

Two-phase flows involving capillary barriers in heterogeneous porous media

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We consider a simplified model of a two-phase flow through a heterogeneous porous medium, in which convection is neglected. This leads to a nonlinear degenerate parabolic problem in a domain divided into an arbitrary finite number of homogeneous porous media. We introduce a new way to connect capillary pressures on the interfaces between the homogeneous domains, which leads to a general notion of solution. We then compare this notion of solution with an existing one, showing that it allows one to deal with a larger class of problems. We prove the existence of such a solution in a general case, and the existence and uniqueness of a regular solution in the one-dimensional case for initial data regular enough.

Keywords: Flows in porous media; capillarity; nonlinear PDE of parabolic type.

1. Presentation of the problem

Models of immiscible two-phase flows are widely used in petroleum engineering, particularly in basin modelling, whose aim can be the prediction of the migration of hydrocarbon components at geological time scale in a sedimentary basin.

The heterogeneity of the porous medium leads to the phenomena of oil-trapping and oil-expulsion, which is modelled with discontinuous capillary pressures between the different geological layers.

The physical and mathematical models can be found in [4, 5, 12, 13, 14]. The phenomenon of capillary trapping has been considered only in simplified cases (see [6]), and several numerical methods have been developed (see e.g. [16, 15]).

The aim of this paper is to introduce a new notion of weak solution, which allows us to deal with more general cases than those treated in [15], while it is equivalent to the notion of weak solution introduced in [15] in the already treated cases. We will consider a simplified model (\mathcal{P}) (page 242), in which convection is neglected.

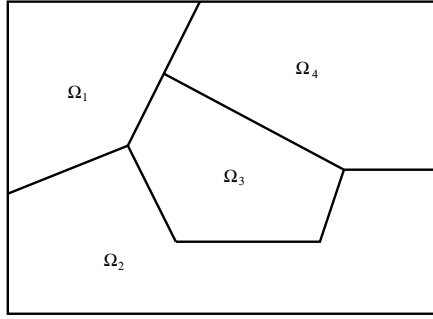
We then give a uniqueness result in the one-dimensional case which is inspired from the result in [6] and extends it to more general situations, by requiring weaker assumptions on the solutions and applying to a larger class of initial data.

We have to make some assumptions on the heterogeneous porous medium:

ASSUMPTIONS 1 (Geometrical assumptions)

1. The heterogeneous porous medium is represented by a polygonal bounded connected domain $\Omega \subset \mathbb{R}^d$ with $\text{meas}_{\mathbb{R}^d}(\Omega) > 0$, where $\text{meas}_{\mathbb{R}^n}$ is the Lebesgue measure on \mathbb{R}^n .
2. There exist a finite number N of polygonal connected subdomains $(\Omega_i)_{1 \leq i \leq N}$ of Ω such that:
 - (a) for all $i \in \llbracket 1, N \rrbracket$, $\text{meas}_{\mathbb{R}^d}(\Omega_i) > 0$,
 - (b) $\bigcup_{i=1}^N \overline{\Omega}_i = \overline{\Omega}$,
 - (c) for $(i, j) \in \llbracket 1, N \rrbracket^2$ with $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$.

Each Ω_i represents a homogeneous porous medium. For all $(i, j) \in \llbracket 1, N \rrbracket^2$, one denotes by $\Gamma_{i,j} \subset \Omega$ the interface between the geological layers Ω_i and Ω_j , defined by $\overline{\Gamma}_{i,j} = \partial\Omega_i \cap \partial\Omega_j$.

FIG. 1. An example of the domain Ω .

We consider an incompressible and immiscible oil-water flow through Ω , and thus through each Ω_i . By Darcy's law, the conservation of oil and water phases is given for all $(x, t) \in \Omega_i \times]0, T[$ by the equations

$$\begin{cases} \phi_i \partial_t u_i(x, t) - \nabla \cdot (\eta_{o,i}(u_i(x, t))(\nabla p_{o,i}(x, t) - \rho_o \mathbf{g})) = 0, \\ -\phi_i \partial_t u_i(x, t) - \nabla \cdot (\eta_{w,i}(u_i(x, t))(\nabla p_{w,i}(x, t) - \rho_w \mathbf{g})) = 0, \\ p_{o,i}(x, t) - p_{w,i}(x, t) = \pi_i(u_i(x, t)), \end{cases} \quad (1)$$

where $u_i \in [0, 1]$ is the oil saturation in Ω_i (and therefore $1 - u_i$ the water saturation), $\phi_i \in]0, 1[$ is the porosity of Ω_i , which is supposed to be constant in each Ω_i for the sake of simplicity, $\pi_i(u_i(x, t))$ is the capillary pressure, and \mathbf{g} is the gravity acceleration. The indices o and w stand for the oil and the water phases respectively. Thus, for $\sigma = o, w$, $p_{\sigma,i}$ is the pressure, $\eta_{\sigma,i}$ is the mobility, and ρ_σ is the density of the phase σ .

We now have to make assumptions on the data to specify the transmission conditions through the interfaces $\Gamma_{i,j}$:

ASSUMPTIONS 2 (Assumptions on the data)

1. For all $i \in \llbracket 1, N \rrbracket$, $\pi_i \in C^1([0, 1], \mathbb{R})$, with $\pi_i'(x) > 0$ for $x \in]0, 1[$.
2. For all $i \in \llbracket 1, N \rrbracket$, $\eta_{o,i} \in C([0, 1], \mathbb{R}_+)$ is an increasing function with $\eta_{o,i}(0) = 0$.
3. For all $i \in \llbracket 1, N \rrbracket$, $\eta_{w,i} \in C([0, 1], \mathbb{R}_+)$ is a decreasing function with $\eta_{w,i}(1) = 0$.
4. The initial data u_0 belongs to $L^\infty(\Omega)$, $0 \leq u_0 \leq 1$.

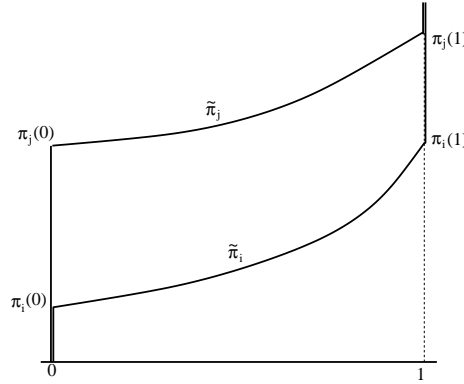


FIG. 2. Graphs for the capillary pressures.

We set $\alpha_i = \lim_{s \rightarrow 0} \pi_i(s)$ and $\beta_i = \lim_{s \rightarrow 1} \pi_i(s)$. We can now define the monotonic graphs $\tilde{\pi}_i$ by

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0, 1[, \\]-\infty, \alpha_i] & \text{if } s = 0, \\ [\beta_i, +\infty[& \text{if } s = 1. \end{cases} \quad (2)$$

As shown in [15], the following conditions must be satisfied on the traces of u_i , $p_{\sigma,i}$ and $\nabla p_{\sigma,i}$ on $\Gamma_{i,j} \times]0, T[$, still denoted respectively u_i , $p_{\sigma,i}$ and $\nabla p_{\sigma,i}$ (see [5]):

1. For any $\sigma = o, w$, and $(i, j) \in \llbracket 1, N \rrbracket^2$ such that $\Gamma_{i,j} \neq \emptyset$, the flux of the phase σ through $\Gamma_{i,j}$ must be continuous:

$$\eta_{\sigma,i}(u_i)(\nabla p_{\sigma,i} - \rho_{\sigma} \mathbf{g}) \cdot \mathbf{n}_i + \eta_{\sigma,j}(u_j)(\nabla p_{\sigma,j} - \rho_{\sigma} \mathbf{g}) \cdot \mathbf{n}_j = 0, \quad (3)$$

where \mathbf{n}_i denotes the outward normal to $\Gamma_{i,j}$ pointing to Ω_i .

