Interfaces and Free Boundaries **11** (2009), 153–176

Level set approach for fractional mean curvature flows

CYRIL IMBERT
CEREMADE, Université Paris-Dauphine,
place de Lattre de Tassigny, 75775 Paris Cedex 16, France
E-mail: imbert@ceremade.dauphine.fr

[Received 10 October 2007 and in revised form 17 July 2008]

This paper is concerned with the study of a geometric flow whose law involves a singular integral operator. This operator is used to define a non-local mean curvature of a set. Moreover, the associated flow appears in two important applications: dislocation dynamics and phasefield theory for fractional reaction-diffusion equations. It is defined by using the level set method. The main results of this paper are: on one hand, the proper level set formulation of the geometric flow; on the other hand, stability and comparison results for the geometric equation associated with the flow.

2000 Mathematics Subject Classification: 35F25, 35K55, 35K65, 49L25, 35A05.

Keywords: Fractional mean curvature; mean curvature; geometric flows; dislocation dynamics; level set approach; stability results; comparison principles; generalized flows.

1. Introduction

In this paper, we define a geometric flow whose law is non-local. We recall that a geometric flow of a set Ω is a family $\{\Omega_t\}_{t>0}$ such that the velocity of a point $x \in \partial\Omega_t$ along its outer normal $n(x)$ is a given function of x and $n(x)$ for instance. In our case, this velocity does not only depend on x and $n(x)$ but also on a *fractional mean curvature* at x. Our motivation comes from two different problems: dislocation dynamics and phasefield theory for fractional reaction-diffusion equations.

1.1 *Motivation and existing results*

Mathematical study of non-local moving fronts recently attracted a lot of attention (see in particular [12] and references therein). An important application is the study of dislocation dynamics [3].

Dislocation dynamics. Dislocations are linear defects in crystals and the study of their motion gives rise to the study of a non-local geometric flow. In recent years, several papers were dedicated to this problem. We next briefly recall the results contained in those papers.

A dislocation creates an elastic field in the whole space $\mathbb{R}^3$ and this field creates a force (called the Peach–Koehler force) that acts not only on the dislocation that created it (self-force) but also on all dislocations in the material. We restrict ourselves here to the case of a single curve. We also assume that this curve moves in a plane (called the slip plane).

The level set approach [24, 13, 16] is a general method for constructing moving interfaces. It consists in representing Ω_t as zero level sets of functions $u(t,\cdot)$. The geometric law satisfied by the interface $\partial\Omega_t$ is thus translated into an evolution equation satisfied by u. This approach is used in [3] to describe the dynamics of a dislocation line. If $\partial\Omega_t$ is the zero level set of a function $u(t,\cdot)$, the following non-local eikonal equation is obtained:

$$\partial_t u = (c_1(x) + \kappa[x,u])|Du|$$

where c_1 is an external force and $\kappa[x,u]$ is the Peach–Koehler force applied to the curve ($N = 2$ in this application).

We briefly mentioned above that the Peach–Koehler force is created by the curve. Let us be a bit more specific. This force is computed through the resolution of an elliptic equation on a half space (corresponding to the law of linear elasticity). This equation is supplemented with Dirichlet boundary conditions. On one hand, the boundary datum equals the indicator function of the interior of the curve. On the other hand, loosely speaking, the force on the curve equals the normal derivative of the solution of the elliptic equation. Hence, the integral operator which defines the Peach–Koehler force is a Dirichet-to-Neumann operator associated with an elliptic equation. In particular, the operator is singular.

In order to define solutions for small times, the authors of [3] consider a physically relevant regularized problem and $\kappa[x,u]$ reduces to

$$\int_{\{z\,:\,u(z)\geqslant 0\}} c_0(z)\,\mathrm{d}z$$

with $c_0 \in W^{1,1}(\mathbb{R}^N)$. *The major technical difficulty of this paper is that c_0 does not have a constant sign, and consequently solutions corresponding to ordered initial data are not ordered; in other words, the comparison principle does not hold true.* In particular, this is one of the reasons why solutions are constructed for small times. If c_1 is assumed to be large enough, Alvarez, Cardaliaguet and Monneau [2] managed to prove the existence and uniqueness for large times.

The difficulty related to the comparison principle is circumvented in [19] by assuming that the negative part of c_0 is concentrated at the origin. The Peach–Koehler force $\kappa[x,u]$ (in the case of a single dislocation line) is defined in [19] as

$$\int \operatorname{sign}(u(x+z)-u(x))c_0(z)\,\mathrm{d}z = \int_{\{z\,:\,u(x+z)\geqslant u(x)\}} c_0(z)\,\mathrm{d}z - \int_{\{z\,:\,u(x+z)<u(x)\}} c_0(z)\,\mathrm{d}z \qquad (1)$$

where $\operatorname{sign}(r)$ equals 1 if $r \geqslant 0$ and -1 if $r < 0$. After an approximation procedure, the problem can be reduced to the study of

$$\partial_t U = \left[c_1(x) + \int (U(x+z) - U(x))c_0(z)\,\mathrm{d}z\right]|DU|$$

where c_0 is smooth, non-negative and of finite mass. We used the letter U instead of u in order to emphasize the fact that a change of unknown function is needed in order to reduce the study of the original equation to the study of this new one.

A second important remark is that solving such non-local eikonal equations does not permit one to construct a geometric flow properly. More precisely, if the initial front $\partial\Omega_0$ is described with two different initial functions u_0 and v_0, it is not sure that the zero level sets of the corresponding solutions u and v coincide. In other words, the invariance principle does not hold true.

Still assuming that the negative part of c_0 is concentrated at the origin, a good geometric definition of the flow is obtained in [17] by considering a formulation "à la Slepčev" of the geometric flow. The equation now becomes

$$\partial_t u = \left[c_1(x) + \int_{\{z\,:\,u(t,x+z)>u(t,x)\}} c_0(z)\,\mathrm{d}z \right] |Du|. \tag{2}$$

We point out that, with such a formulation, we cannot deal with singular potentials c_0.

Notice that in [17], several fronts move, and they are interacting. The motion of a single front is a special case. Eventually, existence results of very weak solutions in a very general setting are obtained in [5] and uniqueness is studied in [6].

In [15], it is proved that if $c_0(z)$ is smooth and regular near the origin and behaves exactly like $|z|^{-N-1}$ at infinity, then a proper rescaling of (2) converges towards the mean curvature motion (see Proposition 1 and Theorem 4 below).

We finally mention that Caffarelli and Souganidis [11] consider a Bence–Merriman–Osher scheme with kernels associated with the fractional heat equation (that is, the heat equation where the usual Laplacian is replaced with the fractional one). They prove that this scheme approximates the geometric flow at stake in this paper.

Phasefield theory for fractional reaction-diffusion equations. Our second main motivation comes from phasefield theory for fractional reaction-diffusion equations [21]. If one considers for instance stochastic Ising models with Kac potentials with very slow decay at infinity (like a power law with proper exponent), then the study of the resulting mean field equation (after proper rescaling) is closely related to phasefield theory for fractional reaction-diffusion equations such as

$$\partial_t u^\varepsilon + (-\Delta)^{\alpha/2} u^\varepsilon + \frac{1}{\varepsilon^{1+\alpha}} f(u^\varepsilon) = 0$$

where $(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian with $\alpha \in (0, 1)$ (in the case presented here) and f is a bistable non-linearity. In particular, it is *essential* in the analysis to deal with singular potentials. Indeed, we have to be able to treat the case where

$$c_0(z) = \frac{1}{|z|^{N+\alpha}}$$

with $\alpha \in (0, 1)$. It is also convenient to use the notion of generalized flows introduced by Barles and Souganidis [9] in order to develop a phasefield theory for such reaction-diffusion equations. See [21] for further details and [18] for analogous problems.

1.2 *A new formulation*

The main contributions of this paper are the following:

- to give a proper level set formulation of dislocation dynamics for singular interaction potentials; in particular, sufficient conditions on the singularity to get stability results and comparison principles are exhibited;
- to shed light on the fact that the integral operator measures in a non-local way the curvature of the interface;
- to study the geometric flow in detail: consistency of the definition, equivalent definition in terms of generalized flows, motion of bounded sets etc.

Because $\nu(\mathrm{d}z) = c_0(z)\,\mathrm{d}z$ is singular, we cannot define $\kappa[x,u]$ as in (2). Indeed, we must *compensate* the singularity as is commonly done in order to get a proper integral representation of the fractional Laplacian. We recall that the fractional Laplacian can be defined as follows:

$$(-\Delta)^{\alpha/2}u(x) = -c_N(\alpha)\int (u(x+z)-u(x))\,\frac{\mathrm{d}z}{|z|^{N+\alpha}}$$

where $c_N(\alpha)$ is a given positive constant depending on N, α. Notice that if $\alpha < 1$ and u is Lipschitz continuous at x and u is globally bounded, the integral is well defined. If $\alpha \geqslant 1$, the integral is not convergent in the neighbourhood of $z = 0$. In this case, the fractional Laplacian is defined either by considering the principal value of the previous singular integral or by writing

$$(-\Delta)^{\alpha/2}u(x) = -c_N(\alpha)\int (u(x+z)-u(x)-Du(x)\cdot z\mathbf{1}_B(z))\,\frac{\mathrm{d}z}{|z|^{N+\alpha}}$$

where $\mathbf{1}_B(z)$ denotes the indicator function of the unit ball B. Notice that we have used the fact that the singular measure

$$\nu(\mathrm{d}z) = \frac{\mathrm{d}z}{|z|^{N+\alpha}} \tag{3}$$

(with $\alpha \in (0,2)$) is even in order to get (at least formally)

$$\int (Du(x)\cdot z\mathbf{1}_B(z))\,\frac{\mathrm{d}z}{|z|^{N+\alpha}} = 0.$$

As far as the fractional mean curvature is concerned, we must compensate the singularity of the measure ν in a geometrical way. We explain how to do it when $\nu(\mathrm{d}z) = c_0(z)\,\mathrm{d}z$ with $c_0(z) = |z|^{-N-\alpha}$. Hence, we start from (1). We use the fact that c_0 is even in order to get (formally)

$$\nu\{z\in\mathbb{R}^N : Du(x)\cdot z \geqslant 0\} = \nu\{z\in\mathbb{R}^N : Du(x)\cdot z < 0\}.$$

Straightforward computations yield

$$\begin{aligned}&\int \mathrm{sign}(u(x+z)-u(x))c_0(z)\,\mathrm{d}z\\ &\quad = \nu\{z : u(x+z)\geqslant u(x),\ Du(x)\cdot z\leqslant 0\} - \nu\{z : u(x+z) < u(x),\ Du(x)\cdot z > 0\}.\end{aligned}$$

We thus define an integral operator $\kappa[x,u]$ for a general singular non-negative measure ν as follows:

$$\kappa[x,u] = \nu\{z : u(x+z)\geqslant u(x), Du(x)\cdot z\leqslant 0\} - \nu\{z : u(x+z) < u(x), Du(x)\cdot z > 0\}. \tag{4}$$

We explain below in detail (see Lemma 2) the rigorous links between the different formulations we have considered up to now.

