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Properties of Sobolev-type metrics in the space of curves

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We define a manifold M where objects $c \in M$ are curves, which we parameterize as $c : S^1 \to \mathbb{R}^n$ ($n \geqslant 2$, S^1 is the circle). We study geometries on the manifold of curves, provided by Sobolev-type Riemannian metrics H^j. These metrics have been shown to regularize gradient flows used in computer vision applications (see [13, 14, 16] and references therein).

We provide some basic results on H^j metrics; and, for the cases $j = 1, 2$, we characterize the completion of the space of smooth curves. We call these completions "H^1 and H^2 Sobolev-type Riemannian manifolds of curves". This result is fundamental since it is a first step in proving the existence of geodesics with respect to these metrics. As a byproduct, we prove that the Fréchet distance of curves (see [7]) coincides with the distance induced by the "Finsler L^∞ metric" defined in §2.2 of [18].

1. Introduction

Suppose that $c : S^1 \to \mathbb{R}^n$ is an immersed curve, where $S^1 \subset \mathbb{R}^2$ is the circle; we want to define a geometry on M, the space of all such immersions c. The tangent space T_cM of M at c contains all the *deformations* $h \in T_cM$ of the curve c, which are all the vector fields along c. An infinitesimal deformation of the curve c in "direction" h will yield (to first order) the curve $c(u) + \varepsilon h(u)$. For simplicity, we postpone the details of the definitions (in particular on the regularity of c and h and on the topology on M) to Section 2.

We would like to define a *Riemannian metric* on the manifold M of immersed curves; this means that, given two deformations $h, k \in T_cM$, we want to define a scalar product $\langle h, k\rangle_c$, possibly dependent on c. The Riemannian metric would then entail a *distance* $d(c_0, c_1)$ between the curves in M, defined as the infimum of the lengths $\mathrm{Len}(\gamma)$ of all smooth paths $\gamma : [0, 1] \to M$ connecting c_0 to c_1. We define a *minimal geodesic* to be a path providing the minimum of $\mathrm{Len}(\gamma)$ in the class of γ with fixed endpoints.[1]

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[1] Note that this is an oversimplification of what we will actually do: compare Definitions 10 and 12.

At the same time, we would like to consider curves as "geometric objects," i.e., curves up to reparameterization; to this end, we will define the space of *geometrical curves* as the quotient space $B := M/\mathrm{Diff}(S^1)$, that is, the space of immersed curves up to reparameterization. For this reason we will ask that the metric defined on M be independent of the parameterization of the curves. B and M are the *shape spaces* that are studied in this paper.

1.i *Shape theory*

There are two different (but interconnected) fields of applications for shape theory in computer vision.

Shape optimization: We want to find a shape that best satisfies a design goal. This is usually done by minimizing a chosen energy that is defined on shapes.

Shape analysis: We study a family of shapes for purposes of computing statistics, (automatic) cataloging, probabilistic modeling, etc., and possibly to create an a-priori model for better shape optimization.

This entails an important remark.

REMARK 1 The aforementioned space B represents "curves up to reparameterization." A different approach would be to define the shape space S as "curves up to reparameterization, rotation, translation, scaling…" which is often more convenient for shape analysis tasks. The Riemannian metrics we study are not defined on curves up to rotation, translation, scaling, etc., but they are invariant with respect to joint application of these actions in the sense that $d(g \circ c_0, g \circ c_1) = d(c_0, c_1)$ where g is the action of rotation, translation, etc. Therefore the metric can be projected from B to S.

If one wishes to have a consistent view of the geometry of the space of curves in both shape optimization and shape analysis, then one should use the same metric when computing distances, averages and morphings between shapes, as when optimizing with respect to shape. Consistency is especially important when optimizing an energy that contains an a-priori model obtained from a shape analysis study. In this case the optimization scheme has natural connections to the geometry of the shape space (see Section IIA in [14] for more details).

1.ii *Notation*

We begin by introducing some notation. For a smooth curve $c : S^1 \to \mathbb{R}^n$, let

$$\mathrm{len}(c) := \int_{S^1} |\dot{c}(\theta)| \, \mathrm{d}\theta \tag{1}$$

be the length of the curve c; we will often write $L = \mathrm{len}(c)$, to shorten formulas.

For $g : S^1 \to \mathbb{R}^k$, we define the integration with respect to arc parameter

$$\int_c g(s) \, \mathrm{d}s := \int_{S^1} g(\theta)|\dot{c}(\theta)| \, \mathrm{d}\theta.$$

Let D_s be the differential operator $\frac{1}{|\partial_\theta c|}\partial_\theta$ (the derivative with respect to arc parameter), so that $D_s c$ is the tangent unit vector, and $D_s^2 c$ is the curvature vector of c.

1.iii *Origin of the problem*

A number of methods have been proposed in shape analysis to define distances between shapes, averages of shapes, and optimal morphings between shapes. At the same time, there has been much previous work in shape optimization (for example image segmentation via active contours and 3D stereo reconstruction via deformable surfaces). In these latter methods, many authors have defined energy functionals $E(c)$ on curves (or on surfaces), whose minima represent the desired segmentation/reconstruction, and have then utilized the calculus of variations to derive curve evolutions toward local minimizers of $E(c)$, often referring to these evolutions as *gradient flows*. The reference to these flows as *gradient flows* implies a certain Riemannian metric on the space of curves, but this fact has been largely overlooked. We call this metric H^0, and define it by

$$\langle h, k\rangle_{H^0} := \frac{1}{L}\int_c \langle h(s), k(s)\rangle \, \mathrm{d}s$$

where $h, k \in T_c M$, L is the length of c, the integration is performed with respect to arc parameter, and $\langle h(s), k(s)\rangle$ is the usual Euclidean scalar product in $\mathbb{R}^n$ (which we sometimes also write as $h(s) \cdot k(s)$).

Unfortunately, gradient flows that are induced by the H^0 metric have many unpleasant properties and limitations.

EXAMPLE 2 Consider a family $C = C(\theta, t)$ evolving by the *geometric heat flow* (also known as *motion by mean curvature*)

$$\frac{\partial C}{\partial t} = D_s^2 C.$$

This well known flow is often referred to as the *gradient flow for length*; indeed, by direct computation we find that the H^0 gradient is

$$\nabla_{H^0} \operatorname{len}(c) = \operatorname{len}(c) D_s^2 c$$

so the previous statement is true up to a conformal factor $1/\operatorname{len}(c)$, that is,

$$\frac{\partial C}{\partial t} = -\frac{1}{\operatorname{len}(C)} \nabla_{H^0} \operatorname{len}(C).$$

It is important to remark that the geometric heat flow is well-posed only for increasing time. This limits the usefulness of H^0 gradient flows in shape optimization, as illustrated in the following example.

EXAMPLE 3 Let $T = D_s c$ be the tangent vector of a planar curve c, and $L = \operatorname{len}(c)$. We define the normal vector N as the unit-length vector obtained by rotating T counterclockwise by the angle $\pi/2$. We define the scalar curvature κ so that $D_s^2 c = \kappa N$. Let

$$\operatorname{avg}(c) := \frac{1}{L}\int_c c(s)\, \mathrm{d}s$$

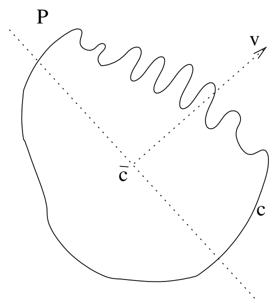
be the *center of mass* of the curve. Let us fix a target point $v \in \mathbb{R}^2$. Let

$$E(c) := \frac{1}{2}|\operatorname{avg}(c) - v|^2$$

be a functional that penalizes the distance from the center of mass to v. By direct computation the H^0 gradient descent flow is

$$\frac{\partial c}{\partial t} = -\nabla_{H^0} E(c) = \langle v - \mathrm{avg}(c), N \rangle N - \kappa N \langle c - \mathrm{avg}(c), v - \mathrm{avg}(c) \rangle. \tag{2}$$

Let $P := \{w : \langle w - \mathrm{avg}(c), v - \mathrm{avg}(c) \rangle \geqslant 0\}$ be the half plane "on the v side". The first term $\langle v - \mathrm{avg}(c), N \rangle N$ in the gradient descent flow moves the whole curve towards v. The second term $-\kappa N \langle c - \mathrm{avg}(c), v - \mathrm{avg}(c) \rangle$ tries to decrease the curve length out of P and increase the curve length in P, *and this is ill-posed.*



This is just one example of a large class of energies that may be of interest in shape optimization but whose H^0 gradient flow is ill-defined. A classical method to overcome such situations is to add a regularization term to the energy; this remedy, though, does change the energy, and we end up solving a different problem (see [13, 15, 16]).