2. For any $\sigma = o, w$, and $(i, j) \in \llbracket 1, N \rrbracket^2$ such that $\Gamma_{i,j} \neq \emptyset$, either p_{σ} is continuous or $\eta_{\sigma} = 0$. Since the saturation is itself discontinuous across $\Gamma_{i,j}$, one must express the mobility at the upstream side of the interface. This gives

$$\eta_{\sigma,i}(u_i)(p_{\sigma,i} - p_{\sigma,j})^+ - \eta_{\sigma,j}(u_j)(p_{\sigma,j} - p_{\sigma,i})^+ = 0. \quad (4)$$

The conditions (4) have direct consequences for the behaviour of the capillary pressures on both sides of $\Gamma_{i,j}$. Indeed, if $0 < u_i, u_j < 1$, then the partial pressures p_o and p_w both have to be continuous, and so we have the continuity of the capillary pressures $\pi_i(u_i) = \pi_j(u_j)$. If $u_i = 0$ and $0 < u_j < 1$, then $p_{o,i} \geq p_{o,j}$ and $p_{w,i} = p_{w,j}$, thus $\pi_j(u_j) \leq \pi_i(0)$. The same way, $u_i = 1$ and $0 < u_j < 1$ implies $\pi_j(u_j) \geq \pi_i(1)$. If $u_i = 0, u_j = 1$, then $p_{o,i} \geq p_{o,j}$ and $p_{w,i} \leq p_{w,j}$, so $\pi_i(0) \geq \pi_j(1)$. Checking that the definition of the graphs $\tilde{\pi}_i$ and $\tilde{\pi}_j$ implies $\tilde{\pi}_i(0) \cap \tilde{\pi}_j(0) \neq \emptyset$, $\tilde{\pi}_i(1) \cap \tilde{\pi}_j(1) \neq \emptyset$, we can claim that (4) leads to

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset. \quad (5)$$

We introduce the global pressure in Ω_i ,

$$\bar{p}_i(x, t) = p_{w,i}(x, t) + \int_0^{u_i(x,t)} \frac{\eta_{o,i}(a)}{\eta_{o,i}(a) + \eta_{w,i}(a)} \pi_i'(a) da \quad (6)$$

(see e.g. [3] or [12]), and the global mobility in Ω_i ,

$$\lambda_i(u_i(x, t)) = \frac{\eta_{o,i}(u_i(x, t))\eta_{w,i}(u_i(x, t))}{\eta_{o,i}(u_i(x, t)) + \eta_{w,i}(u_i(x, t))}, \quad (7)$$

which satisfies $\lambda_i(0) = \lambda_i(1) = 0$, and $\lambda_i(s) > 0$ for $0 < s < 1$. Taking into account (6) and (7) in (1), and adding the conservation laws, leads to, for $(x, t) \in \Omega_i \times]0, T[$,

$$\begin{cases} \phi_i \partial_t u_i(x, t) - \nabla \cdot (\eta_{o,i}(u_i(x, t))(\nabla \bar{p}_i(x, t) - \rho_o \mathbf{g}) - \lambda_i(u_i(x, t))\nabla \pi_i(u_i(x, t))) = 0, \\ -\nabla \cdot \left(\sum_{\sigma=o,w} \eta_{\sigma,i}(u_i(x, t))(\nabla \bar{p}_i(x, t) - \rho_\sigma \mathbf{g}) \right) = 0. \end{cases} \quad (8)$$

We neglect the convective effects, so that we focus on the mathematical modelling of flows with discontinuous capillary pressures, which seems necessary to explain the phenomena of oil trapping. This simplification will allow us to neglect the coupling with the second equation of (8). Then we get the simple degenerate parabolic equation in $\Omega_i \times]0, T[$:

$$\phi_i \partial_t u_i(x, t) - \nabla \cdot (\lambda_i(u_i(x, t))\nabla \pi_i(u_i(x, t))) = 0 \quad \text{in } \Omega_i \times]0, T[. \quad (9)$$

In this simplified framework, the transmission condition (3) on the fluxes through $\Gamma_{i,j}$ can be rewritten

$$\lambda_i(u_i(x, t))\nabla(\pi_i(u_i(x, t))) \cdot \mathbf{n}_i + \lambda_j(u_j(x, t))\nabla(\pi_j(u_j(x, t))) \cdot \mathbf{n}_j = 0 \quad \text{on } \Gamma_{i,j} \times]0, T[. \quad (10)$$

We suppose furthermore that $u_i(x, 0) = u_0(x)$ for $x \in \Omega_i$. In the remainder of this paper, we assume a homogeneous Neumann boundary condition. The existence of a weak solution proven in Section 3 can be extended to the case of nonhomogeneous Dirichlet conditions. Nevertheless, homogeneous Neumann boundary conditions are needed to prove Theorem 4.1, and thus to prove the final Theorem 5.4.

Taking into account the equations (5), (9), (10), the boundary condition, and the initial condition, we can write the problem we aim to solve this way: for all $i \in \llbracket 1, N \rrbracket$, and all $j \in \llbracket 1, N \rrbracket$ such that $\Gamma_{i,j} \neq \emptyset$,

$$\begin{cases} \phi_i \partial_t u_i - \nabla \cdot (\lambda_i(u_i)\nabla \pi_i(u_i)) = 0 & \text{in } \Omega_i \times]0, T[, \\ \tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset & \text{on } \Gamma_{i,j} \times]0, T[, \\ \lambda_i(u_i)\nabla(\pi_i(u_i)) \cdot \mathbf{n}_i + \lambda_j(u_j)\nabla(\pi_j(u_j)) \cdot \mathbf{n}_j = 0 & \text{on } \Gamma_{i,j} \times]0, T[, \\ \lambda_i(u_i)\nabla(\pi_i(u_i)) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times]0, T[, \\ u_i(\cdot, 0) = u_0(x) & \text{in } \Omega_i. \end{cases} \quad (\mathcal{P})$$

All the results presented in this paper still hold if one does not neglect the effect of gravity and if one assumes that the global pressure is known, that is, for problems of the type

$$\begin{cases} \phi_i \partial_t u_i + \nabla \cdot (\mathbf{q}f_i(u_i) + \lambda_i(u_i)(\rho_o - \rho_w)\mathbf{g} - \lambda_i(u_i)\nabla \pi_i(u_i)) = 0 & \text{in } \Omega_i \times]0, T[, \\ \tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset & \text{on } \Gamma_{i,j} \times]0, T[, \\ \sum_{k=i,j} (\mathbf{q}f_k(u_k) + \lambda_k(u_k)(\rho_o - \rho_w)\mathbf{g} - \lambda_k(u_k)\nabla \pi_k(u_k)) \cdot \mathbf{n}_k = 0 & \text{on } \Gamma_{i,j} \times]0, T[, \\ (\mathbf{q}f_i(u_i) + \lambda_i(u_i)(\rho_o - \rho_w)\mathbf{g} - \lambda_i(u_i)\nabla \pi_i(u_i)) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times]0, T[, \\ u_i(\cdot, 0) = u_0(x) & \text{in } \Omega_i, \end{cases}$$

where f_i is supposed to be a $C^1([0, 1], \mathbb{R})$ increasing function, λ_i is also supposed to belong to $C^1([0, 1], \mathbb{R}_+)$ and \mathbf{q} satisfies

- $\forall i, \mathbf{q} \in (C^1(\overline{\Omega}_i \times]0, T]))^d$,
- $\nabla \cdot \mathbf{q} = 0$ in $\Omega_i \times]0, T[$,
- $\mathbf{q}|_{\Omega_i} \cdot \mathbf{n}_i + \mathbf{q}|_{\Omega_j} \cdot \mathbf{n}_j = 0$ on $\Gamma_{i,j} \times]0, T[$,
- $\mathbf{q} \cdot \mathbf{n} = 0$.

In order to ensure the uniqueness result stated in Theorem 5.1, we assume the technical condition (see [2] or [27])

$$\forall i, \quad f_i \circ \varphi_i^{-1}, \lambda_i \circ \varphi_i^{-1} \in C^{0,1/2}([0, \varphi_i(1)], \mathbb{R}).$$

REMARK 1.1 In the modelling of two-phase flows, irreducible saturations are often taken into account. One can suppose that there exist s_i and S_i ($0 < s_i < S_i < 1$) such that $\lambda_i(s) = 0$ if $s \notin]s_i, S_i[$. In such a case, the problem (\mathcal{P}) becomes strongly degenerate, but a convenient scaling eliminates this difficulty (at least if $s_i \leq u_0 \leq S_i$ a.e. in Ω_i). Moreover, the dependence of the capillary pressure on the saturation can be weak, at least for saturations not too close to 0 or 1. Thus the effects of capillarity are often neglected for the study of flows in homogeneous porous media, leading to the Buckley–Leverett equation (see e.g. [19]). Looking for degeneracy of $u \mapsto \pi_i(u)$ is a more complex problem, particularly if convection is not neglected as above. Suppose for example that $\pi_i(u) = \varepsilon u + P_i$, where P_i are constants, and let ε tend 0. Nonclassical shocks can appear at the level of the interfaces $\Gamma_{i,j}$ (see [10]). Thus the notion of entropy solution used by Adimurthi, J. Jaffré, and G. D. Veerappa Gowda [1] is not sufficient to deal with this problem. This difficulty has to be overcome when considering degenerate parabolic problems. But it seems clear that the notion of entropy solution developed by K. H. Karlsen, N. H. Risebro and J. D. Towers [20, 21, 22] is not suitable for our problem.

2. The notion of weak solution

In this section, we introduce the notion of weak solution to the problem (\mathcal{P}), which is more general than the notion of weak solution given in [14, 15]. Indeed, we are able to define such a solution even in the case of an arbitrary finite number of different homogeneous porous media. Furthermore, the notion of weak solution introduced in this paper is still available in cases where the one defined in [15] does not make sense any more. We finally show that the two notions of solution are equivalent in the case where the notion of [15] is well defined. Proving the existence of a weak solution to the problem (\mathcal{P}) in a wider case is the aim of Section 3.