Notice that this definition makes sense even if ν is not even. We recall that the fractional Laplacian is a Lévy operator. Since Lévy operators [4] are defined for singular (Lévy) measures that are not necessarily even, it seems to be relevant to define fractional mean curvature for singular measures that are not necessarily even.

We can say that this singular integral operator measures in a non-local way the curvature of the "curve" $\{u = u(x)\}$. Indeed, loosely speaking, we can say that in (4) the first (resp. second) part measures how concave (resp. convex) the set $\Omega = \{z : u(x+z) > u(x)\}$ is "near x". Moreover, we prove (see Proposition 2 below) that, when ν is given by (3), the function $(1-\alpha)\kappa[x,u]$ converges as $\alpha \in (0,1)$ goes to 1 towards the classical mean curvature of $\{u = u(x)\}$ at x. This is why we refer to $\kappa[x,u]$ as the fractional mean curvature of the curve $\{u = u(x)\}$ at point x.

The variational case. When the singular measure $\nu(\mathrm{d}z)$ has the form

$$\nu(\mathrm{d}z) = -(\nabla \cdot G(z))\,\mathrm{d}z$$

for a vector field G, the previous singular integral operator can be written as follows:

$$\kappa[x,u] = \int_{\{z\,:\,u(x+z)=u(x)\}} \left(G(z) \cdot \frac{\nabla u(x+z)}{|\nabla u(x+z)|} \right) \sigma(\mathrm{d}z) - b_G\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) \cdot \frac{\nabla u(x)}{|\nabla u(x)|} \tag{5}$$

where σ denotes the surface measure on the "curve" $\{z : u(x+z) = u(x)\}$ and where $b_G = \int_{\{z\,:\,\nabla u(x)\cdot z=0\}} G(z)\,\sigma(\mathrm{d}z)$ is a vector field on $\mathbb{R}^N$.

Remark that the example we gave above is of this form. Indeed,

$$\frac{\mathrm{d}z}{|z|^{N+\alpha}} = -\frac{1}{\alpha}\left(\nabla \cdot \frac{z}{|z|^{N+\alpha}}\right)\mathrm{d}z.$$

It is quite clear from this new formula that the singular integral operator is geometric (in the sense that it only depends on the curve and not and its parametrization u) and "fractional".

After this work was finished, we have been told that non-local minimal surfaces are being studied by Caffarelli, Roquejoffre and Savin [10]. Loosely speaking, they study sets whose indicator functions minimize a fractional Sobolev norm $\|\cdot\|_{H^\alpha}$, $\alpha \in (0,1)$. They prove in particular that local minimizers are viscosity solutions of $\kappa[x,u] = 0$.

Comments and related work. We gave two different formulations in the case of singular potentials. *We think that formulation* (4) *is the proper one in order to get a complete level set formulation of the geometric flow* even if formulation (5) is somehow more intuitive since it only involves the curve itself. In particular, the approach proposed by Slepčev [26] can be adapted (see (14) below).

The level set equation we study has the form

$$\partial_t u = \mu(\widehat{Du})[c_1(x) + \kappa[x,u]]|Du| \quad \text{in } (0,+\infty) \times \mathbb{R}^N \tag{6}$$

supplemented with the initial condition

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N \tag{7}$$

where $\hat{p}$ denotes $p/|p|$ if $p \neq 0$, μ denotes the mobility vector field, and $c_1(x)$ is a driving force.

Equation (6) is a non-linear non-local Hamilton–Jacobi equation. A lot of papers are dedicated to the study of such equations. In our case, the main technical issues are the definition of viscosity solutions, the proof of their stability and the proof of a strong uniqueness result. We use some ideas from [26] and combine them with the ones from [7], even if the results of those two papers do not apply to our equation.

From a physical point of view and as far as dislocation dynamics is concerned, the measure $\nu(\mathrm{d}z) = c_0(z)\,\mathrm{d}z$ should be $\nu(\mathrm{d}z) = g(z/|z|)|z|^{-N-1}\,\mathrm{d}z$, but in this case, the fractional mean curvature is not well defined (see Remark 1). It is also physically relevant to say that close to the dislocation line, in the *core* of the dislocation, the potential should be regularized. On the other hand, it is important to assume that $\nu(\mathrm{d}z) \sim g(z/|z|)|z|^{-N-1}\,\mathrm{d}z$ as $|z| \to +\infty$ since this prescribes the long range interaction between dislocation lines. Another way to understand this difficulty is to say that in the core of the dislocation, the potential is very singular and the singularity should be compensated at a higher (second) order. On one hand, this can explain the loss of the inclusion principle for such flows (if one can define them for large times). On the other hand, one can think that in this case, the first term in such an expansion should be a mean curvature term. This can make sense since curvature terms are commonly used to describe dislocation dynamics. It may be relevant to add one in (6). However, we choose not to do so in order to avoid technicalities and keep clear some important points in the proof of the stability result and the comparison principle.

In order to better understand properties of the fractional mean curvature flow, a deterministic zero-sum repeated game is constructed in [20] in the spirit of [23, 22].

Organization of the article. In Section 2, we first give the precise assumptions we make on data. We next give the definition(s) of the fractional mean curvature $\kappa[x, \cdot]$. In Section 3, we first give the definition of viscosity solutions for (6), we then state and prove stability results. We next obtain strong uniqueness results by establishing comparison principles. We also construct solutions of (6) by Perron's method. We finally give two convergence results which explain in which limit one recovers the classical mean curvature equation. In Section 4, we verify that the zero levet set of the solution u we have constructed in the previous section only depends on the zero level set of the initial condition. This provides a level set formulation of the geometric flow. In the last section, we give an alternative geometric definition of the flow in terms of generalized flows in the sense of [9].

Notation. $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$. The ball of radius δ centred at x is denoted by $B_\delta(x)$. If $x = 0$, we simply write B_δ and if moreover $\delta = 1$, we write B. If $p \in \mathbb{R}^N \setminus \{0\}$, $\hat{p}$ denotes $p/|p|$. If A is a subset of $\mathbb{R}^d$ with $d = N, N+1$ for instance, then A^c denotes $\mathbb{R}^d \setminus A$. For two subsets A and B, $A \sqcup B$ denotes $A \cup B$ and means that $A \cap B = \emptyset$. The function $\mathbf{1}_A(z)$ equals 1 if $z \in A$ and 0 if not.

2. Preliminaries

In this section, we make precise the assumptions we need on data and we give several definitions of the fractional mean curvature.

2.1 *Assumptions*

Here are the assumptions we make on the singular measure throughout the paper.

ASSUMPTIONS

(A1) The mobility function $\mu : \mathbb{S}^{N-1} \to (0, +\infty)$ is continuous.

(A2) The driving force $c_1 : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous.

(A3) The singular measure ν is a non-negative Radon measure satisfying

$$\begin{cases} \text{for all } \delta > 0, & \nu(\mathbb{R}^N \setminus B_\delta) < +\infty, \\ \text{for all } r > 0,\ e \in \mathbb{S}^{N-1}, & \nu\{z \in B : r|z \cdot e| \leqslant |z - (z \cdot e)e|^2\} < +\infty, \\ & \delta\nu(\mathbb{R}^N \setminus B_\delta) \to 0 \text{ as } \delta \to 0, \\ \text{for all } e \in \mathbb{S}^{N-1}, & r\nu\{z \in B : r|z \cdot e| \leqslant |z - (z \cdot e)e|^2\} \to 0 \text{ as } r \to 0 \end{cases} \tag{8}$$

(B_δ denotes the ball of radius δ centred at the origin and $B = B_1$), and the last limit is uniform with respect to unit vectors $e \in \mathbb{S}^{N-1}$.

(A4) The initial datum $u_0 : \mathbb{R}^N \to \mathbb{R}$ is bounded and Lipschitz continuous.

We point out that the set $\{z \in B : r|z \cdot e| \leqslant |z - (z \cdot e)e|^2\}$ appearing in the second and the fourth lines of (8) is the region between an upward and downward (with respect to vector e) parabola.

Even if the assumptions on the singular measure look technical at first glance, they are quite natural in the sense that they imply several important properties:

- the measure is bounded away from the origin;
- the singularity at the origin (if any) is a weak singularity in the sense that the fractional mean curvature of regular curves can be defined; if the reader thinks of the example given in (3), this means that we choose $\alpha < 1$;
- the parabolas $\{z : rz_N = |z'|^2\}$ (which are the model regular curves for us) can be handled even when they degenerate ($r \to 0$).

EXAMPLE 1 The Standing Example for the singular measure is

$$\nu_{SE}(\mathrm{d}z) = g\left(\frac{z}{|z|}\right)\frac{\mathrm{d}z}{|z|^{N+\alpha}}$$

with $g : \mathbb{S}^{N-1} \to (0, +\infty)$ continuous and $\alpha \in (0, 1)$. The measure in (3) corresponds to the isotropic case ($g \equiv 1$).

2.2 *Fractional mean curvature*

In this subsection, we make precise the definition of fractional mean curvature. Our definition extends the ones given in [15, 17] where $\nu(\mathrm{d}z) = c_0(z)\,\mathrm{d}z$ to the case of singular measures.

Let us define the fractional curvature of a smooth curve $\Gamma = \{x \in \mathbb{R}^N : u(x) = 0\} = \partial\{x \in \mathbb{R}^N : u(x) > 0\}$ associated with ν. If u is $C^{1,1}$ and $Du(x) \neq 0$, then the following quantities are well defined (see Lemma 1 below):

$$\begin{aligned} \kappa^*[x, \Gamma] &= \kappa^*[x, u] = \kappa^*_+[x, u] - \kappa^-_*[x, u], \\ \kappa_*[x, \Gamma] &= \kappa_*[x, u] = \kappa^+_*[x, u] - \kappa^*_-[x, u], \end{aligned} \tag{9}$$

where

$$\begin{aligned} \kappa^+_*[x, u] &= \nu\{z : u(x+z) > u(x),\ Du(x) \cdot z < 0\}, \\ \kappa^-_*[x, u] &= \nu\{z : u(x+z) < u(x),\ Du(x) \cdot z > 0\}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \kappa^*_+[x, u] &= \nu\{z : u(x+z) \geqslant u(x),\ Du(x) \cdot z \leqslant 0\}, \\ \kappa^*_-[x, u] &= \nu\{z : u(x+z) \leqslant u(x),\ Du(x) \cdot z \geqslant 0\}. \end{aligned}$$

We will see later (see Lemma 3 below) that these functions are semicontinuous, which explains our choice of notation. In order to understand the way these quantities are related to the geometry of the curve $\{u = u(x)\}$, it is convenient to write for instance

$$\kappa^+_*[x, u] = \nu\{z : 0 < -Du(x) \cdot z < u(x+z) - u(x) - Du(x) \cdot z\}.$$

As shown in Figure 1, $\kappa^+_*[x, u]$ measures how concave the curve is near x, and $\kappa^-_*[x, u]$ how convex it is.

