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A moving boundary problem for periodic Stokesian Hele–Shaw flows

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This paper is concerned with the motion of an incompressible, viscous fluid in a Hele–Shaw cell. The free surface is moving under the influence of gravity and the fluid is modelled using a modified Darcy law for Stokesian fluids.

We combine results from the theory of quasilinear elliptic equations, analytic semigroups and Fourier multipliers to prove existence of a unique classical solution to the corresponding moving boundary problem.

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1. Introduction

Starting from a non-Newtonian Darcy law as presented in [9], we derive a mathematical model for the flow of a Stokesian fluid[1] located between the plates of a vertical Hele–Shaw cell. The pressure on the bottom of the cell is assumed to be constant. The corresponding mathematical setting is a fully nonlinear coupled system consisting of a quasilinear elliptic Dirichlet problem for the *velocity potential* and an evolution equation for the free boundary, i.e. the *interface* separating the fluid from the air. The contact angle problem is avoided by considering periodic flows only. The Newtonian case, studied in [3]–[7] in various contexts, is also included in this model. Our setting is general enough to embrace shear thinning fluids, like Oldroyd-B or power law fluids, as well as shear thickening fluids.

We shall attack this problem by transforming it into a problem on a fixed manifold $\mathbb{S}^1 \times (0, 1)$. This will be done in Section 1. In Section 2 we identify the new setting with an abstract Cauchy problem on the unit circle $\mathbb{S}^1$:

$$\partial_t f + \Phi(f) = 0, \qquad f(0) = f_0.$$

Our analysis shows that Φ is a pseudodifferential operator of first order with a symbol depending nonlinearly on the function f modelling the free boundary. Moreover, the operator $f \mapsto \Phi(f)$ is fully nonlinear, in the sense that its nonlinear part is of first order as well. Nevertheless, we prove that given any positive constant c, the Fréchet derivative $-\partial\Phi(c)$ generates a strongly continuous

[1] In a *Stokesian* fluid the stress tensor is a continuous function of the deformation. A *Newtonian* fluid is a linear Stokesian fluid. In particular, the viscosity μ is constant in this case.

analytic semigroup in $\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))$ with dense domain $h^{2+\alpha}(\mathbb{S}^1)$. Working with small Hölder spaces $h^{m+\alpha}(\mathbb{S}^1)$, $m \in \mathbb{N}$ and $\alpha \in (0,1)$, is a significant advantage, because $h^{m_1+\alpha_1}(\mathbb{S}^1)$ is dense and compactly embedded in $h^{m_2+\alpha_2}(\mathbb{S}^1)$ provided $m_1+\alpha_1 > m_2+\alpha_2$. It is known that this property does not hold for the usual Hölder spaces.

The main result, a well-posedness result for the full flow, is proved in Section 3 and is based on a multiplier theorem for periodic Besov spaces. This theorem generalizes a result of Arendt and Bu presented in [2]. As in [2], our multiplier theorem is also based on Marcinkiewicz type conditions.

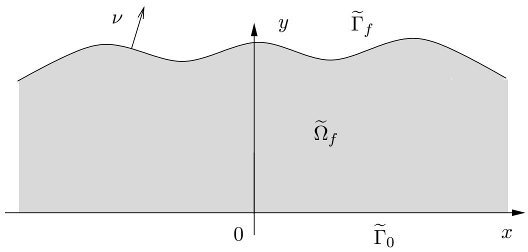
1.1 *The mathematical model*

Given a positive function $f \in C^1(\mathbb{R})$, which is bounded away from 0, we define the set

$$\widetilde{\Omega}_f := \{(x,y) \in \mathbb{R}^2 \,:\, 0 < y < f(x)\},$$

and denote the components of its boundary by

$$\widetilde{\Gamma}_f := \{(x, f(x)) \,:\, x \in \mathbb{R}\}, \quad \widetilde{\Gamma}_0 := \mathbb{R} \times \{0\}.$$



The domain $\widetilde{\Omega}_f$ consists of a Stokesian fluid at pressure p and we denote by v the velocity field inside the fluid's body. The motion of the fluid is governed by the following modified version of Darcy's law:

$$v = -\frac{Du}{\overline{\mu}(|Du|^2)}$$

(cf. [9]), where

$$u(x,y) = \frac{p(x,y)}{g \cdot \rho} + y, \qquad (x,y) \in \widetilde{\Omega}_f,$$

is the so-called *velocity potential* or *piezometric heat*, g is the gravity acceleration, ρ is the density of the fluid and $Du = (\partial_1 u, \partial_2 u)$ is the gradient of u. The effective viscosity $\overline{\mu}$ is defined (see [9]) by

$$\frac{1}{\overline{\mu}(r)} := c_\mu \int_{-1}^{1} \frac{s^2}{\widetilde{\mu}(rs^2)}\,\mathrm{d}s$$

for all $r \geqslant 0$, where c_μ is a positive constant. Denoting by $\mu \in C^\infty([0,\infty),(0,\infty))$ the viscosity of the fluid, we have assumed that the mapping $r \mapsto h(r) := r\mu^2(r)$ is invertible. This is true for example if $\mu(r) + 2r\mu'(r) > 0$ for all $r \geqslant 0$. The mapping $\widetilde{\mu}$ is defined by $\widetilde{\mu} := \mu \circ h^{-1}$.

We assume the fluid is incompressible (div $v = 0$), thus we get

$$\mathrm{div}\left(\frac{Du}{\overline{\mu}(|Du|^2)}\right) = 0 \quad \text{in } \widetilde{\Omega}_f. \tag{1}$$

On the boundary component $\widetilde{\Gamma}_0$ the velocity potential is known, namely

$$u(x,0) = \frac{p(x,0)}{g\cdot\rho} =: b(x), \qquad x\in\mathbb{R}. \tag{2}$$

Moreover, we assume that the fluid is surrounded by air at atmospheric pressure, normalized to be zero. Then $p(x, f(x)) = 0$ for $x \in \mathbb{R}$, and so

$$u(x, f(x)) = f(x), \qquad x\in\mathbb{R}. \tag{3}$$

Set $F(t,z) = y - f(t,x)$ for $z = (x,y) \in \mathbb{R}$ and $t \geqslant 0$. Then the interface $\widetilde{\Gamma}_f$ can be described by the conservative property that F is identically equal to zero on $\widetilde{\Gamma}_f$. Differentiating with respect to the time variable t we get

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t,z) = -\partial_t f(t,x) + (-f_x, 1)\cdot z'.$$

Replacing z' by $-Du/\overline{\mu}(|Du|^2)$, we obtain

$$\partial_t f + \frac{\sqrt{1+\partial_x f^2}}{\overline{\mu}(|Du|^2)}\partial_\nu u = 0 \quad \text{on } \widetilde{\Gamma}_f, \tag{4}$$

with ν denoting the outer normal of $\widetilde{\Gamma}_f$. Finally, we set

$$f(0,\cdot) = f_0, \tag{5}$$

where f_0 corresponds to the initial surface. We shall make the following periodicity requirement on f and u:

$$\begin{aligned} f(t, x+2\pi) &= f(t,x), \quad \forall x\in\mathbb{R},\ t\geqslant 0,\\ u(x+2\pi, y) &= u(x,y), \quad \forall (x,y)\in \widetilde{\Omega}_{f(t)},\ t\geqslant 0. \end{aligned}$$

Thus, instead of (1)–(5) we study

$$\begin{aligned} \operatorname{div}\left(\frac{Du}{\overline{\mu}(|Du|^2)}\right) &= 0 \quad \text{in } \Omega_{f(t)}, t\geqslant 0,\\ u &= b \quad \text{on } \Gamma_0, \quad t\geqslant 0,\\ u &= f \quad \text{on } \Gamma_{f(t)},\ t\geqslant 0,\\ \partial_t f(t,\cdot) + \frac{\sqrt{1+\partial_x f^2(t,\cdot)}}{\overline{\mu}(|Du(\cdot, f(t,\cdot))|^2)}\partial_\nu u(\cdot, f(t,\cdot)) &= 0 \quad \text{on } \mathbb{S}^1, \quad t>0,\\ f(0,\cdot) &= f_0 \quad \text{on } \mathbb{S}^1, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \Omega_{f(t)} &:= \{(x,y)\in\mathbb{S}^1\times\mathbb{R} : 0 < y < f(t,x)\},\\ \Gamma_{f(t)} &:= \{(x, f(t,x)) : x\in\mathbb{S}^1\}, \quad \Gamma_0 = \mathbb{S}^1\times\{0\}. \end{aligned}$$

for $t \geqslant 0$, and $\mathbb{S}^1$ is the unit circle. For the sake of simplicity, we identify periodic functions on $\mathbb{R}$ with functions on $\mathbb{S}^1$, and periodic functions in the x variable on $\widetilde{\Omega}_f$ with functions on Ω_f, for positive functions f on $\mathbb{S}^1$.