We denote by φ_i the $C^1([0, 1], \mathbb{R}_+)$ function which naturally appears in the problem (\mathcal{P}) and which is defined by

$$\varphi_i(s) = \int_0^s \lambda_i(a) \pi_i'(a) da \quad \forall s \in [0, 1]. \tag{11}$$

REMARK 2.1 The assumptions on the data ensure that $\varphi_i' > 0$ on $]0, 1[$, and so we can define an increasing continuous function $\varphi_i^{-1} : [0, \varphi_i(1)] \rightarrow [0, 1]$.

We are now able to define the notion of weak solution to the problem (\mathcal{P}).

DEFINITION 2.1 Under Assumptions 1 and 2, a function u is said to be a *weak solution to the problem (\mathcal{P})* if

1. $u \in L^\infty(\Omega \times]0, T[), 0 \leq u \leq 1$ a.e. in $\Omega \times]0, T[$,
2. for all $i \in \llbracket 1, N \rrbracket$, $\varphi_i(u_i) \in L^2(0, T; H^1(\Omega_i))$, where u_i denotes the restriction of u to $\Omega_i \times]0, T[$,
3. $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times]0, T[$,
4. for all $\psi \in \mathcal{D}(\overline{\Omega} \times]0, T[)$,

$$\sum_{i=1}^N \int_{\Omega_i} \int_0^T \phi_i u_i(x, t) \partial_t \psi(x, t) \, dx \, dt + \sum_{i=1}^N \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) \, dx - \sum_{i=1}^N \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x, t)) \cdot \nabla \psi(x, t) \, dx \, dt = 0. \quad (12)$$

The third point of the definition, which ensures the continuity in the graph sense of the capillary pressures on the interfaces between several porous media, is well defined. Indeed, since $\varphi_i(u_i)$ belongs to $L^2(0, T; H^1(\Omega_i))$, it admits a trace still denoted $\varphi_i(u_i)$ on $\Gamma_{i,j} \times]0, T[$. Thanks to Remark 2.1, we can define the trace of u_i on $\Gamma_{i,j} \times]0, T[$.

REMARK 2.2 One can equivalently replace the third point of Definition 2.1 by the condition

3bis. $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times]0, T[$,

where $\tilde{\pi}_i$ is the monotonic graph given by

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0, 1[, \\ [\min(\alpha_j), \alpha_i] & \text{if } s = 0, \\ [\beta_i, \max(\beta_j)] & \text{if } s = 1. \end{cases} \quad (13)$$

We will now quickly show the equivalence between the notion of weak solution to the problem (\mathcal{P}) and the notion of weak solution given in [15], in the case where the latter is well defined, i.e. $N = 2$ and $\max(\alpha_1, \alpha_2) = \alpha < \beta = \min(\beta_1, \beta_2)$. We denote as in [15] the truncated capillary pressures by $\hat{\pi}_1 = \max(\alpha, \pi_1)$, $\hat{\pi}_2 = \min(\beta, \pi_2)$, and we introduce the problem $(\tilde{\mathcal{P}})$, which is treated in [15]:

$$\begin{cases} \phi_i \partial_t u_i - \nabla \cdot (\lambda_i(u_i) \nabla \pi_i(u_i)) = 0 & \text{in } \Omega_i \times]0, T[, \\ \hat{\pi}_1(u_1) = \hat{\pi}_2(u_2) & \text{on } \Gamma_{i,j} \times]0, T[, \\ \lambda_1(u_1) \nabla(\pi_1(u_1)) \cdot \mathbf{n}_1 + \lambda_2(u_2) \nabla(\pi_2(u_2)) \cdot \mathbf{n}_2 = 0 & \text{on } \Gamma_{i,j} \times]0, T[, \\ \lambda_i(u_i) \nabla(\pi_i(u_i)) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times]0, T[, \\ u_i(\cdot, 0) = u_0(x) & \text{in } \Omega_i. \end{cases} \quad (\tilde{\mathcal{P}})$$

Then it is easy to check that for all $(s_1, s_2) \in [0, 1]^2$,

$$\hat{\pi}_1(s_1) = \hat{\pi}_2(s_2) \Leftrightarrow \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset \Leftrightarrow \check{\pi}_1(s_1) \cap \check{\pi}_2(s_2) \neq \emptyset. \quad (14)$$

In order to recall the definition of weak solution, we have to introduce the function

$$\Psi : [\alpha, \beta] \rightarrow \mathbb{R}, \quad p \mapsto \int_\alpha^p \min_{j=1,2} (\lambda_j \circ \pi_j^{-1}(a)) \, da.$$

Ψ is increasing, and for $i = 1, 2$, $\Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}$ is a Lipschitz continuous function.

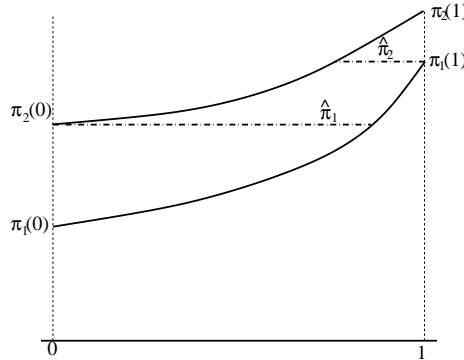


FIG. 3. Truncated capillary pressures.

DEFINITION 2.2 A function u is said to be a *weak solution to the problem* $(\tilde{\mathcal{P}})$ if

1. $u \in L^\infty(\Omega \times]0, T[), 0 \leq u \leq 1$ a.e. in $\Omega \times]0, T[$,
2. $\varphi_i(u_i) \in L^2(0, T; H^1(\Omega_i))$ for $i \in \{1, 2\}$,
3. $w : \Omega \times]0, T[\rightarrow \mathbb{R}$, defined for $(x, t) \in \Omega_i \times]0, T[$ by $w(x, t) = \Psi \circ \hat{\pi}_i(u_i)(x, t)$, belongs to $L^2(0, T; H^1(\Omega))$,
4. for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0, T[)$,

$$\sum_{i=1}^N \int_{\Omega_i} \int_0^T \phi_i u_i(x, t) \partial_t \psi(x, t) \, dx \, dt + \sum_{i=1}^N \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) \, dx - \sum_{i=1}^N \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x, t)) \cdot \nabla \psi(x, t) \, dx \, dt = 0.$$

REMARK 2.3 The notion of weak solution to the problem $(\tilde{\mathcal{P}})$ can be adapted to the case where there are $N > 2$ homogeneous domains, but we then need compatibility conditions on $(\alpha_i)_{1 \leq i \leq N}$ and $(\beta_i)_{1 \leq i \leq N}$.

Proof of the equivalence of weak solutions. If u is a weak solution to the problem $(\tilde{\mathcal{P}})$ in the sense of Definition 2.2, then for a.e. $t \in (0, T)$, $w(\cdot, t) \in H^1(\Omega)$, and in particular $w(\cdot, t)$ admits a trace on $\Gamma_{i,j}$, whose value is at the same time $\Psi(\hat{\pi}_i(u_i(\cdot, t)))$ and $\Psi(\hat{\pi}_j(u_j(\cdot, t)))$. Since Ψ is increasing, $\hat{\pi}_i(u_i(x, t)) = \hat{\pi}_j(u_j(x, t))$ for a.e. $(x, t) \in \Gamma_{i,j} \times]0, T[$. Using (14), we conclude that any weak solution to the problem $(\tilde{\mathcal{P}})$ is a weak solution to the problem (\mathcal{P}) in the sense of Definition 2.1.

Conversely, if u is a weak solution to the problem (\mathcal{P}) in the sense of Definition 2.1, then thanks to (14), for almost every $(x, t) \in \Gamma_{i,j} \times]0, T[$,

$$\hat{\pi}_i(u_i(x, t)) = \hat{\pi}_j(u_j(x, t)) \Leftrightarrow \Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}(\varphi_i(u_i(x, t))) = \Psi \circ \hat{\pi}_j \circ \varphi_j^{-1}(\varphi_j(u_j(x, t))). \quad (15)$$

Since $\Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}$ is a Lipschitz continuous function, the second point in Definition 2.1 ensures that $\Psi \circ \hat{\pi}_i(u_i)$ belongs to $L^2(0, T; H^1(\Omega_i))$ for $i = 1, 2$, and (15) ensures the continuity of the traces on $\Gamma_{i,j} \times]0, T[$, so the third condition of Definition 2.2 is fulfilled and u is a weak solution to the problem $(\tilde{\mathcal{P}})$. \square

REMARK 2.4 We can define a function $\tilde{\pi}_i^{-1}$, $i \in \llbracket 1, N \rrbracket$, which satisfies $\tilde{\pi}_i^{-1} \circ \tilde{\pi}_i(s) = s$ for any $s \in [0, 1]$. Using the function defined on \mathbb{R} by $\tilde{\Psi}(p) = \int_{-\infty}^p \min_{j=1,2}(\lambda_j \circ \tilde{\pi}_j^{-1}(a)) da$, it is easy to check that we can equivalently substitute the function $\tilde{\Psi} \circ \pi_i(u_i)$ for $\Psi \circ \hat{\pi}_i(u_i)$ in the third point of Definition 2.2. This function is still defined if $\alpha \geq \beta$, but it becomes identically 0, so the notion of weak solution to the problem $(\tilde{\mathcal{P}})$ is weaker than the notion of weak solution to the problem (\mathcal{P}) . Indeed, $u(x, t) = u_0(x) = a \in]0, 1[$ for any $(x, t) \in \Omega \times]0, T[$ is a weak solution to the problem $(\tilde{\mathcal{P}})$, but it does not satisfy the third condition in Definition 2.1.