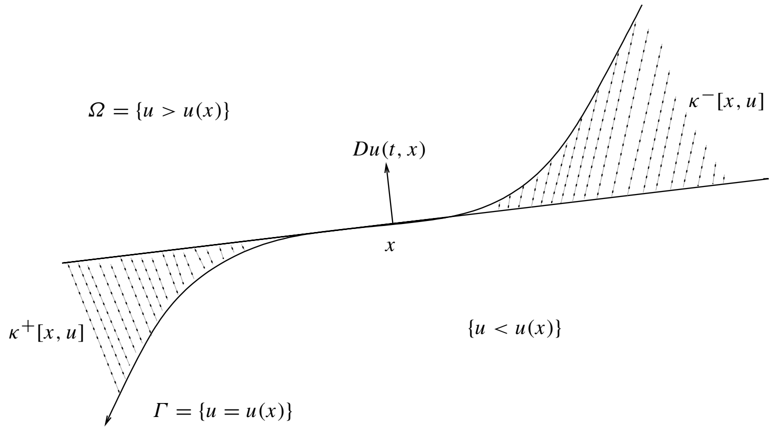

FIG. 1. Fractional mean curvature of a curve

LEMMA 1 (Fractional mean curvature is finite) If u is $C^{1,1}$ at a point x, i.e. there exists a constant $C = C(x) > 0$ such that for all $z \in \mathbb{R}^N$,

$$|u(x+z) - u(x) - Du(x) \cdot z| \leqslant C|z|^2$$

and its gradient $Du(x)$ is not 0, then $\kappa^*_\pm[x,u]$ are finite.

If u is $C^{1,1}$ at x and $Du \neq 0$ everywhere on $\{y \in \mathbb{R}^N : u(y) = u(x)\}$ and ν is absolutely continuous with respect to the Lebesgue measure, then $\kappa^*_\pm[x,u]$ are finite and

$$\kappa^*[x,u] = \kappa_*[x,u].$$

REMARK 1 One can check that this lemma is false if $\alpha = 1$ in the Standing Example 1.

Proof. We only prove the first part of the lemma since the second part is clear. Since ν is bounded on $\mathbb{R}^N \setminus B_\delta$ for all $\delta > 0$, it is enough to consider

$$\begin{aligned}(\kappa^*_+)^{1,\delta}[x,u] &= \nu\{z \in B_\delta : u(x+z) \geqslant u(x),\ Du(x) \cdot z \leqslant 0\} \\ &= \nu\{z \in B_\delta : 0 \leqslant re \cdot z \leqslant u(x+z) - u(x) + re \cdot z\}\end{aligned}$$

where $r = |Du(x)| \neq 0$ and $e = r^{-1} Du(x)$. If now z_N denotes $e \cdot z$ and $z' = z - z_N e$, and if we choose δ such that $r - C\delta > 0$, we can write

$$\begin{aligned}(\kappa^*_+)^{1,\delta}[x,u] &\leqslant \nu\{z \in B_\delta : 0 \leqslant r z_N \leqslant C z_N^2 + C|z'|^2\} \\ &\leqslant \nu\{z \in B_\delta : 0 \leqslant C^{-1}(r - C\delta) z_N \leqslant |z'|^2\},\end{aligned}$$

and the result now follows from (8). □

The following lemma explains rigorously the link between (6) and (2) and the link with the formulation used in [17] in the case where ν is a bounded measure.

LEMMA 2 (Link with regular dislocation dynamics) Consider $c_0 \in L^1(\mathbb{R}^N)$ such that $c_0(x) = c_0(-x)$. Then

$$\int_{\{z\,:\,u(t,x+z)>u(t,x)\}} c_0(z)\,\mathrm{d}z = \frac{1}{2}\int c_0 + \kappa_*[x,u],$$

$$\int \mathrm{sign}^*(u(x+z)-u(x))c_0(z)\,\mathrm{d}z = \frac{1}{2}\kappa^*[x,u],$$

$$\int \mathrm{sign}_*(u(x+z)-u(x))c_0(z)\,\mathrm{d}z = \frac{1}{2}\kappa_*[x,u],$$

with $\mathrm{sign}^*(r) = 1$ (resp. $\mathrm{sign}_*(r) = 1$) if $r \geqslant 0$ (resp. $r > 0$) and -1 if not, and with $\nu(\mathrm{d}z) = c_0(z)\,\mathrm{d}z$.

Since the proof is elementary, we omit it.

We conclude this section by stating two results which explain the link between two special cases of fractional mean curvature operator and the classical mean curvature operator. The first one appears in [15, Corollary 4.2]. We state it in a special case in order to simplify the presentation.

PROPOSITION 1 (From dislocation dynamics to mean curvature flow, [15]) Assume that $\nu = \nu^\varepsilon$ has the form

$$\nu(\mathrm{d}z) = \nu^\varepsilon(\,\mathrm{d}z) = \frac{1}{\varepsilon^{N+1}|\ln\varepsilon|}c_0\left(\frac{z}{\varepsilon}\right)\mathrm{d}z$$

with c_0 even, smooth, non-negative and such that $c_0(z) = |z|^{-N-1}$ for $|z| \geqslant 1$. Assume that $u \in C^2(\mathbb{R}^N)$ and $Du(x) \neq 0$. Then

$$\kappa[x,u] = \kappa^\varepsilon[x,u] \to C\,\mathrm{div}\left(\frac{Du}{|Du|}\right)(x)$$

as $\varepsilon \to 0$ for some constant $C > 0$.

REMARK 2 In [15], general anisotropic mean curvature operators can be obtained by considering anisotropic measures $\nu(\mathrm{d}z)$.

This result can be compared with the following one.

PROPOSITION 2 (From fractional mean curvature to mean curvature) Assume that ν has the form

$$\nu(\mathrm{d}z) = \nu^\alpha(\mathrm{d}z) = (1-\alpha)\frac{\mathrm{d}z}{|z|^{N+\alpha}}$$

with $\alpha \in (0,1)$. Assume that $u \in C^2(\mathbb{R}^N)$ and $Du(x) \neq 0$. Then

$$\kappa[x,u] = \kappa^\alpha[x,u] \to C\,\mathrm{div}\left(\frac{Du}{|Du|}\right)$$

as $\alpha \to 1$, $\alpha < 1$, where C is some positive constant.

REMARK 3 Anisotropic mean curvature can be obtained by considering

$$\nu^\alpha(\mathrm{d}z) = (1-\alpha)g\left(\frac{z}{|z|}\right)\frac{\mathrm{d}z}{|z|^{N+\alpha}}.$$

Sketch of the proof of Proposition 2. For all η, we first choose δ such that

$$\left| u(x+z) - u(x) - Du(x)\cdot z - \frac{1}{2} D^2 u(x) z \cdot z \right| \leqslant \eta |z|^2. \tag{11}$$

If e denotes $-Du(x)$ and $W(z)$ denotes $u(x+z) - u(x) - Du(x)\cdot z$, we have

$$\begin{aligned}\kappa^\alpha[x,u] &= \nu^\alpha\{z \in \mathbb{R}^N : 0 \leqslant e\cdot z \leqslant W(z)\} - \nu^\alpha\{z \in \mathbb{R}^N : W(z) \leqslant e \cdot z \leqslant 0\} \\ &= (1-\alpha)\int_{\{z\in B_\delta\,:\, 0\leqslant e\cdot z\leqslant W(z)\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}} - (1-\alpha)\int_{\{z\in B_\delta\,:\, W(z)\leqslant e\cdot z\leqslant 0\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}} \\ &\quad + O(1-\alpha)\end{aligned}$$

since $|z|^{-N-\alpha}$ is a bounded measure in B_δ^c.

In view of (11), it is enough to prove the result for $W(z) = Bz\cdot z$ where B is a symmetric $N\times N$ matrix. Hence we study the convergence of

$$K^\alpha = (1-\alpha)\int_{\{z\in B_\delta\,:\, 0\leqslant e\cdot z\leqslant Bz\cdot z\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}} - (1-\alpha)\int_{\{z\in B_\delta\,:\, Bz\cdot z\leqslant e\cdot z\leqslant 0\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}}.$$

We next use the following system of coordinates: $z_1 = \hat{e}\cdot z$ and $z = (z_1, z')$. We now write

$$Bz\cdot z = b_1 z_1^2 + z_1(b_1'\cdot z') + B'z'\cdot z'$$

for some $b_1 \in \mathbb{R}$, $b_1' \in \mathbb{R}^{N-1}$ and an $(N-1)\times(N-1)$ symmetric matrix B'. We thus want to prove

$$K^\alpha \to |e|^{-1}\operatorname{tr} B'$$

as $\alpha \to 1$. We can assume without loss of generality that $|e| = 1$. For $z \in B_\delta$, we have

$$\begin{aligned} e\cdot z \leqslant Bz\cdot z \;&\Rightarrow\; z_1 \leqslant (1-C\delta)^{-1} B'z'\cdot z', \\ z_1 \geqslant (1-C\delta)^{-1}B'z'\cdot z' \;&\Rightarrow\; e\cdot z \geqslant Bz\cdot z.\end{aligned}$$

Hence, it is enough to study the convergence of

$$\tilde{K}^\alpha = (1-\alpha)\int_{\{(z_1,z')\in B_\delta\,:\, 0\leqslant z_1\leqslant B'z'\cdot z'\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}} - (1-\alpha)\int_{\{(z_1,z')\in B_\delta\,:\, B'z'\cdot z'\leqslant z_1\leqslant 0\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}}.$$

If $\sigma(\mathrm{d}\theta)$ denotes the measure on the sphere $\mathbb{S}^{N-2}$, we can write

$$\begin{aligned}\tilde{K}^\alpha &= (1-\alpha)\int_{\{(z_1,z')\,:\,|z'|\leqslant\delta, 0\leqslant z_1\leqslant B'z'\cdot z'\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}} - (1-\alpha)\int_{\{(z_1,z')\,:\,|z'|\leqslant\delta, B'z'\cdot z'\leqslant z_1\leqslant 0\}} \frac{\mathrm{d}z}{|z|^{N+\alpha}} \\ &= (1-\alpha)\int_{\{\theta\in\mathbb{S}^{N-2}\,:\, B'\theta\cdot\theta\geqslant 0\}} \int_{r=0}^{\delta}\int_{z_1=0}^{r^2 B'\theta\cdot\theta} \frac{r^{N-2}}{(z_1^2+r^2)^{(N+\alpha)/2}}\,\mathrm{d}z_1\,\mathrm{d}r\,\sigma(\mathrm{d}\theta) \\ &\quad -(1-\alpha)\int_{\{\theta\in\mathbb{S}^{N-2}\,:\, B'\theta\cdot\theta\leqslant 0\}} \int_{r=0}^{\delta}\int_{z_1=r^2B'\theta\cdot\theta}^{0} \frac{r^{N-2}}{(z_1^2+r^2)^{(N+\alpha)/2}}\,\mathrm{d}z_1\,\mathrm{d}r\,\sigma(\mathrm{d}\theta).\end{aligned}$$

