$J_h^S(\Omega, u) = \int_{\mathbb{R}^N \setminus S} \left(|\nabla u|^2 + \mathbf{1}_{\{u>0\}} \left(1 + \frac{1}{h} d_\Omega^s\right)_+ \right),$$

where d_Ω^s is the signed distance to Ω defined by (5), $\mathbf{1}_A$ denotes the indicator function of any set $A \subset \mathbb{R}^N$ and $r_+ = r \vee 0$ for any $r \in \mathbb{R}$. We write $J_h(\Omega, u)$ if there is no ambiguity about S.

Let us recall some existence and regularity results given in [3]:

PROPOSITION 4.1 (Alt and Caffarelli [3]) Let Ω be an open subset of $\mathbb{R}^N$ such that $\Omega \setminus S$ is bounded and with $S \subset\subset \Omega$. Then there is at least a minimizer $u \in E(S)$ to $J_h(\Omega, \cdot)$. Moreover, u is Lipschitz continuous and is harmonic in $\{u > 0\} \setminus S$. Finally, $\mathcal{H}^{N-1}(\partial\{u > 0\}) < +\infty$.

REMARK 4.1 We note that $S \subset\subset \{u > 0\}$ because u is Lipschitz continuous with $u = 1$ in S.

The existence of u and its Lipschitz continuity come from [3, Theorem 1.3 and Corollary 3.3]. The fact that u has a compact support is established in [3, Lemma 2.8], and its harmonicity in [3, Lemma 2.4]. The finiteness of $\mathcal{H}^{N-1}(\partial\{u > 0\})$ is given in [3, Theorem 4.5].

We are now ready to define the discrete motions.

Let $\Omega_0 \supset\supset S$ be a fixed initial condition. We define by induction the sequence (Ω_n^h) of open bounded subsets of $\mathbb{R}^N$ with $\Omega_n^h \supset\supset S$ by setting

$$\Omega_0^h := \Omega_0 \quad \text{and} \quad \Omega_{n+1}^h := \{u_n > 0\} \cup \{x \in \Omega_n^h : d_{\partial\Omega_n^h}(x) > h\},$$

where

$$u_n \in \underset{v \in E(S)}{\operatorname{argmin}}\, J_h^S(\Omega_n^h, v).$$

We call such a family of open sets a *discrete motion*. Of course, it is defined so that it converges to a solution of the front propagation problem (1) (see Theorem 5.2 and Remark 4.2).

In order to investigate the behavior of discrete motions, we need some properties of the minimizers of J_h.

LEMMA 4.2 Let Ω and u be as in Proposition 4.1. Let $\Omega' = \{u > 0\} \cup \hat{\Omega}_h$, where

$$\hat{\Omega}_h := \{y \in \Omega : d_{\partial\Omega}(y) > h\} = \{y \in \mathbb{R}^N : d_\Omega^s(y) < -h\}. \tag{10}$$

Then $|\partial\Omega'| = 0$ and u is the capacity potential of $\overline{\Omega'}$.

REMARK 4.2 We do not claim that u is positive in Ω'. For instance, consider a set Ω with two connected components Ω_1 and Ω_2 such that $S \subset\subset \Omega_1$. In this case, $u \equiv 0$ in Ω_2. Notice that this explains why we define $\Omega_{n+1}^h := \{u_n > 0\} \cup \{x \in \Omega_n^h : d_{\partial\Omega_n^h}(x) > h\}$. Adding the set $\{x \in \Omega_n^h : d_{\partial\Omega_n^h}(x) > h\}$ prevents the discrete motion from a sudden disappearance of a connected component. Indeed, the discrete motion is built in order to approach a solution of the front propagation problem (1), and a connected component which does not contain any part of the source is expected to move with constant normal velocity -1.

Proof of Lemma 4.2. First notice that $|\partial\Omega'| = 0$. Indeed, we already know that $|\partial\{u > 0\}| = 0$ (because its $\mathcal{H}^{N-1}$-measure is finite from Proposition 4.1). On the other hand, $\partial\hat{\Omega}_h \subset \{y \in \Omega : d_{\partial\Omega'}(y) = h\}$ also has a finite $\mathcal{H}^{N-1}$-measure thanks to [4, Lemma 2.4].

Let now $\epsilon > 0$ be fixed and set, for any $\alpha > 0$, $\Omega_\alpha = \{y \in \mathbb{R}^N : d_{\Omega'}(y) < \alpha\}$. The set Ω_α is open, bounded and satisfies $\Omega' \subset\subset \Omega_\alpha$. Moreover, since $\mathbf{1}_{\Omega_\alpha} \to \mathbf{1}_{\overline{\Omega'}}$ and Ω' is bounded with $|\partial\Omega'| = 0$, for $\alpha > 0$ enough small we have

$$\int_{\Omega_\alpha \setminus \Omega'} \left(1 + \frac{1}{h} d^s_{\partial\Omega}\right)_+ \leqslant \epsilon. \tag{11}$$

Let v be the capacity potential of Ω_α and set

$$v_k(x) = v(x) + \frac{1}{k} d_{\mathbb{R}^N \setminus \Omega_\alpha}(x) \quad \forall x \in \mathbb{R}^N.$$

Then (v_k) converges to v in $H^1(\mathbb{R}^N)$ and $|\Omega_\alpha \setminus \{v_k > 0\}| = 0$. Therefore

$$\begin{aligned} J_h(\Omega, v_k) &= \int_{\mathbb{R}^N \setminus S} \left(|\nabla v_k|^2 + \mathbf{1}_{\{v_k > 0\}} \left(1 + \frac{1}{h} d^s_\Omega\right)_+ \right) \\ &\underset{k}{\to} \operatorname{cap}(\Omega_\alpha) + \int_{\mathbb{R}^N \setminus S} \mathbf{1}_{\Omega_\alpha} \left(1 + \frac{1}{h} d^s_\Omega\right)_+ . \end{aligned}$$

Since $J_h(\Omega, v_k) \geqslant J_h(\Omega, u)$, from (11) we get

$$\begin{aligned} \operatorname{cap}(\overline{\Omega'}) &\geqslant \operatorname{cap}(\Omega_\alpha) \geqslant \lim_k J_h(\Omega, v_k) - \int_{\mathbb{R}^N \setminus S} \mathbf{1}_{\Omega_\alpha} \left(1 + \frac{1}{h} d^s_\Omega\right)_+ \\ &\geqslant J_h(\Omega, u) - \int_{\mathbb{R}^N \setminus S} \mathbf{1}_{\Omega_\alpha} \left(1 + \frac{1}{h} d^s_\Omega\right)_+ \geqslant \int_{\mathbb{R}^N \setminus S} \left(|\nabla u|^2 - \mathbf{1}_{\Omega_\alpha \setminus \{u > 0\}} \left(1 + \frac{1}{h} d^s_\Omega\right)_+ \right) \\ &\geqslant \int_{\mathbb{R}^N \setminus S} \left(|\nabla u|^2 - \mathbf{1}_{\Omega_\alpha \setminus \Omega'} \left(1 + \frac{1}{h} d^s_\Omega\right)_+ \right) \geqslant \int_{\mathbb{R}^N \setminus S} |\nabla u|^2 - \epsilon. \end{aligned}$$

Thus $\int_{\mathbb{R}^N \setminus S} |\nabla u|^2 \leqslant \operatorname{cap}(\overline{\Omega'})$, so u is the capacity potential of $\overline{\Omega'}$ by Lemma 3.3. □

Next we need to compare solutions to $J_h(\Omega, \cdot)$ for different S and Ω.