Given $m \in \mathbb{N}$ and $\alpha \in (0,1)$, we define the so-called little Hölder space $h^{m+\alpha}(\mathbb{S}^1)$ as the closure of $C^\infty(\mathbb{S}^1)$ in $C^{m+\alpha}(\mathbb{S}^1)$. If f is a positive function in $C(\mathbb{S}^1)$, then we denote by $buc^{m+\alpha}(\Omega_f)$ the closure of $BUC^\infty(\Omega_f)$ in the Hölder space $BUC^{m+\alpha}(\Omega_f)$. The notation $BUC^{m+\alpha}(\Omega_f)$ stands for the space of all maps from Ω_f to $\mathbb{R}$ which have bounded and uniformly continuous derivatives up to order m, and in addition uniformly α-Hölder continuous derivatives of order m.

Throughout this paper we fix $\alpha \in (0,1)$ and we define

$$\mathcal{U} := \{f \in C^{2+\alpha}(\mathbb{S}^1) \,:\, \min_{x \in \mathbb{S}^1} f(x) > 0\}, \qquad \mathcal{V} := \mathcal{U} \cap h^{2+\alpha}(\mathbb{S}^1).$$

A pair (u, f) is called a *classical Hölder solution* of (6) on $[0, T]$, $T > 0$, if

$$\begin{aligned} &f \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha}(\mathbb{S}^1)),\\ &u(\,\cdot\,, t) \in buc^{2+\alpha}(\Omega_{f(t)}), \qquad t \in [0,T], \end{aligned}$$

and (u, f) satisfies the equations in (6) pointwise. Suppose there exist two positive constants m_μ and M_μ such that

$$\begin{aligned} &(A_1) \quad m_\mu \leqslant \overline{\mu}(r) \leqslant M_\mu, \qquad \forall r \geqslant 0,\\ &(A_2) \quad m_\mu \leqslant \overline{\mu}(r) - 2r\overline{\mu}'(r) \leqslant M_\mu, \qquad \forall r \geqslant 0. \end{aligned}$$

Our main result reads as follows.

THEOREM 1.1 Assume (A_1) and (A_2) hold true. Then we have:

(a) Let c and b be two positive constants. There exists an open neighbourhood $\mathcal{O}$ of c in $\mathcal{V}$ such that, for each $f_0 \in \mathcal{O}$, problem (6) has a classical Hölder solution (u, f) on an interval $[0, T]$, $T > 0$. Moreover, there exists a constant $\gamma \in (0,1)$ such that $f \in C^\gamma_\gamma((0,T], h^{2+\alpha}(\mathbb{S}^1))$.

(b) Let (u_1, f_1) and (u_2, f_2) be solutions of (6) with $f_1 \in C^\gamma_\gamma((0,T], h^{2+\alpha}(\mathbb{S}^1))$, $\gamma \in (0,1)$, and $f_2 \in C^\delta_\delta((0,T], h^{2+\alpha}(\mathbb{S}^1)))$, $\delta \in (0,1)$. If $f_1([0,T]) \subset \mathcal{O}$ and $f_2([0,T]) \subset \mathcal{O}$, then $(u_1, f_1) = (u_2, f_2)$.

For the definition of the weighted Hölder spaces $C^\gamma_\gamma((0,T], h^{2+\alpha}(\mathbb{S}^1))$, $\gamma \in (0,1)$ see [10]. If the viscosity μ is decreasing then the Stokesian fluid is called *shear thinning*. If μ is increasing then the fluid is called *shear thickening*. Notice that, if μ is constant, then $\overline{\mu}$ is also constant. Moreover, if μ is a strictly decreasing or strictly increasing function of its argument, then so is $\overline{\mu}$. The conditions (A_1) and (A_2) ensure that at great velocities the fluid behaves like a Newtonian fluid.

We now look for conditions on μ which imply (A_1) and (A_2). We remark that (A_1) and (A_2) are satisfied iff there exist positive constants c and C with

$$\begin{aligned} &c \leqslant \frac{1}{\overline{\mu}(r)} \leqslant C, \qquad \forall r \geqslant 0,\\ &c \leqslant \frac{1}{\overline{\mu}(r)} + 2r\left(\frac{1}{\overline{\mu}}\right)'(r) \leqslant C, \qquad \forall r \geqslant 0. \end{aligned}$$

Using the definition of $\overline{\mu}$ we compute

$$\frac{1}{\overline{\mu}(r)}+2r\left(\frac{1}{\overline{\mu}}\right)'(r)=c_\mu\int_{-1}^{1}s^2\left[\frac{1}{\widetilde{\mu}(rs^2)}+2(rs^2)\left(\frac{1}{\widetilde{\mu}}\right)'(rs^2)\right]\mathrm{d}s,$$

hence (A_1) and (A_2) are satisfied if there exist positive constants $\widetilde{c}$ and $\widetilde{C}$ with

$$\widetilde{c}\leqslant\frac{1}{\widetilde{\mu}(r)}\leqslant\widetilde{C},\quad \forall r\geqslant 0,$$
$$\widetilde{c}\leqslant\frac{1}{\widetilde{\mu}(r)}+2r\left(\frac{1}{\widetilde{\mu}}\right)'(r)\leqslant\widetilde{C},\quad \forall r\geqslant 0.$$

Further we compute

$$\begin{aligned}\frac{1}{\widetilde{\mu}(r)}+2r\left(\frac{1}{\widetilde{\mu}}\right)'(r) &= \frac{1}{\mu^2(h^{-1}(r))}\big(\mu(h^{-1}(r))-2r\mu'(h^{-1}(r))(h^{-1})'(r)\big)\\ &\overset{h^{-1}(r)=s}{=} \frac{1}{\mu^2(s)}\left(\mu(s)-2h(s)\mu'(s)\frac{1}{h'(s)}\right)\\ &= \frac{1}{\mu^2(s)}\left(\mu(s)-2s\mu^2(s)\mu'(s)\frac{1}{\mu^2(s)+2s\mu(s)\mu'(s)}\right)\\ &= \frac{1}{\mu(s)+2s\mu'(s)},\end{aligned}$$

thus, (A_1) and (A_2) hold if there exist positive constants $\overline{c}$ and $\overline{C}$ such that

$$(V_1)\quad \overline{c}\leqslant\mu(r)\leqslant\overline{C},$$
$$(V_2)\quad \overline{c}\leqslant\mu(r)+2r\mu'(r)\leqslant\overline{C},$$

for all $r\geqslant 0$. The class of fluids with viscosity satisfying (V_1) and (V_2) is quite large.

For *Oldroyd-B fluids*, e.g. blood, the viscosity is given by

$$\mu(r)=\nu_\infty+(\nu_0-\nu_\infty)\frac{1+\ln(1+\lambda r)}{1+\lambda r},\qquad r\geqslant 0,$$

where $\lambda>0$ is a material constant and $\nu_0>\nu_\infty>0$. The conditions (V_1) and (V_2) hold if $(e^2+1)\nu_\infty>\nu_0$. Also, various variants of *power law fluids* belong to this class:

$$\mu(r)=\nu_\infty+\nu_0(1+r^2)^{s/4}\quad\text{or}\quad\mu(r)=\nu_\infty+\nu_0(1+r)^{s/2},$$

for all $r\geqslant 0$, where ν_0 and ν_∞ are positive and $s\leqslant 0$. In this case (V_1) and (V_2) hold if $-1\leqslant s\leqslant 0$. Notice that the above examples are all shear thinning fluids. We now give an example of a shear thickening fluid which can be considered in our model. If

$$\mu(r)=\mu_0\frac{\gamma r+r_0}{r+r_0},\qquad\forall r\geqslant 0,$$

with $r_0>0$, $\gamma\geqslant 1$ and $\mu_0>0$, then (V_1) and (V_2) hold for any choice of the parameters r_0, μ_0 and γ.