We next make the change of variables $z_1 = r^2\tau$ to get

$$\tilde{K}^\alpha = \int_{\{\theta\in\mathbb{S}^{N-2}: B'\theta\cdot\theta\geqslant 0\}} (1-\alpha)\int_{r=0}^{\delta} r^{-\alpha}\int_{\tau=0}^{B'\theta\cdot\theta} \frac{1}{(r^2\tau^2+1)^{(N+\alpha)/2}}\,\mathrm{d}\tau\,\mathrm{d}r\,\sigma(\mathrm{d}\theta) - (\ldots).$$

We finally remark that

$$\forall r \in (0, \delta), \qquad \int_{\tau=0}^{B'\theta\cdot\theta} \frac{\mathrm{d}\tau}{(r^2\tau^2+1)^{(N+\alpha)/2}} \to B'\theta\cdot\theta \quad \text{as } \delta \to 0.$$

In particular, for δ small enough,

$$(1-\eta)B'\theta\cdot\theta \leqslant \int_{\tau=0}^{B'\theta\cdot\theta} \frac{\mathrm{d}\tau}{(r^2\tau^2+1)^{(N+\alpha)/2}} \leqslant (1+\eta)B'\theta\cdot\theta.$$

It is now easy to conclude by remarking that

$$(1-\alpha)\int_0^\delta r^{-\alpha}\,\mathrm{d}r = \delta^{1-\alpha}, \qquad \int_{\mathbb{S}^{N-2}} \theta\otimes\theta\,\sigma(\mathrm{d}\theta) = CI_{N-1},$$

where I denotes the $(N-1)\times(N-1)$ identity matrix and C is a positive constant. □

3. Viscosity solutions for (6)

3.1 *Definitions*

The viscosity solution theory introduced in [26] suggests that the good notion of solution for the fractional equation (6) is the following one.

DEFINITION 1 (i) An upper semicontinuous function $u : [0, T] \times \mathbb{R}^N$ is a *viscosity subsolution* of (6) if for every smooth test function ϕ such that $u - \phi$ admits a global zero maximum at (t, x), we have

$$\partial_t\phi(t,x) \leqslant \mu(\widehat{D\phi(t,x)})\big[c_1(x) + \kappa^*[x, \phi(t,\cdot)]\big]|D\phi|(t,x) \tag{12}$$

if $D\phi(t, x) \neq 0$, and $\partial_t\phi(t, x) \leqslant 0$ if not.

(ii) A lower semicontinuous function u is a *viscosity supersolution* of (6) if for every smooth test function ϕ such that $u - \phi$ admits a global minimum 0 at (t, x), we have

$$\partial_t\phi(t,x) \geqslant \mu(\widehat{D\phi(t,x)})\big[c_1(x) + \kappa_*[x, \phi(t,\cdot)]\big]|D\phi|(x_0,t_0) \tag{13}$$

if $D\phi(t, x) \neq 0$, and $\partial_t\phi(t, x) \geqslant 0$ if not.

(iii) A locally bounded function u is a *viscosity solution* of (6) if u^* (resp. u_*) is a subsolution (resp. supersolution).

REMARK 4 Given $\delta > 0$, the global extrema in Definition 1 can be assumed to be strict in a ball of radius δ centred at (t, x). Such a result is classically expected and the reader can have a look, for instance, at the proof of the stability result in [7].

If one uses the notation introduced in [26], the equation reads

$$\partial_t u + F(x, Du, \{z : u(x+z) \geqslant u(x)\}) = 0 \tag{14}$$

with, for $x, p \in \mathbb{R}^N$ and $K \subset \mathbb{R}^N$,

$$F(x,p,K) = \begin{cases} -\mu(\hat{p})\big[c_1(x) + \nu\{K\cap\{p\cdot z \leqslant 0\}\} - \nu\{K^c \cap \{p\cdot z > 0\}\}\big]|p| & \text{if } p \neq 0,\\ 0 & \text{if not,}\end{cases}$$

where K^c is the complementary set of K. With this notation, one can check that this non-linearity does not satisfy Assumption (F5) of [26]. The idea is to check that Assumption (NLT) in [7] is satisfied and stability results thus hold true.

Let us be more precise. We previously associated with $\kappa[\cdot,\cdot]$ the following non-local operators (see the proof of Lemma 1):

$$\begin{aligned}(\kappa_*^+)^{1,\delta}[x,\phi] &= \nu\{z\in B_\delta : \phi(x+z)>\phi(x),\ z\cdot D\phi(x)<0\},\\ (\kappa_*^+)^{2,\delta}[x,p,\phi] &= \nu\{z\notin B_\delta : \phi(x+z)>\phi(x),\ z\cdot p<0\}.\end{aligned}\tag{15}$$

In the same way, we can define

- the negative non-local curvature operators $(\kappa_*^-)^{i,\delta}$, $i=1,2$,
- upper semicontinuous envelopes of the four integral operators $(\kappa_\pm^*)^{i,\delta}$, $i=1,2$,
- lower/upper semicontinuous total non-local curvature operators $(\kappa_*)^{i,\delta}$, $(\kappa^*)^{i,\delta}$, $i=1,2$.

By using the idea of Lemma 2, it is easy to see that

$$\begin{cases}(\kappa^*)^{2,\delta}[x,p,u] = \nu\{z\notin B_\delta : u(x+z)\geqslant u(x)\} - \nu\{z\notin B_\delta : p\cdot z>0\},\\ (\kappa_*)^{2,\delta}[x,p,u] = \nu\{z\notin B_\delta : u(x+z)> u(x)\} - \nu\{z\notin B_\delta : p\cdot z\geqslant 0\}.\end{cases}\tag{16}$$

We can now state an equivalent definition of viscosity solutions of (6).

DEFINITION 2 (Equivalent definition) (i) An upper semicontinuous function $u:[0,T]\times\mathbb{R}^N$ is a viscosity subsolution of (6) if for every smooth test function ϕ such that $u-\phi$ admits a maximum 0 at (t,x) on $B_\delta(t,x)$, we have

$$\partial_t\phi(t,x)\leqslant \mu(\widehat{D\phi(t,x)})\big[c_1(x)+(\kappa^*)^{1,\delta}[x,\phi(t,\cdot)]+(\kappa^*)^{2,\delta}[x,D\phi(t,x),u(t,\cdot)]\big]|D\phi|(t,x)\tag{17}$$

if $D\phi(t,x)\neq 0$, and $\partial_t\phi(t,x)\leqslant 0$ if not.

(ii) A lower semicontinuous function u is a viscosity supersolution of (6) if for every smooth test function ϕ such that $u-\phi$ admits a global minimum 0 at (t,x), we have

$$\partial_t\phi(t,x)\geqslant \mu(\widehat{D\phi(t,x)})\big[c_1(x)+(\kappa_*)^{1,\delta}[x,\phi(t,\cdot)]+(\kappa_*)^{2,\delta}[x,D\phi(t,x),u(t,\cdot)]\big]|D\phi|(x_0,t_0)\tag{18}$$

if $D\phi(t,x)\neq 0$, and $\partial_t\phi(t,x)\geqslant 0$ if not.

(iii) A continuous function u is a viscosity solution of (6) if it is both a subsolution and supersolution.

REMARK 5 Equivalent definitions of this type first appeared in [25] and since the proof is the same, we omit it.

REMARK 6 Remark 4 applies to the equivalent definition too.

REMARK 7 Definition 2 seems to depend on δ. But since all these definitions are equivalent to Definition 1, it does not depend on it. Hence, when proving that a function is a solution of (6), it is enough to do it for a fixed $\delta>0$.

3.2 *Stability results*

THEOREM 1 (Discontinuous stability) Assume (A1)–(A3).

- Let $(u_n)_{n\geqslant 1}$ be a family of subsolutions of (6) that is locally bounded, uniformly with respect to n. Then its relaxed upper limit u^* is a subsolution of (6).
- If moreover $u_n(0,x) = u_0^n(x)$, then for all $x \in \mathbb{R}^N$,

$$u^*(0,x) \leqslant u_0^*(x)$$

  where u_0^* is the relaxed upper limit of u_0^n.
- Let $(u_\alpha)_{\alpha\in\mathcal{A}}$ be a family of subsolutions of (6) that is locally bounded, uniformly with respect to $\alpha \in \mathcal{A}$. Then $\bar{u}$, the upper semicontinuous envelope of $\sup_\alpha u_\alpha$, is a subsolution of (6).

Even if this result follows from ideas introduced in [7] together with classical ones, we give a detailed proof for the sake of completeness.

Proof. We only prove the first part of the theorem since it is easy to adapt it to get a proof of the third part. The second one is very classical and can be adapted from [1] for instance.

Consider a test function φ such that $u^* - \varphi$ attains a global maximum at (t,x). We can assume (see Remark 4) that $u^* - \varphi$ attains a strict maximum at (t,x) on $B_\delta(t,x)$. Consider a subsequence $p = p(n)$ and (t_p, x_p) such that

$$u^*(t,x) = \lim_{n\to+\infty} u_{p(n)}(t_p, x_p).$$

Classical arguments show that $u_p - \varphi$ attains a maximum on $B_\delta(t,x)$ at $(s_p, y_p) \in B_\delta(t,x)$ and that

$$(s_p, y_p) \to (t,x) \quad \text{and} \quad u_p(s_p,y_p) \to u^*(t,x).$$

Since u_p is a subsolution of (6), we have

$$\partial_t\varphi(s_p,y_p) \leqslant \mu(\widehat{D\varphi(s_p,y_p)})\big[c_1(y_p) + (\kappa^*)^{1,\delta}[y_p,\varphi(s_p,\cdot)] + (\kappa^*)^{2,\delta}[y_p, D_x\varphi(s_p,y_p), u(s_p,\cdot)]\big]|D\varphi|(s_p,y_p)$$

if $D\varphi(t_p,x_p) \neq 0$, and $\partial_t\varphi(t_p,x_p) \leqslant 0$ if not. If there exists a subsequence q of p such that $D\varphi(s_q,y_q) = 0$, then it is easy to conclude the proof. We thus now assume that $D\varphi(s_p,y_p) \neq 0$ for p large enough. In view of the continuity of μ and c_1, the following technical lemma completes the argument. □

LEMMA 3 Assume that $D\varphi(s_p,y_p) \neq 0$ for p large enough.

- Assume moreover that $D\varphi(t,x) \neq 0$. Then

$$(s,y) \mapsto (\kappa^*)^{1,\delta}[y,\varphi(s,\cdot)] \quad \text{and} \quad (s,y) \mapsto (\kappa^*)^{2,\delta}[y, D_x\varphi(s,y), u_p(s,\cdot)]$$

  are well defined for $i = 1,2$ in a neighbourhood of (t,x) and

$$\limsup_p\, (\kappa^*)^{1,\delta}[y_p,\varphi(s_p,\cdot)] \leqslant (\kappa^*)^{1,\delta}[x,\varphi(t,\cdot)],$$

$$\limsup_p\, (\kappa^*)^{2,\delta}[y_p, D_x\varphi(s_p,y_p), u(s_p,\cdot)] \leqslant (\kappa^*)^{2,\delta}[x, D_x\varphi(t,x), \varphi(t,\cdot)],$$

  as soon as $u_p(s_p,y_p) \to u(t,x)$ as $p \to +\infty$.