3. Existence of a weak solution

The aim of this section is to prove the following theorem, which states the existence of a weak solution to the problem (\mathcal{P}) . This result has already been proven in Section 2 in the case $N = 2$ and $\alpha > \beta$, for which the notion of weak solution in the sense of Definition 2.1 is equivalent to the notion of weak solution in the sense of Definition 2.2.

THEOREM 3.1 (Existence of a weak solution) Under Assumptions 1 and 2, there exists a weak solution to the problem (\mathcal{P}) in the sense of Definition 2.1.

Proof. We build a sequence of solutions to approximate problems (16), which converges, up to a subsequence, toward a weak solution to the problem (\mathcal{P}) . The approximate problems do not involve capillary barriers, so existence and uniqueness of such approximate solutions is given in [9]. We leave the proof of the following technical lemma to the reader.

LEMMA 3.2 There exist sequences $(\lambda_{i,n})_n, (\pi_{i,n})_n$ belonging to $(C^\infty([0, 1], \mathbb{R}))^{\mathbb{N}}$ such that, for $i \in \llbracket 1, N \rrbracket$, and for n large enough:

- $\lambda_{i,n}|_{[0,1/n] \cup [1-1/n,1]} = 1/n^2$, $\lambda_{i,n}(s) > 1/2n^2$ for all $s \in [0, 1]$, and $\lambda_{i,n} \rightarrow \lambda_i$ uniformly on $[0, 1]$,
- $\pi_{i,n}(0) = \pi_{j,n}(0) \rightarrow -\infty$, $\pi_{i,n}(1) = \pi_{j,n}(1) \rightarrow +\infty$, $Kn^{3/2} > \pi'_{i,n} \geq 1/n$, $\pi_{i,n} \rightarrow \pi_i$ in $L^1(0, 1)$, $\pi_{i,n} \rightarrow \pi_i$ and $\pi'_{i,n} \rightarrow \pi'_i$ uniformly on any compact subset of $]0, 1[$,
- the function $\varphi_{i,n} : s \mapsto \int_0^s \lambda_{i,n}(a)\pi'_{i,n}(a) da$ satisfies $\varphi_{i,n}([0, 1]) = \varphi_i([0, 1])$ and $\varphi_{i,n} \rightarrow \varphi_i$ in $W^{1,\infty}(0, 1)$.

We also define the increasing functions:

$$\Psi_n : [a_n, b_n] \rightarrow \mathbb{R}, \quad p \mapsto \int_{a_n}^p \min_{j \in \llbracket 1, N \rrbracket} (\lambda_{j,n} \circ \pi_{j,n}^{-1}(a)) da.$$

The conditions on the functions on the intervals $[0, 1/n] \cup [1 - 1/n, 1]$ ensure that for any fixed large n , the functions $(\varphi_{i,n} \circ \pi_{i,n}^{-1} \circ \Psi_n^{-1})'$ are Lipschitz continuous. Then thanks to [9], for all n , the approximate problem

$$\begin{cases} \phi_i \partial_t u_{i,n} - \nabla \cdot (\lambda_{i,n}(u_{i,n}) \nabla \pi_{i,n}(u_{i,n})) = 0 & \text{in } \Omega_i \times]0, T[, \\ \pi_{i,n}(u_{i,n}) = \pi_{j,n}(u_{j,n}) & \text{on } \Gamma_{i,j} \times]0, T[, \\ \lambda_{i,n}(u_{i,n}) \nabla(\pi_{i,n}(u_{i,n})) \cdot \mathbf{n}_i + \lambda_{j,n}(u_{j,n}) \nabla(\pi_{j,n}(u_{j,n})) \cdot \mathbf{n}_j = 0 & \text{on } \Gamma_{i,j} \times]0, T[, \\ \lambda_{i,n}(u_{i,n}) \nabla(\pi_{i,n}(u_{i,n})) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times]0, T[, \\ u_{i,n}(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{16}$$

admits a unique weak solution in the sense of Definition 3.1 below, and this solution belongs to $C([0, T], L^p(\Omega))$ for $1 \leq p < \infty$.

DEFINITION 3.1 A function u_n is said to be a *weak solution to the problem* (16) if:

1. $u_n \in L^\infty(\Omega \times]0, T[)$, $0 \leq u_n \leq 1$ a.e. in $\Omega \times]0, T[$,
2. $\varphi_{i,n}(u_{i,n}) \in L^2(0, T; H^1(\Omega_i))$ for $i \in \{1, 2\}$,
3. $w_n : \Omega \times]0, T[\rightarrow \mathbb{R}$, defined on $\Omega_i \times]0, T[$ by $w_n = \Psi_n \circ \pi_{i,n}(u_{i,n})$, belongs to $L^2(0, T; H^1(\Omega))$,
4. for all $\psi \in \mathcal{D}(\Omega \times]0, T[)$,

$$\sum_{i=1}^N \int_{\Omega_i} \int_0^T \phi_i u_{i,n}(x, t) \partial_t \psi(x, t) \, dx \, dt + \sum_{i=1}^N \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) \, dx - \sum_{i=1}^N \int_{\Omega_i} \int_0^T \nabla \varphi_{i,n}(u_{i,n}(x, t)) \cdot \nabla \psi(x, t) \, dx \, dt = 0. \quad (17)$$

The proof of existence of a weak solution, given in [9], shows that there exists $C_1 > 0$ not depending on n such that, for all $i \in \llbracket 1, N \rrbracket$ and all n ,

$$\|\varphi_{i,n}(u_{i,n})\|_{L^2(0,T;H^1(\Omega_i))}^2 \leq C_1 \|\pi_{i,n}\|_{L^1(0,1)}; \quad (18)$$

thus $(\varphi_{i,n}(u_{i,n}))_n$ is a bounded sequence in $L^2(0, T; H^1(\Omega_i))$ by Lemma 3.2. A study of the proof of the time translate estimate used in [9, 15], and detailed in [17, Lemma 4.6], leads to the existence of C_2 not depending on n such that

$$\|\varphi_{i,n}(u_{i,n}(\cdot, \cdot + \tau)) - \varphi_{i,n}(u_{i,n}(\cdot, \cdot))\|_{L^2(\Omega_i \times]0, T-\tau])}^2 \leq \tau C_2 \|\pi_{i,n}\|_{L^1(0,1)} \|\varphi'_{i,n}\|_{L^\infty(0,1)}. \quad (19)$$

Lemma 3.2 and estimates (18), (19) allow us to apply Kolmogorov’s compactness criterion (see e.g. [8]) to deduce the relative compactness of the sequence $(\varphi_{i,n}(u_{i,n}))_n$ in $L^2(\Omega_i \times]0, T[)$. There exists $f_i \in L^2(0, T; H^1(\Omega_i))$ such that

$$\begin{aligned} \varphi_{i,n}(u_{i,n}) &\rightarrow f_i && \text{in } L^2(\Omega_i \times]0, T[), \\ \varphi_{i,n}(u_{i,n}) &\rightarrow f_i && \text{weakly in } L^2(0, T; H^1(\Omega_i)). \end{aligned}$$

Let us now recall a very useful lemma, classically called Minty’s trick, and introduced in this framework by Leray and Lions in the famous paper [25].

LEMMA 3.3 (Minty’s trick) Let $(\phi_n)_n : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of nondecreasing functions and let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing continuous function such that:

- $\phi_n \rightarrow \phi$ pointwise,
- there exists $g \in L^1_{\text{loc}}(\mathbb{R})$ such that $|\phi_n| \leq g$.

Let \mathcal{O} be an open subset of \mathbb{R}^k , $k \geq 1$. Let $(u_n)_n \in (L^\infty(\mathcal{O}))^\mathbb{N}$, let $u \in L^\infty(\mathcal{O})$ and let $f \in L^1(\mathcal{O})$ be such that:

- $u_n \rightarrow u$ in the $L^\infty(\mathcal{O})$ -weak* sense,
- $\phi_n(u_n) \rightarrow f$ in $L^1(\mathcal{O})$.

Then $f = \phi(u)$.