1.2 *The transformed problem*

For simplification we introduce first the operator $\mathcal{Q} : C^2(\Omega_f) \to C(\Omega_f)$ with

$$\mathcal{Q}u := \operatorname{div}\left(\frac{Du}{\overline{\mu}(|Du|^2)}\right), \quad u \in C^2(\Omega_f).$$

In order to solve the problem we transfer it onto a fixed reference manifold. Let $\Omega := \mathbb{S}^1 \times (0, 1)$. For $f \in \mathcal{U}$ we define $\phi_f \in \mathit{Diff}^{2+\alpha}(\Omega, \Omega_f)$ by

$$\phi_f(x, y) = (x, (1 - y)f(x)), \quad (x, y) \in \Omega.$$

Defining the push-forward and pull-back operators induced by ϕ_f,

$$\begin{aligned} \phi_f^* &: BUC(\Omega_f) \to BUC(\Omega), \quad u \mapsto u \circ \phi_f, \\ \phi_*^f &: BUC(\Omega) \to BUC(\Omega_f), \quad v \mapsto v \circ \phi_f^{-1}, \end{aligned}$$

we introduce the transformed operators $\mathcal{A}(f)$ and $\mathcal{B}$, acting on $BUC^2(\Omega)$ and $\mathcal{U} \times BUC^{2+\alpha}(\Omega)$ respectively by

$$\begin{aligned} \mathcal{A}(f) &:= \phi_f^* \circ \mathcal{Q} \circ \phi_*^f, \\ \mathcal{B}(f, v)(x) &:= \frac{D(\phi_*^f v)}{\overline{\mu}(|D(\phi_*^f v)|^2)}(x, f(x)) \cdot n(x), \quad x \in \mathbb{S}^1, \end{aligned}$$

with $n(x) := (-f'(x), 1),\ x \in \mathbb{S}^1$.

Transformation of (6) to Ω yields

$$\begin{aligned} \mathcal{A}(f)v &= 0 && \text{in } \Omega \times [0, \infty), \\ v &= f && \text{on } \Gamma_0 \times [0, \infty), \\ v &= b && \text{on } \Gamma_1 \times [0, \infty), \\ \partial_t f + \mathcal{B}(f, v) &= 0 && \text{on } \Gamma_0 \times (0, \infty), \\ f(0) &= f_0, \end{aligned} \tag{7}$$

where $v := \phi_f^* u$. A pair (v, f) is called a *classical Hölder solution* of (7) on $[0, T]$, $T > 0$, if

$$\begin{aligned} & f \in C([0, T], \mathcal{V}) \cap C^1([0, T], h^{1+\alpha}(\mathbb{S}^1)), \\ & v(\cdot\,, t) \in buc^{2+\alpha}(\Omega), \quad t \in [0, T], \end{aligned}$$

and (v, f) satisfies the equations in (7) pointwise.

LEMMA 1.2 Let $f_0 \in \mathcal{V}$ and $b \in h^{2+\alpha}(\mathbb{S}^1)$ be given.

(a) If (u, f) is a classical Hölder solution of (6), then $(\phi_f^* u, f)$ is a classical Hölder solution of (7).

(b) If (v, f) is a classical Hölder solution of (7), then $(\phi_*^f v, f)$ is a classical Hölder solution of (6).

Proof. The main difficulty is to show that $\phi_*^f(buc^\alpha(\Omega)) = buc^\alpha(\Omega_f)$ for each $f \in \mathcal{V}$. We show just the inclusion $\phi_*^f(buc^\alpha(\Omega)) \subset buc^\alpha(\Omega_f)$. The proof of $\phi_f^*(buc^\alpha(\Omega_f)) \subset buc^\alpha(\Omega)$ is similar.

Let $f \in \mathcal{V}$ and $v \in buc^\alpha(\Omega)$. We find two sequences $(f_m) \subset C^\infty(\mathbb{S}^1)$ and $(v_n) \subset BUC^\infty(\Omega)$ such that $f_m \searrow f$ in $C^\alpha(\mathbb{S}^1)$ and $v_n \to v$ in $BUC^\alpha(\Omega)$. Let $u := \phi_*^f v$. We show that each neighbourhood of u in $BUC^\alpha(\Omega_f)$ contains a function $u_{n,m}$, $n, m \in \mathbb{N}$, where

$$u_{n,m}(x, y) = v_n(\phi_{f_m}^{-1}(x, y)) = v_n\left(x, 1 - \frac{y}{f_m(x)}\right), \qquad (x, y) \in \Omega_f.$$

are smooth functions on Ω_f. The functions $u_{n,m}$, $n, m \in \mathbb{N}$, are well-defined because $f_m \geqslant f$ for all $m \in \mathbb{N}$. First we have

$$|u_{n,m}(x, y) - u(x, y)| = |v_n(\phi_{f_m}^{-1}(x, y)) - v(\phi_f^{-1}(x, y))| \leqslant \|\partial v_n\|_0 \frac{\|f_m - f\|_0}{\min f} + \|v_n - v\|_0$$

for all $(x, y) \in \Omega_f$. Let now (x, y) and (x', y') be two different points in Ω_f. We have

$$\begin{aligned}
&|(u_{n,m} - u)(x, y) - (u_{n,m} - u)(x', y')| \\
&\quad = |v_n(\phi_{f_m}^{-1}(x, y)) - v(\phi_f^{-1}(x, y)) - v_n(\phi_{f_m}^{-1}(x', y')) + v(\phi_{f_m}^{-1}(x', y'))| \\
&\quad \leqslant |v_n(\phi_f^{-1}(x, y)) - v(\phi_f^{-1}(x, y)) - v_n(\phi_f^{-1}(x', y')) + v(\phi_f^{-1}(x', y'))| \\
&\qquad + |v_n(\phi_{f_m}^{-1}(x, y)) - v_n(\phi_f^{-1}(x, y)) - v_n(\phi_{f_m}^{-1}(x', y')) + v_n(\phi_f^{-1}(x', y'))| \\
&\quad \leqslant \|v_n - v\|_{BUC^\alpha(\Omega)} \cdot |\phi_f^{-1}(x, y) - \phi_f^{-1}(x', y')|^\alpha \\
&\qquad + |v_n(\phi_{f_m}^{-1}(x, y)) - v_n(\phi_f^{-1}(x, y)) - v_n(\phi_{f_m}^{-1}(x', y')) + v_n(\phi_f^{-1}(x', y'))|.
\end{aligned}$$

Since

$$\frac{|\phi_f^{-1}(x, y) - \phi_f^{-1}(x', y')|}{|(x, y) - (x', y')|} \leqslant 1 + \frac{|y'/f(x') - y/f(x)|}{|(x, y) - (x', y')|} \leqslant 1 + \frac{1}{\min f} + \frac{\|f\|_0 \cdot \|f'\|_0}{\min f^2},$$

it remains to estimate the second term on the right hand side. Using the mean value theorem we obtain

$$\begin{aligned}
&|v_n(\phi_{f_m}^{-1}(x, y)) - v_n(\phi_f^{-1}(x, y)) - v_n(\phi_{f_m}^{-1}(x', y')) + v_n(\phi_f^{-1}(x', y'))| \\
&= \left| \int_0^1 \partial v_n(t\phi_{f_m}^{-1}(x, y) + (1 - t)\phi_{f_m}^{-1}(x', y'))\, \mathrm{d}t \cdot (\phi_{f_m}^{-1}(x, y) - \phi_{f_m}^{-1}(x', y')) \right. \\
&\quad \left. - \int_0^1 \partial v_n(t\phi_f^{-1}(x, y) + (1 - t)\phi_f^{-1}(x', y'))\, \mathrm{d}t \cdot (\phi_f^{-1}(x, y) - \phi_f^{-1}(x', y')) \right| \\
&\leqslant \|\partial v_n\|_0 \left| \frac{y'}{f_m(x')} - \frac{y}{f_m(x)} - \frac{y'}{f(x')} + \frac{y}{f(x)} \right| \\
&\quad + \int_0^1 \|\partial^2 v_n\|_0 \left| \frac{ty}{f(x)} + \frac{(1 - t)y'}{f(x')} - \frac{ty}{f_m(x)} - \frac{(1 - t)y'}{f_m(x')} \right| \mathrm{d}t\, |\phi_f^{-1}(x, y) - \phi_f^{-1}(x', y')| \\
&\leqslant \|\partial v_n\|_0 \left| \frac{y'}{f_m(x')} - \frac{y}{f_m(x)} - \frac{y'}{f(x')} + \frac{y}{f(x)} \right| \\
&\quad + \|\partial^2 v_n\|_0 \frac{\|f\|_0 \|f_m - f\|_0}{\min f^2} |\phi_f^{-1}(x, y) - \phi_f^{-1}(x', y')|.
\end{aligned}$$