- Assume now that $D\varphi(t,x)=0$. Then, for $i=1,2$,

$$\big[(\kappa^*)^{1,\delta}[y_p,\varphi(s_p,\cdot)]+(\kappa^*)^{2,\delta}[y_p,D\varphi(s_p,y_p),u(s_p,\cdot)]\big]|D\varphi|(s_p,y_p)\to 0 \quad \text{as } p\to+\infty.$$

As we shall see, this lemma is a consequence of the following one.

LEMMA 4 ([26]) Let f_p and g_p be two sequences of measurable functions on a set U and $f \geqslant \limsup^* f_p$, $g \geqslant \limsup^* g_p$, and a_p, b_p two sequences of real numbers converging to 0. Then

$$\nu(\{f_p\geqslant a_p, g_p\geqslant b_p\}\setminus\{f\geqslant 0, g\geqslant 0\})\to 0 \quad \text{as } n\to+\infty.$$

We mention that in [26], the measure is not singular and there is only one sequence of measurable functions but the reader can check that the slightly more general version we give here can be proven with exactly the same arguments. An immediate consequence of the lemma is the inequality

$$\limsup_p \nu\{f_p\geqslant a_p, g_p\geqslant b_p\}\leqslant \nu\{f\geqslant 0, g\geqslant 0\}.$$

Proof of Lemma 3. Let us first assume that $D\varphi(t,x)\neq 0$. In this case, for (s,y) close to (t,x), $D\varphi(s,y)\neq 0$ and all the integral operators we consider here are well defined (see Lemma 1). Recall next that, for $i=1,2$, $(\kappa^*)^{i,\delta}=(\kappa_+^*)^{i,\delta}-(\kappa_*^-)^{i,\delta}$. Hence, it is enough to prove that

$$\begin{aligned}
\limsup_p (\kappa_+^*)^{1,\delta}[y_p,\varphi(s_p,\cdot)] &\leqslant (\kappa_+^*)^{1,\delta}[x,\varphi(t,\cdot)],\\
\liminf_p (\kappa_*^-)^{1,\delta}[y_p,\varphi(s_p,\cdot)] &\geqslant (\kappa_*^-)^{1,\delta}[x,\varphi(t,\cdot)],\\
\limsup_p (\kappa_+^*)^{2,\delta}[y_p,D_x\varphi(s_p,y_p),u_p(s_p,\cdot)] &\leqslant (\kappa_+^*)^{2,\delta}[x,D_x\varphi(t,x),u^*(t,\cdot)],\\
\liminf_p (\kappa_*^-)^{2,\delta}[y_p,D_x\varphi(s_p,y_p),u_p(s_p,\cdot)] &\leqslant (\kappa_*^-)^{2,\delta}[x,D_x\varphi(t,x),u^*(t,\cdot)].
\end{aligned}$$

In order to prove the first inequality above for instance, choose $f_p(z)=\varphi(s_p,y_p+z)-\varphi(t,x)$, $a_p=\varphi(s_p,y_p)-\varphi(t,x)$, $g_p(z)=-D\varphi(s_p,y_p)\cdot z$, $b_p=0$ in Lemma 4.

We now turn to the case $D\varphi(t,x)=0$. We look for $\delta=\delta_p$ that goes to 0 as $p\to+\infty$ such that $|Du(s_p,y_p)|\leqslant C\delta_p$ and

$$(\kappa_+^*)^{1,\delta_p}[y_p,\varphi(s_p,\cdot)]|D\varphi(s_p,y_p)|\to 0 \quad\text{and}\quad (\kappa_*^-)^{1,\delta_p}[y_p,\varphi(s_p,\cdot)]|D\varphi(s_p,y_p)|\to 0$$

as $p\to+\infty$. This is enough to conclude the proof since condition (8) implies that

$$\begin{aligned}
&(\kappa_+^*)^{2,\delta_p}[y_p,D\varphi(s_p,y_p),u(s_p,\cdot)]|D\varphi(s_p,y_p)|\to 0,\\
&(\kappa_*^-)^{2,\delta_p}[y_p,D\varphi(s_p,y_p),u(s_p,\cdot)]|D\varphi(s_p,y_p)|\to 0.
\end{aligned}$$

We only prove that the first limit equals zero since the argument for the second one is similar. If r_p denotes $|D\varphi(s_p,y_p)|$ and e_p denotes $-r_p^{-1}D\varphi(s_p,y_p)$, and $z_N=e_P\cdot z$ and $z'=z-z_Ne_p$, then

$$\begin{aligned}
&(\kappa_+^*)^{1,\delta}[y_p,\varphi(s_p,\cdot)]|D\varphi(s_p,y_p)|\\
&\qquad = r_p\nu\{z\in B_{\delta_p}: 0\leqslant r_pe_p\cdot z\leqslant \varphi(s_p,y_p+z)-\varphi(s_p,y_p)+r_pe_p\cdot z\}\\
&\qquad \leqslant r_p\nu\{z\in B_{\delta_p}: 0\leqslant r_pz_N\leqslant C|z'|^2+Cz_N^2\}\\
&\qquad \leqslant r_p\nu\{z\in B_{\delta_p}: 0\leqslant r_pz_N\leqslant C|z'|^2+C\delta_pz_N\}
\end{aligned}$$

where C is a bound for the second derivatives of φ around (t, x). Now if we choose $\delta_p = r_p/(2C)$, we get

$$\begin{aligned}(\kappa_+^*)^{1,\delta}[y_p, \varphi(s_p, \cdot)]|D\varphi(s_p, y_p)| &\leqslant r_p \nu\{z \in B_{\delta_p} : 0 \leqslant (r_p/2C) z_N \leqslant |z'|^2\} \\ &\leqslant r_p \nu\{z \in B : 0 \leqslant (r_p/2C) z_N \leqslant |z'|^2\}\end{aligned}$$

and the last limit in (8) permits us now to conclude. □

3.3 *Existence and uniqueness results*

Let us first state a strong uniqueness result.

THEOREM 2 (Comparison principle) Assume (A1)–(A4). Assume moreover

(A3′) For all $e \in \mathbb{S}^{N-1}$ and $r \in (0, 1)$

$$r\nu\{z \in B_\delta : r|z \cdot e| \leqslant |z - (z \cdot e)e|^2\} \to 0 \quad \text{as } \delta \to 0 \tag{19}$$

uniformly in e and $r \in (0, 1)$ and

$$\nu(\mathrm{d}z) = J(z)\,\mathrm{d}z \quad \text{with } J \in W^{1,1}(\mathbb{R}^N \setminus B_\delta) \text{ for all } \delta > 0. \tag{20}$$

Consider a bounded and Lipschitz continuous function u_0. Let u (resp. v) be a bounded subsolution (resp. bounded supersolution) of (6). If $u(0, x) \leqslant u_0(x) \leqslant v(0, x)$, then $u \leqslant v$ on $(0, +\infty) \times \mathbb{R}^N$.

The proof is quite classical. The main difficulty is to deal with the singularity of the measure.

Proof of Theorem 2. We classically consider $M = \sup_{t,x}\{u(t, x) - v(t, x)\}$ and argue towards a contradiction by assuming $M > 0$. We next consider the following approximation of M:

$$\tilde{M}_{\varepsilon,\alpha} = \sup_{t,s>0,\, x,y\in\mathbb{R}^N} \left\{ u(t, x) - v(s, y) - \frac{(t-s)^2}{2\gamma} - e^{Kt}\frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha|x|^2 \right\}.$$

Since u and v are bounded, this supremum is attained at a point $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$. We first observe that $\tilde{M}_{\varepsilon,\alpha} \geqslant M/2 \geqslant 0$ for η and α small enough. Since u and v are bounded, this implies in particular

$$\eta\tilde{t} + e^{Kt}\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon} + \alpha|\tilde{x}|^2 \leqslant C_0 \tag{21}$$

where $C_0 = \|u\|_\infty + \|v\|_\infty$.

Classical results about penalization imply that $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y}) \to (\bar{t}, \bar{t}, \bar{x}, \bar{y})$ as $\gamma \to 0$ and $(\bar{t}, \bar{t}, \bar{x}, \bar{y})$ realizes the following supremum:

$$M_{\varepsilon,\alpha} = \sup_{t>0,\, x,y\in\mathbb{R}} \left\{ u(t, x) - v(t, y) - e^{Kt}\frac{|x-y|^2}{2\varepsilon} - \eta t - \alpha|x|^2 \right\}.$$

It is also classical [14] to get, for ε, η fixed,

$$\alpha|\bar{x}|^2 \to 0 \quad \text{as } \alpha \to 0. \tag{22}$$

We next claim that this supremum cannot be achieved at $t = 0$ if $\varepsilon, \alpha, \eta$ are small enough. To see this, remark first that $M_{\varepsilon,\alpha} \geqslant M/2 \geqslant 0$ for η and α small enough, and if $\bar{t} = 0$, use the fact that u_0 is Lipschitz continuous to get

$$0 < \frac{M}{2} \leqslant \sup_{x,y\in\mathbb{R}^N}\left\{u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon}\right\} \leqslant \sup_{r>0}\left\{C_0 r - \frac{r^2}{2\varepsilon}\right\} = \frac{1}{2}C_0^2\varepsilon,$$

which is obviously false if ε is small enough. We conclude that, if the four parameters are small enough, then $\tilde{t} > 0$ and $\tilde{s} > 0$.

Hence, we can write two viscosity inequalities. In order to clarify computations, we introduce the function $M(p)$ defined as follows:

$$M(p) = \begin{cases} \mu(\hat{p})|p| & \text{if } p \neq 0, \\ 0 & \text{if } p = 0. \end{cases}$$

It is easy to see that M is uniformly continuous and it trivially satisfies

$$|M(p)| \leqslant \|\mu\|_\infty |p|.$$

In the following, ω_M denotes the modulus of continuity of M.