The situation is even worse when we consider shape analysis. Surprisingly, H^0 does not yield a well defined metric structure, since the associated distance is identically zero; this striking fact was first described in [9], and is generalized to spaces of submanifolds in [5]. So H^0 completely fails for our stated goal, which is to provide a geometry of the space of curves usable both for shape optimization and shape analysis.

1.iv *Previous work*

When the above problems were recognized, there were many attempts at finding a better metric for curves.

In [17, 18, 19] we proposed a set of desirable metric properties and discussed some models available in the literature. Eventually we proposed and studied conformal metrics such as

$$\langle h, k \rangle_{H^0_\phi} := \mathrm{len}(c) \int \langle h(s), k(s) \rangle \, \mathrm{d}s \tag{3}$$

and proved results regarding this metric. In particular, we showed that the associated distance is nondegenerate. We also proved that minimal geodesics exist if we restrict ourselves to only unit length curves with an upper bound on curvature.

The same approach was proposed independently by J. Shah [10], who moreover proved that in the simplest case given by (3), minimal geodesics are represented by a curve evolution with constant speed along the normal direction.

Another possible definition appeared in [7] (by Michor and Mumford), who proposed the metric

$$\langle h, k\rangle_{H^0_A} := \int (1 + A\kappa^2(s))\langle h(s), k(s)\rangle \,\mathrm{d}s \tag{4}$$

where κ is the curvature of c, and $A > 0$ is a fixed constant. They proved many results regarding this metric, in particular, that the induced distance is nondegenerate, and that the completion of smooth curves lies between the space Lip of rectifiable curves and the space BV^2 of rectifiable curves whose curvature is a bounded measure.

1.v *Sobolev-type Riemannian metrics*

In [12] we proposed a family of Sobolev-type Riemannian metrics.

DEFINITION 4 Let $c \in M$, L be the length of c, and $h, k \in T_c M$. Let $\lambda > 0$. We define, for $\lambda > 0$ and $j \geqslant 1$ integer,

$$\langle h, k\rangle_{H^j} := \langle h, k\rangle_{H^0} + \lambda L^{2j}\langle D_s^j h, D_s^j k\rangle_{H^0},$$
$$\langle h, k\rangle_{\tilde{H}^j} := \mathrm{avg}(h)\cdot \mathrm{avg}(k) + \lambda L^{2j}\langle D_s^j h, D_s^j k\rangle_{H^0},$$

where again $\mathrm{avg}(h) := L^{-1}\int_c h(s)\,\mathrm{d}s$ and D_s^j is the j-th derivative with respect to arc parameter.

Note that $\langle h, k\rangle_{H^0} = \mathrm{avg}(h\cdot k)$ so the difference in the two metrics above is in substituting the term $\mathrm{avg}(h\cdot k)$ by $\mathrm{avg}(h)\cdot\mathrm{avg}(k)$.

It is easy to verify that the above bilinear forms are inner products. Note that we have introduced length dependent scale factors so that these inner products (and corresponding norms) are independent of curve rescaling.

In the above paper, and in following papers [11, 13, 14, 15, 16] we studied how the use of Sobolev metrics positively impacts shape optimization tasks. Indeed, we remark that changing the metric will change the gradient and thus the gradient descent flow. This change will alter the topology in the space of curves, but the change of topology does not affect the energy to be minimized nor its global minima.

In these papers we showed that the Sobolev-type gradients regularize the gradient flows of energies. This is exhibited in numerical experiments and applications, where it is observed that Sobolev flows will not in general be trapped in local minima due to small scale details or noise. It is also a mathematical property that a Sobolev metric will yield a lower degree gradient descent PDE than H^0 and will thereby be well-posed in many cases where the H^0 flow is ill-posed. We provide two simple examples.

EXAMPLE 5 If $E(c) = \frac{1}{2}|\mathrm{avg}(c) - v|^2$ is defined as in Example 3, the $\tilde{H}^1$ gradient $f = \nabla_{\tilde{H}^1}E(c)$ of E is the unique function f satisfying

$$\mathrm{avg}(f) = \mathrm{avg}(c) - v, \qquad \lambda L^2 D_s f = D_s c\langle \mathrm{avg}(c) - v, c - \mathrm{avg}(c)\rangle + \alpha$$

(where $\alpha \in \mathbb{R}^n$ is the unique constant such that there exists a periodic solution f to the rightmost equation). The gradient descent flow is the solution $C = C(\theta, t)$ of

$$\frac{\partial C}{\partial t} = -\nabla_{\tilde{H}^1}E(C); \tag{5}$$

during this flow, the length of curves $C(\cdot, t)$ is kept constant; the mean part

$$\operatorname{avg}(\partial_t C) = v - \operatorname{avg}(C)$$

of this flow simply moves the whole curve so that the center of mass will move towards v.

EXAMPLE 6 (§4.3 in [13]) In the case of the elastic energy $E(c) = \int \kappa^2 \, \mathrm{d}s = \int |D_s^2 c|^2 \, \mathrm{d}s$, the H^0 gradient is $\nabla_{H^0} E = L D_s (2 D_s^3 c + 3 |D_s^2 c|^2 D_s c)$, which includes fourth order derivatives; whereas the $\tilde{H}^1$ gradient is

$$-\frac{2}{\lambda L} D_s^2 c - 3\lambda L (|D_s^2 c|^2 D_s c) * \tilde{K}_\lambda \tag{6}$$

so that the gradient descent flow is an integro-differential second order PDE. The kernel $\tilde{K}_\lambda$ is defined in equation (17) of [13].

In conclusion, Sobolev gradient methods effectively enlarge the family of energies that may be used in shape optimization—without requiring extra regularization terms.

Since we did find great advantages by using Sobolev-type metrics in shape optimization, we would like to further analyze the properties of the related Riemannian geometry. These metrics may indeed eventually satisfy the goal expressed in Section 1.i, that is, provide a consistent geometry of the space of curves to be used both in shape optimization and in shape analysis.

One question of major interest is whether or not the Riemannian space of curves is complete. How can we characterize the completion of the space of smooth curves in the metric H^j? This question is a fundamental first step if we wish to prove that geodesics do exist, but it is also important in shape optimization since it would be a basic ingredient of any proof of existence and regularity for minimizing gradient flows.

1.vi *Related works.*

A family of metrics similar to what we defined in Definition 4 (up to the length scale factors) was concurrently studied in [6]. In that paper the geodesic equation, horizontality, conserved momenta, lower and upper bounds on the induced distance and scalar curvatures are computed. In a much earlier work, Younes [20] had proposed a computable definition of distance of curves, modeled on elastic curves. This model may be viewed as a Sobolev-type metric in the space of curves up to rotation, translation and scaling and has been studied in depth in a recent paper [8].

1.vii *Paper outline*

In the rest of this paper we present a mathematical study of the Riemannian geometry of curves defined in Definition 4, and specifically the cases $j = 1, 2$. In Section 2 we properly define the model space for the manifold of curves, and discuss benefits and shortcomings of different choices of hypotheses. In Section 2.i we define the Fréchet distance of curves (see [7]) and prove that it coincides with the distance induced by the "Finsler L^∞ metric" defined in Section 2.2 of [18]. In Section 3 we define H^j Sobolev-type Riemannian metrics and prove some of their basic properties. Eventually we characterize the completion of smooth curves in the H^1 and H^2 metric; those complete spaces are the "H^1 and H^2 Sobolev-type Riemannian manifolds of curves".

2. Spaces of curves

As anticipated in the introduction, we want to define a geometry on M, the space of all immersions $c : S^1 \to \mathbb{R}^n$.

We will sometimes specify exactly what M is, choosing between the space $\mathrm{Imm}(S^1, \mathbb{R}^n)$ of immersions, $\mathrm{Imm}_{\mathrm{f}}(S^1, \mathbb{R}^n)$ of *free immersions*, and $\mathrm{Emb}(S^1, \mathbb{R}^n)$ of *embeddings*. We recall that $c : S^1 \to \mathbb{R}^1$ is a *free immersion* when the only diffeomorphism $\phi : S^1 \to S^1$ satisfying $c(u) = c(\phi(u))$ for all u is the identity. More details are in §2.4 and §2.5 of [7].

We will equip M with a topology τ stronger than the C^1 topology; for any such choice, M is an open subset of the vector space $C^1(S^1, \mathbb{R}^n)$ (which is a Banach space), so it is a manifold.

The tangent space $T_c M$ to M at c contains vector fields $h : S^1 \to \mathbb{R}^n$ along c.