Using the estimates

$$\frac{\left|\frac{y'}{f_m(x')} - \frac{y}{f_m(x)} - \frac{y'}{f(x')} + \frac{y}{f(x)}\right|}{|(x,y)-(x',y')|^\alpha} \leqslant \|f\|_0^{1-\alpha} \cdot \frac{\|f_m - f\|_0}{\min f^2} + \|f\|_0 \cdot \frac{\|f_m - f\|_{C^\alpha(\mathbb{S}^1)}}{\min f^2}$$

$$+ \|f\|_0 \cdot \|f_m - f\|_0 \cdot \frac{\|f\|_0 \cdot \|f_m\|_{C^\alpha(\mathbb{S}^1)} + \|f\|_{C^\alpha(\mathbb{S}^1)} \cdot \|f_m\|_0}{\min f^4},$$

$$\frac{\left|\left(x - x', \frac{y'}{f(x')} - \frac{y}{f(x)}\right)\right|}{|(x,y)-(x',y')|^\alpha} \leqslant (2\pi)^{1-\alpha} + \frac{\|f\|_0^{1-\alpha}}{\min f} + \frac{\|f\|_0 \cdot \|f\|_{C^\alpha(\mathbb{S}^1)}}{\min f^2}$$

we obtain the desired conclusion. □

2. The abstract Cauchy problem

We have already noticed that the conditions (A_1) and (A_2) on $\overline{\mu}$ imply the existence of two positive constants c and C such that

$$c \leqslant \frac{1}{\overline{\mu}(r)} \leqslant C, \qquad \forall r \geqslant 0, \tag{8}$$

$$c \leqslant \frac{1}{\overline{\mu}(r)} - \frac{2r\overline{\mu}'(r)}{\overline{\mu}^2(r)} \leqslant C, \qquad \forall r \geqslant 0. \tag{9}$$

Under these assumptions the quasilinear operator $\mathcal{Q}$ is uniformly elliptic in $\mathbb{R}^2$. For $u \in C^2(\Omega_f)$ we compute

$$\mathcal{Q}u = a_{ij}(Du)u_{ij},$$

and the coefficients $(a_{ij})_{1\leqslant i,j\leqslant 2}$ are

$$a_{ij}(p) = \frac{\delta_{ij}}{\overline{\mu}(|p|^2)} - \frac{2p_ip_j\overline{\mu}'(|p|^2)}{\overline{\mu}^2(|p|^2)}, \qquad p = (p_1, p_2) \in \mathbb{R}^2.$$

Actually, the eigenvalues of $(a_{ij})_{1\leqslant i,j\leqslant 2}$ are

$$\lambda_1(p) = \frac{1}{\overline{\mu}(|p|^2)}, \qquad \lambda_2(p) = \frac{1}{\overline{\mu}(|p|^2)} - \frac{2|p|^2\overline{\mu}'(|p|^2)}{\overline{\mu}^2(|p|^2)},$$

and we have

$$c|\xi|^2 \leqslant a_{ij}(p)\xi_i\xi_j \leqslant C|\xi|^2, \qquad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,\ p \in \mathbb{R}^2.$$

LEMMA 2.1 Given $f \in \mathcal{U}$, we have

$$\mathcal{A}(f)v = b_{ij}(y, f, Dv)v_{ij} + b(y, f, Dv)v_2 \quad \text{for } v \in BUC^2(\Omega),$$

where, using the notation

$$D_f v := \left(v_1 + \frac{(1-y)f'}{f}v_2, -\frac{1}{f}v_2\right) \quad \text{for } f \in \mathcal{U},\ v \in BUC^2(\Omega) \text{ and } y \in [0,1],$$

we have

$$b_{11}(y,f,Dv) = a_{11}(D_f v),$$

$$b_{12}(y,f,Dv) = b_{21}(y,f,Dv) = \frac{(1-y)f'}{f}a_{11}(D_f v) - \frac{1}{f}a_{12}(D_f v),$$

$$b_{22}(y,f,Dv) = \frac{(1-y)^2 f'^2}{f^2}a_{11}(D_f v) - \frac{2(1-y)f'}{f^2}a_{12}(D_f v) + \frac{1}{f^2}a_{22}(D_f v),$$

$$b(y,f,Dv) = (1-y)\left(\frac{f''}{f} - \frac{2f'^2}{f^2}\right)a_{11}(D_f v) + \frac{2f'}{f^2}a_{12}(D_f v).$$

Proof. This follows by direct computation. □

Given $f \in \mathcal{U}$, the quasilinear operator $\mathcal{A}(f)$ is uniformly elliptic. Indeed, for $(y,p) \in [0,1] \times \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} b_{ij}(y,f,p)\xi_i\xi_j = {} & a_{11}\left(p_1 + \frac{(1-y)f'}{f}p_2, -\frac{1}{f}p_2\right)\left(\xi_1 + \frac{(1-y)f'}{f}\xi_2\right)^2 \\ & + 2a_{12}\left(p_1 + \frac{(1-y)f'}{f}p_2, -\frac{1}{f}p_2\right)\left(\xi_1 + \frac{(1-y)f'}{f}\xi_2\right)\left(-\frac{\xi_2}{f}\right) \\ & + a_{22}\left(p_1 + \frac{(1-y)f'}{f}p_2, -\frac{1}{f}p_2\right)\left(-\frac{\xi_2}{f}\right)^2, \end{aligned}$$

and the assertion follows from (8) and (9) upon taking also into account that ϕ_f is a diffeomorphism.

Using maximum principle arguments and Morrey and De Giorgi–Nash type estimates as in [8] one can show that, given

$$f \in \mathcal{U}, \qquad q_1, q_2, q_3, g, b \in C^{2+\alpha}(\mathbb{S}^1), \qquad \sigma \in [0,1],$$

there exist constants $\delta > 0$, $\beta \in (0,1)$ and $M > 0$ such that every solution $v \in BUC^2(\Omega)$ of the Dirichlet problem

$$\begin{aligned} \mathcal{A}(f+q_1)v &= 0 && \text{in } \Omega, \\ v &= \sigma g + q_2 && \text{on } \Gamma_0, \\ v &= \sigma b + q_3 && \text{on } \Gamma_1 \end{aligned} \tag{10}$$

satisfies the estimate

$$\|v\|_{BUC^{1+\beta}(\Omega)} \leqslant M$$

provided $\|q_i\|_{C^{2+\alpha}(\mathbb{S}^1)} \leqslant \delta$ for $i \in \{1,2,3\}$. This a priori estimate allows an application of the techniques developed in Chapter 10 of [8] to derive the following existence, uniqueness and regularity result.

LEMMA 2.2 Let $f \in \mathcal{V}$ and $b \in h^{2+\alpha}(\mathbb{S}^1)$. Then there exists a unique solution $\mathcal{T}(f) \in buc^{2+\alpha}(\Omega)$ of the Dirichlet problem

$$\begin{aligned} \mathcal{A}(f)u &= 0 && \text{in } \Omega, \\ u &= f && \text{on } \Gamma_0, \\ u &= b && \text{on } \Gamma_1. \end{aligned} \tag{11}$$

The mapping $[\mathcal{V} \ni f \mapsto \mathcal{T}(f) \in buc^{2+\alpha}(\Omega)]$ is smooth. □

We fix $b \in h^{2+\alpha}(\mathbb{S}^1)$. Replacing v in the fourth equation of (7) by $\mathcal{T}(f)$, the unique solution to (11), we reduce the full problem (7) into an abstract Cauchy problem over $\mathbb{S}^1$,

$$\partial_t f + \Phi(f) = 0, \qquad f(0) = f_0, \tag{12}$$

where $\Phi(f) := \mathcal{B}(f, \mathcal{T}(f))$. The operator Φ is a pseudodifferential operator of the first order, with a symbol depending nonlinearly on the variable f. Further we show that $\Phi \in C^\infty(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ and compute the derivative $\partial\Phi(c)$ in the special case $c, b \in \mathbb{R}_{>0}$.

