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Traveling waves for the Keller–Segel system with Fisher birth terms

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We consider the traveling wave problem for the one-dimensional Keller–Segel system with a birth term of either a Fisher/KPP type or with a truncation for small population densities. We prove that there exists a solution under some stability conditions on the coefficients which enforce an upper bound on the solution and $\dot{H}^1(\mathbb{R})$ estimates. Solutions in the KPP case are built as a limit of traveling waves for the truncated birth rates (similar to ignition temperature in combustion theory).

We also discuss some general bounds and long time convergence for the solution of the Cauchy problem and in particular linear and nonlinear stability of the nonzero steady state.

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1. The main result

The growth of bacterial colonies undergoes complex biomechanical processes which underly the variety of interfaces exhibited by the colonies occupancy region. Usually cells divide and undergo active motion resulting in fronts of bacteria that are propagating and delimit a free boundary between the colonized and uncolonized areas. These fronts may be unstable leading to various patterns that have been studied for a long time, such as, for instance, spiral waves [17], aggregates [19] and dendrites [1, 11]. At least three elementary biophysical processes play commonly a central role in these patterns, and have been used in all modeling: (i) cell division which induces the growth of the colony, (ii) random cell motion—for instance, bacteria can swim in a liquid medium thanks to flagella, and (iii) chemoattraction through different molecules that the cells may release in their environment and that diffuse, leading to some kind of (possibly long distance) communication. Our purpose here is to study the existence of traveling waves and the linear and nonlinear stability of the

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steady states for a simple model combining these three effects. The macroscopic model describes the density of bacteria, denoted by $u(t, x)$ below, and the chemoattractant concentration $v(t, x)$ in the medium. It is a variant of the Keller–Segel system that has been widely studied in various contexts (see [5, 13, 20, 21, 7] and references therein).

We consider the one-dimensional Keller–Segel system with a Fisher–KPP birth term (we will refer to it as the *Keller–Segel–Fisher system*)

$$\begin{cases} u_t - u_{xx} + \chi(uv_x)_x = u(1-u), \\ -dv_{xx} + v = u. \end{cases} \tag{1}$$

Here the notation u_t or u_x means time or space derivatives. The boundary conditions for u and v are

$$v(-\infty) = u(-\infty) = 1, \qquad v(+\infty) = u(+\infty) = 0, \tag{2}$$

that is, there are no bacteria on the right. The two parameters χ and d are, respectively, the sensitivity of the cells to chemoattraction, and the diffusion coefficient of the chemoattractant. The traveling wave solutions moving with a speed c (which becomes a new unknown of the problem) for (1) are special solutions of the form $u(x - ct)$ and $v(x - ct)$ that satisfy

$$\begin{cases} -cu' - u'' + \chi(uv')' = u(1-u), \\ -dv'' + v = u, \end{cases} \tag{3}$$

together with the boundary conditions (2). We prove the following result.

THEOREM 1.1 Let $\chi > 0$ and $d > 0$ satisfy

$$\chi < \min(1, d). \tag{4}$$

Then there exists a traveling wave solution (c_*, u, v) of (3) with the boundary conditions (2) and a constant $K(d, \chi)$ such that the functions $u(x)$ and $v(x)$ and the speed c_* satisfy

$$0 < u(x), v(x) \leqslant \left(1 - \frac{\chi}{d}\right)^{-1}, \tag{5}$$

$$\int u(x)(1-u(x))^2\,\mathrm{d}x + \int |u'(x)|^2\,\mathrm{d}x + \int |v'(x)|^2\,\mathrm{d}x \leqslant K(d, \chi), \tag{6}$$

$$2 \leqslant c_* \leqslant 2 + \frac{\chi\sqrt{d}}{d - \chi}. \tag{7}$$

Writing the second equation as a convolution $v = K_d \star u$, one may view this system as a Fisher equation with a nonlocal drift. Reaction-diffusion with nonlocal reaction or diffusion terms has recently been investigated (see [4, 8, 10, 12]), but, as far as we know, not for a nonlocal drift term. Nonlocal terms may make the homogeneous positive state unstable and then create periodic stable patterns. In this paper, we need some conditions on the coefficients, such as (4), that imply the stability of the state $u = v \equiv 1$.

Other situations where traveling waves appear in chemotaxis have been considered in the literature. For instance, [14] considers a source term for the chemoattractant in the equation on v, and [9] considers existence of traveling fronts by a linearization analysis (for small bacterial diffusion). There are also other related models of biological interest: see for instance the case of haptotaxis in [18]. We also refer to these papers for further references on fronts and waves for cell populations as well as to [22, 23] for the general theory of traveling waves.

Our strategy for the proof of Theorem 1.1 is as follows. We introduce a smooth monotonic cut-off function $g_0(u)$ such that $g_0(u) = 0$ for $u \leqslant 1$ and $g_0(u) = 1$ for $u \geqslant 2$ and set $g(u) = g_0((u - \theta_0)/\theta_0)$—this function has a cut-off $\theta_0 \in (0, 1)$. Consider a regularized system

$$\begin{cases} -cu' - u'' + \chi(g(u)uv')' = g(u)u(1-u), \\ -dv'' + v = u, \end{cases} \tag{8}$$

with the same boundary conditions (2). The system with the cut-off is of independent interest—the cut-off means that bacteria feel the chemoattractant and reproduce only if their density exceeds a critical threshold value. Mathematically, the role of the cut-off is very similar to that of the ignition temperature in combustion theory [15]. The first step in the proof of Theorem 1.1 is to construct a traveling wave solution $(c(\theta_0), u(x;\theta_0), v(x;\theta_0))$ of (8) for $\theta_0 > 0$—as we have mentioned, this result is of independent interest. We do this for $\theta_0 > 0$ sufficiently small and also obtain (uniform in θ_0) bounds on $c(\theta_0)$ and $u(x;\theta_0)$, $v(x;\theta_0)$.

PROPOSITION 1.2 Let $\chi > 0$ and $d > 0$ satisfy

$$\frac{1}{\chi} > \frac{1}{d} + 1. \tag{9}$$

Then there exists $\alpha_0 > 0$ such that for all $\theta_0 \in (0, \alpha_0)$ there exists a traveling wave solution $(c(\theta_0), u(x;\theta_0), v(x;\theta_0))$ of (8), (2). In addition, there exists a constant $K > 0$ which does not depend on θ_0 such that we have the following uniform bounds:

$$\begin{cases} 0 < u(x;\theta_0), v(x;\theta_0) \leqslant (1-\chi/d)^{-1}, \quad 0 < 1/K \leqslant c(\theta_0) \leqslant K < \infty, \\ \displaystyle\int g(u(x;\theta_0))u(x;\theta_0)(1-u(x;\theta_0))^2\,\mathrm{d}x + \int |u'(x;\theta_0)|^2\,\mathrm{d}x + \int |v'(x;\theta_0)|^2\,\mathrm{d}x \leqslant K. \end{cases} \tag{10}$$

Here and throughout the paper we denote by C and K generic constants which may depend on χ and d but not on the cut-off θ_0 or the size a of the approximating finite interval which appears later in the proof. We recall that in the case of a single equation with no chemoatractant coupling ($\chi = 0$) the speed $c(\theta_0)$ is unique for $\theta_0 > 0$ [15].

In this general framework it seems difficult to relax the size condition (9) and to achieve the more general condition (4) that we use in Theorem 1.1. This is possible if we introduce two modifications in the above procedure. First, we consider another regularization of the system:

$$\begin{cases} -cu' - u'' + \chi(g(u)uv')' = g(u)u(1-u), \\ -dv'' + v = g(u)u, \end{cases} \tag{11}$$

that is, the chemoattractant source also now has a small density cut-off. Second, we tune the truncation function appropriately—we now choose it with the following properties:

$$\begin{cases} g(u) = 0 \quad \text{for } u \leqslant \theta_0, \quad g' \geqslant 0, \quad g(u) = 1 \quad \text{for } u \geqslant 1, \\ g(u) + ug'(u) \leqslant 1 + \alpha(\theta_0) \quad \text{with } \alpha(\theta_0) \xrightarrow[\theta_0 \to 0]{} 0, \\ g(u) \text{ increases to 1 for } u \in (0,1) \text{ as } \theta_0 \to 0. \end{cases} \tag{12}$$

The reader can easily check that these conditions are satisfied by the function

$$g(u) = 1 + 2\alpha(1 + \ln(u) - u)$$

with $\alpha(\theta_0)$ normalized so that $g(\theta_0) = 0$.