Note that we represent both curves $c \in M$ and deformations $h \in T_c M$ as functions $S^1 \to \mathbb{R}^n$; this is a special structure that is not usually present in abstract manifolds: so we can easily define "charts" for M:

REMARK 7 (Charts in M) Given a curve c, there is a neighborhood U_c of $0 \in T_c M$ such that for $h \in U_c$, the curve $c + h$ is still immersed; then the map $h \mapsto c + h$ is the simplest natural candidate to be a chart of $\Phi_c : U_c \to M$; indeed, if we pick another curve $\tilde{c} \in M$ and the corresponding $U_{\tilde{c}}$ such that $U_{\tilde{c}} \cap U_c \neq \emptyset$, then the equality $\Phi_c(h) = c + h = \tilde{c} + \tilde{h} = \Phi_{\tilde{c}}(\tilde{h})$ can be solved for h to obtain $h = (\tilde{c} - c) + \tilde{h}$.

The above is trivial but is worth remarking for two reasons: it stresses that the topology τ must be strong enough to maintain immersions; and is a basis block of what we will do in the space $B_{\mathrm{i,f}}$ defined below.

We look mainly for metrics in the space M that are independent of the parameterization of the curves c; to this end, we define the spaces of *geometrical curves*

$$B_{\mathrm{i}} = B_{\mathrm{i}}(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$$

and

$$B_{\mathrm{i,f}} = B_{\mathrm{i,f}}(S^1, \mathbb{R}^2) = \mathrm{Imm}_{\mathrm{f}}(S^1, \mathbb{R}^2)/\mathrm{Diff}(S^1)$$

that are the quotients of the spaces Imm and $\mathrm{Imm}_{\mathrm{f}}$ by $\mathrm{Diff}(S^1)$; alternatively we may quotient by $\mathrm{Diff}^+(S^1)$ (the space of orientation preserving automorphisms of S^1), and obtain spaces of *geometrical oriented curves*.

REMARK 8 (on model spaces and properties) We have two possible choices in mind for the topology τ to put on M: the topology of the Fréchet space of C^∞ functions; or that of a Hilbert space such as standard Sobolev spaces $H^j(S^1 \to \mathbb{R}^n)$.

Suppose we define on M a Riemannian metric: we would like B_{i} to have a nice geometrical structure; we would like our Riemannian geometry to have some useful properties.

Unfortunately, this currently seems an antinomy.

If M is modeled on a Hilbert space H^j, then most of the usual calculus carries over; for example, the exponential map would be locally a diffeomorphism; but the quotient space $M/\mathrm{Diff}(S^1)$ is not a smooth bundle (since the tangent to the orbit contains $\dot{c}$ and this is in H^{j-1} in general!).

If M is modeled on the Fréchet space of C^∞ functions, then the quotient space $M/\mathrm{Diff}(S^1)$ is a smooth bundle; but some of the usual calculus fails: the Cauchy–Lipschitz theorem on existence of local solutions to ODEs does not hold in general; and the exponential map is not locally surjective.

We will suppose in the following that τ is the Fréchet topology of C^∞ functions; then $B_{\mathrm{i,f}}$ is a manifold, the base of a principal fiber bundle, while B_{i} is not (see §2.4.3 in [7] for details).

To define charts on this manifold, we imitate what was done for M:

PROPOSITION 9 (Charts in $B_{\mathrm{i,f}}$) Let Π be the projection from $\mathrm{Imm}_{\mathrm{f}}(S^1, \mathbb{R}^2)$ to the quotient $B_{\mathrm{i,f}}$. Let $[c] \in B_{\mathrm{i,f}}$, and pick a curve c such that $\Pi(c) = [c]$. We represent the tangent space $T_{[c]}B_{\mathrm{i,f}}$ as the space of all $k : S^1 \to \mathbb{R}^n$ such that $k(s)$ is orthogonal to $\dot{c}(s)$. Again we can define a simple natural chart $\Phi_{[c]}$ by projecting the chart Φ_c (defined in Remark 7):

$$\Phi_{[c]}(k) := \Pi(c(\cdot) + k(\cdot)),$$

that is, it moves $c(u)$ in direction $k(u)$; and it is easily seen that the chart does not depend on the choice of c such that $\Pi(c) = [c]$. These maps Φ are a chart of $B_{\mathrm{i,f}}$.

The proof may be found in [7], or in §4.4.7 and §4.6.6 of [2].

We now define a Finsler metric F on M, that is, a lower semicontinuous function $F : TM \to \mathbb{R}^+$ such that $F(c, \cdot)$ is a norm on T_cM for all c.

If $\gamma : [0, 1] \to M$ is a path connecting two curves c_0, c_1, then we may define a homotopy $C : S^1 \times [0, 1] \to \mathbb{R}^n$ associated to γ by $C(\theta, v) = \gamma(v)(\theta)$, and vice versa.

DEFINITION 10 Given a metric F in M, we could define the *standard distance* of two curves c_0, c_1 as the infimum of the lengths

$$\int_0^1 F(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t$$

in the class of all γ connecting c_0, c_1.

This is not, though, the most interesting distance for applications: we are indeed interested in studying metrics and distances in the quotient space $B := M/\mathrm{Diff}(S^1)$.

We add a hypothesis on F.

DEFINITION 11 The metric $F(c, h)$ is *curve-wise parameterization invariant* if it does not depend on the parameterization of the curves c.

If this is satisfied, then F may be projected to $B := M/\mathrm{Diff}(S^1)$; we will say that F is a *geometrical metric*.

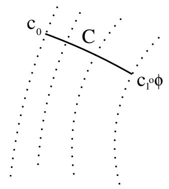
Consider two geometrical curves $[c_0], [c_1] \in B$, and a path $\gamma : [0, 1] \to B_i$ connecting $[c_0], [c_1]$: then we may lift it to a homotopy $C : S^1 \times [0, 1] \to \mathbb{R}^n$; in this case, the homotopy will connect a reparameterization $c_0 \circ \phi_0$ to a reparameterization $c_1 \circ \phi_1$, with $\phi_0, \phi_1 \in \mathrm{Diff}(S^1)$. Since F does not depend on the parameterization, we can factor out ϕ_0 from the definition of the projected length.

To summarize, we introduce

DEFINITION 12 Given c_0, c_1, we define the class $\mathcal{A}$ of homotopies C connecting the curve c_0 to a reparameterization $c_1 \circ \phi$ of the curve c_1, that is, $C(u, 0) = c_0(u)$ and $C(u, 1) = c_1(\phi(u))$. We define the *geometric distance* d_F of $[c_0], [c_1]$ in $B := M/\mathrm{Diff}(S^1)$ as the infimum of the lengths

$$\mathrm{Len}_F(C) := \int_0^1 F(C(\cdot, v), \partial_v C(\cdot, v))\,\mathrm{d}v$$

in the class of all such $C \in \mathcal{A}$.[2] Any homotopy that achieves the minimum of $\mathrm{Len}_F(C)$ is called a *geodesic*.



We call such distances $d_F(c_0, c_1)$, dropping the square brackets for simplicity.[3]

We provide an interesting example of the above ideas in the following section.

2.i *L^∞-type Finsler metric and Fréchet distance*

We digress from the main theme of the paper to prove a result that will be used in the following. For any fixed immersed curve c and $\theta \in S^1$, we define for convenience $\pi_N : \mathbb{R}^n \to \mathbb{R}^n$ to be the projection on the space $N(\theta)$ orthogonal to the tangent vector $D_s c(\theta)$,

$$\pi_{N(\theta)} w = w - \langle w, D_s c(\theta) \rangle D_s c(\theta) \qquad \forall w \in \mathbb{R}^n. \tag{7}$$

Consider two immersed curves c_0 and c_1; the Fréchet distance d_{f} (as found in [7]) is defined by

DEFINITION 13 (Fréchet distance)

$$d_{\mathrm{f}}(c_0, c_1) := \inf_{\phi} \sup_{u} |c_1(\phi(u)) - c_0(u)|$$

where $u \in S^1$ and ϕ runs over the class of diffeomorphisms of S^1.

This is a well defined distance in the space B_{i} (which is not, though, complete with respect to this distance: its completion is the space of Fréchet curves).

Another similar distance was defined in §2.2 of [18] by a different approach, using a Finsler metric:

DEFINITION 14 (Finsler L^∞ metric) If we wish to define a norm on $T_c M$ that is modeled on the norm of the Banach space $L^\infty(S^1 \to \mathbb{R}^n)$, we define

$$F^\infty(c, h) := \|\pi_N h\|_{L^\infty} = \sup_{\theta} |\pi_{N(\theta)} h(\theta)|.$$

We define the distance $d_\infty(c_0, c_1)$ as in Definition 12.