PROPOSITION 4.3 Let S_1 and S_2 be the closures of two open bounded subsets of $\mathbb{R}^N$ with $\mathcal{C}^2$ boundary, and let Ω_1 and Ω_2 be open bounded subsets of $\mathbb{R}^N$ such that $S_1 \subset\subset \Omega_1$ and $S_2 \subset\subset \Omega_2$. Let u_1 and u_2 be, respectively, minimizers of $J_h^{S_1}(\Omega_1, \cdot)$ and $J_h^{S_2}(\Omega_2, \cdot)$. If $S_1 \subset S_2$ and $\Omega_1 \subset \Omega_2$, then $u_1 \wedge u_2$ and $u_1 \vee u_2$ are, respectively, minimizers of $J_h^{S_1}(\Omega_1, \cdot)$ and $J_h^{S_2}(\Omega_2, \cdot)$.

REMARK 4.3 1. In particular, if $J_h^{S_2}(\Omega_2, \cdot)$ has a unique minimizer u_2, then $\{u_1 > 0\} \subset \{u_2 > 0\}$.
2. This proposition still holds true if we replace, for $i = 1, 2$, Ω_i, S_i bounded by $\Omega_i \setminus S_i$ bounded; see Remark 3.1 and Lemma 4.5.

Proof of Proposition 4.3. Let us set

$$I := J_h^{S_1}(\Omega_1, u_1 \wedge u_2) + J_h^{S_2}(\Omega_2, u_1 \vee u_2) - J_h^{S_1}(\Omega_1, u_1) - J_h^{S_2}(\Omega_2, u_2)$$
$$= \int_{\mathbb{R}^N \setminus S_1} \left(|\nabla(u_1 \wedge u_2)|^2 - |\nabla u_1|^2 + (\mathbf{1}_{\{u_1 \wedge u_2 > 0\}} - \mathbf{1}_{\{u_1 > 0\}}) \left(1 + \frac{1}{h} d^s_{\Omega_1}\right)_+ \right)$$
$$+ \int_{\mathbb{R}^N \setminus S_2} \left(|\nabla(u_1 \vee u_2)|^2 - |\nabla u_2|^2 + (\mathbf{1}_{\{u_1 \vee u_2 > 0\}} - \mathbf{1}_{\{u_2 > 0\}}) \left(1 + \frac{1}{h} d^s_{\Omega_2}\right)_+ \right).$$

Since $\Omega_1 \subset \Omega_2$ we have $d^s_{\Omega_2} \leqslant d^s_{\Omega_1}$ in $\mathbb{R}^N$. It follows that

$$I \leqslant \int_{\mathbb{R}^N \setminus S_2} (|\nabla(u_1 \wedge u_2)|^2 - |\nabla u_1|^2 + |\nabla(u_1 \vee u_2)|^2 - |\nabla u_2|^2)$$
$$+ \int_{S_2 \setminus S_1} (|\nabla(u_1 \wedge u_2)|^2 - |\nabla u_1|^2)$$
$$+ \int_{\mathbb{R}^N \setminus S_1} (\mathbf{1}_{\{u_1 \wedge u_2 > 0\}} - \mathbf{1}_{\{u_1 > 0\}}) \left(1 + \frac{1}{h} d^s_{\Omega_1}\right)_+$$
$$+ \int_{\mathbb{R}^N \setminus S_2} (\mathbf{1}_{\{u_1 \vee u_2 > 0\}} - \mathbf{1}_{\{u_2 > 0\}}) \left(1 + \frac{1}{h} d^s_{\Omega_1}\right)_+$$

Moreover, by classical results,

$$|\nabla(u_1 \wedge u_2)|^2 + |\nabla(u_1 \vee u_2)|^2 = |\nabla u_1|^2 + |\nabla u_2|^2 \quad \text{a.e. in } \mathbb{R}^N. \tag{12}$$

So we get

$$I \leqslant \int_{S_2 \setminus S_1} \left(|\nabla(u_1 \wedge u_2)|^2 - |\nabla u_1|^2 + (\mathbf{1}_{\{u_1 \wedge u_2 > 0\}} - \mathbf{1}_{\{u_1 > 0\}}) \left(1 + \frac{1}{h} d^s_{\Omega_1}\right)_+ \right).$$

But $u_1 \wedge u_2 = u_1$ on $S_2 \setminus S_1$, which gives $I \leqslant 0$, and thus

$$J_h^{S_1}(\Omega_1, u_1 \wedge u_2) + J_h^{S_2}(\Omega_2, u_1 \vee u_2) \leqslant J_h^{S_1}(\Omega_1, u_1) + J_h^{S_2}(\Omega_2, u_2). \tag{13}$$

Since u_1 and u_2 are minimizers we have

$$J_h^{S_1}(\Omega_1, u_1) \leqslant J_h^{S_1}(\Omega_1, u_1 \wedge u_2) \quad \text{and} \quad J_h^{S_2}(\Omega_2, u_2) \leqslant J_h^{S_2}(\Omega_2, u_1 \vee u_2). \tag{14}$$

The inequalities in (13) and (14) are therefore equalities. Hence $u_1 \wedge u_2$ and $u_1 \vee u_2$ are respectively minimizers of $J_h^{S_1}(\Omega_1, \cdot)$ and $J_h^{S_2}(\Omega_2, \cdot)$. □

We define the energy $\mathcal{E}(\overline{\Omega})$ by

$$\mathcal{E}(\overline{\Omega}) = |\Omega| + \mathrm{cap}(\overline{\Omega})$$

(cf. (4)).

LEMMA 4.4 Let (Ω_n^h) be a discrete motion with $|\partial \Omega_0^h| = 0$. Then the energy $\mathcal{E}(\overline{\Omega_n^h})$ is non-increasing with respect to n. More precisely,

$$\mathcal{E}(\overline{\Omega_{n+1}^h}) - \mathcal{E}(\overline{\Omega_n^h}) \leqslant \int_{\mathbb{R}^N} (\mathbf{1}_{\Omega_n^h \setminus \{d^s_{\partial\Omega_n^h} < -h\}} - \mathbf{1}_{\{u_n > 0\} \setminus \{d^s_{\partial\Omega_n^h} < -h\}}) \frac{1}{h} d^s_{\Omega_n^h} \leqslant 0,$$

where u_n is a minimizer for $J_h(\Omega_n^h, \cdot)$.