Since $0 \leq u_{i,n} \leq 1$, $(u_{i,n})_n$ converges up to a subsequence to u_i in the $L^\infty(\Omega_i \times]0, T[)$ -weak* sense. $(\varphi_{i,n})_n$ converges uniformly toward φ_i on $[0, 1]$, and we can easily check, using Minty's trick, that $f_i = \varphi_i(u_i) \in L^2(0, T; H^1(\Omega_i))$. Thus we can pass to the limit in the formulation (17) to obtain the required weak formulation:

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega_i} \int_0^T \phi_i u_i(x, t) \partial_t \psi(x, t) \, dx \, dt + \sum_{i=1}^N \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) \, dx \\ - \sum_{i=1}^N \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x, t)) \cdot \nabla \psi(x, t) \, dx \, dt = 0. \end{aligned}$$

To complete the proof of Theorem 3.1 we need the convergence of the traces of the approximate solutions $(u_{i,n})_n$ on $\Gamma_{i,j} \times]0, T[$ toward the trace of u_i , and to verify that $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times]0, T[$.

Since Ω_i has a Lipschitz boundary, there exists an operator P , continuous from $H^1(\Omega_i)$ into $H^1(\mathbb{R}^d)$, and also from $L^2(\Omega_i)$ into $L^2(\mathbb{R}^d)$, such that $Pv|_{\Omega_i} = v$ for all $v \in L^2(\Omega_i)$. Then P is continuous from $H^s(\Omega_i)$ into $H^s(\mathbb{R}^d)$ for all $s \in [0, 1]$. One has, for all $v \in H^s(\Omega_i)$,

$$\|v\|_{H^s(\Omega_i)} \leq \|Pv\|_{H^s(\mathbb{R}^d)} \leq \|Pv\|_{H^1(\mathbb{R}^d)}^s \|Pv\|_{L^2(\mathbb{R}^d)}^{1-s} \leq C \|v\|_{H^1(\Omega_i)}^s \|v\|_{L^2(\Omega_i)}^{1-s}.$$

One deduces from this inequality and from (19) that for all $s \in]0, 1[$ and $\tau \in]0, T[$, there exists C_3 not depending on n, τ such that

$$\|\varphi_{i,n}(u_{i,n}(\cdot, \cdot + \tau)) - \varphi_{i,n}(u_{i,n}(\cdot, \cdot))\|_{L^2(0, T-\tau; H^s(\Omega_i))}^2 \leq \tau^{1-s} C_3. \tag{20}$$

For $s_1 > s_2$, H^{s_1} is compactly imbedded in H^{s_2} , and so estimate (20) implies that the sequence $(\varphi_{i,n}(u_{i,n}))_n$ is relatively compact in $L^2(0, T; H^s(\Omega_i))$ for all $s \in]0, 1[$. In particular, one can extract a subsequence converging toward $\varphi_i(u_i)$ in $L^2(0, T; H^s(\Omega_i))$. We can claim, using Minty's trick once again, that the traces of $(\varphi_{i,n}(u_{i,n}))_n$ on $\Gamma_{i,j}$ also converge toward the trace of $\varphi_i(u_i)$, still denoted $\varphi_i(u_i)$, in $L^2(0, T; H^{s-1/2}(\Gamma_{i,j}))$, and in particular for almost every $(x, t) \in \Gamma_{i,j} \times]0, T[$. Since φ_i is increasing, $(u_{i,n}(x, t))_n$ converges almost everywhere on $\Gamma_{i,j} \times]0, T[$ toward $u_i(x, t)$.

Let us now check that $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times]0, T[$. For almost every $(x, t) \in \Gamma_{i,j} \times]0, T[$ the sequence $(\pi_{i,n}(u_{i,n}(x, t)))_n$ converges (up to a new subsequence) to $\gamma_i(x, t) \in \overline{\mathbb{R}}$. Since $\pi_{i,n}(u_{i,n}(x, t)) = \pi_{j,n}(u_{j,n}(x, t))$ for all n , one has

$$\gamma_i(x, t) = \gamma_j(x, t) \quad \text{a.e. on } \Gamma_{i,j} \times]0, T[. \tag{21}$$

If $u_i(x, t) \in]0, 1[$, then $\gamma_i(x, t) = \pi_i(u_i(x, t))$. If $u_i(x, t) = 0$, then $\gamma_i(x, t) \leq \alpha_i$, and $\gamma_i(x, t) \in \tilde{\pi}_i(0)$. In the same way, if $u_i(x, t) = 1$, then $\gamma_i(x, t) \in \tilde{\pi}_i(1)$.

This completes the proof of Theorem 3.1, because relation (21) ensures the connection of the traces in the sense of

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset \quad \text{a.e. on } \Gamma_{i,j} \times]0, T[. \quad \square$$

4. A regularity result

In this section and in Section 5, we show the existence and uniqueness of a solution with bounded flux to the problem (P) in the one-dimensional case. We give the proofs in the case where there

are only two subdomains $\Omega_1 =]-1, 0[$ and $\Omega_2 =]0, 1[$, but a straightforward adaptation gives the same result for any finite number of Ω_i , each of an arbitrary finite measure. We now state the main result of this section, which states the existence of a solution with bounded spatial derivatives on \mathcal{Q}_i , where $\mathcal{Q}_i = \Omega_i \times]0, T[$. We also set $\mathcal{Q} =]-1, 1[\times]0, T[$ and $\Gamma = \{x = 0\}$.

THEOREM 4.1 (Existence of a bounded flux solution) Let $u_0 \in L^\infty(-1, 1)$, $0 \leq u_0 \leq 1$, be such that

- $\varphi_i(u_0) \in W^{1,\infty}(\Omega_i)$,
- $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$ on Γ .

Then there exists a weak solution u to the problem (\mathcal{P}) such that $\partial_x \varphi_i(u_i) \in L^\infty(\mathcal{Q}_i)$.

All the section will be devoted to the proof of this theorem. As in Section 3, we will get this existence result by taking the limit of a sequence of solutions to approximate problems (16) involving no capillary barriers, whose data have the properties stated in Lemma 3.2.

Proof. We will build a sequence of approximate initial data $(u_{0,n})$ adapted to the sequence of approximate problems.

LEMMA 4.2 Let u_0 be as in Theorem 4.1. Then there exists $(u_{0,n})_n$ such that, for all n :

- $0 \leq u_{0,n} \leq 1$,
- $\pi_{1,n}(u_{0,n,1}) = \pi_{2,n}(u_{0,n,2})$ on Γ ,

and furthermore

$$\lim_{n \rightarrow \infty} \|u_{0,n} - u_0\|_\infty = 0, \quad \|\partial_x \varphi_{i,n}(u_{0,n})\|_{L^\infty(\Omega_i)} \leq \|\partial_x \varphi_i(u_0)\|_{L^\infty(\Omega_i)}. \quad (22)$$

Proof. Since $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, there exists $(a_{1,n}, a_{2,n}) \in [0, 1]^2$ such that $\pi_{1,n}(a_{1,n}) = \pi_{2,n}(a_{2,n})$ and $|a_{1,n} - u_{0,1}| + |a_{2,n} - u_{0,2}| \rightarrow 0$. We set, for $x \in \Omega_i$,

$$u_{0,n}(x) = \varphi_{i,n}^{-1}(T_{\varphi_i}[\varphi_i(u_0) + \varphi_{i,n}(a_{i,n}) - \varphi_i(u_{0,i})])$$

where

$$T_{\varphi_i}(s) = \begin{cases} s & \text{if } s \in [0, \varphi_i(1)] = [0, \varphi_{i,n}(1)], \\ \varphi_{i,n}(1) & \text{if } s > \varphi_i(1), \\ 0 & \text{if } s < 0. \end{cases}$$

Then the sequence $(u_{0,n})$ converges uniformly toward u_0 . Moreover, for all n , $0 \leq u_{0,n} \leq 1$ and either $\partial_x \varphi_{i,n}(u_{0,n}) = \partial_x \varphi_i(u_0)$, or $\partial_x \varphi_{i,n}(u_{0,n}) = 0$. \square

By [9], the approximate problem (16) admits a unique solution u_n which belongs to $C([0, T], L^1(\Omega))$. Now, in order to get an $L^\infty(\mathcal{Q}_i)$ -estimate of the sequence $(\partial_x \varphi_{i,n}(u_n))_n$, we introduce a new family of approximate problems (23) for which the spatial dependence of the data is smooth.

Let $\theta \in C^\infty(\mathbb{R})$ with $0 \leq \theta \leq 1$, $\theta(x) = 0$ if $x < -1$, and $\theta(x) = 1$ if $x > 1$. For all $k \in \mathbb{N}^*$, we set

$$\begin{aligned} \phi^k(x) &= (1 - \theta(kx))\phi_1 + \theta(kx)\phi_2, \\ \lambda_{n,k}(s, x) &= (1 - \theta(kx))\lambda_{1,n}(s) + \theta(kx)\lambda_{2,n}(s), \\ \pi_{n,k}(s, x) &= (1 - \theta(kx))\pi_{1,n}(s) + \theta(kx)\pi_{2,n}(s). \end{aligned}$$

We will now take a new approximation of the initial data. We set

$$u_{0,n,k}(x) = \begin{cases} u_{0,n} \left(\frac{k}{k-1} \left(x + \frac{1}{k} \right) \right) & \text{if } x < -1/k, \\ u_{0,n} \left(\frac{k}{k-1} \left(x - \frac{1}{k} \right) \right) & \text{if } x > 1/k. \end{cases}$$

In the layer $[-1/k, 1/k]$, $u_{0,n,k}$ is defined by the relation

$$(1 - \theta(kx))\pi_{1,n}(u_{0,n,k}(x)) + \theta(kx)\pi_{2,n}(u_{0,n,k}(x)) = \pi_{1,n}(a_{1,n}) = \pi_{2,n}(a_{2,n}),$$

so that the approximate capillary pressure $\pi_{n,k}(u_{0,n,k}, \cdot)$ is constant throughout the layer.