Section §2.2.1 in [18] discusses the relationship between the distance d_∞ and the Hausdorff distance of compact sets. We discuss here the relationship between d_{f} and d_∞; indeed, we prove that $d_{\mathrm{f}} = d_\infty$.

THEOREM 15 $d_{\mathrm{f}} = d_\infty$.

[2] Note the difference in notation between $\mathrm{Len}(C)$ and $\mathrm{len}(c)$, defined in (1).

[3] We are abusing notation: these d_F are not, properly speaking, distances in the space M, since the distance between c and a reparameterization $c \circ \phi$ is zero.

Proof. Fix c_0 and c_1, and define $\mathcal{A}$ as in Definition 12. We recall that d_∞ is also equal to the infimum of

$$d_\infty(c_0, c_1) = \inf_{C \in \mathcal{A}} \int_0^1 \sup_\theta \left| \frac{\partial C}{\partial v}(\theta, v) \right| \mathrm{d}v$$

(the proof follows immediately from Prop. 3.10 in [18]).

Consider a homotopy $C = C(u, v) \in \mathcal{A}$ connecting the curve c_0 to a reparameterization $c_1 \circ \phi$ of the curve c_1:

$$\begin{aligned} \sup_u |c_1(\phi(u)) - c_0(u)| &= \sup_u |C(u, 1) - C(u, 0)| \\ &= \sup_u \left| \int_0^1 \frac{\partial C}{\partial v}(u, v)\, \mathrm{d}v \right| \leqslant \int \sup_u \left| \frac{\partial C}{\partial v}(u, v) \right| \mathrm{d}v \end{aligned}$$

so that $d_\mathrm{f} \leqslant d_\infty$.

On the other hand, let

$$C^\phi(\theta, v) := (1 - v)c_0(\theta) + v c_1(\phi(\theta))$$

be the linear interpolation. Then

$$\frac{\partial C^\phi}{\partial v}(u, v) = c_1(\phi(u)) - c_0(u)$$

(which does not depend on v) so that

$$\sup_u \left| \int_0^1 \frac{\partial C^\phi}{\partial v}(u, v)\, \mathrm{d}v \right| = \int \sup_u \left| \frac{\partial C^\phi}{\partial v}(u, v) \right| \mathrm{d}v$$

and then, for that particular homotopy C^ϕ,

$$\mathrm{Len}_\infty(C^\phi) = \sup_u |c_1(\phi(u)) - c_0(u)|.$$

We compute the infimum over all possible choices of ϕ to get

$$d_\infty(c_0, c_1) = \inf_C \mathrm{Len}_\infty(C) \leqslant \inf_\phi \mathrm{Len}_\infty(C^\phi) = \inf_\phi \sup_u |c_1(\phi(u)) - c_0(u)| = d_\mathrm{f}(c_0, c_1). \qquad \square$$

The conclusion of the theorem holds as well if we use orientation preserving diffeomorphisms $\mathrm{Diff}^+(S^1)$ both in the definition of the Fréchet distance and in the definition of L^∞.

3. Sobolev-type H^j metrics

We start by generalizing Definition 4. Fix $\lambda > 0$. Suppose that $h \in L^2$. Then we can expand it in Fourier series:

$$h(s) = \sum_{l \in \mathbb{Z}} \widehat{h}(l) \exp\left(\frac{2\pi i}{L} l s \right) \tag{8}$$

where $\widehat{h} \in \ell^2(\mathbb{Z} \to \mathbb{C})$.

For any $\alpha > 0$, given the Fourier coefficients $\widehat{h}, \widehat{k} : \mathbb{Z} \to \mathbb{C}$ of h, k, we define the fractional Sobolev inner product

$$\langle h, k\rangle_{H_0^\alpha} := \sum_{l\in\mathbb{Z}} (2\pi l)^{2\alpha} \widehat{h}(l) \cdot \overline{\widehat{k}(l)}, \tag{9}$$

independent of curve scaling; here $\overline{\widehat{k}(l)}$ is the complex conjugate of $\widehat{k}(l)$. Then we can define

$$\begin{aligned} \langle h, k\rangle_{H^\alpha} &:= \operatorname{avg}(h \cdot k) + \lambda \langle h, k\rangle_{H_0^\alpha}, \\ \langle h, k\rangle_{\tilde{H}^\alpha} &:= \operatorname{avg}(h) \cdot \operatorname{avg}(k) + \lambda \langle h, k\rangle_{H_0^\alpha}. \end{aligned} \tag{10}$$

When $\alpha = j$ is an integer, these definitions coincide with those in Definition 4. So, for any $\alpha > 0$, we represent the Sobolev-type metrics by

$$\langle h, k\rangle_{H^\alpha} = \sum_{l\in\mathbb{Z}} (1 + \lambda(2\pi l)^{2\alpha}) \widehat{h}(l) \cdot \overline{\widehat{k}(l)}, \tag{11}$$

$$\langle h, k\rangle_{\tilde{H}^\alpha} = \widehat{h}(0) \cdot \overline{\widehat{k}(0)} + \sum_{l\in\mathbb{Z}} \lambda(2\pi l)^{2\alpha} \widehat{h}(l) \cdot \overline{\widehat{k}(l)}. \tag{12}$$

REMARK 16 Unfortunately, for j that is not an integer, the inner products (therefore, norms) are not local, that is, they cannot be written as integrals of derivatives of the curves. An interesting representation is by kernel convolution: given $r \in \mathbb{R}^+$, we can represent them, for j integer $> r + 1/4$, as

$$\langle h, k\rangle_{\tilde{H}^r} = \int_c \int_c D^j h(s) K(s - \tilde{s}) D^j k(\tilde{s}) \, \mathrm{d}s \, \mathrm{d}\tilde{s},$$

that is, $\langle h, k\rangle_{\tilde{H}^r} = \langle D^j h, K * D^j k\rangle_{H^0}$, for a specific kernel K. Here $*$ denotes convolution in S^1 with respect to arc parameter.

REMARK 17 The norm $\|h\|_{\tilde{H}^j}$ has an interesting interpretation in connection with applications in computer vision. Consider a deformation $h \in T_c M$ and write it as $h = \operatorname{avg}(h) + \tilde{h}$; this decomposes

$$T_c M = \mathbb{R}^n \oplus D_c M \tag{13}$$

with

$$D_c M := \{h : S^1 \to \mathbb{R}^n \mid \operatorname{avg}(h) = 0\}.$$

If we equip $\mathbb{R}^n$ with its usual Euclidean norm, and $D_c M$ with the scale-invariant H_0^α norm defined in (9), then we are naturally led to decompose, as in (10),

$$\|h\|^2_{\tilde{H}^\alpha} = |\operatorname{avg}(h)|^2_{\mathbb{R}^n} + \lambda \|\tilde{h}\|^2_{H_0^\alpha}. \tag{14}$$

This means that $\mathbb{R}^n$ and $D_c M$ are orthogonal with respect to $\tilde{H}^\alpha$.

In the above, $\mathbb{R}^n$ is akin to *the space of translations* and $D_c M$ to the *space of non-translating deformations*. That labeling is not rigorous, though, since the subspace of $T_c M$ that does not move the center of mass $\operatorname{avg}(c)$ is not $D_c M$, but rather

$$\left\{ h : \int_{S^1} h + (c - \operatorname{avg}(c)) \langle D_s h, T\rangle \, \mathrm{d}s = 0 \right\}.$$

Note that $\sqrt{\langle h, h\rangle_{H_0^\alpha}}$ is a norm on $D_c M$ (by (16)), and it is a seminorm and not a norm on $T_c M$.

We define Finsler norms as

$$F_{H^j}(c, h) = \|h\|_{H^j} = \sqrt{\langle h, h\rangle_{H^j}}, \qquad F_{\tilde{H}^j}(c, h) = \|h\|_{\tilde{H}^j} = \sqrt{\langle h, h\rangle_{\tilde{H}^j}},$$

and then we define distances d_{H^j} and $d_{\tilde{H}^j}$ as explained in Definition 12.

3.i *Preliminary results*

We improve a result from [12]:[4] we show that the norms associated with the inner products H^j and $\tilde{H}^j$ are equivalent. We first prove

LEMMA 18 (Poincaré inequalities) Let $h : [0, L] \to \mathbb{R}^n$ be weakly differentiable, with $h(0) = h(L)$ (so h is periodically extensible). Then

$$\sup_u |h(u) - \operatorname{avg}(h)| \leqslant \frac{1}{2}\int_0^L |h'(s)|\,\mathrm{d}s. \tag{15}$$

The constant $1/2$ is optimal and is approximated by a family of h such that

$$h'(s) = a(\mathbb{1}_{[0,\varepsilon)}(s) - \mathbb{1}_{[\varepsilon,2\varepsilon)}(s))$$

when $\varepsilon \to 0$ (for a fixed $a \in \mathbb{R}^n$); here $\mathbb{1}_A$ is the characteristic function of A.