Proof. Let us fix n. In order to simplify the notations, let us set

$$\Omega := \Omega_n^h, \qquad \hat{\Omega}_h := \{x \in \Omega : d_\Omega^s(x) < -h\} = \{x \in \Omega : d_{\partial\Omega}(x) > h\} .$$

Let u_0 be the capacity potential of $\overline{\Omega}$ and u be a minimizer of $J_h(\Omega, \cdot)$. We finally set $\Omega' := \Omega_{n+1}^h = \{u > 0\} \cup \hat{\Omega}_h$. Recall that $\Omega' \in \mathcal{D}$ and $|\partial\Omega'| = 0$: indeed, this is true for $n = 0$ from the assumption and by Lemma 4.2 for $n \geqslant 1$. With these notations we have to prove that

$$\mathcal{E}(\overline{\Omega'}) \leqslant \mathcal{E}(\overline{\Omega}).$$

For this we introduce for any $k \geqslant 1$ the function u_k defined by

$$u_k(x) = \begin{cases} u_0(x) + \frac{1}{k} d_{\partial\Omega}(x) & \text{if } x \in \Omega, \\ u_0(x) & \text{otherwise.} \end{cases}$$

Then (u_k) converges to u_0 in $H^1(\mathbb{R}^N)$ and $\{u_k > 0\} = \Omega$ a.e. because $\{u_0 > 0\} \subset \overline{\Omega}$ and $|\partial\Omega| = 0$. Hence

$$\begin{aligned} \lim_k J_h(\Omega, u_k) &= \lim_k \int_{\mathbb{R}^N \setminus S} \left(|\nabla u_k|^2 + \mathbf{1}_{\{u_k > 0\}} \left(1 + \frac{1}{h} d_\Omega^s \right)_+ \right) \\ &= \operatorname{cap}(\overline{\Omega}) + \int_{\mathbb{R}^N \setminus S} \mathbf{1}_\Omega \left(1 + \frac{1}{h} d_\Omega^s \right)_+ \\ &= \mathcal{E}(\overline{\Omega}) - |\Omega| + \int_{\Omega \setminus \hat{\Omega}_h} \left(1 + \frac{1}{h} d_\Omega^s \right) = \mathcal{E}(\overline{\Omega}) + \int_{\Omega \setminus \hat{\Omega}_h} \frac{1}{h} d_\Omega^s - |\hat{\Omega}_h|. \end{aligned}$$

On the other hand, since $\operatorname{cap}(\overline{\Omega'}) = \int_{\mathbb{R}^N \setminus S} |\nabla u|^2$ from Lemma 4.2, and since $|\overline{\Omega'}| = |\Omega'|$, we also have

$$\begin{aligned} J_h(\Omega, u) &= \int_{\mathbb{R}^N \setminus S} \left(|\nabla u|^2 + \mathbf{1}_{\{u > 0\}} \left(1 + \frac{1}{h} d_\Omega^s \right)_+ \right) \\ &= \mathcal{E}(\overline{\Omega'}) - |\Omega'| + \int_{\{u>0\} \setminus \hat{\Omega}_h} \left(1 + \frac{1}{h} d_\Omega^s \right) = \mathcal{E}(\overline{\Omega'}) + \int_{\{u>0\} \setminus \hat{\Omega}_h} \frac{1}{h} d_\Omega^s - |\hat{\Omega}_h|. \end{aligned}$$

Noting that $J_h(\Omega, u) \leqslant J_h(\Omega, u_k)$, we get the desired claim. □

Next we show that the solution does not blow up when h becomes small.

LEMMA 4.5 Let $R > 0$ and $r_0 \in (0, R/2^{1/(N-2)})$ be fixed. Let us also fix M such that $\sqrt{1 + M} \geqslant 4(N-2)/r_0$. Then there is some $h_0 = h_0(N, r_0, R, M)$ such that, for any $h \in (0, h_0)$ and $r \in (r_0, R/2^{1/(N-2)})$, for any $\Omega \in \mathcal{D}$ open bounded, for any $x \notin \overline{\Omega}$ with $r \leqslant d_\Omega(x)$, $R \leqslant d_S(x)$ and for any u minimizer of $J_h(\Omega, \cdot)$, we have

$$d_{\{u>0\} \cup \hat{\Omega}_h}(x) \geqslant r - Mh,$$

where $\hat{\Omega}_h$ is defined by (10).

Proof. The idea is to compare the solution with radial ones. For simplicity we assume that $N \geqslant 3$, the computation in the case $N = 2$ being similar. We also suppose without loss of generality that $x = 0$.

Let us first investigate the problem of minimizing $J_h^{B_R^c}(B_r^c, \cdot)$, where $B_r = B(0, r)$ and $B_R = B(0, R)$. Notice that neither the source B_R^c nor the subset B_r^c is bounded, but $B_r^c \setminus B_R^c = B_R \setminus B_r$ is bounded, so the previous results on the minimization problem apply (see Remark 3.1). Standard symmetrization arguments show that a minimizer v of $J_h^{B_R^c}(B_r^c, \cdot)$ must be radially symmetric. For $\rho \in (0, R)$, let us denote by v_ρ the (radial) harmonic function which vanishes on ∂B_ρ and is equal to 1 on ∂B_R. We also set $J_h(\rho) := J_h^{B_R^c}(B_r^c, v_\rho)$. Notice that a minimizer of $J_h^{B_R^c}(B_r^c, \cdot)$ has to be either of the form v_ρ with ρ a minimizer of $J_h(\cdot)$, or constant equal to $v_0 := 1$. Let us fix h_0 small enough that $r + h < R$ for $h \in (0, h_0)$. We have

$$J_h(0^+) = J_h^{B_R^c}(B_r^c, v_0) = \frac{\alpha_{N-1}(r+h)^{N+1}}{hN(N+1)},$$

where α_{N-1} is the volume of the unit sphere of $\mathbb{R}^N$. For $J_h(\rho)$ with $\rho > 0$, we distinguish two cases. If $r + h < \rho < R$, then

$$J_h(\rho) = \frac{\alpha_{N-1}(N-2)}{\rho^{2-N} - R^{2-N}}.$$

If $0 < \rho \leqslant r + h$, then

$$\frac{J_h(\rho)}{\alpha_{N-1}} = \frac{N-2}{\rho^{2-N} - R^{2-N}} + \frac{1}{h}\left(\frac{(r+h)^{N+1}}{N(N+1)} + \frac{\rho^{N+1}}{N+1} - \frac{(r+h)\rho^N}{N}\right).$$

We show that v_0 cannot be a minimizer by comparing $J_h(0^+)$ with $J_h(\rho)$ for $0 < \rho \leqslant r + h$. Choosing $\rho = \beta\sqrt{h}$ with $\beta > 0$, we have

$$\begin{aligned}\frac{1}{\alpha_{N-1}}(J_h(\rho) - J_h(0) &= \rho^{N-2}\left(\frac{N-2}{1-(\rho/R)^{N-2}} + \frac{\rho^3}{h(N+1)} - \frac{(r+h)\rho^2}{hN}\right)\\ &\leqslant \rho^{N-2}\left(\frac{N-2}{1-\beta^{N-2}h^{(N-2)/2}/R^{N-2}} + \frac{\beta^3 h^{1/2}}{N+1} - \frac{r\beta^2}{N}\right).\end{aligned} \tag{15}$$

Recalling that $r_0 \in (0, R/2^{1/(N-2)})$ is fixed, we choose

$$\beta > \frac{N(2(N-2)+1)}{r_0} \tag{16}$$

and then $h_0 = h_0(N, \beta, r_0, R) > 0$ small enough that

$$1 - \frac{\beta^{N-2}h_0^{(N-2)/2}}{R^{N-2}} > \frac{1}{2} \quad \text{and} \quad \frac{\beta^3 h_0^{1/2}}{N+1} < 1. \tag{17}$$

For all $h \in (0, h_0)$, we see that (15) is negative, which proves that v_0 is not a minimizer.