We now write viscosity inequalities: for all $\delta > 0$,

$$\begin{aligned} \eta + \frac{\tilde{t}-\tilde{s}}{\gamma} + Ke^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{2\varepsilon} &\leqslant \big(c_1(\tilde{x}) + (\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)]\big) \\ &\quad \times M(\tilde{p} + 2\alpha\tilde{x}), \\ \frac{\tilde{t}-\tilde{s}}{\gamma} &\geqslant \big(c_1(\tilde{y}) + (\kappa_*)^{1,\delta}[\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s}, \cdot)]\big)M(\tilde{p}), \end{aligned}$$

where $\tilde{p} = e^{K\tilde{t}}(\tilde{x} - \tilde{y})/\varepsilon$ and

$$\begin{aligned} \phi_u(t, x) &= v(\tilde{s}, \tilde{y}) + \frac{(t-\tilde{s})^2}{2\gamma} + e^{Kt}\frac{|x-\tilde{y}|^2}{2\varepsilon} + \eta t + \alpha|x|^2, \\ \phi_v(s, y) &= u(\tilde{t}, \tilde{x}) - \frac{(s-\tilde{t})^2}{2\gamma} - e^{K\tilde{t}}\frac{|y-\tilde{x}|^2}{2\varepsilon} - \eta\tilde{t} - \alpha|\tilde{x}|^2. \end{aligned}$$

Subtracting these inequalities yields

$$\eta + Ke^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{2\varepsilon} \leqslant \|\mu\|_\infty\|Dc_1\|_\infty e^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon} + \|c_1\|_\infty\omega_M(2\sqrt{C_0\alpha}) + T_{nl} \tag{23}$$

(we used (21)) where

$$\begin{aligned} T_{nl} = {} & \big((\kappa^*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t}, \cdot)] + (\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)]\big)M(\tilde{p} + 2\alpha\tilde{x}) \\ & - \big((\kappa_*)^{1,\delta}[\tilde{y}, \phi_v(\tilde{s}, \cdot)] + (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s}, \cdot)]\big)M(\tilde{p}). \end{aligned}$$

Our task is now to find $\delta = \delta(\alpha, \varepsilon)$ so that the right hand side of this inequality is small when the four parameters are small. We distinguish two cases.

Assume first that there exist sequences $\alpha_n \to 0$ and $\varepsilon_n \to 0$ such that

$$\tilde{p} = \tilde{p}_n \to 0.$$

In this case, we simply choose $\delta = 1$, $K = 2\|\mu\|_\infty \|Dc_1\|_\infty$ and we let $n \to +\infty$ in (23) to get the desired contradiction: $\eta \leqslant 0$.

Assume now that for α and ε small enough, we have a constant C_ε independent of α such that

$$|\tilde{p}| \geqslant C_\varepsilon > 0. \tag{24}$$

In this case, the following technical lemma holds true.

LEMMA 5 By using (19), we have

$$T_{nl} \leqslant \frac{1}{\varepsilon} o_\delta(1) + \frac{1}{\delta}\omega_M(2\sqrt{C_0\alpha}) + o_\alpha(1)[\varepsilon] + C_\delta e^{K\tilde{t}} \frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon}$$

where C_0 appears in (21) and C_δ only depends on J, $\|\mu\|_\infty$ and δ (we emphasize that the third term goes to 0 as $\alpha \to 0$ for fixed ε).

The proof of this lemma is postponed. We thus get (recall that $\tilde{p} = e^{K\tilde{t}}(\tilde{x}-\tilde{y})/\varepsilon$)

$$\eta + Ke^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{2\varepsilon} \leqslant Ce^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon} + C\left(1+\frac{1}{\delta}\right)\omega_M(2\sqrt{C_0\alpha}) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)[\varepsilon] + C_\delta e^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{\varepsilon}$$

where C only depends on c_1, ν and $\|u\|_\infty + \|v\|_\infty$, and C_δ is given by the lemma. By choosing $K = 2(C + C_\delta)$, we get

$$\eta \leqslant C\left(1+\frac{1}{\delta}\right)\omega_M(2\sqrt{C_0\alpha}) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)[\varepsilon].$$

By letting successively α and δ go to 0, we thus get a contradiction. This completes the proof of the comparison principle. □

Proof of Lemma 5. We first write

$$\begin{aligned}
T_{nl} \leqslant\ & \|\mu\|_\infty |(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t},\cdot)]|\tilde{p} + 2\alpha\tilde{x}| + \|\mu\|_\infty |(\kappa_*)^{1,\delta}|[\tilde{y}, \phi_v(\tilde{s},\cdot)]|\tilde{p}| \\
& + |(\kappa^*)^{2,\delta}|[\tilde{x}, u(\tilde{t},\cdot)]\omega_M(|2\alpha\tilde{x}|) \\
& + \big((\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t},\cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s},\cdot)]\big)M(\tilde{p}).
\end{aligned}$$

We now estimate the right hand side of the previous inequality. We start with the first two integral terms. First

$$\begin{aligned}
|(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t},\cdot)] &\leqslant (\kappa^*_+)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t},\cdot)] + (\kappa^-_*)^{1,\delta}[\tilde{x}, \phi_u(\tilde{t},\cdot)] \\
&\leqslant \nu\{z \in B_\delta : 0 \leqslant -(\tilde{p} + 2\alpha\tilde{x})\cdot z \leqslant (\alpha + e^{K\tilde{t}}/(2\varepsilon))|z|^2\} \\
&\quad + \nu\{z \in B_\delta : 0 > -(\tilde{p} + 2\alpha\tilde{x})\cdot z > (\alpha + e^{K\tilde{t}}/(2\varepsilon))|z|^2\} \\
&\leqslant \nu\{z \in B_\delta : |\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}|\,|e\cdot z| \leqslant C(\eta)|z|^2\}
\end{aligned}$$

where we use (21) to ensure, for α, ε small enough,

$$\frac{e^{K\tilde{t}}}{2} + \alpha\varepsilon \leqslant \frac{1}{2}e^{KC_0/\eta} + 1 =: C(\eta).$$

If now $r_{\alpha,\varepsilon}$ denotes $|\varepsilon\tilde{p} + 2\varepsilon\alpha\tilde{x}|$ and we choose $\delta \leqslant r_{\alpha,\varepsilon}/(2C(\eta))$, we use (19) to write

$$\begin{aligned}
|(\kappa^*)^{1,\delta}|[\tilde{x}, \phi_u(\tilde{t}, \cdot)]|\tilde{p} + 2\alpha\tilde{x}| &= \frac{1}{\varepsilon} r_{\alpha,\varepsilon}\nu\{z \in B_\delta : r_{\alpha,\varepsilon}|e \cdot z| \leqslant C(\eta)|z|^2\} \\
&\leqslant \frac{1}{\varepsilon} r_{\alpha,\varepsilon}\nu\left\{z \in B_\delta : \frac{1}{2} r_{\alpha,\varepsilon}|e \cdot z| \leqslant C(\eta)|z - (e \cdot z)e|^2\right\} \\
&\leqslant \frac{2C(\eta)}{\varepsilon}\left\{\sup_{e \in \mathbb{S}^{N-1},\, r \in (0,1)} r\nu\{z \in B_\delta : r|e \cdot z| \leqslant |z - (e \cdot z)e|^2\}\right\} \\
&= \frac{1}{\varepsilon} o_\delta(1).
\end{aligned}$$

Since $\alpha\tilde{x} \to 0$ (see (21)), we choose for instance

$$\delta \leqslant \frac{\varepsilon|\tilde{p}|}{4C(\eta)}.$$

Arguing similarly, we get

$$|(\kappa_*)^{1,\delta}|[\tilde{y}, \phi_v(\tilde{s}, \cdot)]|\tilde{p}| \leqslant \frac{1}{\varepsilon} o_\delta(1).$$

As far as the third integral term is concerned, we simply write

$$|(\kappa^*)^{2,\delta}|[\tilde{x}, u(\tilde{t}, \cdot)]\omega_M(|2\alpha\tilde{x}|) \leqslant \nu(B_\delta^c)\omega_M(|2\alpha\tilde{x}|) \leqslant \frac{1}{\delta}\omega_M(2\sqrt{C_0\alpha})$$

(we used (21)). We now turn to the last two integral terms. In view of (16), we can write

$$\begin{aligned}
\tilde{T}_{nl} &= (\kappa^*)^{2,\delta}[\tilde{x}, \tilde{p} + 2\alpha\tilde{x}, u(\tilde{t}, \cdot)] - (\kappa_*)^{2,\delta}[\tilde{y}, \tilde{p}, v(\tilde{s}, \cdot)] \\
&= \nu\{z \notin B_\delta : u(\tilde{t}, \tilde{x} + z) \geqslant u(\tilde{t}, \tilde{x})\} - \nu\{z \notin B_\delta : v(\tilde{s}, \tilde{y} + z) > v(\tilde{s}, \tilde{y})\} \\
&\quad -\nu\{z \notin B_\delta : (\tilde{p} + 2\alpha\tilde{x}) \cdot z > 0\} + \nu\{z \notin B_\delta : \tilde{p} \cdot z \geqslant 0\}.
\end{aligned}$$

Now, we use (20) to get

$$\tilde{T}_{nl} = \int_{B_\delta^c} J(z - \tilde{x})\mathbf{1}_{\{u(\tilde{t},\cdot) > u(\tilde{t},\tilde{x})\}}(z)\,\mathrm{d}z - \int_{B_\delta^c} J(z - \tilde{y})\mathbf{1}_{\{v(\tilde{s},\cdot) > u(\tilde{s},\tilde{y})\}}(z)\,\mathrm{d}z + o_\alpha(1).$$

Remark next that the definition of $(\tilde{t}, \tilde{s}, \tilde{x}, \tilde{y})$ implies the following inequality: for all $z \in \mathbb{R}^N$,

$$u(\tilde{t}, z) - u(\tilde{t}, \tilde{x}) \leqslant v(\tilde{s}, z) - v(\tilde{s}, \tilde{y}) + \alpha(|z|^2 - |\tilde{x}|^2) - e^{K\tilde{t}}\,\frac{|\tilde{x} - \tilde{y}|^2}{2\varepsilon}.$$

This implies that for $|z| \leqslant R_{\alpha,\varepsilon}$, we have

$$\mathbf{1}_{\{u(\tilde{t},\cdot) > u(\tilde{t},\tilde{x})\}}(z) \leqslant \mathbf{1}_{\{v(\tilde{s},\cdot) > u(\tilde{s},\tilde{y})\}}(z)$$

where

$$R^2_{\alpha,\varepsilon} = \frac{1}{\alpha}\left(\alpha|\tilde{x}|^2 + e^{K\tilde{t}}\frac{|\tilde{x}-\tilde{y}|^2}{2\varepsilon}\right) = \frac{1}{\alpha}\left(o_\alpha(1) + \frac{\varepsilon C_\varepsilon^2}{4C(\eta)}\right) \geqslant \frac{\varepsilon C_\varepsilon^2}{8C(\eta)\alpha}$$

where C_ε appears in (24). We have used (22) here. Hence, we have

$$\begin{aligned}\tilde{T}_{nl} &\leqslant \int_{|z|\geqslant R_{\alpha,\varepsilon}} J(z-\tilde{x})\,\mathrm{d}z + \int_{z\in B_\delta^c} |J(z-\tilde{x}) - J(z-\tilde{y})|\,\mathrm{d}z \\ &\leqslant \int_{|\tilde{z}|\geqslant\sqrt{\varepsilon}C_\varepsilon/(2\sqrt{8C(\eta)\alpha})} J(\tilde{z})\,\mathrm{d}\tilde{z} + C_\delta|\tilde{x}-\tilde{y}| = o_\alpha(1)[\varepsilon] + C_\delta|\tilde{x}-\tilde{y}|\end{aligned}$$

where we have used (22) once again. It is now easy to conclude the proof. □

We now turn to the existence result.