Moreover, one has either

$$\lambda_{n,k}(u_{0,n,k}, x)\partial_x(\pi_{n,k}(u_{0,n,k}, x)) = \frac{k}{k-1}\partial_x\varphi_{i,n}(u_{0,n}) \quad \text{if } |x| > 1/k,$$

or

$$\partial_x(\pi_{n,k}(u_{0,n,k}, x)) = 0 \quad \text{if } |x| < 1/k.$$

So from the definition of $u_{0,n,k}$ we directly deduce the following lemma:

LEMMA 4.3 Let $n \geq 1$ and $0 \leq u_{0,n} \leq 1$ with $\varphi_{i,n}(u_{0,n}) \in W^{1,\infty}(\Omega_i)$ and $\pi_{1,n}(u_{0,n,1}) = \pi_{2,n}(u_{0,n,2})$. Then there exists a sequence $(u_{0,n,k})_k$ such that, for all $k \geq 2$, $0 \leq u_{0,n,k} \leq 1$ and

$$\|\lambda_{n,k}(u_{0,n,k}, \cdot)\partial_x(\pi_{n,k}(u_{0,n,k}, \cdot))\|_\infty \leq 2 \max_{i=1,2} \|\partial_x\varphi_{i,n}(u_{0,n})\|_\infty,$$

and

$$u_{0,n,k} \rightarrow u_{0,n} \quad \text{in } L^1(\Omega) \text{ as } k \rightarrow \infty.$$

For any fixed $k \geq 2$ and n large enough, we can now introduce the smooth nondegenerate parabolic problem

$$\begin{cases} \phi^k(x)\partial_t u_{n,k} - \partial_x(\lambda_{n,k}(u_{n,k}, x)\partial_x\pi_{n,k}(u_{n,k}, x)) = 0, \\ \partial_x u_{n,k}(-1, t) = \partial_x u_{n,k}(1, t) = 0, \\ u_{n,k}(x, 0) = u_{0,n,k}(x). \end{cases} \quad (23)$$

Moreover, one can furthermore suppose, up to a new regularization, that $u_{0,n,k} \in C^\infty([-1, 1])$. Then (23) admits a unique strong solution $u_{n,k} \in C^\infty([0, T] \times [-1, 1])$ (see for instance [18, 24]).

Now one sets $f_{n,k}(x, t) = \lambda_{n,k}(u_{n,k}, x)\partial_x\pi_{n,k}(u_{n,k}, x)$, so the main equation of (23) can be rewritten

$$\phi^k \partial_t u_{n,k} = \partial_x f_{n,k}.$$

A short calculation shows that $f_{n,k}(x, t)$ is the solution of the problem

$$\begin{cases} \partial_t f_{n,k} = a_{n,k}\partial_{xx}^2 f_{n,k} + b_{n,k}\partial_x f_{n,k}, \\ f_{n,k}(-1, t) = f_{n,k}(1, t) = 0, \\ f_{n,k}(x, 0) = \lambda_{n,k}(u_{0,n,k}, \cdot)\partial_x(\pi_{n,k}(u_{0,n,k}, \cdot)), \end{cases} \quad (24)$$

where $a_{n,k}, b_{n,k}$ are the regular functions defined by

$$a_{n,k} = \lambda_{n,k}(u_{n,k}, x) \frac{(\pi_{n,k})'(u_{n,k}, x)}{\phi^k(x)} > 0,$$

$$b_{n,k} = (\lambda_{n,k})'(u_{n,k}, x) \frac{\partial_x [\pi_{n,k}(u_{n,k}, x)]}{\phi^k(x)} + \lambda_{n,k}(u_{n,k}, x) \partial_x \left[\frac{(\pi_{n,k})'(u_{n,k}, x)}{\phi^k(x)} \right].$$

The fact that $u_{0,n,k}$ is supposed to be regular allows us to write the problem (24) in a strong sense (this is necessary, because this problem cannot be written in a conservative form). In particular, $f_{n,k}$ satisfies the maximum principle, and thus

$$\|f_{n,k}\|_{L^\infty([-1, 1] \times]0, T])} \leq \|\lambda_{n,k}(u_{0,n,k}, \cdot) \partial_x (\pi_{n,k}(u_{0,n,k}, \cdot))\|_{L^\infty(-1, 1)}.$$

Thanks to Lemmas 4.3 and 4.2, we have a uniform bound on $(f_{n,k})$:

$$\|f_{n,k}\|_{L^\infty([-1, 1] \times]0, T])} \leq 2 \max_{i=1,2} \|\partial_x \varphi_i(u_0)\|_\infty. \tag{25}$$

Since the problem (23) is fully nondegenerate (recall that $\lambda_{i,n} > 1/2n^2$ and $\pi'_{i,n} \geq 1/n$) it follows that $\partial_x u_{n,k}$ and $\partial_t u_{n,k}$ are uniformly bounded respectively in $L^\infty(Q_i)$ and in $L^2(0, T; H^{-1}(\Omega_i))$ with respect to k ; then the sequence $(u_{n,k})_k$ converges toward u_n in $L^2(Q_i)$, and the limit u_n satisfies, thanks to the estimate (25),

$$\|\partial_x \varphi_{i,n}(u_n)\|_{L^\infty(Q_i)} \leq 2 \max_{i=1,2} \|\partial_x \varphi_i(u_0)\|_\infty. \tag{26}$$

For all $\psi \in \mathcal{D}([-1, 1] \times [0, T])$, one has

$$\int_0^T \int_{-1}^1 \phi^k u_{n,k} \partial_t \psi + \int_{-1}^1 \phi^k u_{0,n}^k \psi_0 - \int_0^T \int_{-1}^1 f_{n,k} \partial_x \psi = 0. \tag{27}$$

Thanks to (25),

$$\lim_{k \rightarrow \infty} \int_0^T \int_{-1/k}^{1/k} f_{n,k} \partial_x \psi = 0.$$

One has $u_{n,k} \rightarrow u_n$ in the $L^\infty(Q)$ -weak* and $L^2(Q)$ senses, and $u_{0,n,k} \rightarrow u_{0,n}$ in $L^1(-1, 1)$ thanks to Lemma 4.3. Moreover, by (25), $\partial_x \pi_{i,n,k}(u_{n,k}) \rightarrow \partial_x \pi_{i,n}(u_{n,k})$ in the $L^\infty(Q)$ -weak* sense. Thus we can let k tend toward ∞ in (27) to get

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u_n \partial_t \psi + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_{0,n} \psi_0 - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \lambda_{i,n}(u_n) \partial_x \pi_{i,n}(u_n) \partial_x \psi = 0. \tag{28}$$

Furthermore, using the fact that $\pi_{n,k}(u_{n,k}, x)$ belongs to $L^2(0, T; H^1(\Omega))$ and, even more, that $\partial_x (\pi_{n,k}(u_{n,k}, x))$ is bounded uniformly in k , we can claim that $\pi_{1,n}(u_{1,n}) = \pi_{2,n}(u_{2,n})$, and so u_n is the unique weak solution to the approximate problem (16) with initial data $u_{0,n}$.

When n tends toward ∞ , the sequence $(u_n)_n$ converges, up to a subsequence, toward a weak solution to the problem (\mathcal{P}) , as seen in Section 3, and the estimate (26) ensures that

$$\partial_x \varphi_i(u) \in L^\infty(Q_i).$$

This completes the proof of Theorem 4.1. □

5. A uniqueness result

In this section, we give a uniqueness result in the one-dimensional case in a framework where the existence results are stronger than the general existence result stated in Theorem 3.1. Under a regularity assumption on the initial data u_0 , we have proved in Section 4 the existence of a solution having bounded flux, for which we give a uniqueness result in Theorem 5.1 and Corollary 5.2. The bound on the flux will be necessary to prove that the contraction property is also available in the neighbourhood of the interface $\{x = 0\}$. Then we show in Theorem 5.4 the existence and uniqueness of the weak solution which is the limit of bounded flux solutions for any initial data u_0 with $0 \leq u_0 \leq 1$. Indeed, the set of initial data giving a bounded flux solution is dense in $L^\infty(\Omega)$ for the $L^1(\Omega)$ topology, and Theorem 5.1 implies that the contraction property can be extended to a larger class of solutions, defined for all initial data in $L^\infty(\Omega)$. Unfortunately, we can only characterize them by a limit of bounded flux solutions, and we cannot exhibit a weak solution which is not the limit of bounded flux solutions.