Therefore minimizers have to be of the form v_ρ for some $\rho \in (0, R)$. On $(r + h, R)$, $J_h(\rho)$ is increasing. For $\rho \in (0, r + h)$, we have

$$\frac{J_h'(\rho)}{\alpha_{N-1}} = \frac{(N-2)^2\rho^{1-N}}{(\rho^{2-N} - R^{2-N})^2} + \frac{\rho^N}{h} - \frac{(r+h)\rho^{N-1}}{h}.$$

The stationary points of J_h on $(0, r+h]$ satisfy

$$f(\rho) := \frac{(N-2)^2}{[\rho(1-(\rho/R)^{N-2})]^2} - \frac{1}{h}(r+h-\rho) = 0. \tag{18}$$

Notice that $\rho \mapsto f(\rho)$ is convex on $(0, r+h]$ and tends to $+\infty$ as $\rho \to 0^+$ and as $\rho \to R^-$. If we find some value ρ for which $f(\rho)$ is negative, then there are exactly two solutions to (18).

For this, let us choose $\rho = \beta\sqrt{h}$ with $\beta > 0$. Then

$$hf(\beta\sqrt{h}) = \frac{(N-2)^2}{\beta(1-(\beta h^{1/2}/R)^{N-2})^{N-2})^2} - r - h + \beta h^{1/2}.$$

Choosing $\beta > 0$ satisfying (16) and

$$\beta > 4(N-2)/r_0^{1/2}$$

and h_0 satisfying (17) and

$$\beta h_0^{1/2} < r_0/2, \tag{19}$$

we find that $f(\beta\sqrt{h}) < 0$ for $h \in (0, h_0)$.

Let us fix $h \in (0, h_0)$ and let ρ_1 and ρ_2 be respectively the smallest and largest solutions to (18). By the above arguments, $\rho_1 \leqslant \beta\sqrt{h} \leqslant \rho_2$, where β is defined as above. Since $J_h'(\rho) = \alpha_{N-1}\rho^{N-1}f(\rho)$, we have

$$\begin{aligned}
J_h''(\rho_1) &= \alpha_{N-1}\rho_1^{N-1}f'(\rho_1)\\
&= \alpha_{N-1}\rho_1^{N-1}\left[\frac{-2(N-2)^2(1-(N-1)(\rho_1/R)^{N-2})}{(\rho_1(1-(\rho_1/R)^{N-2}))^3} + \frac{1}{h}\right]\\
&\leqslant \alpha_{N-1}\rho_1^{N-1}\left[-\frac{2(N-2)^2}{\rho_1^3}\left(1-(N-1)\left(\frac{\rho_1}{R}\right)^{N-2}\right) + \frac{\beta^2}{\rho_1^2}\right].
\end{aligned}$$

If we choose $h_0 > 0$ satisfying (17), (19) and furthermore

$$(N-1)\left(\frac{\beta h_0^{1/2}}{R}\right)^{N-2} < \frac{1}{2} \quad \text{and} \quad h_0^{1/2} < \frac{N-2}{\beta^3}, \tag{20}$$

we deduce that $J_h''(\rho_1) < 0$ for $h \in (0, h_0)$ and ρ_1 is not a minimum to J_h. Therefore, J_h is increasing on $(0, \rho_1)$, decreasing on (ρ_1, ρ_2) and increasing on (ρ_2, R). The minimum is achieved at $\rho = \rho_2$.

Let us now estimate ρ_2. We suppose that h_0 satisfies (17), (19), (20) and

$$h_0 \leqslant \frac{r_0}{2M} \quad \text{where} \quad 1 + M \geqslant \frac{16(N-2)^2}{r_0^2}.$$

Then, for all $h \in (0, h_0)$ and $r \in (r_0, R/2^{1/(N-2)})$, we have $r - Mh \geqslant r_0/2 > 0$ and we compute

$$\begin{aligned}
f(r-Mh) &= \frac{(N-2)^2}{(r-Mh)^2(1-((r-Mh)/R)^{N-2})^2} - (1+M)\\
&\leqslant \frac{4(N-2)^2}{(r-Mh)^2} - (1+M) \leqslant 0.
\end{aligned}$$

Therefore $\rho_2 \geqslant r - Mh$.

To summarize, we know that, setting $h_0 = h_0(N, r_0, R, M)$ small enough, for all $h \in (0, h_0)$ and $r \in (r_0, R/2^{1/(N-2)})$, the problem of minimizing $J_h^{B_R^c}(B_r^c, \cdot)$ has a unique solution v_{ρ_2}, which is radially symmetric and such that $\rho_2 \geqslant r - Mh$.

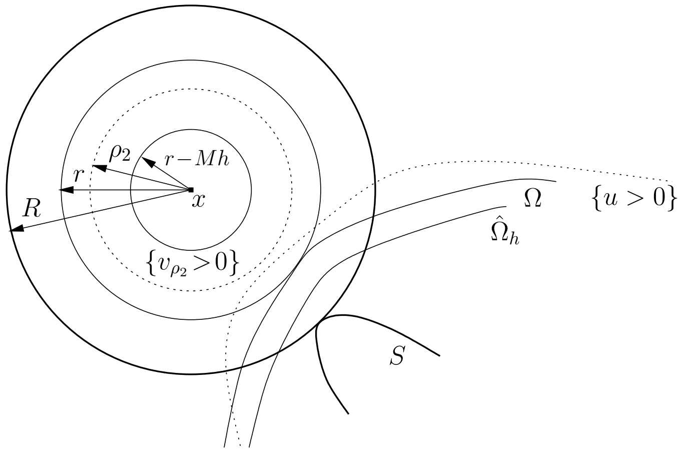

FIG. 1. Illustration of the proof of Lemma 4.5.

Let now $\Omega \in \mathcal{D}$, $x \notin \overline{\Omega}$ with $R \leqslant d_S(x)$, $r \leqslant d_\Omega(x)$, and let u be a minimizer of $J_h(\Omega, \cdot)$. Since $S \subset B_R^c(x)$ and $\Omega \subset B_r^c(x)$, Proposition 4.3 states that $\{u > 0\} \subset \{v_{\rho_2} > 0\} \subset B_{r-Mh}^c(x)$ (see Figure 1). Since $\hat{\Omega}_h \subset \Omega \subset B_r^c$, we conclude that $d_{\{u>0\}\cup\hat{\Omega}_h}(x) \geqslant r - Mh$. □

Finally, we show that the set $\{u > 0\}$ satisfies some inequalities in the viscosity sense. Here again the regularity results of Alt and Caffarelli [3] play a crucial role. Let Σ be an open set with $\mathcal{C}^{1,1}$ boundary such that $S \subset\subset \Sigma$ and $\Sigma \setminus S$ is bounded. We denote by u_S^Σ the (classical) solution to (3) (replacing Ω by Σ), i.e., the capacity potential of Σ with respect to S.

LEMMA 4.6 Let Ω be a bounded open subset of $\mathbb{R}^N$ with $S \subset\subset \Omega$ and u be a minimizer of $J_h(\Omega, \cdot)$. Set

$$\hat{\Omega}_h = \{x \in \Omega : d_{\partial\Omega}(x) > h\} \quad \text{and} \quad \Omega' = \{u > 0\} \cup \Omega_h.$$

Let Σ be an open bounded subset of $\mathbb{R}^N$ with $\mathcal{C}^{1,1}$ boundary.

