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Linear stability of selfsimilar solutions of unstable thin-film equations

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We study the linear stability of selfsimilar solutions of long-wave unstable thin-film equations with power-law nonlinearities

$$u_t = -(u^n u_{xxx} + u^m u_x)_x \qquad \text{for } 0 < n < 3,\ n \leqslant m.$$

Steady states, which exist for all values of m and n above, are shown to be stable if $m \leqslant n+2$ when $0 < n \leqslant 2$, marginally stable if $m \leqslant n+2$ when $2 < n < 3$, and unstable otherwise. Dynamical selfsimilar solutions are known to exist for a range of values of n when $m = n+2$. We carry out the analysis of the stability of these solutions when $n = 1$ and $m = 3$. Spreading selfsimilar solutions are proven to be stable. Selfsimilar blowup solutions with a single local maximum are proven to be stable, while selfsimilar blowup solutions with more than one local maximum are shown to be unstable.

The equations above are gradient flows of a nonconvex energy on formal infinite-dimensional manifolds. In the special case $n = 1$ the equations are gradient flows with respect to the Wasserstein metric. The geometric structure of the equations plays an important role in the analysis and provides a natural way to approach a family of linear stability problems.

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Thin-film equations model the evolution of a thin layer of viscous fluid on a solid substrate. They are derived from the Navier–Stokes equations in the limit of low Reynolds number, assuming the separation of the horizontal and the vertical length scales—the so called lubrication approximation. Overviews on thin-film equations and further references can be found in [6] and [26]. The general form of a thin-film equation is

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u + g(u)\nabla u) \tag{1}$$

where u is the height of the fluid. The leading order term, containing Δu, describes the effects of surface tension. The particular form of the function $f(u)$ depends on the boundary condition between the fluid and the substrate. In particular, $f(u) = u^3$ models the no-slip boundary condition. The equations with $f(u) = u$ are obtained in lubrication approximation of a Hele–Shaw cell [1], [16] in the case that the friction between the side-wall and the fluid is negligible. The term containing $g(u)$ models the effects of additional forces acting on the fluid, such as gravity or intermolecular forces (e.g. van der Waals forces). We consider these equations in the regime of complete wetting in which the angle at the edge of the support of the liquid (i.e. the contact angle) is zero.

In this paper we consider one-dimensional thin-film equations with power-law nonlinearities and destabilizing lower-order terms:

$$u_t = -(u^n u_{xxx} + u^m u_x)_x, \qquad x \in \mathbb{R}, \tag{2}$$

where $0 < n < 3$ and $n \leqslant m$. The existence of weak solutions was studied by Bertozzi and Pugh [7, 8]. The condition $0 < n < 3$ is needed for the existence of compactly supported solutions even if the destabilizing term is not present, due to the so called contact line singularity. The condition $n \leqslant m$ is also required for the existence due to considerations near the contact line [7, Thm. 4.4]. For that reason we only consider the powers $m \geqslant n$. Bertozzi and Pugh have shown that the power $m = n + 2$ is critical in the sense that for $m < n + 2$ there exist global in time solutions for initial data in

$$X := \{u \in H^1 \cap L^1 \mid u \geqslant 0\},$$

while for $m \geqslant n + 2$ finite-time blowup is possible. The existence of solutions that blow up in finite time has been shown in the special cases $n = 1$ and $m \geqslant 3$ in [7] and for $0 < n < 3/2$ and $m = n + 2$ in [29].

Structurally these equations represent a higher-order analogue of the second-order nonlinear heat equations with destabilizing lower-order terms. Furthermore, the equations have strong structural analogies with the nonlinear Schrödinger equations with destabilizing lower order terms. The blowup of these equations has been studied extensively. Details and further references can be found in [12], [13], [14], [17], [25], and [30]. Blowup in related fourth-order parabolic equations has also been investigated in [2], [5], [9], [34], and [35]. For a large family of equations blowup generically occurs at a point as the solution grows and focuses at the point. The blowup is often, at least locally, selfsimilar. In the case that the equation is invariant under appropriate scaling it can have global in space selfsimilar blowup solutions that, if stable, govern the local structure of the blowup.

In the critical case, $m = n+2$, the scaling of the equation (2) suggests that there exist dynamical selfsimilar solutions. Beretta [3] has shown that there exist source-type (spreading) selfsimilar solutions for $0 < n < 3$, and that there are no source-type selfsimilar solutions when $n \geqslant 3$. Selfsimilar blowup solutions were considered in [29]. Their existence was shown for $0 < n < 3/2$ and nonexistence for $n \geqslant 3/2$. It was shown that while spreading selfsimilar solutions have only one local maximum, the blowup selfsimilar solutions can have one or more local maxima. The solutions with one maximum are called the *single-bump* solutions while the solutions with more than one local maximum are called the *multi-bump* solutions.

The critical case $n = 1$ and $m = 3$:

$$u_t = -(u u_{xxx} + u^3 u_x)_x \tag{3}$$

was investigated by Witelski, Bernoff, and Bertozzi [33]. By numerically computing the spectrum of the linearized operator they have demonstrated that spreading selfsimilar solutions are linearly stable as are the single-bump blowup selfsimilar solutions. On the other hand, the multi-bump selfsimilar solutions were demonstrated to be linearly unstable. Nonlinear stability was also apparent from numerical simulations of the Cauchy problem.