PROPOSITION 1.3 Assume that the cut-off g has the properties (12) and that χ and d satisfy the condition (4). Then there exists $\alpha_0 > 0$ such that for all $\theta_0 \in (0, \alpha_0)$ there exists a traveling wave solution $(c(\theta_0), u(x; \theta_0), v(x; \theta_0))$ of (11) with the boundary conditions (2) which satisfies the estimates (5), (10) and

$$K \leqslant c(\theta_0) \leqslant 2 + (1+\alpha)\frac{\chi\sqrt{d}}{d-\chi}, \tag{13}$$

with a constant $K > 0$ which does not depend on $\theta_0 \in (0, \alpha_0)$.

This proposition allows us to pass to the limit $\theta_0 \to 0$ and obtain a traveling wave solution of the original problem (3) without a cut-off as stated in Theorem 1.1 and with the smallness condition (4) on the chemotaxis. The traveling waves for a positive cut-off $\theta_0 > 0$ in Propositions 1.2 and 1.3 are constructed by first building an approximate solution on a finite interval $-a \leqslant x \leqslant a$ and then letting $a \to +\infty$, the strategy originated in [3].

This method enables us to find *one* speed $c(\theta_0)$ for a traveling wave. This is what we expect when there is a positive cut-off θ_0 since there is a unique speed of propagation for the Fisher equation with ignition type nonlinearity. In the case of the Fisher equation with positive nonlinearity, it is well-known that there exists a half-line $[c_*, \infty)$ of speeds associated with traveling waves. The minimal speed gives the only stable wave, since the solution of the Cauchy problem with compactly supported initial data converges to this wave. In this paper we only prove that there exists at least one speed associated with a traveling wave for the Keller–Segel–Fisher equation. We do not know if this speed is minimal and if there exist any traveling waves with higher speeds.

By construction, the traveling wave solutions in Theorem 1.1 have a nonlinear stability property with respect to the perturbations of the birth term, under condition (4). This condition arises several times in our proof but we do not know if it is sharp: it implies the less restrictive condition $\chi < d$ which provides us with the maximum principle for u, but it is also instrumental in deriving the other fundamental a priori estimates in (10). It is interesting that the linear stability condition of the steady state solutions (1, 1) of (1) is much weaker than (4). To see that, we linearize the problem in the neighborhood of (1, 1) and write

$$u = 1 + U, \qquad v = 1 + V, \qquad \text{where } U, V \ll 1.$$

One finds the linearized equations

$$\begin{cases} U_t - U_{xx} + \chi V_{xx} = -U, \\ -dV_{xx} + V = U. \end{cases} \tag{14}$$

Taking the Fourier transform we obtain

$$\begin{cases} \widehat{U}_t + k^2\widehat{U} - \chi k^2\widehat{V} = -\widehat{U}, \\ dk^2\widehat{V} + \widehat{V} = \widehat{U}, \end{cases}$$

and since $\widehat{V}$ can be explicitly computed in terms of $\widehat{U}$, we reduce it to

$$\widehat{U}_t + \left[k^2 + 1 - \frac{\chi k^2}{1+dk^2}\right]\widehat{U} = 0.$$

This equation is *linearly stable* if and only if

$$k^2 + 1 - \frac{\chi k^2}{1+dk^2} \geqslant 0 \quad \text{for all } k \in \mathbb{R}.$$

Setting $X = k^2 \geqslant 0$, we find the equivalent condition $1 + X(d + 1 - \chi) + dX^2 \geqslant 0$ for $X \geqslant 0$, which in turn is equivalent to

$$\chi \leqslant (1 + \sqrt{d})^2. \tag{15}$$

In this case the steady state (1, 1) is linearly stable. When this condition is violated as in [9] unstable patterns arise. Condition (4) is of course stronger than (15) and even the sufficient condition $\chi < d$ for the uniform upper bound on u and v in (10) is still stronger than (15). This leaves open the question of the optimal condition for the existence of traveling waves.

The organization of this paper is as follows. We first consider the problem with cut-off on an interval $[-a, a]$ and prove the existence by a homotopy argument in Section 2. We also establish the main estimates in that section. In Section 3 we remove the cut-off and let the interval length a tend to infinity, the main difficulty being to show that the states (1, 1) and (0, 0) are indeed connected by the solution obtained by this procedure. In the last section we establish some general bounds on the solution of the Cauchy problem and prove that the homogeneous solution is stable as soon as it is linearly stable.

2. The problem on a finite interval $[-a, a]$

Our approach follows the traditional methods (see [3] and [23], for instance), which we adapt to our specific situation. In particular, as usual, specific difficulties arise in showing that the speed c is controlled from below and above, and that the states $u = 1$ and $u = 0$ are indeed reached at infinity (see [2, 6, 16] for an example where this question is left open in the construction of travelling waves for a reactive Boussinesq system).

The finite interval approximation

In order to prove Proposition 1.2, we first construct an approximation (c_a, u_a, v_a) (we drop θ_0 in the notation for the traveling wave for the moment) on a finite interval $-a \leqslant x \leqslant a$:

$$\begin{cases} -c_a u_a' - u_a'' + \chi (g(u_a) u_a v_a')' = g(u_a) u_a (1 - u_a), \\ -d v_a'' + v_a = u_a. \end{cases} \tag{16}$$

The boundary conditions for u_a are

$$u_a(-a) = 1, \qquad u_a(a) = 0. \tag{17}$$

Instead of imposing the boundary conditions for v_a at $x = \pm a$, we extend u_a to the whole real line as

$$\bar{u}_a(x) = \begin{cases} 1, & x < -a, \\ u_a(x), & -a \leqslant x \leqslant a, \\ 0, & x \geqslant a, \end{cases} \tag{18}$$

and then we set

$$v_a(x) = \int_{-\infty}^{\infty} K_d(|x - \xi|) \bar{u}_a(\xi)\, \mathrm{d}\xi, \qquad K_d(\xi) = \frac{e^{-|\xi|/\sqrt{d}}}{2\sqrt{d}}, \qquad \int K_d(\xi)\, \mathrm{d}\xi = 1. \tag{19}$$

The function $v_a(x)$ is defined for all $x \in \mathbb{R}$ and satisfies

$$-dv_a'' + v_a = \bar{u}_a, \qquad v_a(-\infty) = 1,\ v_a(+\infty) = 0. \tag{20}$$

Three consequences of the representation formula (19) are the bounds

$$\begin{aligned}
&|v_a(x)| \leqslant \|u_a\|_\infty,\\
&|v_a'(x)| = \frac{1}{2d}\left|\int e^{-|x-\xi|/\sqrt{d}}\,\mathrm{sgn}(\xi - x)u_a(\xi)\,\mathrm{d}\xi\right| \leqslant \frac{1}{\sqrt{d}}\|u_a\|_\infty,\\
&|v_a''(x)| \leqslant \frac{C}{d}\|u_a\|_\infty,
\end{aligned} \tag{21}$$

which we frequently use.

In order to ensure that the solution u_a has a nontrivial limit as $a \to +\infty$, we normalize it so that

$$\max_{x \geqslant 0} u_a(x) = \theta_0. \tag{22}$$

This constraint indirectly fixes the speed c_a. It follows from the maximum principle and (22) that $u_a(0) = \theta_0$ and thus u_a satisfies the boundary value problem on $[0, a]$:

$$-c_a u_a'(x) - u_a''(x) = 0, \quad 0 \leqslant x \leqslant a, \qquad u_a(0) = \theta_0,\ u_a(a) = 0.$$

PROPOSITION 2.1 With the assumption (9), there exists a solution (c_a, u_a, v_a) of (16), (17), (19), (22) with nonnegative functions u_a and v_a, which in addition satisfies the uniform bounds (10).

The rest of this section is devoted to the proof of this proposition, which uses a homotopy argument. Accordingly, we introduce the homotopy parameter $\tau \in [0, 1]$ and consider a family of problems

$$\begin{cases} -c_{\tau,a}u_{\tau,a}' - u_{\tau,a}'' + \chi\tau(g(u_{\tau,a})u_{\tau,a}v_{\tau,a}')' = \tau g(u_{\tau,a})u_{\tau,a}(1 - u_{\tau,a}),\\ -dv_{\tau,a}'' + v_{\tau,a} = \tau u_{\tau,a}, \end{cases} \tag{23}$$

together with the boundary conditions (17), the first relation (19) (with the right side multiplied by the factor τ) and normalization (22). To simplify the notation we drop the subscript τ below.