1. [Outward estimate] Suppose that Σ is such that

$$\{u > 0\} \subset \Sigma \quad \text{and} \quad \exists x \in \partial\Sigma \cap \partial\{u > 0\}.$$

Then

$$|\nabla u_S^\Sigma(x)| \geqslant \left(1 + \frac{1}{h} d_\Omega^s(x)\right)_+^{1/2}.$$

2. [Inward estimate] Now assume that Σ is such that

$$S \subset\subset \Sigma, \quad \Sigma \subset \Omega' \quad \text{and} \quad \exists x \in \partial\Sigma \cap \partial\Omega'.$$

Then

$$|\nabla u_S^\Sigma(x)| \leqslant \left(1 + \frac{1}{h} d_\Omega^s(x)\right)_+^{1/2}.$$

Proof. Let us set $g_\Omega(x) = (1 + d_\Omega^s(x)/h)_+$. We first prove the outward estimate. From [3, Lemma 4.10] we have

$$\limsup_{x' \to x,\, x' \in \{u>0\}} \frac{u(x')}{d_B(x')} \geqslant \sqrt{g_\Omega(x)}$$

for any ball B contained in $\{u = 0\}$ and tangent to $\{u > 0\}$ at x. Let ν be the outward unit normal to Σ at x, and $r > 0$ be such that the ball $B := B(x + r\nu, r)$ is tangent to Σ at x. Then B is also tangent to $\{u > 0\}$ at x. Since by the maximum principle, $u \leqslant u_S^\Sigma$, we have

$$|\nabla u_S^\Sigma(x)| = \limsup_{x' \to x,\, x' \in \{u>0\}} \frac{u_S^\Sigma(x')}{d_B(x')} \geqslant \limsup_{x' \to x,\, x' \in \{u>0\}} \frac{u(x')}{d_B(x')} \geqslant \sqrt{g_\Omega(x)}.$$

We now turn to the proof of the inward estimate. We first prove that $u_S^\Sigma \leqslant u$ in $\{u_S^\Sigma > 0\}$. Indeed, from Lemma 4.2, u is the capacity potential of $\overline{\Omega'}$. In particular, u is harmonic in $\Omega' \setminus S \supset \Sigma \setminus S$, $u = u_S^\Sigma$ on ∂S and $0 = u_S^\Sigma \leqslant u$ on $\partial\{u_S^\Sigma > 0\}$. Hence $u_S^\Sigma \leqslant u$ in $\{u_S^\Sigma > 0\}$. Let us note that $u = 0$ on $\partial\Omega'$. Therefore $u(x) = u_S^\Sigma(x) = 0$.

We now consider two cases. If $x \notin \partial\{u > 0\}$, then $x \in \partial\hat{\Omega}_h$; thus $d_\Omega^s(x) = -h$ and $g_\Omega(x) = 0$. But $0 \leqslant u_S^\Sigma \leqslant u = 0$ in a neighborhood of x so that $\nabla u_S^\Sigma(x) = 0$. Therefore

$$|\nabla u_S^\Sigma(x)| = 0 = g_\Omega(x).$$

Let us now consider the case $x \in \partial\{u > 0\}$. Then [3, Theorem 6.3] states that

$$\sup_{B(x,r)} |\nabla u| \leqslant \sqrt{g_\Omega(x)} + m(r),$$

where $m(r) \to 0$ as $r \to 0^+$. Since we want to prove that $|\nabla u_S^\Sigma(x)| \leqslant \sqrt{g_\Omega(x)}$, we can assume without loss of generality that $\nabla u_S^\Sigma(x) \neq 0$. Let ν be the outward unit normal to Σ at x. Since $\nu = -\nabla u_S^\Sigma(x)/|\nabla u_S^\Sigma(x)|$, for $r > 0$ sufficiently small, the segment $]x, x - r\nu[$ is contained in Σ and in $\{u_S^\Sigma > 0\}$, and thus in $\{u > 0\}$. So u is smooth at each point of this segment. Since moreover $u \geqslant u_S^\Sigma$, we have, for some $\xi \in (x, x - r\nu)$,

$$u_S^\Sigma(x - r\nu) \leqslant u(x - r\nu) = u(x) + \langle \nabla u(\xi), -r\nu \rangle \leqslant r(\sqrt{g_\Omega(x)} + m(r)).$$

Therefore

$$|\nabla u_S^\Sigma(x)| = \lim_{r \to 0^+} \frac{u_S^\Sigma(x - r\nu)}{r} \leqslant \sqrt{g_\Omega(x)}. \qquad \square$$

5. Discrete motions and viscosity solutions

Let us fix Ω_0 open and bounded such that $S \subset\subset \Omega_0$. Let $(\Omega_n^h)_n$ be a discrete motion with $\Omega_0^h = \Omega_0$.

Let us now introduce a lower and upper envelope for the sequences $(\Omega_n^h)_n$ as the time step h tends to 0^+: the *upper envelope* $\mathcal{K}^*$ is

$$\mathcal{K}^*(t) := \left\{ x \in \mathbb{R}^N : \begin{array}{l} \exists h_k \to 0^+,\ n_k \to +\infty,\ x_k \in \Omega_{n_k}^{h_k} \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\}, \tag{21}$$

while the *lower envelope* $\mathcal{K}_*$ is defined by its complement:

$$\mathbb{R}^N \setminus \mathcal{K}_*(t) = \left\{ x \in \mathbb{R}^N : \begin{array}{l} \exists h_k \to 0^+,\ n_k \to +\infty,\ x_k \notin \Omega_{n_k}^{h_k} \\ \text{with } x_k \to x \text{ and } h_k n_k \to t \end{array} \right\}. \tag{22}$$

LEMMA 5.1 The set $\mathcal{K}^*$ is closed while $\mathcal{K}_*$ is open. Moreover, the maps $t \mapsto \mathcal{K}^*(t)$ and $t \mapsto \widehat{\mathcal{K}}_*(t)$ are left lower semicontinuous on $(0, +\infty)$.

Proof. The fact that $\mathcal{K}^*$ is closed comes from its construction since the upper limit of sets is always closed. The argument works in a symmetric way for $\mathcal{K}_*$.

We now prove that $t \mapsto \mathcal{K}^*(t)$ is left lower semicontinuous on $(0, +\infty)$ (see Section 2 for the definition). We proceed by contradiction supposing there exist $t > 0$, $x \in \mathcal{K}^*(t)$, $\rho > 0$ and a sequence $t_p \to t^-$ such that $B(x, \rho) \cap \mathcal{K}^*(t_p) = \emptyset$. Therefore $d_{\mathcal{K}^*(t_p)}(x) \geqslant \rho > 0$ for all p.

Set $R = d_S(x)$ and choose $r_0 < \min\{\rho, R/2^{1/(N-2)}\}$ and $M > 0$ with $\sqrt{1+M} \geqslant 8(N-2)/r_0$. Then Lemma 4.5 states that there is some $h_0 = h_0(N, r_0, R, M)$ with the following property: for any $r \in (r_0/2, R/2^{1/N-2})$ and $h \in (0, h_0)$, for any Ω with $r \leqslant d_\Omega(x)$ and for any minimizer u of $J_h(\Omega, \cdot)$, we have $d_{\{u>0\}\cup\hat{\Omega}_h}(x) \geqslant r - Mh$, where $\hat{\Omega}_h = \{d_\Omega^s < -h\}$.