Proof. Since $h(0) = h(L)$, we have

$$h(u) - h(0) = \int_0^u h'(s)\,\mathrm{d}s = -\int_u^L h'(s)\,\mathrm{d}s.$$

Consequently,

$$h(u) - h(0) = \frac{1}{2}\left(\int_0^u h'(s)\,\mathrm{d}s - \int_u^L h'(s)\,\mathrm{d}s\right),$$

hence

$$\operatorname{avg}(h) - h(0) = \frac{1}{2L}\int_0^L \left(\int_0^u h'(s)\,\mathrm{d}s - \int_u^L h'(s)\,\mathrm{d}s\right)\mathrm{d}u,$$

and therefore

$$\begin{aligned}|\operatorname{avg}(h) - h(0)| &\leqslant \frac{1}{2L}\int_0^L \left(\int_0^u |h'(s)|\,\mathrm{d}s + \int_u^L |h'(s)|\,\mathrm{d}s\right)\mathrm{d}u \\ &= \frac{1}{2L}\int_0^L \left(\int_0^L |h'(s)|\,\mathrm{d}s\right)\mathrm{d}u = \frac{1}{2}\int_0^L |h'(s)|\,\mathrm{d}s,\end{aligned}$$

so that (by extending h and replacing 0 with an arbitrary point) we obtain (15). □

By using the Hölder inequality we can then derive many useful Poincaré inequalities of the form $\|h - \operatorname{avg}(h)\|_p \leqslant c_{p,q,j}\|h'\|_q$, like the one below.

[4] And we provide a better version that unfortunately was prepared too late for the printed version of [12].

COROLLARY 19 For $p = q = 2$,

$$\int_0^L |h(s) - \text{avg}(h)|^2 \, \mathrm{d}s \leqslant \frac{L^{2j}}{(2\pi)^{2j}} \int_0^L |h^{(j)}(s)|^2 \, \mathrm{d}s \tag{16}$$

where the constant $c_{2,2,j} = (L/2\pi)^{2j}$ is optimal and is achieved by $h(s) = a \sin(2\pi s/L)$ (with $a \in \mathbb{R}^n$).

The proof may be obtained by expanding in Fourier series.

PROPOSITION 20

$$\|h\|_{\tilde{H}^j} \leqslant \|h\|_{H^j} \leqslant \sqrt{\frac{1 + (2\pi)^{2j}\lambda}{(2\pi)^{2j}\lambda}} \|h\|_{\tilde{H}^j}.$$

Proof. Fix a smooth immersed curve $c : S^1 \to \mathbb{R}^n$. Let $L = \text{len}(c)$. By Hölder's inequality, we have $|\text{avg}(h)|^2 \leqslant L^{-1} \int_0^L |h(s)|^2 \, \mathrm{d}s$ so that $\|h\|_{\tilde{H}^j} \leqslant \|h\|_{H^j}$. On the other hand,

$$\frac{1}{L} \int_0^L |h(s) - \text{avg}(h)|^2 \, \mathrm{d}s = \frac{1}{L} \int_0^L |h(s)|^2 \, \mathrm{d}s - |\text{avg}(h)|^2, \tag{17}$$

so that (by the Poincaré inequality (16)),

$$\begin{aligned}
\|h\|_{H^j}^2 &= \int_0^L \left(\frac{1}{L} |h(s)|^2 + \lambda L^{2j-1} |h^{(j)}(s)|^2 \right) \mathrm{d}s \\
&= \frac{1}{L} \int_0^L |h(s) - \text{avg}(h)|^2 \, \mathrm{d}s + \int_0^L \lambda L^{2j-1} |h^{(j)}(s)|^2 \, \mathrm{d}s + |\text{avg}(h)|^2 \\
&\leqslant |\text{avg}(h)|^2 + L^{2j-1} \left(\frac{1}{(2\pi)^{2j}} + \lambda \right) \int_0^L |h^{(j)}(s)|^2 \, \mathrm{d}s \leqslant \frac{1 + (2\pi)^{2j}\lambda}{(2\pi)^{2j}\lambda} \|h\|_{\tilde{H}^j}^2.
\end{aligned}$$

□

More generally, we have

PROPOSITION 21 For $i = 0, \ldots, j$, choose $\tilde{a}_0 \geqslant 0$ and $a_i \geqslant 0$ with $a_0 + \tilde{a}_0 > 0$ and $a_j > 0$. Define an H^j-type Riemannian norm[5]

$$\|h\|_{(a),j}^2 := \tilde{a}_0 |\text{avg}(h)|^2 + \sum_{i=0}^{j} a_i L^{2i-1} \int_0^L |h^{(i)}(s)|^2 \, \mathrm{d}s. \tag{18}$$

Then all such norms are equivalent.

Moreover, choose r with $1 \leqslant r \leqslant j$, and choose $\tilde{b}_0 \geqslant 0$, $b_i \geqslant 0$ with $\tilde{b}_0 + b_0 > 0$ and $b_r > 0$. Then the norm $\|h\|_{(a),j}$ is stronger than $\|h\|_{(b),r}$.

Proof. The proof is just an application of (17) and of (16) (repeatedly); note also that for $1 \leqslant i < j$ the inequality (16) becomes

$$\int_0^L |h^{(i)}(s)|^2 \, \mathrm{d}s \leqslant \frac{L^{2j-2i}}{(2\pi)^{2j-2i}} \int_0^L |h^{(j)}(s)|^2 \, \mathrm{d}s \tag{19}$$

since $\text{avg}(h^{(i)}) = 0$. □

[5] The scalar product can be easily inferred, by using polarization.

So our definitions of $\|\cdot\|_{H^j}$ and $\|\cdot\|_{\tilde{H}^j}$ are in a sense the simplest choices of a Sobolev-type norm that are scale invariant; in particular:

REMARK 22 The H^j type metric

$$\|h\|_M^2 := \int \sum_{i=0}^{j} |h^{(i)}(s)|^2 \, \mathrm{d}s$$

studied in [6] is equivalent to our choices:

$$b_1 \|\cdot\|_{\tilde{H}^j} \leqslant \|\cdot\|_M \leqslant b_2 \|\cdot\|_{\tilde{H}^j},$$

but the constants b_1, b_2 depend on the length of the curve.

From these propositions, we will deduce some properties of the H^1 metric, and we will find that they can be extended to $\tilde{H}^1$ and to more general H^j-type metrics defined as in (18).

We now prove a fundamental inequality:

PROPOSITION 23 Let $C(u, v)$ be a smooth homotopy of immersed curves $C(\cdot, v)$. Then

$$\|\partial_v C(\cdot, v)\|_{H^1} \geqslant \sqrt{\lambda} \int |\partial_{uv} C(u, v)| \, \mathrm{d}u. \tag{20}$$

Proof. Fix a smooth immersed curve $c : S^1 \to \mathbb{R}^n$, and let $L = \mathrm{len}(c)$. Let $h : S^1 \to \mathbb{R}^n$ be a vector field. We rewrite for convenience

$$\|h\|_{H^1}^2 \geqslant \lambda L^2 \langle h', h' \rangle_{H^0} = \lambda L \int_0^L |h'(s)|^2 \, \mathrm{d}s = \lambda \int |\dot{c}(u)| \, \mathrm{d}u \int |h'(u)|^2 |\dot{c}(u)| \, \mathrm{d}u$$

where $h' = D_s h$; then by Cauchy–Schwarz,

$$\int |\dot{c}(u)| \, \mathrm{d}u \int |h'(u)|^2 |\dot{c}(u)| \, \mathrm{d}u \geqslant \left(\int |h'(u)| \, |\dot{c}(u)| \, \mathrm{d}u \right)^2.$$

To conclude, set $h(u, v) = \partial_v C(u, v)$ so that $D_s h = D_s \partial_v C = \partial_{uv} C / |\partial_u C|$. □

As argued in Proposition 21, the above result extends to all H^j-type norms (18).

We now relate the H^1-type metric to the L^∞-type metrics.

PROPOSITION 24 The $\tilde{H}^1$ metric is stronger than the L^∞ metric defined in Definition 14. As a consequence, by Proposition 20 and Theorem 15, the H^j and $\tilde{H}^j$ distances are lower bounded by the Fréchet distance (with appropriate constants depending on λ).