A uniform upper bound for the traveling speed

We begin with an upper bound for the speed.

LEMMA 2.2 If $d > \chi$, then any solution of (17), (19), (22), (23) satisfies

$$0 \leqslant u_a(x), v_a(x) \leqslant (1 - \chi/d)^{-1}, \qquad |v_a'(x)| \leqslant C, \tag{24}$$

with the constant $C > 0$ which depends only on d and χ. In addition, there exists a constant $a_0(\theta_0) > 0$, and a constant $K > 0$ which depends only on d and χ but not on a, $\tau \in [0, 1]$, or $\theta_0 \in (0, 1)$, such that for all $a > a_0(\theta_0)$ we have

$$c_a \leqslant K < \infty. \tag{25}$$

Proof. Let us rewrite the equation (23) for u_a as

$$-c_a u_a' - u_a'' + \tau\chi g(u_a) v_a' u_a' + \tau\chi g'(u_a) u_a v_a' u_a' = \tau g(u_a) u_a (1 - u_a) - \frac{\tau\chi}{d} g(u_a) u_a (v_a - u_a)$$
$$= \tau g(u_a) u_a \left(1 - u_a + \frac{\chi}{d} u_a - \frac{\chi}{d} v_a\right) \leqslant \tau g(u_a) u_a \left(1 - u_a + \frac{\chi}{d} u_a\right). \quad (26)$$

The last inequality holds if $v_a \geqslant 0$. As $g(u) = 0$ for $u \leqslant 0$, it follows that u_a cannot attain an interior negative minimum on $(-a, a)$ and thus $u_a \geqslant 0$, which, in turn, implies that $v_a \geqslant 0$ and (26) indeed holds. It also follows from (26) that u_a cannot attain an interior maximum at a point where $u_a \geqslant (1 - \chi/d)^{-1}$. Therefore, $0 \leqslant u_a \leqslant (1 - \chi/d)^{-1}$ and hence the same bound holds for v_a. The bound for $|v_a'(x)|$ in (24) is then a consequence of (21).

Next, we show that the speed c_a is uniformly bounded from above by using a supersolution argument. The function $u_a(x)$ satisfies the inequality

$$-c_a u_a' - u_a'' + \tau\chi[g(u_a) + g'(u_a)u_a] v_a' u_a' \leqslant \tau u_a,$$

which follows from (26) and the condition $\chi/d < 1$. Let us set $\psi_M(x) = Me^{-x}$. Then the function ψ_M satisfies

$$-c_a \psi_M' - \psi_M'' + \tau\chi[g(u_a) + g'(u_a)u_a] v_a' \psi_M' = (c_a - 1 - \tau\chi g(u_a) v_a' - \tau\chi g'(u_a) u_a v_a') \psi_M$$
$$\geqslant (c_a - 1 - K_0)\psi_M,$$

with the constant $K_0 = K_0(\chi, d)$, which is uniformly bounded in a, τ and θ_0, chosen so that (using (21))

$$\chi[g(u_a) + g'(u_a)u_a]|v_a'| \leqslant \frac{\chi}{\sqrt{d}} \|g(\sigma) + g'(\sigma)\sigma\|_\infty \|u_a\|_\infty \leqslant K_0. \quad (27)$$

This is possible because of the uniform bounds in (24) and since for $u \notin (\theta_0, 2\theta_0)$ we have $g'(u) = 0$, while for $u \in (\theta_0, 2\theta_0)$ the following estimate holds:

$$|g'(u)u| = \frac{u}{\theta_0} g_0'\left(\frac{u - \theta_0}{\theta_0}\right) \leqslant 2 \max_{1 \leqslant u \leqslant 2} |g_0'(u)|.$$

Now, suppose by contradiction that

$$c_a > 2 + K_0. \quad (28)$$

Then ψ_M satisfies

$$-c_a \psi_M' - \psi_M'' + \tau\chi[g(u_a) + g'(u_a)u_a] v_a' \psi_M' \geqslant \psi_M \geqslant \tau\psi_M.$$

Note that the upper bound on $u_a(x)$ in (24) implies that $\psi_M(x) > u_a(x)$ for $M \geqslant e^a/(1 - \chi/d)$. Let us define

$$M_0 = \inf\{M : \psi_M(x) > u_a(x) \text{ for all } x \in [-a, a]\}.$$

Then $M_0 > 0$ and, in addition, $\psi_{M_0}(x) \geqslant u_a(x)$ for all $x \in [-a, a]$ and there exists $x_0 \in [-a, a]$ such that $\psi_{M_0}(x_0) = u_a(x_0)$. However, the difference $\psi_{M_0}(x) - u_a(x)$ may not attain an interior minimum at x_0 and $\psi_{M_0}(a) > 0 = u_a(a)$. Therefore, $x_0 = -a$ and thus $M_0 = e^{-a}$. As a consequence, $\theta_0 = u_a(0) \leqslant \psi_{M_0}(0) = e^{-a}$, which is a contradiction if a is sufficiently large. We conclude that (28) is impossible and thus (25) holds with $K = 2 + K_0$. □

A lower bound for the traveling speed

Now, we need a lower bound for c_a and an upper bound for $\|u_a'\|_2$.

LEMMA 2.3 With the assumptions of Lemma 2.2 and (9), there exists a constant $a_0(\theta_0) > 0$ and $K > 0$ which depends only on d and χ but not on $a > a_0$, $\theta_0 \in (0, 1)$ and $\tau \in [0, 1]$, such that for all $a > a_0$ and $\theta_0 < 1/3$ we have

$$c_a \geqslant \frac{\sqrt{\tau}}{K} - \frac{K\theta_0}{a}, \tag{29}$$

$$\tau \int_{-a}^{a} g(u_a)u_a(1-u_a)^2\,\mathrm{d}x + \int_{-a}^{a} |u_a'(x)|^2\,\mathrm{d}x + \int_{-a}^{a} |v_a'(x)|^2\,\mathrm{d}x \leqslant K. \tag{30}$$

Proof. Start with

$$-c_a u_a' - u_a'' + \tau\chi(g(u_a)u_a v_a')' = \tau g(u_a)u_a(1-u_a), \tag{31}$$

and integrate on $[-a, a]$:

$$c_a - u_a'(a) + u_a'(-a) - \tau\chi v_a'(-a) = \tau \int g(u_a)u_a(1-u_a). \tag{32}$$

Now, multiply (31) by u_a and integrate:

$$\frac{c_a}{2} + u_a'(-a) + \int_{-a}^{a} |u_a'|^2 - \tau\chi v_a'(-a) - \tau\chi \int_{-a}^{a} g(u_a)u_a u_a' v_a' = \tau \int_{-a}^{a} g(u_a)u_a^2(1-u_a).$$

Combining the last two equalities, we get

$$\frac{c_a}{2} - u_a'(a) - \int_{-a}^{a} |u_a'|^2 + \tau\chi \int g(u_a)u_a u_a' v_a' = \tau \int_{-a}^{a} g(u_a)(u_a - u_a^2)(1-u_a).$$

This can be written as

$$\tau \int_{-a}^{a} g(u_a)u_a(1-u_a)^2 + \int_{-a}^{a} |u_a'|^2 + u_a'(a) = \frac{c_a}{2} + \tau\chi \int_{-a}^{a} g(u_a)u_a u_a' v_a'. \tag{33}$$

However, on the interval $(0, a)$ we have $g(u_a) = 0$, and we can find u_a explicitly:

$$u_a(x) = \theta_0 \frac{e^{-c_a x} - e^{-c_a a}}{1 - e^{-c_a a}}, \tag{34}$$

so that

$$u_a'(a) = -\frac{c_a \theta_0 e^{-c_a a}}{1 - e^{-c_a a}}.$$

Note that for $c_a > 0$ we have

$$0 \leqslant \frac{c_a}{e^{c_a a} - 1} \leqslant \frac{1}{a},$$

while for $c_a < 0$ we have

$$0 \leqslant \frac{c_a}{e^{c_a a} - 1} = \frac{a|c_a|}{a(1 - e^{-|c_a|a})} \leqslant \frac{1 + |c_a|a}{a} = \frac{1}{a} + |c_a|.$$

Therefore, for all $c_a \in \mathbb{R}$ we have

$$|u_a'(a)| \leqslant \frac{\theta_0}{a} + |c_a|\theta_0. \tag{35}$$

We note that the special case $c_a = 0$ that we did not treat above can be easily considered separately.