The restriction of the operator $\mathcal{B}$ defined in Section 1 to the set $\mathcal{V} \times buc^{2+\alpha}(\mathbb{S}^1)$ satisfies

$$\mathcal{B}(f, v) = -\frac{1}{\overline{\mu}(|\gamma_0 D_f v|^2)}\left(f'\gamma_0 v_1 + \frac{1}{f}(1 + f'^2)\gamma_0 v_2\right) \quad \text{for } (f, v) \in \mathcal{V} \times buc^{2+\alpha}(\Omega),$$

where γ_0 is the trace operator on Γ_0. Together with the relation

$$|\gamma_0 D_f v|^2 = \gamma_0 v_1^2 + 2\frac{f'}{f}\gamma_0 v_1 v_2 + \frac{1 + f'^2}{f^2}\gamma_0 v_2^2$$

we conclude that the operator $\mathcal{B}$ defined above is smooth. More precisely, we have:

LEMMA 2.3 The mapping $\mathcal{B} : \mathcal{V} \times buc^{2+\alpha}(\Omega) \to h^{1+\alpha}(\mathbb{S}^1)$ is smooth. The Fréchet derivative of $\mathcal{B}$ at $(f, v) \in \mathcal{V} \times buc^{2+\alpha}(\Omega)$ is given by

$$\begin{aligned}
\partial\mathcal{B}(f, v)[h, u] = &- \frac{1}{\overline{\mu}}(|\gamma_0 D_f v|^2)\bigg[f'\gamma_0 u_1 + h'\gamma_0 v_1 + \frac{1}{f}(1 + f'^2)\gamma_0 u_2 \\
&- \left(\frac{h}{f^2} - \frac{2f'h'}{f} + \frac{hf'^2}{f^2}\right)\gamma_0 v_2\bigg] \\
&- 2\left(\frac{1}{\overline{\mu}}\right)'(|\gamma_0 D_f v|^2)\left(f'\gamma_0 v_1 + \frac{1}{f}(1 + f'^2)\gamma_0 v_2\right)\bigg[\gamma_0 v_1 u_1 + \frac{h'}{f}\gamma_0 v_1 v_2 \\
&+ \frac{f'}{f}\gamma_0 u_1 v_2 + \frac{f'}{f}\gamma_0 v_1 u_2 - \frac{f'h}{f^2}\gamma_0 v_1 v_2 + \frac{f'h'}{f^2}\gamma_0 v_2^2 + \frac{f'^2}{f^2}\gamma_0 v_2 u_2 \\
&- \frac{hf'^2}{f^3}\gamma_0 v_2^2 + \frac{1}{f^2}\gamma_0 v_2 u_2 - \frac{h}{f^3}\gamma_0 v_2^2\bigg]
\end{aligned}$$

for all $[h, u] \in h^{2+\alpha}(\mathbb{S}^1) \times buc^{2+\alpha}(\Omega)$.

Combining Lemmas 2.2 and 2.3 we conclude that $\Phi \in C^\infty(\mathcal{V}, h^{1+\alpha}(\mathbb{S}^1))$. Since

$$\Phi(f) = \mathcal{B} \circ [f \mapsto (f, \mathcal{T}(f))],$$

the chain rule implies that $\partial\Phi(f) = \partial\mathcal{B}(f, \mathcal{T}(f)) \circ (\mathrm{id}_{h^{2+\alpha}(\mathbb{S}^1)}, \partial\mathcal{T}(f))$ for $f \in \mathcal{V}$. We are thus left with the task of computing the derivative $\partial\mathcal{T}(f)$.

LEMMA 2.4 Given $f \in \mathcal{V}$ and $h \in h^{2+\alpha}(\mathbb{S}^1)$ the mapping $\partial\mathcal{T}(f)[h]$ is the unique solution of the linear Dirichlet problem

$$
\begin{aligned}
& b_{ij}w_{ij} + bw_2 + D_f w\bigg[u_{11}\partial a_{11}(D_f u) + 2u_{12}\bigg(\frac{(1-y)f'}{f}\partial a_{11}(D_f u) - \frac{1}{f}\partial a_{12}(D_f u)\bigg) \\
&\quad + u_{22}\bigg(\frac{(1-y)^2 f'^2}{f^2}\partial a_{11}(D_f u) - 2\frac{(1-y)f'}{f^2}\partial a_{12}(D_f u) + \frac{1}{f^2}\partial a_{22}(D_f u)\bigg) \\
&\quad + u_2\bigg((1-y)\bigg(\frac{f''}{f} - 2\frac{f'^2}{f^2}\bigg)\partial a_{11}(D_f u) + 2\frac{f'}{f^2}\partial a_{12}(D_f u)\bigg)\bigg] \\
&= -u_2\bigg((1-y)\frac{fh' - f'h}{f^2}, \frac{h}{f^2}\bigg)\cdot\bigg[u_{11}\partial a_{11}(D_f u) + 2u_{12}\bigg(\frac{(1-y)f'}{f}\partial a_{11}(D_f u) - \frac{1}{f}\partial a_{12}(D_f u)\bigg) \\
&\quad + u_{22}\bigg(\frac{(1-y)^2 f'^2}{f^2}\partial a_{11}(D_f u) - 2\frac{(1-y)f'}{f^2}\partial a_{12}(D_f u) + \frac{1}{f^2}\partial a_{22}(D_f u)\bigg) \\
&\quad + u_2\bigg((1-y)\bigg(\frac{f''}{f} - 2\frac{f'^2}{f^2}\bigg)\partial a_{11}(D_f u) + 2\frac{f'}{f^2}\partial a_{12}(D_f u)\bigg)\bigg] \\
&\quad - 2u_{12}\bigg((1-y)\frac{fh' - f'h}{f^2}a_{11}(D_f u) + \frac{h}{f^2}a_{12}(D_f u)\bigg) \\
&\quad - 2u_{22}\bigg(\frac{(1-y)^2(ff'h' - f'^2 h)}{f^3}a_{11}(D_f u) - (1-y)\frac{fh' - 2f'h}{f^3}a_{12}(D_f u) - \frac{h}{f^3}a_{22}(D_f u)\bigg) \\
&\quad - u_2\bigg((1-y)\bigg(\frac{fh'' - f''h}{f^2} - 4\frac{ff'h' - f'^2 h}{f^3}\bigg)a_{11}(D_f u) + 2\frac{fh' - 2f'h}{f^3}a_{12}(D_f u)\bigg) \quad \text{in } \Omega,
\end{aligned}
$$

$$
\begin{aligned}
w &= h \quad \text{on } \Gamma_0, \\
w &= 0 \quad \text{on } \Gamma_1,
\end{aligned}
$$

where $u := \mathcal{T}(f)$ and $b_{ij} = b_{ij}(y, f, Du)$, $b = b(y, f, Du)$ are the coefficients of $\mathcal{A}(f)$.

Our next goal is to compute $\partial\Phi(c)$ when c and b are positive constant functions. More precisely, we would like to know how it acts on Fourier series. The solution $\mathcal{T}(c)$ of the Dirichlet problem (11) is

$$\mathcal{T}(c)(x, y) = (1-y)c + yb, \quad (x, y) \in \Omega.$$

Given $(h, u) \in h^{2+\alpha}(\mathbb{S}^1) \times buc^{2+\alpha}(\Omega)$, we therefore get

$$\partial\mathcal{B}(c, \mathcal{T}(c))[h, u] = -\frac{1}{c}\zeta\gamma_0 u_2 + \frac{b-c}{c^2}\zeta h,$$

where

$$\zeta := \frac{1}{\mu}\bigg(\bigg(\frac{b-c}{c}\bigg)^2\bigg) + 2\bigg(\frac{b-c}{c}\bigg)^2\bigg(\frac{1}{\mu}\bigg)'\bigg(\bigg(\frac{b-c}{c}\bigg)^2\bigg) > 0.$$

Consequently,

$$\partial\Phi(c)[h] = -\frac{1}{c}\zeta\gamma_0 w_2 + \frac{b-c}{c^2}\zeta h,$$

where $w := \partial\mathcal{T}(c)[h] \in buc^{2+\alpha}(\Omega)$ denotes the solution of the linear Dirichlet problem

$$
\begin{aligned}
w_{11} + \beta^2 w_{22} &= \frac{c-b}{c}(1-y)h'' && \text{in } \Omega, \\
w &= h && \text{on } \Gamma_0, \\
w &= 0 && \text{on } \Gamma_1,
\end{aligned}
\tag{13}
$$

and where

$$\beta^2 := \frac{1}{c^2}\left(1 - 2\left(\frac{c-b}{c}\right)^2 \frac{\overline{\mu}'\big(\big(\frac{c-b}{c}\big)^2\big)}{\overline{\mu}\big(\big(\frac{c-b}{c}\big)^2\big)}\right) > 0.$$

We now expand h and w in the following way:

$$h(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \qquad w(x,y) = \sum_{k\in\mathbb{Z}} C_k(y)e^{ikx}.$$

Substituting these expressions into equations (13) and comparing the coefficients of e^{ikx} for every k, we get the following equations for $C_k(y)$:

$$\begin{aligned} \beta^2 C_k'' - k^2 C_k &= \frac{b-c}{c}k^2 c_k(1-y), \qquad 0 < y < 1,\\ C_k(0) &= c_k,\\ C_k(1) &= 0, \end{aligned} \tag{14}$$

for $k \in \mathbb{Z}\setminus\{0\}$, and

$$\begin{aligned} C_0'' &= 0, \qquad 0 < y < 1,\\ C_0(0) &= c_0,\\ C_0(1) &= 0. \end{aligned} \tag{15}$$

One can easily verify that the solution of (15) is $C_0(y) = (1-y)c_0$. The solutions of (14) are given by

$$C_k(y) = c_k d_k(y)$$

with

$$d_k(y) = \frac{c-b}{c}(1-y) + \frac{b}{c}\left(\frac{e^{ky/\beta}}{1-e^{2k/\beta}} + \frac{e^{-ky/\beta}}{1-e^{-2k/\beta}}\right).$$