THEOREM 5.1 (L^1 -contraction principle for bounded flux solutions) Let u, v be two weak solutions to the problem (\mathcal{P}) for the initial data u_0, v_0 . Then, if $\partial_x \phi_i(u_i)$ and $\partial_x \phi_i(v_i)$ belong to $L^\infty(\mathcal{Q}_i)$, we have the following L^1 -contraction principle: for all $t \in [0, T]$,

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, t))^\pm dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^\pm dx. \tag{29}$$

The first part of this section is devoted to the proof of Theorem 5.1, which, with Theorem 4.1, admits the following straightforward consequence:

COROLLARY 5.2 (Uniqueness of the bounded flux solution) For all $u_0 \in L^\infty(-1, 1)$ with $0 \leq u_0 \leq 1$ such that $\phi_i(u_0) \in W^{1,\infty}(\Omega_i)$ for $i = 1, 2$ and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, there exists a unique weak solution to the problem (\mathcal{P}) in the sense of Definition 2.1 and such that $\partial_x \phi_i(u) \in L^\infty(\mathcal{Q}_i)$; moreover, $u \in C([0, T], L^p(\Omega))$ for all $1 \leq p < \infty$.

Proof. The proof of Theorem 5.1 is based on entropy inequalities, obtained through the method of doubling variables, first introduced by S. Kruřkov [23] for first order equations, and then adapted by J. Carrillo [11] to degenerate parabolic problems. Note that in the present setting, we only need doubling with respect to the time variable, as, for instance, in F. Otto [27] for elliptic-parabolic problems (or in [7] for Stefan-type problems).

We will only give the comparison

$$\sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, t))^+ dx \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ dx.$$

The comparison with $(\cdot)^-$ instead of $(\cdot)^+$ can be proven exactly the same way.

Let u be a bounded flux solution to the one-dimensional problem, i.e. $\partial_x \phi_i(u) \in L^\infty(\mathcal{Q}_i)$, $i = 1, 2$. The weak formulation of Definition 2.1 adapted to the one-dimensional framework of this section can be rewritten, for all $\psi \in \mathcal{D}(\overline{\Omega}) \times [0, T]$,

$$\begin{aligned} \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u(x, t) \partial_t \psi(x, t) dx dt + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) dx \\ - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x \phi_i(u)(x, t) \partial_x \psi(x, t) dx dt = 0. \end{aligned} \tag{30}$$

This formulation clearly implies, for $i = 1, 2$, and all $\psi \in C_c^\infty(\overline{\Omega}_i \times [0, T[)$ with $\psi(0, t) = 0$,

$$\begin{aligned} \int_0^T \int_{\Omega_i} \phi_i u(x, t) \partial_t \psi(x, t) \, dx \, dt + \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) \, dx \\ - \int_0^T \int_{\Omega_i} \partial_x \phi_i(u)(x, t) \partial_x \psi(x, t) \, dx \, dt = 0. \end{aligned} \quad (31)$$

Classical computations (see e.g. [7, 11, 27]) on equation (31) lead to the following entropy inequalities: for all weak solutions u, v , initial data u_0, v_0 , and $\xi \in \mathcal{D}^+(\overline{\Omega}_i \times [0, T[\times [0, T[)$ such that $\xi(0, t, s) = 0$,

$$\begin{aligned} \int_0^T \int_0^T \int_{\Omega_i} \phi_i(u(x, t) - v(x, s))^+ (\partial_t \xi(x, t, s) + \partial_s \xi(x, t, s)) \, dx \, dt \, ds \\ + \int_0^T \int_{\Omega_i} \phi_i(u_0(x) - v(x, s))^+ \xi(x, 0, s) \, dx \, ds \\ + \int_0^T \int_{\Omega_i} \phi_i(u(x, t) - v_0(x))^+ \xi(x, t, 0) \, dx \, dt \\ - \int_0^T \int_0^T \int_{\Omega_i} \partial_x (\phi_i(u)(x, t) - \phi_i(v)(x, s))^+ \partial_x \xi(x, t, s) \, dx \, dt \, ds \geq 0. \end{aligned} \quad (32)$$

Let us note here an important consequence of the entropy inequality (32) (and of the corresponding one for $(u - v)^-$), namely that u can be proved to satisfy

$$\text{ess-lim}_{t \rightarrow 0} \int_{\Omega_i} |u(x, t) - u_0(x)| \, dx = 0. \quad (33)$$

Indeed, this follows by taking v to be a constant in (32) and using an approximation argument (see e.g. Lemma 7.41 in [26]). We deduce the time continuity at $t = 0$ for any solution and in particular for both u and v taken above.

Now, let $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\text{supp}(\rho) \subset [-1, 1]$ and $\int_{\mathbb{R}} \rho(t) \, dt = 1$. Define $\rho_m(t) = m\rho(mt)$. Let $\psi \in \mathcal{D}^+([-1, 1] \times [0, T[)$ with $\psi(0, \cdot) = 0$. For m large enough, $\xi(x, t, s) = \psi(x, t) \rho_m(t - s)$ belongs to $\mathcal{D}^+([-1, 1] \times [0, T[\times [0, T[)$, and we can take it as a test function in (32). Then summing on $i = 1, 2$ leads to

$$\begin{aligned} \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, s))^+ \partial_t \psi(x, t) \rho_m(t - s) \, dx \, dt \, ds \\ + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v(x, s))^+ \psi(x, 0) \rho_m(-s) \, dx \, ds \\ + \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v_0(x))^+ \psi(x, t) \rho_m(t) \, dx \, dt \\ - \int_0^T \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x (\phi_i(u)(x, t) - \phi_i(v)(x, s))^+ \partial_x \psi(x, t) \rho_m(t - s) \, dx \, dt \, ds \geq 0. \end{aligned} \quad (34)$$

We can now let m tend toward ∞ in (34), and using (33) for u and v , and the theorem of continuity in the mean, we get: for all $\psi \in \mathcal{D}^+(\overline{\Omega} \times [0, T])$ such that $\psi(0, t) = 0$,

$$\begin{aligned} & \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, t))^+ \partial_t \psi(x, t) \, dx \, dt \\ & \quad + \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ \psi(x, 0) \, dx \\ & \quad - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \partial_x (\phi_i(u)(x, t) - \phi_i(v)(x, t))^+ \partial_x \psi(x, t) \, dx \, dt \geq 0. \end{aligned} \quad (35)$$

We now aim to extend the inequality (35) to the case where $\psi(0, t) \neq 0$, and in particular to the case $\psi(x, t) = \theta(t)$, so that the third term disappears in (35).

To this end, let $u_i(t) = u_i(0, t)$ denote the trace of u_i at the interface Γ (and correspondingly, $v_i(t) = v_i(0, t)$). We introduce the following subsets of $]0, T[$:

$$\begin{aligned} E_{u>v} &= \{t \in [0, T] \mid u_1(t) > v_1(t) \text{ or } u_2(t) > v_2(t)\}, \\ E_{u\leq v} &= \{t \in [0, T] \mid u_1(t) \leq v_1(t) \text{ and } u_2(t) \leq v_2(t)\}, \end{aligned}$$

so that $E_{u\leq v}$ is the complement of $E_{u>v}$ in $[0, T]$.

For all $\varepsilon > 0$, one defines $\psi_\varepsilon(x) = \max(1 - |x|/\varepsilon, 0)$. For all $\theta \in \mathcal{D}^+([0, T])$, we take $(x, t) \mapsto \theta(t)(1 - \psi_\varepsilon(x))$ instead of $\psi(x, t)$ as a test function in (35), to get

$$\begin{aligned} & \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, t))^+ \partial_t \theta(t)(1 - \psi_\varepsilon(x)) \, dx \, dt \\ & \quad + \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ (1 - \psi_\varepsilon(x)) \theta(0) \, dx \\ & \quad - \int_0^T \frac{\theta(t)}{\varepsilon} \left((\varphi_1(u)(-\varepsilon, t) - \varphi_1(v)(-\varepsilon, t))^+ - (\varphi_1(u_1)(t) - \varphi_1(v_1)(t))^+ \right. \\ & \quad \quad \left. + (\varphi_2(u)(\varepsilon, t) - \varphi_2(v)(\varepsilon, t))^+ - (\varphi_2(u_2)(t) - \varphi_2(v_2)(t))^+ \right) dt \geq 0. \end{aligned}$$

For almost every $t \in E_{u\leq v}$, the function $(\phi_i(u) - \phi_i(v))^+(\cdot, t)$ admits a nil trace on $\{x = 0\}$, thus the third term in the previous inequality can be reduced to the set $E_{u>v}$, yielding

$$\begin{aligned} & \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, t))^+ \partial_t \theta(t)(1 - \psi_\varepsilon(x)) \, dx \, dt \\ & \quad + \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ (1 - \psi_\varepsilon(x)) \theta(0) \, dx \\ & \quad + \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x (\phi_i(u)(x, t) - \phi_i(v)(x, t))^+ \partial_x \psi_\varepsilon(x) \, dx \, dt \geq 0. \end{aligned} \quad (36)$$

We now turn to the crucial point of the uniqueness proof, which is the following lemma.