For $h \in (0, h_0)$, let $n_h = [t_p/h]$ be the integer part of t_p/h. From the definition of $\mathcal{K}^*(t_p)$ and r_0, we can find some $h_1 \in (0, h_0)$ such that $d_{\Omega_{n_h}^h}(x) \geqslant r_0$ for any $h \in (0, h_1)$. We are going to prove by induction that

$$d_{\Omega_{n_h+kh}^h}(x) \geqslant r_0 - Mkh \quad \text{for all } k \in \{0, \ldots, k_0^h\}, \tag{23}$$

where $k_0^h = [r_0/(2Mh)]$. Indeed, inequality (23) holds for $k = 0$. Assume that it holds for some $k < k_0^h$. Let u be a minimizer for $J_h(\Omega_{n_h+kh}^h, \cdot)$ and define

$$\Omega_{n_h+(k+1)h}^h = \{u > 0\} \cup \{y \in \Omega_{n_h+kh}^h : d_{\Omega_{n_h+kh}^h}(y) > h\}.$$

Then since $r_0 - Mkh \geqslant r_0/2$ and $r_0 - Mkh \leqslant r_0 \leqslant R/2^{1/(N-2)}$, Lemma 4.5 recalled above implies that

$$d_{\Omega_{n_h+(k+1)h}^h}(x) \geqslant r_0 - Mkh - Mh.$$

So (23) is proved.

Let us set $\tau = r_0/(4M)$ and fix $s \in (0, \tau)$. Let (k_h) be such that $k_h h \to s$ as $h \to 0^+$. We notice that $k_h \in \{0, \ldots, k_0^h\}$ for h sufficiently small. Letting $h \to 0^+$ in inequality (23) for any such (k_h) implies that

$$d_{\mathcal{K}^*(t_p+s)}(x) \geqslant r_0 - Ms \geqslant r_0/2 > 0. \tag{24}$$

Since τ does not depend on x and t_p and since $t_p \to t^-$, for p large enough, we have $s = t - t_p \leqslant \tau$. Therefore, from (24), we obtain $d_{\mathcal{K}^*(t)}(x) = d_{\mathcal{K}^*(t_p+s)}(x) \geqslant r_0/2 > 0$, which contradicts the assumption $x \in \mathcal{K}^*(t)$.

The proof of the left lower semicontinuity of $\widehat{\mathcal{K}}_*$ is simpler. As above, we proceed by contradiction supposing that there exists $x \in \widehat{\mathcal{K}}_*(t)$ for $t > 0$ and a sequence $t_p \to t^-$ with $d_{\widehat{\mathcal{K}}_*(t_p)}(x) \geqslant \rho > 0$ for all p. From the definition of $\Omega_{n_h+1}^h$, for (n_h) such that $n_h h \to t_p$ and h sufficiently small, we have $B_{\rho/2}(x) \subset \Omega_{n_h}^h$. From the definition of $\Omega_{n_h+1}^h$, we have therefore

$$B_{\rho/2-h}(x) \subset \{y \in \Omega_{n_h}^h : d_{\partial\Omega_{n_h}^h}(x) > h\} \subset \Omega_{n_h+1}^h.$$

By induction we prove in a similar way that, for any $k \leqslant \rho/(4h)$,

$$B_{\rho/2-kh}(x) \subset \{y \in \Omega^h_{n_h+(k-1)h} : d_{\partial\Omega^h_{n_h+(k-1)h}}(x) > h\} \subset \Omega^h_{n_h+k}.$$

Letting now $h \to 0^+$ we get

$$B_{\rho/4}(x) \cap \widehat{\mathcal{K}}_*(t_p + s) = \emptyset \quad \text{for all } s \in [0, \rho/4].$$

Since ρ is independent of p, we get a contradiction by taking p so large that $t - t_p = s \leqslant \rho/4$. $\square$

THEOREM 5.2 The tube $\mathcal{K}^*$ (respectively $\mathcal{K}_*$) is a viscosity subsolution (respectively supersolution) to the front propagation problem $V = h(x, \Omega)$, where

$$h(x, \Omega) = -1 + \bar{h}(x, \Omega)$$

and $\bar{h}$ is defined by (2).

Proof. Let us set $\Omega^h := \bigcup_n \{nh\} \times \Omega^h_n$. Let $(t_0, x_0) \in \mathcal{K}^*$ with $t_0 > 0$ be such that there is a smooth regular tube $\mathcal{K}_r$ with $\mathcal{K}^* \subset \mathcal{K}_r$ and $x_0 \in \partial\mathcal{K}_r(t_0)$. Without loss of generality we can assume that $\mathcal{K}^* \cap \partial\mathcal{K}_r = \{(t_0, x_0)\}$. Then by standard stability arguments (see [7]), one can find a sequence of smooth regular tubes $\mathcal{K}^k_r$ converging to $\mathcal{K}_r$ in the $\mathcal{C}^{1,\mathrm{b}}$ sense (see Section 2 for the definition), and sequences $h_k \to 0$ and $n_k \to +\infty$ such that $\Omega^{h_k} \subset \mathcal{K}^k_r$, $(n_k h_k, x_k) \to (t_0, x_0)$, $x_k \in \partial\Omega^{h_k}_{n_k}$ and $x_k \in \partial\mathcal{K}^k_r(n_k h_k)$.

Let u be a minimizer of $J_{h_k}(\Omega^{h_k}_{n_k-1}, \cdot)$. By definition of the discrete motion, we have

$$\Omega^{h_k}_{n_k} = \{u > 0\} \cup \{y \in \Omega^{h_k}_{n_k-1} : d^s_{\Omega^{h_k}_{n_k-1}}(y) < -h_k\}. \tag{25}$$

Let $v_k := u_S^{\mathcal{K}^k_r(n_k h_k)}$ be the capacity potential of $\mathcal{K}^k_r(n_k h_k)$.

Let us first assume that $x_k \in \partial\{u > 0\}$ for some subsequence of (x_k) (still denoted by (x_k)). The case $x_k \in \mathrm{int}\{u = 0\}$ for any k is treated later. From the discrete viscosity condition in Lemma 4.6 and the inclusion $\Omega^{h_k}_{n_k-1} \subset \mathcal{K}^k_r((n_k - 1)h_k)$, we know that

$$|\nabla v_k(x_k)| \geqslant \left(1 + \frac{1}{h_k} d^s_{\Omega^{h_k}_{n_k-1}}(x_k)\right)^{1/2}_+ \geqslant \left(1 + \frac{1}{h_k} d^s_{\mathcal{K}^k_r((n_k-1)h_k)}(x_k)\right)^{1/2}_+.$$

Hence

$$\frac{1}{h_k} d^s_{\mathcal{K}^k_r((n_k-1)h_k)}(x_k) \leqslant -1 + |\nabla v_k(x_k)|^2. \tag{26}$$