Thus we obtain

$$w(x,y) = (1-y)c_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k(y)c_k e^{ikx}, \qquad \forall (x,y)\in\Omega, \tag{16}$$

and

$$\partial\Phi(c)\Big[\sum_{k\in\mathbb{Z}} c_k e^{ikx}\Big] = \sum_{k\in\mathbb{Z}} \lambda_k c_k e^{ikx} \tag{17}$$

for all $h = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \in h^{2+\alpha}(\mathbb{S}^1)$, with

$$\lambda_0 := \frac{b\zeta}{c^2}, \qquad \lambda_k = \frac{b\zeta}{\beta c^2}k\frac{e^{2k/\beta}+1}{e^{2k/\beta}-1}, \quad k \neq 0. \tag{18}$$

Notice that equations (14) and (15) have been obtained formally by differentiating w with respect to the variables x and y. Thus, it remains to show that the mapping w, given by (16), is the solution of the Dirichlet problem (13). Since $h \in h^{2+\alpha}(\mathbb{S}^1)$, there is a positive constant L such that

$$|c_k| \leqslant \frac{L}{k^2}, \qquad \forall k \in \mathbb{Z}\setminus\{0\}.$$

The functions $d_k,\ k \in \mathbb{Z} \setminus \{0\}$, are uniformly bounded on [0, 1], i.e.

$$M := \sup_{k\in\mathbb{Z}\setminus\{0\}} \max_{[0,1]} |d_k| < \infty.$$

Therefore $w \in BUC(\Omega)$. Let $\overline{w}$ denote the solution of (13). Pick further a sequence $(h_p)_p \subset C^\infty(\mathbb{S}^1)$ which converges to h in $C^{2+\alpha}(\mathbb{S}^1)$, and denote by $w_p \in BUC^\infty(\Omega)$ the solution of (13) which corresponds to h_p. Then

$$w_p \to \overline{w} \quad \text{in } BUC^{2+\alpha}(\Omega). \tag{19}$$

Using the Fourier expansions

$$h_p = \sum_{k\in\mathbb{Z}} c_{p,k} e^{ikx},$$

we find for each $l \in \mathbb{N}$ a constant $L_{p,l} > 0$ such that

$$|k|^l |c_{p,k}| \leqslant L_{p,l}, \qquad \forall k \in \mathbb{Z},$$

and, as before, we obtain

$$w_p(x, y) = (1 - y)c_{p,0} + \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k(y) c_{p,k} e^{ikx}, \quad \forall (x, y) \in \Omega.$$

Notice that these Fourier series are smooth for all p. Fix now $y \in [0, 1]$. Given $p \in \mathbb{N}$, we have

$$w_p(x, y) - w(x, y) = (1 - y)(c_{p,0} - c_0) + \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k(y)(c_{p,k} - c_k) e^{ikx},$$

and so

$$\begin{aligned}\|w_p(\cdot\, y) - w(\cdot\, y)\|^2_{L^2(\mathbb{S}^1)} &= (1 - y)^2 (c_{p,0} - c_0)^2 + \sum_{k\in\mathbb{Z}\setminus\{0\}} d_k^2(y)|c_{p,k} - c_k|^2 \\ &\leqslant M^2 \sum_{k\in\mathbb{Z}} |c_{p,k} - c_k|^2 = M^2 \|h_p - h\|^2_{L^2(\mathbb{S}^1)}.\end{aligned}$$

Observing $h_p \to h$ in $C^{2+\alpha}(\mathbb{S}^1)$ and invoking (19), we see that the previous inequality implies that

$$w(\cdot\,, y) = \overline{w}(\cdot\,, y) \quad \text{in } L^2(\mathbb{S}^1)$$

for all $y \in [0, 1]$. Using the continuity of w and $\overline{w}$, we conclude that $w = \overline{w}$, and formula (17) is proved.

3. The proof of the main result

In this section we regard the spaces $h^{m+\alpha}(\mathbb{S}^1)$, $m = 1, 2$, as Banach spaces over the complex numbers. In order to prove Theorem 1.1 we have to show that the complexification of $-\partial\Phi(c)$, which we also denote by $-\partial\Phi(c)$, considered as an operator in $h^{1+\alpha}(\mathbb{S}^1)$ with domain $h^{2+\alpha}(\mathbb{S}^1)$, generates a strongly continuous analytic semigroup in $\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))$, i.e. $\partial\Phi(c) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$.

Using the same notations as in [1], we have $h^{2+\alpha}(\mathbb{S}^1) \overset{d}{\hookrightarrow} h^{1+\alpha}(\mathbb{S}^1)$ and, given $\kappa \geqslant 1$ and $\omega > 0$, we write

$$\partial\Phi(c) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1), \kappa, \omega)$$

if $\omega + \partial\Phi(c) \in \mathcal{L}is(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ and

$$\kappa^{-1} \leqslant \frac{\|(\lambda + \partial\Phi(c))h\|_{h^{1+\alpha}(\mathbb{S}^1)}}{|\lambda|\,\|h\|_{h^{1+\alpha}(\mathbb{S}^1)} + \|h\|_{h^{2+\alpha}(\mathbb{S}^1)}} \leqslant \kappa, \quad h \in h^{2+\alpha}(\mathbb{S}^1) \setminus \{0\}, \quad \operatorname{Re}\lambda \geqslant \omega.$$

Since

$$\mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)) = \bigcup_{\substack{\kappa\geqslant 1\\ \omega>0}} \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1), \kappa, \omega),$$

it is sufficient to show that $\partial\Phi(c) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)), \kappa, \omega)$ for some $\kappa \geqslant 1$ and $\omega > 0$. In fact, it is enough to find $\kappa \geqslant 1$ and $\omega > 0$ such that

$$\lambda + \partial\Phi(c) \in \mathcal{L}is(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)), \tag{20}$$

$$|\lambda| \cdot \|R(\lambda, -\partial\Phi(c))\|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))} \leqslant \kappa, \tag{21}$$

for all $\operatorname{Re}\lambda \geqslant \omega$.

3.1 *Sobolev spaces over the unit circle*

Let us recall that the Fréchet derivative $\partial\Phi(c) \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ is defined by

$$\partial\Phi(c) \sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} = \sum_{k\in\mathbb{Z}} \lambda_k \widehat{h}(k)e^{ikx}$$

for all $h = \sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} \in h^{2+\alpha}(\mathbb{S}^1)$, with $(\lambda_k)_{k\in\mathbb{Z}}$ given by (18). We denote here by $\widehat{h}(k)$ the k-th Fourier coefficient of $h \in h^{2+\alpha}(\mathbb{S}^1)$. For $r \geqslant 0$ we introduce the Sobolev space

$$H^r(\mathbb{S}^1) := \Big\{ f \in L^2(\mathbb{S}^1) \,:\, \sum_{k\in\mathbb{Z}} (1+k^2)^r |\widehat{f}(k)|^2 < \infty \Big\},$$

equipped with the scalar product $\langle f, g\rangle := \sum_{k\in\mathbb{Z}} (1+k^2)^r \widehat{f}(k)\overline{\widehat{g}(k)}$. The smooth functions are dense in $H^r(\mathbb{S}^1)$, and the Sobolev embedding

$$H^{m+\sigma}(\mathbb{S}^1) \hookrightarrow C^m(\mathbb{S}^1) \tag{22}$$

holds for all $m \in \mathbb{N}$ provided $\sigma > 1/2$.

PROPOSITION 3.1

$$H^{m+s}(\mathbb{S}^1) \overset{d}{\hookrightarrow} h^{m+\alpha}(\mathbb{S}^1)$$

for all $m \in \mathbb{N}$, $\alpha \in [0, 1]$ and $s > 3/2$.

Proof. Given $m \in \mathbb{N}$, $\alpha \in [0, 1]$ and $s > 3/2$ we have the embeddings

$$C^\infty(\mathbb{S}^1) \subset H^{m+s}(\mathbb{S}^1) \hookrightarrow C^{m+\alpha}(\mathbb{S}^1), \tag{23}$$

thus $h^{m+\alpha}(\mathbb{S}^1) = \overline{C^\infty(\mathbb{S}^1)}^{\|\cdot\|_{C^{m+\alpha}(\mathbb{S}^1)}} \subset \overline{H^{m+s}(\mathbb{S}^1)}^{\|\cdot\|_{C^{m+\alpha}(\mathbb{S}^1)}}$.