We may use the representation formula (19) for v to obtain

$$v' = \tau K_d * \bar{u}', \qquad \|v'\|_{L^2} \leqslant \tau \|\bar{u}'\|_{L^2} \leqslant \tau \|u'\|_{L^2}. \tag{36}$$

Using this in (33) we obtain

$$\tau \int_{-a}^{a} g(u_a)u_a(1-u_a)^2 + \int_{-a}^{a} |u_a'|^2 \leqslant \frac{c_a}{2} - u_a'(a) + \frac{\chi\tau}{1-\chi/d}\int_{-a}^{a} |u_a'|^2. \tag{37}$$

It follows that for $0 \leqslant \tau \leqslant 1$ we have, thanks to (35),

$$\tau \int_{-a}^{a} g(u_a)u_a(1-u_a)^2 + M\int_{-a}^{a} |u_a'|^2 \leqslant \frac{c_a}{2} + |u_a'(a)| \leqslant \frac{c_a}{2} + \frac{\theta_0}{a} + \theta_0|c_a|, \tag{38}$$

with, according to condition (9),

$$M = 1 - \frac{\chi}{1-\chi/d} > 0.$$

In addition, as $u_a(-a) = 1$ and $u_a(0) = \theta_0$, there exists a constant $K > 0$ which does not depend on $\theta_0 \in (0, 1/3)$ such that

$$\left(\int g(u_a)u_a(1-u_a)^2\right)\left(\int |u_a'|^2\right) \geqslant K.$$

Therefore, provided that $a > a_0$ and $\theta_0 \in (0, 1/3)$, we have a lower bound for c_a:

$$c_a \geqslant c_0\sqrt{\tau} - \frac{C\theta_0}{a},$$

with the constants $c_0 > 0$ and $C > 0$ which do not depend on the cut-off θ_0. This is the bound in (29), while the bounds in (30) follow from the upper bound (25) for the speed, (36) and (38). □

The homotopy argument

We may now finish the proof of Proposition 2.1 using a homotopy argument. The a priori bounds obtained in Lemmas 2.2 and 2.3 allow us to use the Leray–Schauder topological degree argument to prove existence of solutions to the problem (16), (17), (19) with the normalization (22) on the bounded interval $D_a = (-a, a)$. This method of construction of traveling wave solutions goes back to [3]. We introduce a map (we suppress the subscript a now, resurrecting the subscript τ for the homotopy parameter)

$$\mathcal{K}_\tau : (c, u, v) \to (\theta_\tau, U_\tau, V_\tau)$$

as the solution operator of the linear system

$$\begin{cases} -cU_\tau' - U_\tau'' + \tau\chi(g(u)U_\tau v')' = \tau g(u)u(1-u), \\ -dV_\tau'' + V_\tau = \tau\bar{u}. \end{cases} \tag{39}$$

The boundary conditions for U_τ are as in (17):

$$U_\tau(-a) = 1, \quad U_\tau(a) = 0, \tag{40}$$

while V_τ is given explicitly as before by

$$V_\tau(x) = \tau \int_{-\infty}^{\infty} K_d(|x-\xi|)\bar{u}(\xi)\,\mathrm{d}\xi, \quad K_d(\xi) = \frac{e^{-|\xi|/\sqrt{d}}}{2\sqrt{d}}, \tag{41}$$

where $\bar{u}(x)$ is again the extension of $u(x)$ to the whole real line as in (18).

The number θ_τ is defined by

$$\theta_\tau = \theta_0 - \max_{x \geqslant 0} u(x) + c.$$

The operator $\mathcal{K}_\tau$ is a mapping of the Banach space $X = \mathbb{R} \times C^{1,\alpha}(D_a) \times C^{2,\alpha}(D_a)$, equipped with the norm $\|(c,u,v)\|_X = \max(|c|, \|u\|_{C^{1,\alpha}(D_a)}, \|v\|_{C^{1,\alpha}(D_a)})$, onto itself. A solution $s_\tau = (c_\tau, u_\tau, v_\tau)$ of the finite interval problem (16), (17), (19), (22) is a fixed point of $\mathcal{K}_\tau$ and satisfies $\mathcal{K}_\tau s_\tau = s_\tau$, and vice versa: a fixed point of $\mathcal{K}_\tau$ provides a solution. Hence, in order to establish the existence of a solution to (16), (17), (19) together with the normalization (22), it suffices to show that the kernel of the operator $\mathcal{F}_\tau = \mathrm{Id} - \mathcal{K}_\tau$ is not trivial. The standard elliptic regularity theory implies that the operator $\mathcal{K}_\tau$ is compact and depends continuously on the parameter $\tau \in [0,1]$. Thus we may apply the Leray–Schauder topological degree theory. Let us introduce a ball $B_M = \{\|(c,u,v)\|_X \leqslant M\}$. Then Lemmas 2.2 and 2.3 show that the operator $\mathcal{F}_\tau$ does not vanish on the boundary ∂B_M with M sufficiently large for any $\tau \in [0,1]$. It remains to show that the degree $\deg(\mathcal{F}_1, B_M, 0)$ in $\bar{B}_M$ is not zero. However, the homotopy invariance property of the degree implies that $\deg(\mathcal{F}_\tau, B_M, 0) = \deg(\mathcal{F}_0, B_M, 0)$ for all $\tau \in [0,1]$. Moreover, the degree at $\tau = 0$ can be computed explicitly as the operator $\mathcal{F}_0$ is given by

$$\mathcal{F}_0(c,u,v) = (\max_{x \geqslant 0} u(x) - \theta_0, u - u_0^c, v).$$

Here the function $u_0^c(x)$ solves

$$\frac{\mathrm{d}^2 u_0^c}{\mathrm{d}x^2} + c\frac{\mathrm{d}u_0^c}{\mathrm{d}x} = 0, \quad u_0^c(-a) = 1,\ u_0^c(a) = 0,$$

and is given by

$$u_0^c(x) = \frac{e^{-cx} - e^{-ca}}{e^{ca} - e^{-ca}}.$$

The mapping $\mathcal{F}_0$ is homotopic to

$$\Phi(c,u,v) = (\max_{x \geqslant 0} u_0^c(x) - \theta_0, u - u_0^c, v),$$

which in turn is homotopic to

$$\tilde{\Phi}(c,u,v) = (u_0^c(0) - \theta_0, u - u_0^{c_*^0}, v),$$

where c_*^0 is the unique number so that $u_0^{c_*}(0) = \theta_0$. The degree of the mapping $\tilde{\Phi}$ is the product of the degrees of each component. The last two have degree 1, and the first -1, as the function $u_0^c(0)$ is decreasing in c. Thus $\deg \mathcal{F}_0 = -1$ and hence $\deg \mathcal{F}_1 = -1$ so that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is not empty. This finishes the proof of Proposition 2.1. □

3. Identification of the limit as $a \to +\infty$

In this section we first let $a \to +\infty$ constructing traveling waves with a positive cut-off $\theta_0 > 0$. In the second step we remove the cut-off and obtain traveling waves for the Fisher–KPP birth rate. At this stage we only prove a loose lower bound on c_*; the more precise bound stated in Theorem 1.1 is proved in Section 4.

Passage to the whole line with a cut-off

We now prove Proposition 1.2. Having established the existence of a solution (c_a, u_a, v_a) of (16), (17), (19), (22) on a finite interval we now let $a \to +\infty$ and show that (c_a, u_a, v_a) converges to a traveling wave (c, u, v). The L^2-bound for $u'(x)$ and $v'(x)$ in Lemma 2.3 together with the uniform bounds in Lemma 2.2 and the elliptic regularity imply that there exists a sequence $a_n \to +\infty$ so that $c_n = c_{a_n}$ converges to a limit $c_*(\theta_0)$ and the functions $u_n = u_{a_n}$ and $v_n = v_{a_n}$ converge locally uniformly together with their derivatives to the limits $u(x;\theta_0)$ and $v(x;\theta_0)$. The functions $u(x)$ and $v(x)$ satisfy (we drop the dependence on θ_0 in the notation)

$$\begin{cases} -c_* u' - u'' + \chi(g(u)uv')' = g(u)u(1-u), \\ -dv'' + v = u, \end{cases} \tag{42}$$

and

$$v(x) = \int_{-\infty}^{\infty} K_d(|x-\xi|)u(\xi)\,\mathrm{d}\xi, \qquad K_d(\xi) = \frac{e^{-|\xi|/\sqrt{d}}}{2\sqrt{d}}. \tag{43}$$

Furthermore, the lower bound of Lemma 2.3 yields $c_*(\theta_0) \geqslant 1/K$, where K is a positive constant that only depends on d and χ. In particular, c_* is positive.