Let us now recall that the normal velocity of $\mathcal{K}^k_r$ at a point $(t, x) \in \partial\mathcal{K}^k_r$ is given by $-\frac{\partial}{\partial t} d^s_{\mathcal{K}^k_r(t)}(x)$. Since $x_k \in \partial\mathcal{K}^k_r(n_k h_k)$, since $(n_k h_k, x_k) \to (t_0, x_0)$ and since $\mathcal{K}^k_r$ converges to $\mathcal{K}_r$, we have

$$d^s_{\mathcal{K}^k_r((n_k-1)h_k)}(x_k) = d^s_{\mathcal{K}^k_r(n_k h_k)}(x_k) - h_k \frac{\partial}{\partial t} d^s_{\mathcal{K}^k_r(n_k h_k)}(x_k) + h_k \epsilon(k) = h_k V^{\mathcal{K}_r}_{(t_0,x_0)} + h_k \epsilon(k),$$

where $\epsilon(k) \to 0$ as $k \to +\infty$ and $V^{\mathcal{K}_r}_{(t_0,x_0)}$ is the normal velocity of $\mathcal{K}_r$ at (t_0, x_0). From (26) we get, for k large enough,

$$h(x_k, \mathcal{K}^k_r(n_k h_k)) = -1 + |\nabla v_k(x_k)|^2 \geqslant V^{\mathcal{K}_r}_{(t_0,x_0)} + \epsilon(k).$$

Letting $k \to +\infty$, we obtain

$$h(x_0, \mathcal{K}_r(t_0)) = \lim_k h(x_k, \mathcal{K}_r^k(n_k h_k)) \geqslant V_{(t_0,x_0)}^{\mathcal{K}_r}.$$

The above equality is a straightforward application of [13, Theorem 8.33] since $\mathcal{K}_r^k$ converges to $\mathcal{K}_r$ in the $\mathcal{C}^{1,\mathrm{b}}$ sense (see Section 2 for the definition).

We now assume that $x_k \in \mathrm{int}\{u = 0\}$ for any k. Then from (25) we have

$$d^s_{\Omega^{h_k}_{n_k-1}}(x_k) = -d_{\partial\Omega^{h_k}_{n_k-1}}(x_k) = -h_k.$$

Arguing as above we get

$$-h_k = d^s_{\Omega^{h_k}_{n_k-1}}(x_k) \geqslant d^s_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k) = h_k V_{(t_0,x_0)}^{\mathcal{K}_r} + h_k \epsilon(k),$$

where $\epsilon(k) \to 0$. Dividing by h_k and letting $k \to +\infty$ gives

$$V_{(t_0,x_0)}^{\mathcal{K}_r} \leqslant -1 \leqslant -1 + |\nabla u_S^{\mathcal{K}_r(t_0)}(x_0)|^2 = h(x_0, \mathcal{K}_r(t_0)).$$

So we have finally proved that $\mathcal{K}^*$ is a subsolution.

We now show that $\mathcal{K}_*$ is a supersolution. The proof starts exactly as above: if there is a smooth regular tube $\mathcal{K}_r$ with $\mathcal{K}_r \subset \mathcal{K}_*$ and some $(t_0, x_0) \in \partial\mathcal{K}_*$ with $t_0 > 0$ and $x_0 \in \partial\mathcal{K}_r(t_0)$, then one can find a sequence of smooth regular tubes $\mathcal{K}_r^k$ converging to $\mathcal{K}_r$ in the $\mathcal{C}^{1,\mathrm{b}}$ sense and sequences $h_k \to 0$ and $n_k \to +\infty$ such that $\mathcal{K}_r^k(nh_k) \subset \Omega_n^{h_k}$ for any n, $(n_k h_k, x_k) \to (t_0, x_0)$, and $x_k \in \partial\Omega^{h_k}(n_k h_k) \cap \partial\mathcal{K}_r^k(n_k h_k)$. Let u be a minimizer of $J_{h_k}(\Omega^{h_k}_{n_k-1}, \cdot)$. Then (25) holds for $\Omega^{h_k}_{n_k}$.

Then using Lemma 4.6 we get

$$|\nabla v_k(x_k)| \leqslant \left(1 + \frac{1}{h_k} d^s_{\Omega^{h_k}_{n_k-1}}(x_k)\right)_+^{1/2} \leqslant \left(1 + \frac{1}{h_k} d_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k)\right)_+^{1/2}, \tag{27}$$

where $v_k := u_S^{\mathcal{K}_r^k(n_k h_k)}$ is the capacity potential of $\mathcal{K}_r^k(n_k h_k)$ with respect to S. Since $x_k \in \partial\Omega^{h_k}(n_k h_k)$, we have from (25) that $d^s_{\Omega^{h_k}(n_k h_k)}(x_k) \geqslant -h_k$. Therefore inequality (27) can also be written as

$$\frac{1}{h_k} d_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k) \geqslant -1 + |\nabla v_k(x_k)|^2.$$

As before we have

$$d^s_{\mathcal{K}_r^k((n_k-1)h_k)}(x_k) = h_k V_{(t_0,x_0)}^{\mathcal{K}_r} + h_k \epsilon(k).$$

Hence

$$h(x_k, \mathcal{K}_r^k(n_k h_k)) = -1 + |\nabla v_k(x_k)|^2 \leqslant V_{(t_0,x_0)}^{\mathcal{K}_r} + \epsilon(k) \to V_{(t_0,x_0)}^{\mathcal{K}_r}.$$

Then we can complete the proof as above to get the required condition:

$$h(x_0, \mathcal{K}_r(t_0)) \leqslant V_{(t_0,x_0)}^{\mathcal{K}_r}. \qquad \square$$

In particular we immediately obtain the following theorem:

THEOREM 5.3 Let Ω_0 be an open bounded subset of $\mathbb{R}^N$ such that $S \subset\subset \Omega_0$. Let $\mathcal{K}^+$ and $\mathcal{K}^-$ be, respectively, the largest and smallest viscosity solutions to the front propagation problem (1) with initial position Ω_0. Then

$$\mathcal{K}^- \subset \mathcal{K}_* \subset \mathcal{K}^* \subset \mathcal{K}^+.$$

In particular, if the problem has a unique solution, i.e., $\overline{\mathcal{K}^-} = \mathcal{K}^+$, then

$$\overline{\mathcal{K}^-} = \overline{\mathcal{K}}_* = \mathcal{K}^* = \mathcal{K}^+.$$

Proof. Since $\mathcal{K}^+$ contains any subsolution and $\mathcal{K}^-$ is contained in any supersolution (see [8]), we have $\mathcal{K}^* \subset \mathcal{K}^+$ and $\mathcal{K}^- \subset \mathcal{K}_*$. The inclusion $\mathcal{K}_* \subset \mathcal{K}^*$ holds by construction. Hence the result follows. □

6. The energy is non-increasing along the flow

Let Ω_0 be a bounded open subset of $\mathbb{R}^N$. In this section we assume that the front propagation problem (1) with initial position Ω_0 satisfies

$$|\partial\Omega_0| = 0 \quad \text{and} \quad |\mathcal{K}^+ \setminus \mathcal{K}^-| = 0, \tag{28}$$

where $\mathcal{K}^+$ and $\mathcal{K}^-$ denote the maximal and minimal solutions respectively.

REMARK 6.1 Assumption (28)—which is a strong uniqueness condition for the viscosity solution of the front propagation problem (1)—is not too restrictive. Indeed, it is generic in the following sense: let $(\Omega_0^\lambda)_{\lambda>0}$ be a strictly increasing family of bounded open initial positions containing the source, i.e.,

$$\text{for all } 0 < \lambda < \lambda', \quad S \subset\subset \Omega_0^\lambda \subset\subset \Omega_0^{\lambda'}.$$

If $\mathcal{K}_\lambda^+$ (respectively $\mathcal{K}_\lambda^-$) is the maximal (respectively minimal) viscosity solution to (1) with initial position Ω_0^λ, then (28) holds for all $\lambda > 0$ except for a countable subset. See [8, Props. 4 and 5] for details. For simplicity of notation, we have chosen to consider the case $\lambda = 1$ and to assume that (28) holds for the initial position Ω_0.