Proof. Indeed, by (15),

$$\begin{aligned} \sup_\theta |\pi_{N(\theta)} h(\theta)| &\leqslant \sup_\theta |h(\theta)| \, |\mathrm{avg}(h)| + \frac{1}{2} \int |h'| \, \mathrm{d}s \\ &\leqslant |\mathrm{avg}(h)| + \frac{\sqrt{L}}{2} \sqrt{\int |h'|^2 \, \mathrm{d}s} \leqslant \sqrt{2} \sqrt{|\mathrm{avg}(h)|^2 + \frac{L}{4} \int |h'|^2 \, \mathrm{d}s} \end{aligned}$$

(π_N was defined in (7)). For example, choosing $\lambda = 1/4$, we obtain

$$F_\infty(c, h) \leqslant \sqrt{2} \|h\|_{\tilde{H}^1}.$$

□

We also establish the relationship between the length $\mathrm{len}(c)$ of a curve and the Sobolev metrics.

PROPOSITION 25 Suppose again that $C(u, v)$ is a smooth homotopy of immersed curves, and let $L(v) := \text{len}(C(\cdot, v))$ be the length at time v. Then

$$\partial_v L = \int \left\langle \partial_{uv} C, \frac{\partial_u C}{|\partial_u C|} \right\rangle \mathrm{d}u \leqslant \int |\partial_{uv} C| \, \mathrm{d}u \leqslant \frac{1}{\sqrt{\lambda}} \|C_v(\cdot, v)\|_{H^1}$$

by (20).

There are many interesting consequences:

- We have
$$|L(1) - L(0)| \leqslant \frac{1}{\sqrt{\lambda}} \text{Len}(C) \tag{21}$$
where the length $\text{Len}(C)$ of the homotopy/path C is computed using either H^1 or $\tilde{H}^1$ (or using any metric as in (18) above, but in this case the constant in (21) would change).
- Define the length functional $c \mapsto \text{len}(c)$ on our space of curves; endow the space of curves with the H^1 metric; then the length functional is Lipschitz.
- The "zero curves" are the constant curves (of zero length); these are points in the space of curves where the space of curves is, in a sense, singular; by the above, the "zero curves" form a closed set in the H^1 space of curves, and an immersed curve c is at distance at least $\text{len}(c)\sqrt{\lambda}$ from the "zero curves".

But the most interesting consequence is

THEOREM 26 (Completion of B_{i} with respect to H^1) Let d_{H^1} be the distance induced by H^1. Then the metric completion of the space of curves is contained in the space of all rectifiable curves.

Proof. This statement is a bit fuzzy: indeed, d_{H^1} is not a distance on M, whereas in B_{i} objects are not functions, but classes of functions. So it must be understood "up to reparameterization of curves", as follows.[6]

Let $(c_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Since d_{H^1} does not depend on parameterization, we assume that all c_n are parameterized by arc parameter, that is, $|\partial_\theta c_n| = l_n$ constant in θ. By Proposition 24, all curves are contained in a bounded region; since $\text{len}(c_n) = 2\pi l_n$ by Proposition 25 above, the sequence l_n is bounded. So the (reparameterized) family (c_n) is equibounded and equilipschitz; by the Ascoli–Arzelà theorem, up to a subsequence, c_n converges uniformly to a Lipschitz curve c, and $|\partial_\theta c| \leqslant \lim_n l_n$. □

We also prove

THEOREM 27 Any rectifiable planar curve is approximable by smooth curves according to the distance induced by H^1.

To prove this theorem, we will need a lemma from [18]. Let c be a rectifiable curve, and assume that it is non-constant. We identify S^1 with $[0, 2\pi)$. Let also $L^2 = L^2([0, 2\pi])$.

LEMMA 28 Suppose that $|\partial_\theta c| = 1$. Define the *measurable angle function* to be a measurable function $\tau : [0, 2\pi) \to [0, 2\pi)$ such that $\partial_\theta c(\theta) = (\cos\tau(\theta), \sin\tau(\theta))$. Set

$$S = \{\tau \in L^2([0, 2\pi]) \mid \phi(\tau) = (0, 0)\}$$

[6] The concept is clarified by introducing the concept of *horizontality* in M, which we must unfortunately skip for brevity.

where $\phi : L^2 \to \mathbb{R}^2$ is defined by

$$\phi_1(\tau) = \int_0^{2\pi} \cos\tau(s)\,\mathrm{d}s, \qquad \phi_2(\tau) = \int_0^{2\pi} \sin\tau(s)\,\mathrm{d}s$$

(this is similar to what was done in Srivastava et al. works on “shape representation using direction functions”, see [3]).

(i) Assume that c is not *flat*, that is, the image of c is not contained in a line in the plane. Then, by the implicit function theorem, S is a smooth immersed submanifold of codimension 2 in L^2, locally near τ.

(ii) Moreover, there exists a smooth projection $\pi : V \to S$ defined in a neighborhood $V \subset L^2$ of τ such that, if $f(s)$ is smooth in s, then $\pi(f)(s)$ is smooth in s.

Proof. (i) The proof is a simple adaptation of the proof of Proposition 2.12 in [18]. Suppose by contradiction that $\nabla\phi_1, \nabla\phi_2$ are linearly dependent at $\theta \in M$, that is, there exists $a \in \mathbb{R}^2$, $a \neq 0$, such that

$$a_1 \cos(\tau(\theta)) + a_2 \sin(\tau(\theta(s))) = 0.$$

This means that, at all θ, $\partial_\theta c$ is orthogonal to (a_1, a_2), which implies that c is a flat curve. So, if c is not flat, then by the implicit function theorem (5.9 in [4]) S is a smooth immersed submanifold of L^2, locally near τ.

(ii) We adapt part of the proof of Proposition 2.15 in [18]. Fix $\tau_0 \in S$ associated to a non-flat curve c_0. Let $T = T_{\tau_0}S$ be the tangent at τ_0. It is the vector space orthogonal to $\nabla\phi_i(\tau_0)$ for $i = 1, 2$. Let $e_i = e_i(s) \in L^2 \cap C^\infty$ be near $\nabla\phi_i(\tau_0)$ in L^2, so that the map $(x, y) : T \times \mathbb{R}^2 \to L^2$,

$$(x, y) \mapsto \tau = \tau_0 + x + \sum_{i=1}^{2} e_i y_i, \tag{22}$$

is a linear isomorphism. Let M' be M in these coordinates; by the implicit function theorem, there exists an open set $U' \subset T$ with $0 \in U'$, an open $V' \subset \mathbb{R}^2$ with $0 \in V'$, and a smooth function $f : U \to \mathbb{R}^2$ such that $M' \cap (U' \times V')$ is the graph of $y = f(x)$.

We define a smooth projection $\pi' : U' \times V' \to M'$ by setting $\pi'(x, y) = (x, f(x))$. This may be expressed in L^2. Let $(x(\tau), y(\tau))$ be the inverse of (22) and $U = x^{-1}(U')$; we define the projection $\pi : U \to M$ by setting

$$\pi(\tau) = \tau_0 + x + \sum_{i=1}^{2} e_i f_i(x(\tau)).$$

Then

$$\pi(\tau)(s) - \tau(s) = \sum_{i=1}^{2} e_i(s) a_i, \qquad a_i := f_i(x(\tau)) - y_i \in \mathbb{R}, \tag{23}$$

so if $\tau(s)$ is smooth, then $\pi(\tau)(s)$ is smooth. $\square$

Proof of Theorem 27. We sketch how we can approximate c by smooth curves. Since the metric is independent of reparameterization and rescaling, we rescale c, and assume that $|\partial_\theta c| = 1$.

As a first step, we assume that c is not *flat*; then, by Lemma 28, S is a manifold near τ; and let π be as in the above lemma.

Let f_n be a sequence of smooth functions with $f_n \to \tau$ in L^2; then $g_n := \pi(f_n) \to \tau$. Let then

$$G_n(\theta, t) := \pi(t\tau + (1-t)g_n)(\theta)$$

be the projection on S of the linear path connecting g_n to τ. Since S is smooth in V, the L^2 distance $\|\tau - g_n\|$ is equivalent to the geodesic induced distance; in particular,

$$\lim_n \mathbb{E}_S(G_n) = 0$$

where

$$\mathbb{E}_S(G) := \int_0^1 \|\partial_t G(\cdot, t)\|_{L^2}^2 \,\mathrm{d}t$$

is the *action of the path* G in $S \subset L^2$.

The above G_n can be associated to a homotopy by defining

$$C_n(s,t) := c(0) + \int_0^s (\cos(G_n(\theta,t)), \sin(G_n(\theta,t)))\,\mathrm{d}\theta;$$

note that $C_n(s,0) = c(s)$ and $C_n(s,1)$ is a smooth closed curve.