It remains to prove that $u(x)$ and $v(x)$ satisfy the boundary conditions (2) and, because of (43), it is sufficient to verify them for the function $u(x)$ only. The L^2-bound for the gradient of u in Lemma 2.3 and elliptic regularity imply that the function $u(x)$ has limits as $x \to \pm\infty$:

$$u_l = \lim_{x\to-\infty} u(x), \qquad u_r = \lim_{x\to+\infty} u(x).$$

The functions $u_a(x)$ are given by an explicit expression (34) on the interval $0 \leqslant x \leqslant a$. Therefore, the limit $u(x)$ is given by

$$u(x) = \theta_0 e^{-c_* x} \quad \text{for all } x \geqslant 0. \tag{44}$$

As $c_* > 0$, it follows that $u_r = 0$.

Next, we show that $u_l = 1$ when θ_0 is sufficiently small. We first note that according to the maximum principle the function u_a cannot attain a minimum at a point x where $u_a(x) \leqslant \theta_0$. Therefore, $u_a \geqslant \theta_0$ for $x \in (-a, 0)$ and thus $u_l \geqslant \theta_0$. On the other hand, the uniform bound

$$\int_{-a}^{a} g(u_a)u_a(1-u_a)^2\,\mathrm{d}x \leqslant K$$

in Lemma 2.3 implies that the limit $u(x)$ satisfies

$$\int_{-\infty}^{\infty} g(u)u(1-u)^2\,\mathrm{d}x \leqslant K. \tag{45}$$

Therefore, either $u_l = 1$ or $u_l \in [0, \theta_0]$. The previous argument implies that the only two possibilities are $u_l = \theta_0$ and $u_l = 1$. Let us assume that $u_l = \theta_0$ and find a contradiction when θ_0 is sufficiently small. With this assumption we integrate the first equation in (42) once to get

$$c_*\theta_0 = \int_{-\infty}^{\infty} g(u)u(1-u)\,\mathrm{d}x. \tag{46}$$

Multiplying the same equation by u and integrating leads to

$$\begin{aligned}\frac{c_*\theta_0^2}{2} + \int_{-\infty}^{\infty} |u'|^2\,\mathrm{d}x - \chi\int_{-\infty}^{\infty} g(u)uu'v'\,\mathrm{d}x &= \int_{-\infty}^{\infty} g(u)u^2(1-u)\,\mathrm{d}x \\ &= c_*\theta_0 - \int_{-\infty}^{\infty} g(u)u(1-u)^2\,\mathrm{d}x.\end{aligned} \tag{47}$$

Using the L^∞-bound for u and since $\|v'\|_2 \leqslant \|u'\|_2$ we get, still using condition (9),

$$\frac{c_*\theta_0^2}{2} + K\int_{-\infty}^{\infty} |u'|^2\,\mathrm{d}x + \int_{-\infty}^{\infty} g(u)u(1-u)^2\,\mathrm{d}x \leqslant c_*\theta_0, \tag{48}$$

with $K > 0$, as in the computation leading to (38). Note that since $u_l = u(0) = \theta_0$ and $u(x)$ cannot attain a local minimum at a value below θ_0, the function $u(x)$ attains its maximum at some point x_M—otherwise, $g(u) \equiv 0$ and $c_* = 0$, which would be a contradiction. For the same reason, $u_M = u(x_M) > \theta_0$ since the integral on the right side of (46) is positive because $c_* > 0$. Observe that if $u_M > 1/2$ and $u_l = \theta_0 < 1/3$, then there exists $K_1 > 0$ which does not depend on θ_0 so that

$$\int_{-\infty}^{\infty} |u'|^2 + \int_{-\infty}^{\infty} g(u)u(1-u)^2 \geqslant K_1.$$

Therefore, as c_* is bounded from above, it follows from (48) that there exists $\alpha_0 > 0$ so that if $\theta_0 \in (0, \alpha_0)$ then $\theta_0 < u_M < 1/2$.

Next, assume that $\theta_0 \in (0, \alpha_0)$ and integrate the first equation in (42) between $-\infty$ and x_M to get

$$-c_*(u_M - \theta_0) + \chi g(u_M)u_M v'(u_M) = \int_{-\infty}^{x_M} g(u)u(1-u)\,\mathrm{d}x. \tag{49}$$

As $u_M < 1/2$, the right side above is positive. In addition, we have $\|v'\|_{L^\infty} \leqslant C\|u\|_\infty = Cu_M$ and

$$g(u) = g_0\left(\frac{u-\theta_0}{\theta_0}\right) \leqslant \frac{C(u-\theta_0)}{\theta_0}$$

for $u \geqslant \theta_0$. Then (49) implies

$$-c_*(u_M - \theta_0) + \frac{C\chi(u_M - \theta_0)u_M^2}{\theta_0} \geqslant 0.$$

Therefore, as $c_* > 0$ and $u_M > \theta_0$, we have

$$u_M^2 \geqslant K\theta_0 \quad \text{with } K > 0. \tag{50}$$

In particular, $u_M \geqslant 2\theta_0$ when θ_0 is sufficiently small. Let x_0 be the first point to the left of x_M such that $u(x_0) = u_M/2$, that is, $u(x) \in [u_M/2, u_M]$ for all $x \in (x_0, x_M)$ and $g(u(x)) = 1$ on this interval. Set $L = x_M - x_0$. Then we have, using (48),

$$c_*\theta_0 \geqslant K \int_{x_0}^{x_M} |u'|^2 + \int_{x_0}^{x_M} g(u)u(1-u)^2 \geqslant C\left[\frac{u_M^2}{L} + u_M L\right] \geqslant C u_M^{3/2}.$$

It follows that $u_M \leqslant C\theta_0^{2/3}$, which contradicts (50). This contradiction shows that $u_l = \theta_0$ is impossible when θ_0 is sufficiently small. Therefore, $u_l = 1$. This finishes the proof of Proposition 1.2. □

Proof of Proposition 1.3. We now indicate the additional arguments necessary to arrive at the statement of Proposition 1.3, that is, how existence of traveling waves can be deduced under the weaker restriction (4) on the chemotaxis parameter χ.

The entire proof above of Proposition 1.2 goes through with the general assumption (12) on g. We now indicate how we can take advantage of the property

$$g + \sigma g' \leqslant 1 + \alpha. \tag{51}$$

First, the upper bound on c_* in (7) follows clearly from the value K_0 computed in (27).

Now, we prove gradient and “reaction” bounds in (6). To do that we use equation (33), and the key point is to handle the right hand side more carefully with the help of (51): we split the integral as

$$\chi \int_{-a}^{a} \tau g(u_a)u_a u_a' v_a' = \chi\tau \int_{-a}^{a} [g(u_a)u_a - 1]u_a' v_a' + \chi\tau \int_{-a}^{a} u_a' v_a'.$$

We treat separately the two terms on the right side.

Using the equation on v in (11), which now also has the small density cut-off, we have

$$\begin{aligned}\left(\chi\tau \int_{-a}^{a} u_a' v_a'\right)^2 &\leqslant \chi^2 \int_{-a}^{a} (u_a')^2 \int_{-a}^{a} (v_a')^2 \leqslant \chi^2 \int_{-a}^{a} (u_a')^2 \int_{-a}^{a} [(\tau g(u_a) + u_a \tau g'(u_a))u_a']^2 \\ &\leqslant \chi^2 (1+\alpha)^2 \left(\int_{-a}^{a} (u_a')^2\right)^2.\end{aligned}$$

This term is nicely absorbed for $\chi < 1$ and α (or, equivalently, θ_0) small enough by the corresponding term on the left hand side of (33).