Fix now $u \in \overline{H^{m+s}(\mathbb{S}^1)}^{\|\cdot\|_{C^{m+\alpha}(\mathbb{S}^1)}}$ and choose $\varepsilon > 0$. We can find $u_0 \in H^{m+s}(\mathbb{S}^1)$ with $\|u - u_0\|_{C^{m+\alpha}(\mathbb{S}^1)} \leqslant \varepsilon/2$. Due to (23) there is a constant $C > 0$ such that

$$\|v\|_{C^{m+\alpha}(\mathbb{S}^1)} \leqslant C\|v\|_{H^{m+s}(\mathbb{S}^1)}, \qquad \forall v \in H^{m+s}(\mathbb{S}^1).$$

Let $u_1 \in C^{\infty}(\mathbb{S}^1)$ be a smooth function with $\|u_0 - u_1\|_{H^{m+s}(\mathbb{S}^1)} \leqslant \varepsilon/2C$. Combining these last inequalities, we get $\|u - u_1\|_{C^{m+\alpha}(\mathbb{S}^1)} \leqslant \varepsilon$ and the proof is complete. □

Let us now consider the coefficients λ_k, $k \in \mathbb{Z}$. We notice that $\lambda_k = \lambda_{-k}$ and that λ_k is positive for every $k \in \mathbb{Z}$. Moreover,

$$\lim_{k\to\infty} \frac{\lambda_k}{k} = \frac{b\zeta}{\beta c^2}. \tag{24}$$

We now fix

$$\omega := 1. \tag{25}$$

PROPOSITION 3.2 Given $r \geqslant 0$ and $\operatorname{Re}\lambda \geqslant \omega$, we have $\lambda + \partial\Phi(c) \in \mathcal{L}is(H^{r+1}(\mathbb{S}^1), H^r(\mathbb{S}^1))$.

Proof. We first prove that $\partial\Phi(c)$ is well-defined. Due to (24) there is a constant $M > 0$ such that

$$|\lambda_k| \leqslant M(1 + k^2)^{1/2}, \qquad \forall k \in \mathbb{Z}.$$

Given $h = \sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} \in H^{r+1}(\mathbb{S}^1)$, we have

$$\begin{aligned}\Big\|\partial\Phi(c)\sum_{k\in\mathbb{Z}}\widehat{h}(k)e^{ikx}\Big\|_{H^r(\mathbb{S}^1)} &= \sum_{k\in\mathbb{Z}}(1+k^2)^r|\lambda_k\widehat{h}(k)|^2 \leqslant M^2\sum_{k\in\mathbb{Z}}(1+k^2)^{r+1}|\widehat{h}(k)|^2 \\ &= M^2\Big\|\sum_{k\in\mathbb{Z}}\widehat{h}(k)e^{ikx}\Big\|_{H^{r+1}(\mathbb{S}^1)}.\end{aligned}$$

Thus $\partial\Phi(c)$ is well-defined. For $\operatorname{Re}\lambda \geqslant \omega$ we have $\lambda + \lambda_k \geqslant 1$, and therefore $\lambda + \partial\Phi(c)$ is injective. In order to show that $\lambda + \partial\Phi(c)$ is onto, we have to show that for $h = \sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} \in H^r(\mathbb{S}^1)$, the function $\sum_{k\in\mathbb{Z}}(1/(\lambda + \lambda_k))\widehat{h}(k)e^{ikx}$ is in $H^{r+1}(\mathbb{S}^1)$. Invoking again (24), we find $M_\lambda > 0$ such that

$$|\lambda + \lambda_k|^2 \geqslant M_\lambda(1 + k^2), \qquad \forall k \in \mathbb{Z}.$$

Now

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}}\frac{1}{\lambda+\lambda_k}\widehat{h}(k)e^{ikx}\Big\|_{H^{r+1}(\mathbb{S}^1)} &= \sum_{k\in\mathbb{Z}}(1+k^2)^{r+1}\left|\frac{\widehat{h}(k)}{\lambda+\lambda_k}\right|^2 \leqslant \frac{1}{M_\lambda}\sum_{k\in\mathbb{Z}}(1+k^2)^r|\widehat{h}(k)|^2 \\ &= \frac{1}{M_\lambda}\Big\|\sum_{k\in\mathbb{Z}}\widehat{h}(k)e^{ikx}\Big\|_{H^r(\mathbb{S}^1)},\end{aligned}$$

and the proof is complete. □

Combining these two propositions we obtain the following result.

COROLLARY 3.3 Let $m \in \{1, 2\}$ and suppose $R(\lambda, -\partial\Phi(c)) \in \mathcal{L}(C^{1+\alpha}(\mathbb{S}^1), C^{m+\alpha}(\mathbb{S}^1))$ for some $\operatorname{Re}\lambda \geqslant \omega$. Then $R(\lambda, -\partial\Phi(c)) \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1), h^{m+\alpha}(\mathbb{S}^1))$.

Proof. We prove just the case $m = 2$. The proof in the case $m = 1$ is similar. By assumption, $R(\lambda, -\partial\Phi(c)) \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1), C^{2+\alpha}(\mathbb{S}^1))$. Given $f \in h^{1+\alpha}(\mathbb{S}^1)$, Proposition 3.1 ensures the existence of a sequence $(f_n)_n \subset H^r(\mathbb{S}^1)$, $r > 3$, such that $f_n \to f$ in $C^{1+\alpha}(\mathbb{S}^1)$. Thus

$$R(\lambda, -\partial\Phi(c))f_n \to R(\lambda, -\partial\Phi(c))f \quad \text{in } C^{2+\alpha}(\mathbb{S}^1).$$

We know that $R(\lambda, -\partial\Phi(c))f_n \in H^{r+1}(\mathbb{S}^1)$. Consequently,

$$R(\lambda, -\partial\Phi(c))f \in \overline{H^{r+1}(\mathbb{S}^1)}^{\|\cdot\|_{C^{2+\alpha}(\mathbb{S}^1)}} = h^{2+\alpha}(\mathbb{S}^1). \qquad \square$$

3.2 *Periodic Besov spaces*

Let $(\phi_j)_{j\geqslant 0} \subset \mathcal{S}(\mathbb{R})$ be a sequence with the following properties:

$$\text{(i)} \quad \operatorname{supp}\phi_0 \subset [-2, 2], \quad \operatorname{supp}\phi_j \subset \{x : 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}, \quad j \geqslant 1,$$

$$\text{(ii)} \quad \sum_{j\in\mathbb{N}} \phi_j = 1 \quad \text{in } \mathbb{R},$$

$$\text{(iii)} \quad \forall k \in \mathbb{N}\ \exists c_k > 0 : \quad 2^{kj}\|\phi_j^{(k)}\|_0 \leqslant c_k,\ \forall j \in \mathbb{N}.$$

Further, let $\mathcal{D}'(\mathbb{S}^1)$ denote the topological dual of $\mathcal{D}(\mathbb{S}^1)$. The Fourier coefficients of $f \in \mathcal{D}'(\mathbb{S}^1)$ are $\widehat{f}(k) := (2\pi)^{-1} f(e^{-ikx})$, $k \in \mathbb{Z}$, and the series $\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikx}$ converges to f in $\mathcal{D}'(\mathbb{S}^1)$. The Besov spaces $B^s_{\infty,\infty}(\mathbb{S}^1)$, $s \geqslant 0$, are defined as follows:

$$B^s_{\infty,\infty}(\mathbb{S}^1) := \Big\{ f \in \mathcal{D}'(\mathbb{S}^1) : \|f\|_{B^s_{\infty,\infty}(\mathbb{S}^1)} := \sup_{j\in\mathbb{N}} 2^{sj} \Big\| \sum_{k\in\mathbb{Z}} \phi_j(k)\widehat{f}(k)e^{ikx} \Big\|_{C(\mathbb{S}^1)} < \infty \Big\}.$$

If $s > 0$ is not an integer, then $B^s_{\infty,\infty}(\mathbb{S}^1) = C^s(\mathbb{S}^1)$. For details see e.g. [11]. As one sees from previous computations, the operators $R(\lambda, -\partial\Phi(c))$ are Fourier multiplier operators. In order to prove (20) and (21) we can use, due to former considerations, multiplier theorems for operators between Besov spaces. Using the techniques of [2], it is not difficult to prove the following generalization of a result presented there.

THEOREM 3.4 Let r, s be positive constants and let $(M_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence satisfying the following conditions:

$$\text{(i)} \quad \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s}|M_k| < \infty,$$

$$\text{(ii)} \quad \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+1}|M_{k+1} - M_k| < \infty,$$

$$\text{(iii)} \quad \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+2}|M_{k+2} - 2M_{k+1} + M_k| < \infty.$$

Then the mapping

$$\sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} \mapsto \sum_{k\in\mathbb{Z}} M_k\widehat{h}(k)e^{ikx}$$

belongs to $\mathcal{L}(B^s_{\infty,\infty}(\mathbb{S}^1), B^r_{\infty,\infty}(\mathbb{S}^1))$.