THEOREM 3 (Existence) Assume (A1)–(A4) and (A3′). Then there exists a unique bounded uniformly continuous viscosity solution u of (6), (7).

Proof. We first construct a solution for regular initial data. Precisely, we first assume that $u_0 \in C_b^2(\mathbb{R}^N)$ (the function and its first and second derivatives are bounded).

Because we can apply Perron's method, it is enough to construct a subsolution and a supersolution $u^\pm$ to (6) such that $(u^+)_*(0,x) = (u^-)^*(0,x) = u_0(x)$. We assert that $u^\pm(t,x) = u_0(x) \pm Ct$ are respectively a supersolution and a subsolution of (6) for C large enough. To see this, we first prove that there exists $C_0 = C_0(\|D^2u_0\|_\infty)$ such that for all $x \in \mathbb{R}^N$ with $Du_0(x) \neq 0$, we have

$$(|\kappa_*|[x,u_0] + |\kappa^*|[x,u_0])|Du_0(x)| \leqslant C_0. \tag{25}$$

In order to prove this estimate, we simply write, for x such that $Du_0(x) \neq 0$,

$$\begin{aligned}((\kappa_+^*)^{1,\delta}[x,u_0] + (\kappa_-^*)^{1,\delta}[x,u_0])|Du_0(x)| &\leqslant 2\nu\left\{z \in B_\delta : r|e\cdot z| \leqslant \frac{1}{2}\|D^2u_0\|_\infty|z|^2\right\}r \\ &\leqslant 2\nu\{z \in B_\delta : r|e\cdot z| \leqslant C|z-(e\cdot z)e|^2\}r \leqslant C_\nu\end{aligned}$$

where $r = |Du_0(x)|$, $C = \max(\|Du_0\|_\infty, 1)$, $e = Du_0(x)/r$, $\delta = r/(2C)$ and C_ν is given by (8). On the other hand,

$$((\kappa_+^*)^{2,\delta}[x,u_0] + (\kappa_-^*)^{2,\delta}[x,u_0])|Du_0(x)| \leqslant \frac{C_\nu}{\delta}r = 2C_\nu C.$$

We thus get estimate (25).

If now u_0 is not regular, we approximate it with $u_0^n \in C_b^2(\mathbb{R}^N)$ and prove that the corresponding sequence of solutions u_n converges locally uniformly towards a solution u. Since this is very classical, we omit the details (see for instance [1]). □

We now explain in which limit one recovers the mean curvature flow. To do so, we state two convergence results. Their proofs rely on Propositions 1 and 2. The first one (Theorem 4) appears in [15] and the second one can be proved by using Proposition 2.

THEOREM 4 ([15]) Assume that $\mu \equiv 1$, $c_1 \equiv 0$, u_0 is Lipschitz continuous and bounded and

$$\nu(\mathrm{d}z) = \nu^\varepsilon(\mathrm{d}z) = \frac{1}{\varepsilon^{N+1}|\ln\varepsilon|}c_0\left(\frac{z}{\varepsilon}\right)\mathrm{d}z$$

with c_0 even, smooth, non-negative and such that $c_0(z) = |z|^{-N-1}$ for $|z| \geqslant 1$. Then the viscosity solution u^ε of (6), (7) converges locally uniformly as $\varepsilon \to 0$ towards the viscosity solution u of

$$\partial_t u = C|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

(C is a positive constant) supplemented with the initial condition (7).

THEOREM 5 Assume that $\mu \equiv 1$, $c_1 \equiv 0$, u_0 is Lipschitz continuous and bounded and

$$\nu(\mathrm{d}z) = \nu^\alpha(\mathrm{d}z) = (1-\alpha)\frac{\mathrm{d}z}{|z|^{N+\alpha}}$$

with $\alpha \in (0, 1)$. Then the viscosity solution u^α of (6), (7) converges locally uniformly as $\alpha \to 1$ towards the viscosity solution u of

$$\partial_t u = C|Du| \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

(C is a positive constant) supplemented with the initial condition (7).

4. The level set approach

In the previous section, we constructed a unique solution of (6) in the case of singular measures satisfying (A3) and (A3′) and for bounded Lipschitz continuous initial data (see (A4)). In the present section, we explain how to define a geometric flow by using these solutions of (6). Precisely, we first prove (Theorem 6) that if u and v are solutions of (6) associated with two different initial data u_0 and v_0 that have the same zero level sets, then so have u and v. Hence, the geometric flow is obtained by considering the zero level sets of the solution u of (6) for any (Lipschitz continuous) initial datum. We also describe (Theorem 7) the maximal and minimal discontinuous solutions of (6) associated with an important class of discontinuous initial data.

THEOREM 6 (Consistency of the definition) Assume (A1)–(A3) and (A3′). Let u_0 and v_0 be two bounded Lipschitz continuous functions and consider the viscosity solutions u, v associated with these initial conditions. If

$$\{x \in \mathbb{R}^N : u_0(x) > 0\} = \{x \in \mathbb{R}^N : v_0(x) > 0\},$$
$$\{x \in \mathbb{R}^N : u_0(x) < 0\} = \{x \in \mathbb{R}^N : v_0(x) < 0\},$$

then, for all times $t > 0$,

$$\{x \in \mathbb{R}^N : u(t,x) > 0\} = \{x \in \mathbb{R}^N : v(t,x) > 0\},$$
$$\{x \in \mathbb{R}^N : u(t,x) < 0\} = \{x \in \mathbb{R}^N : v(t,x) < 0\}.$$

In view of the techniques used to prove the consistency of the definition of local geometric fronts (see for instance [8]), it is clear that this result is a straightforward consequence of the following proposition.

PROPOSITION 3 (Equation (6) is geometric) Let $u : [0, +\infty) \times \mathbb{R}^N$ be a bounded subsolution of (6) and $\theta : \mathbb{R} \to \mathbb{R}$ an upper semicontinuous non-decreasing function. Then $\theta(u)$ is also a subsolution of (6).

Such a proposition is classical by now. It is proved by regularizing θ (in a proper way) with a strictly increasing function θ^n, by remarking that $\kappa^*[x,\theta^n(u)] = \kappa^*[x,u]$ in this case, and by using discontinuous stability. Details are left to the reader.

Thanks to Theorem 6, we can define a geometric flow in the following way. Given (Γ_0, D_0^+, D_0^-) such that Γ_0 is closed, $D_0^\pm$ are open and $\mathbb{R}^N = \Gamma_0 \sqcup D_0^+ \sqcup D_0^-$, we can write

$$D_0^+ = \{x \in \mathbb{R}^N : u_0(x) > 0\}, \quad D_0^- = \{x \in \mathbb{R}^N : u_0(x) < 0\}, \quad \Gamma_0 = \{x \in \mathbb{R}^N : u_0(x) = 0\}$$

for some bounded Lipschitz continuous function u_0 (for instance the signed distance function). If u is the solution of (6) with the initial condition $u(0,x) = u_0(x)$ for $x \in \mathbb{R}^N$, then Theorem 6 precisely says that the sets

$$D_t^+ = \{x \in \mathbb{R}^N : u(t,x) > 0\}, \quad D_t^- = \{x \in \mathbb{R}^N : u(t,x) < 0\}, \quad \Gamma_t = \{x \in \mathbb{R}^N : u(t,x) = 0\}$$

do not depend on the choice of u_0.

The next theorem states that there exists a maximal subsolution and a minimal supersolution of (6) associated with the apropriate discontinuous initial data.

THEOREM 7 (Maximal subsolution and minimal supersolution) Assume (A1)–(A3) and (A3′). Then the function $\mathbf{1}_{D_t^+\cup\Gamma_t} - \mathbf{1}_{D_t^-}$ (resp. $\mathbf{1}_{D_t^+} - \mathbf{1}_{D_t^-\cup\Gamma_t}$) is the maximal subsolution (resp. minimal supersolution) of (6) subject to the initial datum $\mathbf{1}_{D_0^+\cup\Gamma_0} - \mathbf{1}_{D_0^-}$ (resp. $\mathbf{1}_{D_0^+} - \mathbf{1}_{D_0^-\cup\Gamma_0}$).

This result is a consequence of Proposition 3 together with discontinuous stability and the comparison principle. See [8, p. 445] for details.

We conclude this section by showing that a bounded front propagates with finite speed.

PROPOSITION 4 (Evolution of bounded sets) Assume (A1)–(A3) and (A3′). Let Ω_0 be a bounded open set of $\mathbb{R}^N$, i.e. there exists $R > 0$ such that $\Omega_0 \subset B_R$. Then the level set evolution (Γ_t, D_t^+, D_t^-) of $(\partial\Omega_0, \Omega_0, (\bar{\Omega}_0)^c)$ satisfies $D_t^+ \cup \Gamma_t \subset \bar{B}_{R+Ct}$ with

$$C = \|c_1\|_\infty - \inf_{e\in\mathbb{S}^{N-1}} \nu\{z \in \mathbb{R}^N : 0 \leqslant e\cdot z \leqslant |z|^2\}$$

as long as $R + Ct > 0$.

REMARK 8 Another consequence of this proposition is that, if there is no driving force ($c_1 = 0$), then the set shrinks till it disappears.

Proof. The proof consists in constructing a supersolution of (6), (7). It is easy to check that C is chosen such that

$$u(t,x) = Ct + \sqrt{\varepsilon^2 + R^2} - \sqrt{\varepsilon^2 + |x|^2}$$

is a supersolution of (6). Since $\bar{B}_R = \{x \in \mathbb{R}^N : u(0,x) \geqslant 0\}$, we conclude that $D_t^+ \cup \Gamma_t \subset \{x \in \mathbb{R}^N : u(t,x) \geqslant 0\} = \bar{B}_{R^\varepsilon(t)}$ with $R^\varepsilon(t) = \sqrt{(Ct + \sqrt{\varepsilon^2 + R^2})^2 - \varepsilon^2}$. Hence, $D_t^+ \cup \Gamma_t \subset \bigcap_{\varepsilon>0} \bar{B}_{R^\varepsilon(t)} = \bar{B}_{R+Ct}$. □

5. Generalized flows

In this section, we follow [9] and give an equivalent definition of the flow by, roughly speaking, replacing smooth test functions with smooth test fronts.

In order to give this equivalent definition, we use the geometrical non-linearities we partially introduced in Section 2. For all $x, p \in \mathbb{R}^N$ and any closed set $\mathcal{F} \subset \mathbb{R}^N$ and open set $\mathcal{O} \subset \mathbb{R}^N$, set

$$F_*(x, p, \mathcal{F}) = \begin{cases} -\mu(\hat{p})\big[c_1(x) + \nu(\mathcal{F} \cap \{p \cdot z \leqslant 0\}) - \nu(\mathcal{F}^c \cap \{p \cdot z > 0\})\big]|p| & \text{if } p \neq 0, \\ 0 & \text{if not,} \end{cases}$$

$$F^*(x, p, \mathcal{O}) = \begin{cases} -\mu(\hat{p})\big[c_1(x) + \nu(\mathcal{O} \cap \{p \cdot z < 0\}) - \nu(\mathcal{O}^c \cap \{p \cdot z \geqslant 0\})\big]|p| & \text{if } p \neq 0, \\ 0 & \text{if not.} \end{cases}$$

We can now give the definition of a generalized flow.