For the other term, we introduce the function

$$h(u) = \int_1^u [g(\sigma)\sigma - 1]\,\mathrm{d}\sigma \quad \text{ for } 0 \leqslant u \leqslant 1$$

and with $h(u) = 0$ for $u \geqslant 1$. Note that

$$0 \leqslant h(u) \leqslant \tfrac{1}{2}(1-u)^2, \quad h(1) = 0.$$

We write

$$\chi\tau\int_{-a}^{a}[g(u_a)u_a-1]u_a'v_a' = \chi\tau\int_{-a}^{a}h(u_a)'v_a'$$
$$= \chi\tau\int_{-a}^{a}h(u_a)(-v_a)''+\chi\tau h(u_a)v_a'|_{x=a}-\chi\tau h(u_a)v_a'|_{x=-a}$$
$$\leqslant \frac{\chi\tau}{d}\int_{-a}^{a}h(u_a)(g(u_a)u_a-v_a)\leqslant \tau\frac{\chi}{2d}\int_{-a}^{a}(1-u_a)^2g(u_a)u_a,$$

because $v_a'(a)=(K_d'*\bar{u}_a)(a)\leqslant 0$ for a sufficiently large. Consequently,

$$\tau\int_{-a}^{a}g(u_a)u_a(1-u_a)^2+\int_{-a}^{a}|u_a'|^2+u_a'(a)=\frac{c_a}{2}+\tau\chi\int_{-a}^{a}g(u_a)u_au_a'v_a'$$
$$\leqslant \frac{c_a}{2}+\chi(1+\alpha)\int_{-a}^{a}(u_a')^2+\tau\frac{\chi}{2d}\int_{-a}^{a}(1-u_a)^2g(u_a)u_a. \tag{52}$$

It follows that

$$\tau\left(1-\frac{\chi}{2d}\right)\int_{-a}^{a}g(u_a)u_a(1-u_a)^2+(1-\chi(1+\alpha))\int_{-a}^{a}|u_a'|^2+u_a'(a)\leqslant\frac{c_a}{2}, \tag{53}$$

and $u_a'(a)$ is still bounded by (35).

Thus if $\chi<\min(1,d)$ (recall the upper control by d is needed for the L^∞ bound) and θ_0 is small enough such that $\chi(1+\alpha)<1$, the quantities of the left hand side are controlled by that of the right hand side and we can go on with the proof and conclude as before. □

4. Removal of the cut-off

Here we remove the cut-off, letting the parameter θ_0 vanish, and prove Theorem 1.1. The traveling waves $(c(\theta_0), u(x;\theta_0), v(x;\theta_0))$, constructed in Proposition 1.3 for $\theta_0>0$, are translationally invariant and have the left and right limits $u_l=v_l=1$, $u_r=v_r=0$. Therefore, we may translate them and fix the shift so that $u(0;\theta_0)=1/2$. The uniform estimates in the same proposition allow us to let $\theta_{0,n}\to 0$ along a subsequence, so that the traveling wave speeds $c_n=c_*(\theta_{0,n})$ converge to a limit $c_*>0$, and the functions $u(x;\theta_{0,n})$ and $v(x;\theta_{0,n})$ converge to $u(x)$ and $v(x)$. We also have $g(u_n)\to\Psi(x)$ with $\Psi(x)\equiv 1$ on the set $\{u(x)\neq 0\}$. In addition, the limits satisfy the system (3):

$$\begin{aligned}&-c_*u'-u''+\chi(\Psi(x)uv')'=\Psi(x)u(1-u),\\&-dv''+v=\Psi(x)u,\end{aligned} \tag{54}$$

and the functions u and v are still related by (43). Moreover, as the function $p(u)=g(u)u$ is globally Lipschitz, the functions $u(x;\theta_{0,n})$ and $v(x;\theta_{0,n})$ are uniformly bounded in $C^{2,\alpha}(\mathbb{R})$ and thus so are the limits u and v. Therefore, we have $u>0$ and thus $\Psi(x)\equiv 1$ and u and v actually satisfy the system (3):

$$\begin{aligned}&-c_*u'-u''+\chi(uv')'=u(1-u),\\&-dv''+v=u.\end{aligned} \tag{55}$$

It remains to verify that u and v satisfy the boundary conditions (2) at infinity. As in the case with $\theta_0 > 0$ it suffices to ensure that $u(x)$ has the left and right limits $u_l = 1$ and $u_r = 0$, respectively. Once again, existence of the limits at infinity follows from the L^2-bound on the gradient

$$\int_{-\infty}^{\infty} |u'(x)|^2 \,\mathrm{d}x \leqslant K,$$

and standard elliptic regularity estimates. Moreover, in the limit $\theta_0 \to 0$ the estimate (45) becomes

$$\int_{-\infty}^{\infty} u(1-u)^2 \,\mathrm{d}x \leqslant K < \infty.$$

As a consequence, the only possible values for u_l and u_r are 0 and 1, hence in order to show that $u_l = 1$ and $u_r = 0$ it suffices to show that $u_l > u_r$. Integrating the first equation in (55) we obtain

$$c_*(u_l - u_r) = \int u(1-u),$$

while multiplying the same equation by u and integrating leads to

$$\frac{c_*(u_l^2 - u_r^2)}{2} + \int |u'|^2 - \chi \int u u' v' = \int u^2(1-u) = c_*(u_l - u_r) - \int u(1-u)^2.$$

As before, we conclude that

$$\frac{c_*(u_l^2 - u_r^2)}{2} + \int u(1-u)^2 + M \int |u'|^2 \leqslant c_*(u_l - u_r),$$

which may be rewritten as

$$\int u(1-u)^2 + M \int |u'|^2 \leqslant c_*(u_l - u_r)\left(1 - \frac{u_l + u_r}{2}\right).$$

As $u(0) = 1/2$ the left side is strictly positive. Moreover, $c_* > 0$ and $(u_l + u_r)/2 \leqslant 1$. As a consequence, $u_l > u_r$, thus $u_l = 1$, $u_r = 0$, and the proof of the existence part of Theorem 1.1 is complete.

A lower bound for the traveling speed

We now obtain a more precise lower bound for the propagation speed c_* in Theorem 1.1. To do so, we consider a more general birth term $f(u)$ in place of $u(1-u)$ in equation (3). We do not expect more difficulties in the proof of the existence part of Theorem 1.1 as long as $f(u)$ is of the KPP type:

$$f(0) = f(1) = 0,\ f(u) > 0 \text{ for } 0 \leqslant u \leqslant 1,\ f(u) < 0 \text{ for } u \geqslant 1 \text{ and } f'(0) = \sup_{u \geqslant 0} \frac{f(u)}{u} > 0. \tag{56}$$

Then we have

PROPOSITION 4.1 Any traveling wave solution of (2)–(3) in $\dot{H}^1(\mathbb{R})$ with the nonlinearity f satisfying (56) and such that $u, v \geqslant 0$, and

$$\int u(1-u)^2 \,\mathrm{d}x < \infty, \tag{57}$$

satisfies $c \geqslant 2\sqrt{f'(0)}$.

Proof. Consider a traveling wave (c, u, v) and choose a sequence x_n that increases to ∞ as $n \to \infty$. Note that $u(x + x_n) \to 0$ uniformly in $x \in [R, \infty)$ for all $R \in \mathbb{R}$ as $n \to \infty$. Indeed, choosing $A > 0$ large enough so that $u(x) \leqslant 1/2$ for $x \geqslant A$, we deduce from (57) that $u \in L^1(A, \infty)$ and thus we may write

$$u^2(x) = -2\int_x^\infty u' \leqslant 2\Big(\int_A^\infty u^2\Big)^{1/2}\Big(\int_A^\infty u'^2\Big)^{1/2} \xrightarrow[\theta_0 \to 0]{} 0,$$

so that in particular $u(x_n) \to 0$ as $n \to \infty$.

Next, set $u_n(x) = u(x + x_n)/u(x_n)$ and $v_n(x) = v(x + x_n)$. These functions satisfy

$$\begin{cases} -u_n'' - cu_n' + \chi(v_n' u_n)' = f(u(x_n)u_n)/u(x_n), \\ -dv_n'' + v_n = u(x + x_n). \end{cases} \tag{58}$$

The right side in the equation on u_n in (58) is bounded by $f'(0)u_n$. Therefore we use elliptic regularity and, up to extracting a subsequence, we know that $u_n \to u_\infty$ and $v_n \to v_\infty$ as $n \to \infty$ in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R})$. These functions satisfy

$$\begin{cases} -u_\infty'' - cu_\infty' + \chi(v_\infty' u_\infty)' = f'(0)u_\infty, \\ -dv_\infty'' + v_\infty = 0. \end{cases} \tag{59}$$

As v_∞ is nonnegative and bounded, we necessarily have $v_\infty \equiv 0$.