Proof. The case $r = s$ is proved in [2]. For $r \neq s$ the proof is similar, with obvious modifications. $\square$

COROLLARY 3.5

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega\} \subset \rho(-\partial\Phi(c)).$$

Proof. Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant \omega$. Due to Corollary 3.3, it is enough to show that $R(\lambda, -\partial\Phi(c)) \in \mathcal{L}(C^{1+\alpha}(\mathbb{S}^1), C^{2+\alpha}(\mathbb{S}^1))$. Here $R(\lambda, -\partial\Phi(c))$ denotes the multiplier operator

$$\sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} \mapsto \sum_{k\in\mathbb{Z}} M_k^\lambda \widehat{h}(k)e^{ikx}$$

with $M_k^\lambda = 1/(\lambda + \lambda_k)$ for $k \in \mathbb{Z}$. In order to prove this assertion, we use the previous theorem with $r := 2 + \alpha$ and $s := 1 + \alpha$. Using relation (24), we obtain

$$\lim_{|k|\to\infty} |k|\,|M_k^\lambda| = \frac{\beta c^2}{b\zeta},$$

thus condition (i) in Theorem 3.4 is satisfied. Given $k \neq 0$, we have

$$k^2|M_{k+1}^\lambda - M_k^\lambda| = \frac{|k|}{|\lambda + \lambda_{k+1}|}\frac{|k|}{|\lambda + \lambda_k|}|\lambda_{k+1} - \lambda_k| \xrightarrow[|k|\to\infty]{} \frac{\beta c^2}{b\zeta},$$

and (ii) is verified. Furthermore, we have

$$\begin{aligned}|k|^3|M_{k+2}^\lambda - 2M_{k+1}^\lambda - M_k^\lambda| = \frac{|k|}{|\lambda + \lambda_{k+2}|}\frac{|k|}{|\lambda + \lambda_{k+1}|}\frac{|k|}{|\lambda + \lambda_k|}| - \lambda(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k)\\ + \lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k)|,\end{aligned}$$

with $(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k) \to 0$ as $|k| \to \infty$. One can easily verify that

$$\lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k) \xrightarrow[|k|\to\infty]{} 2\left(\frac{b\zeta}{\beta c^2}\right)^2,$$

and the proof is complete. □

It remains to prove assertion (21). We shall make again use of Theorem 3.4, but now in the special case $r = s = 1 + \alpha$. Notice that for $k \in \mathbb{Z}$ and $\operatorname{Re}\lambda \geqslant \omega$ we have

$$\lambda + \lambda_k \geqslant \max\{1, \lambda, \lambda_k\}. \tag{26}$$

COROLLARY 3.6 There exists $\kappa \geqslant 1$ such that

$$|\lambda| \cdot \|R(\lambda, -\partial\Phi(c))\|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))} \leqslant \kappa$$

for all $\operatorname{Re}\lambda \geqslant \omega$.

Proof. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant \omega$. Then $|\lambda|R(\lambda, -\partial\Phi(c))$ belongs to $\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))$. We regard $|\lambda|R(\lambda, -\partial\Phi(c))$ as a multiplier operator,

$$\sum_{k\in\mathbb{Z}} \widehat{h}(k)e^{ikx} \mapsto \sum_{k\in\mathbb{Z}} M_k^\lambda \widehat{h}(k)e^{ikx},$$

with

$$M_k^\lambda = \frac{|\lambda|}{\lambda + \lambda_k}, \qquad \forall k \in \mathbb{Z},$$

and we wish to find positive real numbers s_1, s_2 and s_3 such that

$$\begin{aligned}
&\text{(i)} \quad \sup_{k\in\mathbb{Z}} |M_k^\lambda| \leqslant s_1, \\
&\text{(ii)} \quad \sup_{k\in\mathbb{Z}} |k|\,|M_{k+1}^\lambda - M_k^\lambda| \leqslant s_2, \\
&\text{(iii)} \quad \sup_{k\in\mathbb{Z}} |k|^2 |M_{k+2}^\lambda - 2M_{k+1}^\lambda + M_k^\lambda| \leqslant s_3,
\end{aligned}$$

for all $\operatorname{Re}\lambda \geqslant \omega$. The existence of such constants is equivalent to the uniform boundedness of the family $\{|\lambda| R(\lambda, -\partial\Phi(c))\}_{\operatorname{Re}\lambda\geqslant\omega} \subset \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))$. For details see [2]. From relation (26) we obtain

$$|M_k^\lambda| = \frac{|\lambda|}{|\lambda + \lambda_k|} \leqslant 1$$

for all $k \in \mathbb{Z}$ and $\operatorname{Re}\lambda \geqslant \omega$. We also have

$$|k|\,|M_{k+1}^\lambda - M_k^\lambda| = \frac{|\lambda|}{|\lambda + \lambda_{k+1}|}\,\frac{|k|}{|\lambda + \lambda_k|}|\lambda_{k+1} - \lambda_k| \leqslant \frac{|k|}{\lambda_k}|\lambda_{k+1} - \lambda_k|,$$

which, together with (24), implies estimate (ii). Further,

$$\begin{aligned}
|k|^2|M_{k+2}^\lambda - 2M_{k+1}^\lambda - M_k^\lambda| &= \frac{|\lambda|}{|\lambda + \lambda_{k+2}|}\,\frac{|k|}{|\lambda + \lambda_{k+1}|}\,\frac{|k|}{|\lambda + \lambda_k|}| - \lambda(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k) \\
&\quad + \lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k)| \\
&\leqslant \frac{|k|}{\lambda_k}|k|\,|\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k| \\
&\quad + \frac{|k|}{\lambda_{k+1}}\,\frac{|k|}{\lambda_k}|\lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k)|.
\end{aligned}$$

The relation

$$|k|\,|\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k| \xrightarrow[|k|\to\infty]{} 0$$

completes the proof. □

We have proved that for every positive constant c, the complexification of the derivative $\partial\Phi(c)$ generates a strongly continuous analytic semigroup in $\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))$, i.e. it belongs to $\mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$. It is known that $\mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ is an open subset in $\mathcal{L}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ (see [1]), and because $\partial\Phi$ is continuous, there is a neighbourhood $\mathcal{O}$ of c in $\mathcal{V}$ such that the complexification of $\partial\Phi(f_0)$ is an element of $\mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ for all $f_0 \in \mathcal{O}$. The proof of Theorem 1.1 is now similar to the proof of Theorem 8.1.1 in [10].

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References

1. AMANN, H. *Linear and Quasilinear Parabolic Problems*. Volume I, Birkhäuser, Basel (1995). Zbl 0819.35001 MR 1345385
2. ARENDT, W., & BU, S. Operator-valued Fourier multipliers on periodic Besov spaces and applications. *Proc. Edinburgh Math. Soc.* **47** (2004), 15–33. Zbl 1083.42009 MR 2064734
3. ESCHER, J. On moving boundaries in deformable media. *Adv. Math. Sci. Appl.* **7** (1997), 275–316. Zbl 0884.35174 MR 1454668
4. ESCHER, J., & PROKERT, G. Stability of the equilibria for spatially periodic flows in porous media. *Nonlinear Anal.* **45** (2001), 1061–1080. Zbl 0986.35017 MR 1846423
5. ESCHER, J., & SIMONETT, G. Maximal regularity for a free boundary problem. *Nonlinear Differential Equations Appl.* **2** (1991), 463–510. Zbl 0842.35083 MR 1356871
6. ESCHER, J., & SIMONETT, G. Analyticity of the interface in a free boundary problem, *Math. Ann.* **305** (1996), 439–459. Zbl 0857.76086 MR 1397432
7. ESCHER, J., & SIMONETT, G. Classical solutions of multidimensional Hele–Shaw models. *SIAM J. Math. Anal.* **28** (1997), 1028–1047. Zbl 0888.35142 MR 1466667
8. GILBARG, D., & TRUDINGER, N. S. *Elliptic Partial Differential Equations of Second Order*. Springer, New York (1977). Zbl 0361.35003 MR 0473443
9. KONDIC, L., PALFFY-MAHORNY, P., & SHELLEY, M. J. Models of non-Newtonian Hele–Shaw flow. *Phys. Rev. E* **54** (1996), R4536–R4539.
10. LUNARDI, A. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser, Basel (1995). Zbl 0816.35001 MR 1329547
11. SCHMEISSER, H.-J., & TRIEBEL, H. *Topics in Fourier Analysis and Function Spaces*. Wiley, New York (1987). Zbl 0661.46025 MR 0891189