DEFINITION 3 The family $(\mathcal{O}_t)_{t\in(0,T)}$ of open subsets of $\mathbb{R}^N$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$ of closed subsets of $\mathbb{R}^N$) is a *generalized superflow* (resp. *subflow*) of (6) if for all $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$, $r > 0$, $h > 0$, and for any smooth function $\phi : (0; +\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that

1. $\partial_t\phi + F^*(x, D\phi, \{z : \phi(t, x + z) > \phi(t, x)\}) \leqslant -\delta_\phi$ in $[t_0, t_0 + h] \times \bar{B}(x_0, r)$
   (resp. $\partial_t\phi + F_*(x, D\phi, \{z : \phi(t, x + z) \geqslant \phi(t, x)\}) \geqslant -\delta_\phi$ in $[t_0, t_0 + h] \times \bar{B}(x_0, r)$)
2. $D\phi \neq 0$ in $\{(s, y) \in [t_0, t_0 + h] \times \bar{B}(x_0, r) : \phi(s, y) = 0\}$,
3. $\{y \in \mathbb{R}^N : \phi(t_0, y) \geqslant 0\} \subset \mathcal{O}^1_{t_0}$
   (resp. $\{y \in \mathbb{R}^N : \phi(t_0, y) \leqslant 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0}$),
4. $\{y \notin \bar{B}(x_0, r) : \phi(s, y) \geqslant 0\} \subset \mathcal{O}^1_s$ for all $s \in [t_0, t_0 + h]$
   (resp. $\{y \notin \bar{B}(x_0, r) : \phi(s, y) \leqslant 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_s$ for all $s \in [t_0, t_0 + h]$),

we have $\{y \in \bar{B}(x_0, r) : \phi(t_0 + h, y) > 0\} \subset \mathcal{O}^1_{t_0+h}$ (resp. $\{y \in \bar{B}(x_0, r) : \phi(t_0 + h, y) < 0\} \subset \mathbb{R}^N \setminus \mathcal{F}_{t_0+h}$).

Loosely speaking, for generalized superflows, condition 1 says that in a prescribed neighbourhood $\mathcal{V}$ of (t_0, x_0), the normal velocity of the test front $\{\phi > 0\}$ is strictly smaller than the one of the front $\mathcal{O}$; condition 2 asserts that the front $\{\phi = 0\}$ is smooth in $\mathcal{V}$; conditions 3 and 4 assert that the test front is inside the front $\mathcal{O}$ outside $\mathcal{V}$. The conclusion is that the test front is inside the neighbourhood $\mathcal{O}$ at time $t + h$.

REMARK 9 As far as local geometric fronts are concerned, conditions 3 and 4 require that the test front is inside $\mathcal{O}$ on the parabolic boundary of the neighbourhood. Here, because the front is not local, the test front has to be inside $\mathcal{O}$ everywhere outside the neighbourhood.

The next theorem asserts that Definition 3 of the flow coincides with the level set formulation of Section 4.

THEOREM 8 (Generalized flows and level set approach) Assume (A1)–(A3) and (A3′). Let $(\mathcal{O}_t)_{t\in(0,T)}$ be a family of open subsets of $\mathbb{R}^N$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$ of closed subsets of $\mathbb{R}^N$) such that $\bigcup_{t\in(0,T)}\{t\} \times \mathcal{O}_t$ is open in $[0, T] \times \mathbb{R}^N$ (resp. $\bigcup_{t\in(0,T)}\{t\} \times \mathcal{F}_t$ is closed in $[0, T] \times \mathbb{R}^N$). Then $(\mathcal{O}_t)_{t\in(0,T)}$ (resp. $(\mathcal{F}_t)_{t\in(0,T)}$) is a generalized superflow (resp. subflow) of (6) if and only if $\chi(t, x) = \mathbf{1}_{\mathcal{O}_t}(x) - \mathbf{1}_{\mathbb{R}^N\setminus\mathcal{O}_t}(x)$ (resp. $\chi(t, x) = \mathbf{1}_{\mathcal{F}_t}(x) - \mathbf{1}_{\mathbb{R}^N\setminus\mathcal{F}_t}(x)$) is a viscosity supersolution (resp. subsolution) of (6), (7).

Since the proof of [9] can be readily adapted, we omit it. We give a straightforward corollary of Theorems 7 and 8 that is used in [21].

COROLLARY 1 (Abstract method) Assume (A1)–(A3) and (A3′). Assume that $(\mathcal{O}_t)_t$ and $(\mathcal{F}_t)_t$ are respectively a generalized superflow and generalized subflow and suppose there exist two open sets

D_0^+, D_0^- such that $\mathbb{R}^N = \partial\mathcal{O}_0 \sqcup D_0^+ \sqcup D_0^-$ and $D_0^+ \subset \mathcal{O}_0$ and $D_0^- \subset \mathcal{F}_0^c$. Then if (Γ_t, D_t^+, D_t^-) denotes the level set evolution of $(\partial\mathcal{O}_0, D_0^+, D_0^-)$, we have, for all times $t > 0$,

$$D_t^+ \subset \mathcal{O}_t \subset D_t^+ \cup \Gamma_t, \quad D_t^- \subset \mathcal{F}_t^c \subset D_t^- \subset \Gamma_t.$$

REMARK 10 One can check that under the assumptions of the previous corollary, we have in fact $D_0^+ = \mathcal{O}_0$ and $D_0^- = \mathcal{F}_0^c$.

Acknowledgements

This paper is partially supported by the ANR grant "MICA". The author thanks R. Monneau and P. E. Souganidis for fruitful discussions. He also thanks G. Barles and the two referees for their attentive reading of this paper before its publication.

REFERENCES

1. ALIBAUD, N., & IMBERT, C. Fractional semi-linear parabolic equations with unbounded data. *Trans. Amer. Math. Soc.* **361** (2009), 2527–2566.
2. ALVAREZ, O., CARDALIAGUET, P., & MONNEAU, R. Existence and uniqueness for dislocation dynamics with nonnegative velocity. *Interfaces Free Bound.* **7** (2005), 415–434. Zbl 1099.35148 MR 2191694
3. ALVAREZ, O., HOCH, P., LE BOUAR, Y., & MONNEAU, R. Dislocation dynamics: short-time existence and uniqueness of the solution. *Arch. Ration. Mech. Anal.* **181** (2006), 449–504. Zbl pre05051264 MR 2231781
4. APPLEBAUM, D. *Lévy Processes and Stochastic Calculus*. Cambridge Stud. Adv. Math. 93, Cambridge Univ. Press, Cambridge (2004). Zbl 1073.60002 MR 2072890
5. BARLES, G., CARDALIAGUET, P., LEY, O., & MONNEAU, R. Global existence results and uniqueness for dislocation equations. *SIAM J. Math. Anal.* **40** (2008), 44–69. MR 2403312
6. BARLES, G., CARDALIAGUET, P., LEY, O., & MONTEILLET, A. Uniqueness results for nonlocal Hamilton–Jacobi equations. Submitted (2008).
7. BARLES, G., & IMBERT, C. Second-order elliptic integro-differential equations: Viscosity solutions' theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), 567–585. Zbl pre05290961 MR 2422079
8. BARLES, G., SONER, H. M., & SOUGANIDIS, P. E. Front propagation and phase field theory. *SIAM J. Control Optim.* **31** (1993), 439–469. Zbl 0785.35049 MR 1205984
9. BARLES, G., & SOUGANIDIS, P. E. A new approach to front propagation problems: theory and applications. *Arch. Rational Mech. Anal.* **141** (1998), 237–296. Zbl 0904.35034 MR 1617291
10. CAFFARELLI, L., ROQUEJOFFRE, J.-M., & SAVIN, O. Personal communication (2008).
11. CAFFARELLI, L., & SOUGANIDIS, P. E. Convergence of nonlocal threshhold dynamics approximations to front propagation. *Arch. Ration. Mech. Anal.*, to appear.
12. CARDALIAGUET, P. Front propagation problems with nonlocal terms. II. *J. Math. Anal. Appl.* **260** (2001), 572–601. Zbl 1002.35064 MR 1845570
13. CHEN, Y. G., GIGA, Y., & GOTO, S. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* **33** (1991), 749–786. Zbl 0696.35087 MR 1100211
14. CRANDALL, M. G., ISHII, H., & LIONS, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), 1–67. Zbl 0755.35015 MR 1118699

15. DA LIO, F., FORCADEL, N., & MONNEAU, R. Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics. *J. Eur. Math. Soc.* **10** (2008), 1061–1104. Zbl pre05365145 MR 2443929

16. EVANS, L. C., & SPRUCK, J. Motion of level sets by mean curvature. I. *J. Differential Geom.* **33** (1991), 635–681. Zbl 0726.53029 MR 1100206s

17. FORCADEL, N., IMBERT, C., & MONNEAU, R. Homogenization of some particle systems with two-body interactions and of the dislocation dynamics. *Discrete Contin. Dynam. Systems* (2008), to appear.

18. FORCADEL, N., & MONNEAU, R. Existence of solutions for a model describing the dynamics of junctions between dislocations. Submitted (2008).

19. IMBERT, C., MONNEAU, R., & ROUY, E. Homogenization of first order equations with u/ϵ-periodic hamiltonians. Part II: Application to dislocation dynamics. *Comm. Partial Differential Equations* **33** (2008), 479–516. Zbl 1143.35005 MR 2398239

20. IMBERT, C., & SERFATY, S. Repeated games for eikonal equations, dislocation dynamics and parabolic integro-differential equations. In preparation (2008).

21. IMBERT, C., & E. SOUGANIDIS, P. Phasefield theory for fractional reaction-diffusion equations and applications. In preparation (2008).

22. KOHN, R., & SERFATY, S. A deterministic-control-based approach to fully non-linear parabolic and elliptic equations. Tech. report, Courant Inst., New York Univ. (2007).

23. KOHN, R. V., & SERFATY, S. A deterministic-control-based approach to motion by curvature. *Comm. Pure Appl. Math.* **59** (2006), 344–407. Zbl pre05023699 MR 2200259

24. OSHER, S., & SETHIAN, J. A. Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations. *J. Comput. Phys.* **79** (1988), 12–49. Zbl 0659.65132 MR 0965860

25. SAYAH, A. Équations d'Hamilton–Jacobi du premier ordre avec termes intégro-différentiels. I, Unicité des solutions de viscosité. *Comm. Partial Differential Equations* **16** (1991), 1057–1074. Zbl 0742.45004

26. SLEPČEV, D. Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions. *Nonlinear Anal.* **52** (2003), 79–115. Zbl 1028.35068 MR 1938652