Furthermore, as $u_\infty(0) = 1$ and $u_\infty \geqslant 0$, the maximum principle yields $u_\infty > 0$. Thus we can explicitly solve the first equation and the solution can only be of the exponential type: $u_\infty(x) = \mu e^{-\lambda x}$. Inserting such a λ in the equation for u_∞ we find $-\lambda^2 + c\lambda = f'(0)$. Hence we have proved that necessarily $c \geqslant 2\sqrt{f'(0)}$. □

5. Time evolution problem

We now consider the problem

$$\begin{cases} u_t - u_{xx} + \chi(uv_x)_x = u(1-u), \\ -dv_{xx} + v = u, \\ u(t=0) = u_0, \quad \text{with compact support}, \quad 0 \leqslant u_0(x) \leqslant (1 - \chi/d)^{-1}. \end{cases} \tag{60}$$

The maximum principle, as already used earlier, implies that we have the uniform bounds

$$0 \leqslant u(x), v(x) \leqslant \frac{d}{d-\chi}, \qquad |v_x(t,x)|, |v_{xx}(t,x)| \leqslant K.$$

Our goal in this section is to prove two kinds of results on this problem. First, we assume that χ satisfies the conditions of existence of traveling waves. Then we derive some bounds expressing that in the long time limit, the solution converges to 1 on compact sets. Secondly, we show that, under the (weaker) linear stability condition on χ, the state 1 is in fact nonlinearly asymptotically stable.

5.1 *The long time limit of $u(t, x)$*

We have the following

THEOREM 5.1 Assume $\chi \leqslant \min(1, d)$. There exist $C > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a time t_0 such that for all $T > t_0$ the following holds. There exists a set $B \subset [T, 2T]$ of exceptional times with $|B| \leqslant C/\varepsilon$ such that for all nonexceptional $t \in [T, 2T] \cap B^c$ and all $p \in [0, 1)$ we have

$$|\{x : u(t,x)|1 - u(t,x)|^2 \geqslant \varepsilon^p\}| \leqslant C\varepsilon^{1-p} \int u(t,x)\,\mathrm{d}x. \tag{61}$$

The constant $C > 0$ in Theorem 5.1 does not depend on the time T. Therefore, the total set B of "bad" times between a (large) time T and $2T$ is bounded independent of T. The right side of (61) may be loosely interpreted as the size of the support of the function $u(t, x)$ (disregarding the fact that $u(t, x)$ has an infinite support). Thus, (61) may be interpreted as saying that for large times the fraction of the support of $u(t, x)$ where $u(t, x)$ is far from 1 is negligible, except for a (relatively) small set of bad times.

We first prove the following proposition.

PROPOSITION 5.2 Assume that (4) holds and let the initial data $u_0(x) \not\equiv 0$ be compactly supported, with $0 \leqslant u_0(x) \leqslant 1$. There exist two constants K_1 and K_2 which do not depend on the initial data, and a time t_0 such that

$$K_1(t - t_0) \leqslant \int u(t,x)\,\mathrm{d}x \leqslant K_2(t_0 + t).$$

Proof. First, let u and v be solutions of (60) and consider the function $\psi(t,x) = Me^{-\lambda(x-\xi t)}$. It satisfies the inequality

$$\begin{aligned}\psi_t - \psi_{xx} + \chi v_x \psi_x + \chi v_{xx}\psi - \psi(1-\psi) &\geqslant \psi_t - \psi_{xx} + \chi v_x \psi_x - \frac{\chi}{d}(u - v)\psi - \psi \\ &\geqslant \psi_t - \psi_{xx} - K|\psi_x| - K\psi = (\lambda\xi\psi - \lambda^2 - K - K\lambda)\psi \geqslant 0.\end{aligned}$$

This last inequality holds provided that ξ is sufficiently large and λ is chosen appropriately. Therefore, we may also take M large enough so that $\psi_M(t, x)$ is a supersolution for $u(t, x)$. Similarly, $\phi_M(t,x) = Me^{\lambda(x+\xi t)}$ is a supersolution for u. Therefore,

$$u(t,x) \leqslant \min(Me^{-\lambda(x-\xi t)}, Me^{\lambda(x+\xi t)}),$$

and integrating in x gives

$$\int_{\mathbb{R}} u(t,x)\,\mathrm{d}x \leqslant C(t + t_0). \tag{62}$$

Thus the upper bound of the proposition is proved.

To obtain a lower bound on $\|u(t)\|_{L^1}$ we proceed as in the traveling wave case. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (u - u^2/2) = \int u_x^2 + \int u(1-u)^2 - \chi \int u_x v_x u. \tag{63}$$

The last integral on the right side may be split as

$$\chi \int u u_x v_x \, \mathrm{d}x = \chi \int (u-1) u_x v_x \, \mathrm{d}x + \chi \int u_x v_x \, \mathrm{d}x.$$

The second term is bounded as

$$\left(\int u_x v_x \, \mathrm{d}x \right)^2 \leqslant \int u_x^2 \, \mathrm{d}x \int v_x^2 \, \mathrm{d}x \leqslant \left(\int u_x^2 \, \mathrm{d}x \right)^2,$$

while the first one satisfies

$$\chi \int (u-1) u_x v_x \, \mathrm{d}x = \frac{\chi}{2} \int ((u-1)^2)_x v_x \, \mathrm{d}x = \frac{\chi}{2d} \int (u-1)^2 (u-v) \, \mathrm{d}x \leqslant \frac{\chi}{2d} \int (u-1)^2 u \, \mathrm{d}x.$$

Using the last two inequalities in (63) leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int (u - u^2/2) &\geqslant \int u_x^2 + \int u(1-u)^2 - \chi \int u_x^2 - \frac{\chi}{2d} \int (u-1)^2 u \, \mathrm{d}x \\ &\geqslant M \int u_x^2 + M \int u(1-u)^2. \end{aligned} \tag{64}$$

Integrating in time and combining this with the upper bound in (62) we obtain

$$\int_0^T \int u_x^2 + \int_0^T \int u(1-u)^2 \leqslant \frac{1}{M} \left[\int u(T,x) \, \mathrm{d}x - \int (u_0 - u_0^2/2) \right] \leqslant C(1+T). \tag{65}$$

Note that if at some time $t \in [0, T]$ there exists x_0 such that $u(t, x_0) > 1/2$ then

$$M \int u_x^2(t,x) \, \mathrm{d}x + M \int u(t,x)(1-u(t,x))^2 \, \mathrm{d}x \geqslant K.$$

On the other hand, if $0 \leqslant u(t,x) \leqslant 1/2$ for all $x \in \mathbb{R}$ then

$$\int u(t,x)(1-u(t,x))^2 \, \mathrm{d}x \geqslant \frac{1}{4} \int u(t,x) \, \mathrm{d}x.$$

Let $\mathcal{A}_T = \{t \in [0,T] : 0 \leqslant u(t,x) \leqslant 1/2 \text{ for all } x \in \mathbb{R}\}$. It follows from the above that there exists a constant $K > 0$ so that

$$\int u(T,x) \geqslant \int_{\mathcal{A}_T^c} K \, \mathrm{d}x + K \int_{\mathcal{A}_T} \left(\int u(t,x) \, \mathrm{d}x \right) \mathrm{d}t. \tag{66}$$

As a consequence, the function

$$W(t) = \int u(t,x) \, \mathrm{d}x$$

satisfies

$$W(T) \geqslant \int_{\mathcal{A}_T^c} K \, \mathrm{d}x + K \int_{\mathcal{A}_T} W(t) \, \mathrm{d}t \geqslant K \int_0^T \min(1, W(t)) \, \mathrm{d}t \tag{67}$$

for all $T \geqslant 0$. In addition, $W(t)$ is locally Lipschitz in time:

$$|W_t(t)| = \left| \int u(t,x)(1-u(t,x))\,\mathrm{d}x \right| \leqslant M \int u(t,x)\,\mathrm{d}x \leqslant C(1+t).$$

Therefore, in particular, there exists τ_0 so that $W(t) \geqslant W(0)/2 > 0$ for $0 \leqslant t \leqslant \tau_0$ and thus there exists $k_0 > 0$ (which depends on the initial data) such that for $T \geqslant \tau_0$,

$$W(T) \geqslant K \int_0^T \min(1, W(t))\,\mathrm{d}t \geqslant K \int_0^{\tau_0} \min(1, W(t))\,\mathrm{d}t \geqslant k_0.$$