We now compute the H^1 action of C_n,

$$\mathbb{E}_{H^1}(C_n) := \int_0^1 \|\partial_t C_n\|_{H^1}^2 \,\mathrm{d}t = \int_0^1 \int_0^{2\pi} (|\partial_t C_n|^2 + |D_s \partial_t C_n|^2)\,\mathrm{d}s\,\mathrm{d}t.$$

Since any $C_n(\cdot, t)$ is parameterized by arc parameter, we have $D_s \partial_t C_n = \partial_{st} C_n$ so

$$D_s \partial_t C_n = N(s) \partial_t G_n(s,t)$$

where

$$N(s) := (-\sin(G_n(s,t)), \cos(G_n(s,t)))$$

is the normal to the curve; so the second term in the action $\mathbb{E}_{H^1}(C_n)$ is exactly equal to $\mathbb{E}_S(G_n)$, that is,

$$\mathbb{E}_{H^1}(C_n) = \int_0^1 \|\partial_t C_n\|_{H^1}^2 \,\mathrm{d}t = \int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 \,\mathrm{d}s\,\mathrm{d}t + \mathbb{E}_S(G_n).$$

We can also prove that $\int_0^1 \int_0^{2\pi} |\partial_t C_n|^2 \,\mathrm{d}s\,\mathrm{d}t \to 0$, so $\mathbb{E}_{H^1}(C_n) \to 0$, and then

$$\lim_n \mathrm{Len}_{H^1}(C_n) = 0.$$

Now we assume that c is *flat*, that is, the image of c is contained in a line in the plane; then, up to translation and rotation,

$$c(\theta) = (c_1(\theta), 0);$$

since c is parametrized by arc parameter, $\dot{c}_1 = \pm 1$. Let then $f : [0, 2\pi] \to \mathbb{R}$ be smooth and with support in $[1,3]$ and $f(2) = 1$; let moreover

$$C(\theta, t) := (c_1(\theta), t f(\theta))$$

so

$$|\partial_\theta C| = \sqrt{1 + (t f'(\theta))^2} \geqslant 1.$$

Then, by direct computation, we can prove that $\mathrm{Len}_{H^1}(C) < \infty$; moreover, any curve $C(\cdot, t)$ for $t > 0$ is not flat, so it can be approximated by smooth curves. $\square$

3.ii *The completion of M according to H^2 distance*

Let $d(c_0, c_1)$ be the geometric distance induced by H^2 on M (Definition 12). Let $E(c) := \int |D_s^2 c|^2 \,\mathrm{d}s$ be defined on non-constant smooth curves. We prove that

THEOREM 29 (i) E is locally Lipschitz in M with respect to d; and the local Lipschitz constant depends on the length of c.

(ii) As a corollary, all non-constant curves in the completion of $C^\infty(S^1 \to \mathbb{R}^n)$ according to the metric H^2 admit curvature as a measurable function, and the energy of the curvature is finite, that is, $E(c) < \infty$.

(iii) Vice versa, any non-constant curve admitting curvature in a weak sense and satisfying $E(c) < \infty$ is approximable by smooth curves.

The rest of this section is devoted to proving the above three statements.

Proof of 29(i). Fix a curve c_0; let $L_0 := \operatorname{len} c_0$ be its length. By (21) and Proposition 21, the length function $c \mapsto \operatorname{len}(c)$ is Lipschitz in M with respect to the distance d, that is,

$$|\operatorname{len}(c_0) - \operatorname{len}(c_1)| \leqslant a_1 d(c_0, c_1)$$

where a_1 is a positive constant (dependent on λ).

Choose any c_1 with $d(c_0, c_1) < L_0/(4a_1)$. Let $C(\theta, t)$ be a smooth homotopy connecting c_0 to (a reparameterization of) c_1; choose it so that $\operatorname{Len} C < 2d(c_0, c_1)$. Then $\operatorname{Len} C < L_0/(2a_1)$.

Let $L(t) := \operatorname{len} C(\cdot, t)$ be the length of the curve at time t. Since at all times $t \in [0, 1]$, $d(c_0, C(\cdot, t)) < L_0/(2a_1)$, we have $|L(t) - L_0| < a_1 L_0/(2a_1) = L_0/2$; in particular,

$$L_0/2 < L(t) < L_0 3/2.$$

By using this last inequality, we are allowed to discard $L(t)$ in most of the following estimates.

We set $\|f\| := \sqrt{\int |f(s)|^2 \,\mathrm{d}s}$ and

$$N(t) := \|D_s^2 \partial_t C(\cdot, t)\| = \sqrt{\int |D_s^2 \partial_t C|^2 \,\mathrm{d}s}$$

for convenience; using this notation, we recall that

$$\|\partial_t C\|_{H^2} = \sqrt{\lambda L(t)^3 N(t)^2 + \frac{1}{L(t)} \|\partial_t C\|^2};$$

so $\|\partial_t C\|_{H^2} \geqslant \sqrt{\lambda} L^{3/2} N(t)$.

Up to reparameterization in the t parameter, we can suppose that the path $t \mapsto C(\cdot, t)$ in M is parametrized by (approximate) arc parameter, that is, $\|\partial_t C\|_{H^2}$ is (almost) constant in t; so we assume, with no loss of generality, that $\|\partial_t C\|_{H^2} \leqslant 2d(c_0, c_1)$ for all $t \in [0, 1]$, and then $N(t) \leqslant a_2 d(c_0, c_1)$ where $a_2 = 2/\sqrt{(L_0/2)^3 \lambda}$.

We want to prove that

$$E(c_1) - E(c_0) \leqslant a_5 d(c_0, c_1)$$

where the constant a_5 will depend on L_0 and λ.

By direct computation

$$\partial_t E(C(\cdot,t)) = \int |D_s^2 C|^2 \langle D_s \partial_t C, D_s C\rangle \, \mathrm{d}s + 2\int \langle D_s^2 C, \partial_t D_s^2 C\rangle \, \mathrm{d}s.$$

We deal with the two terms separately.

By the Poincaré inequality (15) we deduce

$$\sup_\theta |D_s \partial_t C| \leqslant \frac{1}{2}\int |D_s^2 \partial_t C| \, \mathrm{d}s \leqslant \sqrt{L(t)}\sqrt{\int |D_s^2 \partial_t C|^2 \, \mathrm{d}s} = \sqrt{L(t)} N(t)$$

since $\operatorname{avg}(D_s \partial_t C) = 0$. So we estimate the first term as

$$\int |D_s^2 C|^2 \langle D_s \partial_t C, D_s C\rangle \, \mathrm{d}s \leqslant E(C)\sqrt{L(t)} N(t).$$

On the other hand, the commutator of D_s and ∂_t is $\langle D_s \partial_t c, D_s c\rangle D_s$: indeed,

$$\begin{aligned}\partial_t D_s &= \frac{1}{|\partial_\theta c|}\partial_\theta \partial_t + \left(\partial_t \frac{1}{|\partial_\theta c|}\right)\partial_\theta = D_s \partial_t - \frac{\langle \partial_t \partial_\theta c, \partial_\theta c\rangle}{|\partial_\theta c|^3}\partial_\theta \\ &= D_s \partial_t - \langle D_s \partial_t c, D_s c\rangle D_s,\end{aligned}$$

so

$$\begin{aligned}\partial_t D_s^2 C &= D_s \partial_t D_s C - \langle D_s \partial_t C, D_s C\rangle D_s^2 C \\ &= D_s^2 \partial_t C - D_s(\langle D_s \partial_t C, D_s C\rangle D_s C) - \langle D_s \partial_t C, D_s C\rangle D_s^2 C \\ &= D_s^2 \partial_t C - \langle D_s^2 \partial_t C, D_s C\rangle D_s C - \langle D_s \partial_t C, D_s^2 C\rangle D_s C - 2\langle D_s \partial_t C, D_s C\rangle D_s^2 C,\end{aligned}$$

so (since $|D_s C| = 1$)

$$\|\partial_t D_s^2 C\| \leqslant 2\|D_s^2 \partial_t C\| + 3\|D_s^2 C\| \sup |D_s \partial_t C|,$$

which yields an estimate of the second term:

$$\int \langle D_s^2 C, \partial_t D_s^2 C\rangle \, \mathrm{d}s \leqslant \sqrt{E(C)}(2N(t) + 3\sqrt{E(C)}\sqrt{L(t)} N(t))$$

by using Cauchy–Schwarz.