LEMMA 5.3 For all $\theta \in \mathcal{D}^+([0, T])$, if u, v are both bounded flux solutions, that is, $\partial_x \varphi_i(u), \partial_x \varphi_i(v) \in L^\infty(\mathcal{Q}_i)$, then

$$\limsup_{\varepsilon \rightarrow 0} \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x, t) - \varphi_i(v)(x, t))^+ \partial_x \psi_\varepsilon(x) \, dx \, dt \leq 0.$$

Using the weak formulation (30), we can claim that for any $\vartheta \in \mathcal{D}([0, T])$,

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \vartheta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u) - \varphi_i(v)) \partial_x \psi_\varepsilon(x) \, dx \, dt = 0. \tag{37}$$

Since $\partial_x(\varphi_i(u) - \varphi_i(v))$ belongs to $L^\infty(\Omega_i \times]0, T])$ for $i = 1, 2$, one has

$$\left| \int_0^T \vartheta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u) - \varphi_i(v)) \partial_x \psi_\varepsilon(x) \, dx \, dt \right| \leq C \|\vartheta\|_{L^1(0, T)}.$$

Then a density argument shows that (37) still holds for any $\vartheta \in L^1(0, T)$, and in particular for $\vartheta(t) = \theta(t) \mathbb{1}_{E_{u>v}}(t)$. Thus there exists $A(\varepsilon)$ tending to 0 as ε tends to 0 such that

$$\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x, t) - \varphi_i(v)(x, t)) \partial_x \psi_\varepsilon(x) \, dx \, dt = A(\varepsilon). \tag{38}$$

If we split up $\varphi_i(u)(x, t) - \varphi_i(v)(x, t)$ into the positive and negative parts, (38) becomes

$$\begin{aligned} & \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x, t) - \varphi_i(v)(x, t))^+ \partial_x \psi_\varepsilon(x) \, dx \, dt \\ &= \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x, t) - \varphi_i(v)(x, t))^- \partial_x \psi_\varepsilon(x) \, dx \, dt + A(\varepsilon). \end{aligned} \tag{39}$$

It is at this point that we actually use the monotonicity of the transmission condition, i.e. condition 3 in Definition 2.1. Indeed, the conditions $\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset$ and $\tilde{\pi}_1(v_1(t)) \cap \tilde{\pi}_2(v_2(t)) \neq \emptyset$ ensure that

$$u_1 > v_1 \Rightarrow u_2 \geq v_2 \quad \text{and} \quad u_1 < v_1 \Rightarrow u_2 \leq v_2. \tag{40}$$

Therefore, recalling the definition of the set $E_{u>v}$ and of ψ_ε , the first term in the right member of (39) is nonpositive, and we conclude

$$\limsup_{\varepsilon \rightarrow 0} \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x, t) - \varphi_i(v)(x, t))^+ \partial_x \psi_\varepsilon(x) \, dx \, dt \leq 0.$$

This completes the proof of Lemma 5.3, and allows us to let $\varepsilon \rightarrow 0$ in inequality (36). Then for all $\psi \in \mathcal{D}^+([0, T])$, one gets

$$- \int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x, t) - v(x, t))^+ \partial_t \psi(t) \, dx \, dt \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ \psi(0) \, dx. \tag{41}$$

One can also prove in exactly the same way that

$$-\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^- \partial_t \psi(t) \, dx \, dt \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^- \psi(0) \, dx. \quad (42)$$

These inequalities still hold for $\psi = T - t$, and then if $u_0 = v_0$, one has $u = v$ almost everywhere in \mathcal{Q} . Moreover, we can take $\psi(t) = \mathbb{1}_{[0,s]}(t)$ as a test function in (41) to get the L^1 -contraction principle (29) stated in Theorem 5.1. \square

In the following, we prove that for any u_0 in $L^\infty(-1, 1)$ with $0 \leq u_0 \leq 1$, there exists a unique weak solution of the problem (\mathcal{P}) which is the limit of a sequence $(u_n)_n$ of bounded flux solutions, i.e. $\partial_x \phi_i(u_n) \in L^\infty(\mathcal{Q}_i)$ for all $n \geq 1$.

THEOREM 5.4 (Existence and uniqueness of the SOLA) Let $u_0 \in L^\infty(-1, 1)$, $0 \leq u_0 \leq 1$, and let $(u_{0,n})_{n \geq 1}$ be a sequence of bounded flux initial data, i.e. for all $n \geq 1$,

- $0 \leq u_{0,n} \leq 1$,
- $\phi_i(u_{0,n}) \in W^{1,\infty}(\Omega_i)$,
- $\tilde{\pi}_1(u_{0,n,1}) \cap \tilde{\pi}_2(u_{0,n,2}) \neq \emptyset$,

such that

$$\lim_{n \rightarrow \infty} \|u_{0,n} - u_0\|_{L^1(\Omega)} = 0.$$

Let $(u_n)_{n \geq 1}$ be the sequence of the bounded flux solutions to the problem (\mathcal{P}) for $u_{0,n}$ as initial data. Then the sequence $(u_n)_{n \geq 1}$ converges toward u in $C([0, T], L^p(-1, 1))$, $1 \leq p < \infty$, where u is a solution to the problem (\mathcal{P}) , called a Solution Obtained as the Limit of Approximations (SOLA). Furthermore, if u, v are two SOLAs, for initial data u_0, v_0 , one has the following L^1 -contraction principle: for all $t \in [0, T]$,

$$\sum_{i=1}^N \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^\pm \, dx \leq \sum_{i=1}^N \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^\pm \, dx. \quad (43)$$

This in particular leads to the uniqueness of the SOLA.

Proof. Let $(u_{0,n})$ be a regular sequence of initial data converging toward u_0 in $L^1(-1, 1)$; one can take e.g. $u_{0,n} \in C_c^\infty(-1, 0[\cup]0, 1[)$. Then $(u_{0,n})$ is a Cauchy sequence, and thanks to (29), for all $t \in [0, T]$,

$$\sum_{i=1}^N \int_{\Omega_i} \phi_i |u_n(x,t) - u_m(x,t)| \, dx \leq \sum_{i=1}^N \int_{\Omega_i} \phi_i |u_{0,n}(x) - u_{0,m}(x)| \, dx.$$

Thus $(u_n)_n$ is a Cauchy sequence in $C([0, T], L^1(\Omega))$ and converges to a function u in $C([0, T], L^1(\Omega))$. Since $(u_n)_n$ is bounded in $L^\infty(\mathcal{Q})$, one has $u_n \rightarrow u$ in $C([0, T], L^p(-1, 1))$.

We now have to check that u is a weak solution to the problem (\mathcal{P}) . It is easy to check, using the L^∞ -bound of u_n , that $\phi_i(u_n)$ tends toward $\phi_i(u)$ in $L^p(\Omega_i \times]0, T[)$ for all $p \in [1, \infty[$. Thanks to (18), the sequence $(\phi_i(u_n))_n$ is bounded in $L^2(0, T; H^1(\Omega_i))$, and thus $\phi_i(u_n) \rightarrow \phi_i(u)$ weakly in $L^2(0, T; H^1(\Omega_i))$, and $\phi_i(u_n)$ converges in $L^2(0, T; H^s(\Omega_i))$, for all $s \in]0, 1[$, still toward

$\varphi_i(u)$. In particular, $u_{n,i}(t)$ tends toward $u_i(t)$. Since the set $\{(a, b) \in [0, 1]^2 \mid \tilde{\pi}_1(a) \cap \tilde{\pi}_2(b) \neq \emptyset\}$ is closed, we can claim that

$$\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset \quad \text{for a.e. } t \in [0, T].$$

We can also pass to the limit in the weak formulation in order to conclude that u is a weak solution to the problem (\mathcal{P}) , proving the existence of a SOLA u .

Let now v be another SOLA, obtained through a sequence $(v_{0,n})_n$ of regular initial data converging toward v_0 . Thanks to (29), one has

$$\sum_{i=1}^N \int_{\Omega_i} \phi_i |u_n(x, t) - v_n(x, t)| \, dx \leq \sum_{i=1}^N \int_{\Omega_i} \phi_i |u_{0,n}(x) - v_{0,n}(x)| \, dx,$$

whose limit as n tends toward ∞ gives the desired L^1 -contraction principle:

$$\sum_{i=1}^N \int_{\Omega_i} \phi_i |u(x, t) - v(x, t)| \, dx \leq \sum_{i=1}^N \int_{\Omega_i} \phi_i |u_0(x) - v_0(x)| \, dx,$$

and so the uniqueness of the SOLA, completing the proof of Theorem 5.4. \square

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