Going back to (66) we see that

$$W(T) \geqslant K|\mathcal{A}_T^c| + k_0 K|\mathcal{A}_T| \geqslant k_0 K T.$$

In order to get rid of the dependence on the initial data observe that, as a consequence, $W(T) \geqslant 1$ for all $T \geqslant t_0$ (the time t_0 does depend on the initial data). Hence, the second inequality in (67) yields

$$W(T) \geqslant K \int_0^T \min(1, W(t))\,\mathrm{d}t \geqslant K(T - t_0). \tag{68}$$

This finishes the proof of Proposition 5.2. □

Proof of Theorem 5.1. Theorem 5.1 is an easy consequence of Proposition 5.2 and its proof. Let us start with the inequality (64),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (u - u^2/2)\,\mathrm{d}x \geqslant M \int u_x^2\,\mathrm{d}x + M \int u(1-u)^2\,\mathrm{d}x. \tag{69}$$

Consider the set $B \subset [T, 2T]$ of times $t \in [T, 2T]$ such that

$$M \int u(t,x)(1-u(t,x))^2\,\mathrm{d}x \geqslant \varepsilon \int (u(t,x) - u^2(t,x)/2)\,\mathrm{d}x.$$

Let us set

$$Q(t) = \int (u(t,x) - u^2(t,x)/2)\,\mathrm{d}x.$$

Exactly as in the proof of Proposition 5.2 we deduce that

$$C_1(t - t_0) \leqslant Q(t) \leqslant C_2(t_0 + t). \tag{70}$$

As $Q(t)$ is increasing in time, integrating (69) over B we obtain

$$Q(2T) \geqslant Q(T)e^{\varepsilon|B|}.$$

For $T > 10t_0$ it follows that

$$4C_2 T \geqslant C_1 T e^{\varepsilon|B|},$$

so that $|B| \leqslant K/\varepsilon$ with the constant K independent of $T > t_0$. On the other hand, for $t \in [T, 2T] \cap B^c$ we have

$$\varepsilon^p |\{x : u(t,x)|1-u(t,x)|^2 \geqslant \varepsilon^p\}| \leqslant C \int u(t,x)(1-u(t,x))^2\,\mathrm{d}x \leqslant C\varepsilon \int u(t,x)\,\mathrm{d}x,$$

and (61) follows. □

5.2 *Nonlinear asymptotic stability of the homogeneous state* (1, 1)

In this section, we consider the Keller–Segel–Fisher system and we consider the stability of the state (1, 1) as discussed in the introduction (see (14)–(15)). Therefore, we set $u = 1 + U$ and $v = 1 + V$ and the system (60) reads

$$\begin{cases} U_t - U_{xx} + \chi(V_x U)_x = -U(1+U) - \chi V_{xx}, \\ -dV_{xx} + V = U, \\ U(t=0,x) = U_0(x) := u^0 - 1, \quad x \in \mathbb{R}. \end{cases} \tag{71}$$

We prove that the linear stability of the homogeneous equilibrium state ($u = 1, v = 1$) implies its nonlinear asymptotic stability. More precisely

THEOREM 5.3 For $\chi < (1+\sqrt{d})^2$, there is a positive constant $\delta > 0$ such that for any initial data $u_0 = 1 + U_0$ with $\int_{\mathbb{R}} U_0^2 < \delta$, the solution u of the Cauchy problem (60) converges to 1 in the L^2 norm with an exponential rate:

$$\int_{\mathbb{R}} (u(t,x) - 1)^2 \, dx \to 0 \quad \text{as } t \to +\infty. \tag{72}$$

Proof of Theorem 5.3. For $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, we set $U(t,x) := u(t,x) - 1$, $V(t,x) := v(t,x) - 1$ and $\lambda := (1+\sqrt{d})^2 - \chi > 0$. Multiplying equation (71) by U and integrating over $\mathbb{R}$, we find

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} U^2 \, dx + \int_{\mathbb{R}} (U_x^2 - \chi U_x V_x + U^2) \\ &= -\int_{\mathbb{R}} U^3 + \chi \int_{\mathbb{R}} U U_x V_x = -\int_{\mathbb{R}} U^3 - \frac{\chi}{2}\int_{\mathbb{R}} U^2 V_{xx} \\ &= \left(\frac{\chi}{2d} - 1\right)\int U^3 - \frac{\chi}{2d}\int_{\mathbb{R}} U^2 V \leqslant \left|\frac{\chi}{2d} - 1\right| \int |U|^3 + \frac{\chi}{2d}\int_{\mathbb{R}} U^2 |V|. \end{aligned} \tag{73}$$

The second term on the left side of (73) can be written as

$$\begin{aligned} \int_{\mathbb{R}} (U_x^2 - \chi U_x V_x + U^2) \, dx &= \int_{\mathbb{R}} \left(\xi^2 + 1 - \frac{\chi \xi^2}{1 + d\xi^2}\right) |\hat{U}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}} \frac{P(\xi)}{(1 + d\xi^2)(1+\xi^2)} (1+\xi^2) |\hat{U}(\xi)|^2 \, d\xi, \end{aligned}$$

where P is a fourth order poynomial function which is positive since $\chi < (1+\sqrt{d})^2$.

As $p(\xi) = (1 + d\xi^2)(1+\xi^2)$ is also a positive fourth order polynomial function, the quotient $P(\xi)/[(1 + d\xi^2)(1+\xi^2)]$ has a positive infimum $\lambda > 0$. This gives

$$\int_{\mathbb{R}} [|U_x|^2 - \chi U_x V_x + U^2] \, dx \geqslant \lambda \int_{\mathbb{R}} (1+\xi^2)|\hat{U}(\xi)|^2 \, d\xi \geqslant \lambda \int_{\mathbb{R}} (U_x^2 + U^2) \, dx. \tag{74}$$

Next, set $I(t) = \int_{\mathbb{R}} U^2 \, dx$. The above computation yields

$$\frac{1}{2}\frac{d}{dt} I(t) + \lambda I(t) + \lambda \int_{\mathbb{R}} U_x^2 \leqslant \left|\frac{\chi}{2d} - 1\right| \int |U|^3 + \frac{\chi}{2d}\int_{\mathbb{R}} U^2 |V|. \tag{75}$$

We treat the two terms of the right side separately using Gagliardo–Nirenberg–Sobolev type inequalities:

$$\int_{\mathbb{R}} |U|^3 \leqslant C\Big(\int_{\mathbb{R}} U_x^2\Big)^{1/4}\Big(\int_{\mathbb{R}} U^2\Big)^{5/4} \leqslant \frac{\lambda}{2|\chi/(2d)-1|}\int_{\mathbb{R}} U_x^2 + M\Big(\int_{\mathbb{R}} U^2\Big)^{5/3} \tag{76}$$

(the second inequality follows from the Minkowski inequality). In the same way we obtain

$$\begin{aligned}\int_{\mathbb{R}} |U|^2|V| &\leqslant \Big(\int_{\mathbb{R}} |U|^4 \int_{\mathbb{R}} V^2\Big)^{1/2} \leqslant C_1\Big(\int_{\mathbb{R}} U_x^2\Big)^{1/4}\Big(\int_{\mathbb{R}} U^2\Big)^{3/4}\Big(\int_{\mathbb{R}} U^2\Big)^{1/2}\\ &\leqslant \frac{\lambda d}{\chi}\int_{\mathbb{R}} U_x^2 + M'\Big(\int_{\mathbb{R}} U^2\Big)^{5/3},\end{aligned}$$

where M' is a constant that only depends on C_1, χ, d and λ. This finally gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}I(t) + \lambda I(t) + \lambda\int_{\mathbb{R}} U_x^2 \leqslant \lambda\int_{\mathbb{R}} U_x^2 + (M+M')I^{5/3}(t), \tag{77}$$

and thus for some constant M'' we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}I(t) + \lambda I(t) \leqslant M'' I^{5/3}(t). \tag{78}$$

Set now $\delta = (\lambda/M'')^{3/2}$. Then, for $I(0) < \delta$, the differential inequality (78) implies that $t \mapsto I(t)$ decreases. As it is a nonnegative function, it converges to the equilibrium state $I \equiv 0$. Also, there is an exponential decay (with rate as close to 2λ as we wish), and the proof is complete. □

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