Summing up,

$$|\partial_t E(C(\cdot,t))| \leqslant 2\sqrt{E(C)} N(t) + 4E(C)\sqrt{L(t)} N(t)$$

or, since $\sqrt{x} \leqslant 1 + x$,

$$|\partial_t E(C(\cdot,t))| \leqslant 2N(t) + 2E(C)N(t) + 4E(C)\sqrt{L(t)} N(t).$$

We recall that $N(t) \leqslant a_2 d(c_0, c_1)$, $L(t) \leqslant L_0 3/2$, so we rewrite the above as

$$|\partial_t E(C(\cdot,t))| \leqslant 2a_2 d(c_0, c_1) + 2E(C)a_2 d(c_0, c_1) + 4E(C)a_4 a_2 d(c_0, c_1)$$

with $a_4 = \sqrt{L_0 3/2}$. Apply Gronwall's lemma to obtain

$$E(c_1) \leqslant (E(c_0) + 2a_2 d(c_0, c_1)) \exp((2 + 4a_4)a_2 d(c_0, c_1)).$$

Let

$$g(y) := (E(c_0) + 2a_2 y) \exp((2 + 4a_4)a_2 y).$$

Then $E(c_1) \leqslant g(d(c_0, c_1))$; since g is convex, and $g(0) = E(c_0)$, there exists $a_5 > 0$ such that $g(y) \leqslant E(c_0) + a_5 y$ when $0 \leqslant y \leqslant L_0/(4a_1)$; since we assumed that $d(c_0, c_1) < L_0/(4a_1)$, it follows that

$$E(c_1) \leqslant E(c_0) + a_5 d(c_0, c_1).$$

Note that a_5 is ultimately dependent on L_0 and λ.

This ends the proof of the first statement of Theorem 29.

Proof of Theorem 29(ii). To prove the second statement, let $(c_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Since $d_{H^1} \leqslant a d_{H^2}$, as in the proof of Theorem 26 we assume that, up to reparameterization and a choice of subsequence, c_n converges uniformly to a Lipschitz curve c.

Let $L_0 = \operatorname{len} c$. We have assumed in the statement that c is non-constant, so $L_0 > 0$.

Again, the length function $c \mapsto \operatorname{len}(c)$ is Lipschitz, so the sequence $\operatorname{len}(c_n)$ is Cauchy in $\mathbb{R}$, hence it converges; moreover, $c \mapsto \operatorname{len}(c)$ is lower semicontinuous with respect to uniform convergence, so $\lim_n \operatorname{len}(c_n) \geqslant \operatorname{len}(c) > 0$. Consequently, we assume that, up to taking a subsequence, $2L_0 \geqslant \operatorname{len}(c_n) \geqslant L_0$.

We proved above that, in a neighborhood of c of size $L_0/(8a_1)$, the function $E(c) := \int |D_s^2 c|^2 \,\mathrm{d}s$ is Lipschitz; so the sequence $E(c_n)$ is bounded, and therefore (since curves are parameterized by arc parameter and $\operatorname{len}(c_n) \geqslant L_0$) the energy $\int |\partial_\theta^2 c|^2 \,\mathrm{d}s$ is bounded. Then $\partial_\theta c_n$ are uniformly Hölder continuous, so by the Ascoli–Arzelà compactness theorem, up to a subsequence, $\partial_\theta c_n(\theta)$ converges.

As a corollary, $\lim_n \operatorname{len}(c_n) = \operatorname{len}(c)$, c is parameterized by arc parameter, and $D_s c_n(\theta)$ converges to $D_s c(\theta)$.

Since the functional $\int |\partial_\theta^2 c_n|^2 \,\mathrm{d}s$ is bounded in n, a theorem in [1] shows that c admits weak derivative $\partial_\theta^2 c$ and $\int |\partial_\theta^2 c|^2 \,\mathrm{d}s < \infty$, and equivalently, $\int |D_s^2 c|^2 \,\mathrm{d}s < \infty$.

Proof of 29(iii). For the third statement, let c be a rectifiable curve, and assume that it is non-constant, and $E(c) < \infty$. Since the metric is independent of rescaling, we rescale c, and assume that $|\partial_u c(u)| = 1$.

We expand in Fourier series

$$c(u) = \sum_{n\in\mathbb{Z}} l_n \exp(inu) \tag{24}$$

(by viewing $S^1 = \mathbb{R}/2\pi$), and set

$$C(u, t) := \sum_{n\in\mathbb{Z}} l_n \exp(inu - f(n)t) \tag{25}$$

with $f(n) = f(-n) \geqslant 0$ and $\lim f(n)/\log(n) = \infty$ (for example, $f(n) = |n|$ or $f(n) = (\log(|n| + 2))^2$). Then $C(\cdot, t)$ is smooth for any $t > 0$.

We want to prove that, for t small, $C(\cdot, t)$ is near c in the H^2 metric. To this end, let $\tilde{C}$ be the linear interpolator

$$\tilde{C}(u, t, \tau) := (1 - \tau)c(u) + \tau C(u, t) = \sum_{n\in\mathbb{Z}} l_n e^{inu}(1 - \tau + \tau e^{-f(n)t}). \tag{26}$$

We will prove that

$$\int_0^1 \left(\int_{S^1} |\partial_\tau \tilde{C}|^2 \, \mathrm{d}s + \lambda L^4 \int_{S^1} |D_s^2 \partial_\tau \tilde{C}|^2 \, \mathrm{d}s \right) \mathrm{d}\tau \; < \delta(t) \tag{27}$$

where $\lim_{t \to 0} \delta(t) = 0$, and L is the length of $\tilde{C}(\cdot, t, \tau)$.

We need some preliminary results:

- We prove that

$$\int_{S^1} |\partial_{uu} c - \partial_{uu} \tilde{C}|^2 \, \mathrm{d}u \; < \delta_1(t) \tag{28}$$

where $\lim_{t \to 0} \delta_1(t) = 0$, uniformly in $\tau \in [0, 1]$; indeed,

$$\int_{S^1} |\partial_{uu} c - \partial_{uu} \tilde{C}|^2 \, \mathrm{d}u = 2\pi \tau^2 \sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4 (1 - e^{-f(n)t})^2$$

and since

$$2\pi \sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4 = E(c) = \int_{S^1} |\partial_{uu} c|^2 \, \mathrm{d}s \; < \infty$$

and $\lim_{t \to 0} (1 - e^{-f(n)t})^2 = 0$, we can apply Lebesgue's dominated convergence theorem.

- We prove that

$$|\partial_u c - \partial_u \tilde{C}| \; < \delta_2(t) \tag{29}$$

where $\lim_{t \to 0} \delta_2(t) = 0$ uniformly in u and $\tau \in [0, 1]$; indeed,

$$\begin{aligned} |\partial_u c - \partial_u \tilde{C}| &\leqslant \tau \sum_{n \in \mathbb{Z}} |l_n| \, |n| (1 - e^{-f(n)t}) \\ &\leqslant \sqrt{\sum_{n \in \mathbb{Z}} |l_n|^2 |n|^4} \sqrt{\sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{n^2} (1 - e^{-f(n)t})^2} \end{aligned}$$

and again we apply Lebesgue's dominated convergence theorem.

- By the above we also see that for t small,

$$3/2 \geqslant |\partial_u \tilde{C}| \geqslant 1/2 \quad \text{uniformly in } \tau, u. \tag{30}$$

- We can similarly prove that

$$|c - \tilde{C}| \; < \delta_3(t). \tag{31}$$

Returning to the proof of (27), by direct computation, we have

$$D_s^2 \partial_\tau \tilde{C} = \frac{\partial_{uu\tau} \tilde{C}}{|\partial_u \tilde{C}|^2} + \frac{\langle \partial_{uu} \tilde{C}, \partial_u \tilde{C} \rangle \partial_{u\tau} \tilde{C}}{|\partial_u \tilde{C}|^4}.$$

Then, for t small, by (30),

$$|D_s^2 \partial_\tau \tilde{C}| \leqslant 4 |\partial_{uu\tau} \tilde{C}| + 24 |\partial_{uu} \tilde{C}| \, |\partial_{u\tau} \tilde{C}|.$$

We use the fact that

$$\partial_{uu\tau}\tilde{C} = \partial_{uu}C - \partial_{uu}c, \qquad \partial_{u\tau}\tilde{C} = \partial_{u}C - \partial_{u}c, \qquad \partial_{\tau}\tilde{C} = C - c,$$

so by (28) and (29),

$$\iint |D_s^2 \partial_\tau \tilde{C}|^2 \,\mathrm{d}s\,\mathrm{d}\tau \leqslant a_1(\delta_1(t) + E(c)\delta_2(t)),$$

and by (31), $\iint |\partial_\tau \tilde{C}|^2 \,\mathrm{d}s\,\mathrm{d}\tau \leqslant 8\delta_3(t)$. Finally, we combine all the above to obtain (27) with $\delta(t) = a_2\delta_3(t) + \lambda a_2(\delta_1(t) + E(c)\delta_2(t))$.

This concludes the proof.

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