THEOREM 6.1 Under assumption (28), there is a set $\mathcal{T} \subset [0, +\infty)$ of full measure such that

$$\mathcal{E}(\mathcal{K}^+(t)) \leqslant \mathcal{E}(\mathcal{K}^+(s)) \quad \text{for all } s, t \in \mathcal{T},\ s < t.$$

Proof. Let (Ω_n^h) be a discrete motion starting from Ω_0. Recall for later use that, from Lemma 4.4,

$$\mathcal{E}(\overline{\Omega_n^h}) \leqslant \mathcal{E}(\overline{\Omega}_0) \quad \forall n \geqslant 0,\ \forall h > 0, \tag{29}$$

because we have assumed that $|\partial\Omega_0| = 0$. Let $\mathcal{K}^*$ and $\mathcal{K}_*$ be the associated generalized evolutions defined by (21) and (22). We have

$$\mathcal{K}^- \subset \mathcal{K}_* \subset \mathcal{K}^* \subset \mathcal{K}^+.$$

Let

$$\mathcal{T} := \{t \in [0, +\infty) : |\mathcal{K}^+(t) \setminus \mathcal{K}^-(t)| = 0\}.$$

From assumption (28) and the Fubini theorem, the set $\mathcal{T}$ is of full measure in $[0, +\infty)$.

We first prove that

$$\mathcal{E}(\mathcal{K}^+(t)) \leqslant \mathcal{E}(\overline{\Omega}_0) \quad \forall t \in \mathcal{T}. \tag{30}$$

For this, let $t \in \mathcal{T}$, and let $h_k \to 0^+$ and $n_k \to +\infty$ be such that $h_k n_k \to t$. For simplicity we set $\Omega_k := \Omega_{n_k}^{h_k}$. Since the Kuratowski upper limit of (Ω_k) is contained in $\mathcal{K}^+(t)$, which is a compact subset, the sequence (Ω_k) is bounded. Since moreover the upper limit of $(\mathbb{R}^N \setminus \Omega_k)$ is contained in $\mathbb{R}^N \setminus \mathcal{K}^-(t)$, the latter with boundary at a positive distance from S, there is some $r > 0$ such that $S_r \subset \Omega_k$ for any k sufficiently large (see (8) for the definition of S_r). Since finally the capacity is non-increasing with respect to the inclusion, Lemma 3.2 yields

$$\liminf_k \operatorname{cap}(\overline{\Omega}_k) \geqslant \operatorname{cap}(\mathcal{K}^+(t)). \tag{31}$$

The next step towards (30) amounts to showing that

$$|\mathcal{K}^+(t)| \leqslant \liminf |\Omega_k|. \tag{32}$$

Let $R > 0$ be so large that $\mathcal{K}^+(t) \subset\subset B_R$, where $B_R = B(0, R)$. By definition of the Kuratowski upper limit and the construction of $\mathcal{K}_*$, we have

$$\mathbf{1}_{B_R \setminus \mathcal{K}_*(t)} \geqslant \limsup_k \mathbf{1}_{B_R \setminus \Omega_k}.$$

The Fatou lemma then states that

$$|B_R \setminus \mathcal{K}_*(t)| \geqslant \limsup |B_R \setminus \Omega_k|,$$

whence (32) follows since $\mathcal{K}^-(t) \subset \mathcal{K}_*(t)$ and $|\mathcal{K}^+(t)| = |\mathcal{K}^-(t)|$ because $t \in \mathcal{T}$.

Combining (31), (32) and (29) finally gives

$$\mathcal{E}(\mathcal{K}^+(t)) \leqslant \liminf_k \mathcal{E}(\overline{\Omega}_k) \leqslant \mathcal{E}(\overline{\Omega}_0) \quad \forall t \in \mathcal{T}.$$

This proves (30).

Let now $0 \leqslant s \leqslant t$ with $s, t \in \mathcal{T}$. From the uniqueness of the solution starting from K_0, the maximal solution to the front propagation problem starting at time s from $\mathcal{K}^+(s)$ is equal at time t to $\mathcal{K}^+(t)$. Since $|\partial\mathcal{K}^+(s)| = 0$, because $s \in \mathcal{T}$, inequality (30) states that

$$\mathcal{E}(\mathcal{K}^+(t)) \leqslant \mathcal{E}(\mathcal{K}^+(s)),$$

which is the desired result. □

Acknowledgments

We wish to thank Luis Caffarelli, Antonin Chambolle and Marc Dambrine for fruitful discussions. The authors are partially supported by the ACI grant JC 1041 “Mouvements d’interface avec termes non-locaux” from the French Ministry of Research.

REFERENCES

1. ALLAIRE, G., JOUVE, F., & TOADER, A.-M. Structural optimization using sensitivity analysis and a level-set method. *J. Comput. Phys.* **194** (2004), 363–393. Zbl pre02056194 MR 2033390
2. ALMGREN, F., TAYLOR, J., & WANG, L. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.* **31** (1993), 387–438. Zbl 0783.35002 MR 1205983
3. ALT, H. W., & CAFFARELLI, L. A. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981), 105–144. Zbl 0449.35105 MR 0618549
4. ALVAREZ, O., CARDALIAGUET, P., & MONNEAU, R. Existence and uniqueness for dislocation dynamics with nonnegative velocity. *Interfaces Free Bound.* **7** (2005), 415–434. Zbl 1099.35148 MR 2191694
5. AMBROSIO, L. Minimizing movements. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)* **19** (1995), 191–246. Zbl 0957.49029 MR 1387558
6. CARDALIAGUET, P. On front propagation problems with nonlocal terms. *Adv. Differential Equations* **5** (2000), 213–268. Zbl 1029.53081 MR 1734542
7. CARDALIAGUET, P. Front propagation problems with nonlocal terms. II. *J. Math. Anal. Appl.* **260** (2001), 572–601. Zbl 1002.35064 MR 1845570
8. CARDALIAGUET, P., & LEY, O. On some flows in shape optimization. *Arch. Ration. Mech. Anal.* **183** (2007), 21–58. MR 2259339
9. CHAMBOLLE, A. An algorithm for mean curvature motion. *Interfaces Free Bound.* **6** (2004), 195–218. Zbl 1061.35147 MR 2079603
10. EVANS, L. C., & SPRUCK, J. Motion of level sets by mean curvature. I. *J. Differential Geom.* **33** (1991), 635–681. Zbl 0726.53029 MR 1100206
11. EVANS, L. C., & SPRUCK, J. Motion of level sets by mean curvature. IV. *J. Geom. Anal.* **5** (1995), 77–114. Zbl 0829.53040 MR 1315658
12. FLUCHER, M., & RUMPF, M. Bernoulli's free-boundary problem, qualitative theory and numerical approximation. *J. Reine Angew. Math.* **486** (1997), 165–204. Zbl 0909.35154 MR 1450755
13. GILBARG, D., & TRUDINGER, N. S. *Elliptic Partial Differential Equations of Second Order*. 2nd ed., Springer, Berlin (1983). Zbl 0562.35001 MR 0737190
14. HENROT, A., & PIERRE, M. *Variation et optimisation de formes—Une analyse géométrique*. Springer, Paris (2005). Zbl 1098.49001