Our aim is to verify the linear stability analysis rigorously. In the case of dynamical selfsimilar solutions we rescale the equation spatially and temporally so that the stability analysis of the blowup solutions reduces to the stability analysis of the steady states of the rescaled equation. An important element of this work is in the viewpoint we take. Namely the original equation and the rescaled equation if $n = 1$ have the formal structure of a gradient flow on an infinite-dimensional manifold. Using the gradient-flow structure of the equation suggests the space in which to consider the linearized equation, in particular the one in which the linearized operator is symmetric. This observation follows from the fact that the linearization of a gradient flow at a steady state is determined by the Hessian of the energy, which is always a symmetric operator. Furthermore, the structure of the metric on the manifold suggests a natural choice of coordinates, in which the operator becomes more transparent.

We recall facts about the selfsimilar solutions in Section 1. The gradient-flow structure of the equations is described in Section 2. Stability analysis of steady states is carried out in Section 3. We prove that the steady states are linearly stable when $0 < n \leqslant 2$ and $n+2 \geqslant m \geqslant n$. That is, we show that, modulo zero eigenvalue(s) which correspond to the invariance(s) of the equation, the spectrum of the linearized operator has a positive lower bound. The invariances mentioned are translations, and in the critical case $m = n + 2$ also dilations. When $2 < n < 3$ and $n \leqslant m \leqslant n + 2$ the steady states are shown to be marginally stable, but not stable in the above sense. That is, if $2 < n < 5/2$, modulo translations (and dilations if $m = n+2$) the spectrum is nonnegative but there is no positive lower bound. If $n \geqslant 5/2$ the steady states are again marginally stable, but translations and dilations are no longer allowed perturbations, that is, they are not in the domain of the linearized operator. This is not surprising since as n increases moving the contact line requires greater energy dissipation. In particular, when $n > 5/2$ translating the steady state (even in an arbitrarily small neighborhood of the contact line) would require infinite energy-dissipation rate (11).

Let us also remark here that power $n = 5/2$ is not universal in the sense that not all compactly supported functions with zero contact angle would require infinite energy dissipation to move. The critical power depends on the rate at which a solution touches down. Steady states that we consider touch down quadratically, while for example selfsimilar solutions of the thin-film equation without destabilizing terms touch down like $(L - x)^{3/n}$ [4], where L is the touchdown point. The critical power for such touchdown is $n = 3$.

Steady states are shown to be linearly unstable in the supercritical case $m > n + 2$. The stability analysis of the selfsimilar blowup solutions and the selfsimilar source type is done in Section 4. In the Appendix we prove some facts about the weighted Sobolev space we use, in particular Hardy-type inequalities with constraints, needed for establishing the positive lower bound on the spectrum. Let us also point out that a careful analysis of the linearized operator for a thin-film equation using weighted spaces was recently carried out by Giacomelli, Knüpfer and Otto [15].

1. Steady states, selfsimilar solutions and similarity variables

Steady states of equation (2), both on bounded domains and on $\mathbb{R}$, were investigated in a series of papers by Laugesen and Pugh [20–23]. The simplest steady states on $\mathbb{R}$ with finite mass are the so called *droplet steady states*, which have connected, compact support, have one local maximum and are symmetric. By combining (adding) several droplet steady states with disjoint supports one can obtain more complex steady states, the so called *droplet configurations*.

When $m \neq n+2$ there are droplet steady states of any mass (L^1-norm), while in the critical case $m = n + 2$ all droplet steady states have the same mass, $M_c = 2\pi\sqrt{2/3}$. The value of the constant M_c was determined in [33].

To consider dynamical selfsimilar solutions, observe that the equation (2) is invariant under the scaling $x \to \lambda x$, $u \to u/\lambda$ and $t \to \lambda^{n+4} t$ when $m = n + 2$. If $m \neq n + 2$ there is no such scaling invariance. The scaling suggests that the equation in the critical case could possess solutions of the form

$$u(x,t) = \lambda(t)\rho(\lambda(t)x).$$

These solutions, should they exist, are called *selfsimilar solutions*. Substituting in the equation yields

$$\lambda'(t) = \sigma\lambda^{n+5}(t) \tag{4}$$

for some constant σ. This implies

$$\lambda(t) = (\lambda(0)^{-(n+4)} - (n+4)\sigma t)^{-1/(n+4)}. \tag{5}$$

Spreading selfsimilar solutions. Setting $\sigma < 0$ asks for a solution that is spreading and exists for all $t > 0$. It is convenient to set $\lambda(0) = 1$ and normalize the solution by setting $\sigma := -1/(n+4)$. One should note that had we picked $\lambda(0) = 0$ we would indeed get a true source type selfsimilar solution, with delta mass initial data.

The function ρ is called the *similarity profile* and satisfies an ODE (with σ as a parameter). Propertie, and existence of solutions of the appropriate ODE were studied by Beretta [3]. It was shown that there exists one family of source-type selfsimilar solutions, these solutions are even, have one local maximum (at zero), and have compact support.

To study their stability we introduce, as is customary, a time-dependent rescaling (change of variables) that transforms the selfsimilar solutions into steady states (of a new equation). In particular, looking for a substitution that agrees with the scaling above: $u(x,t) = \lambda(t)v(\lambda(t)x, s(t))$ one obtains

$$\lambda'(t)(yv(y,s))_y + \lambda(t)s'(t)v_s(y,s) = -\lambda(t)^{n+5}(v(y,s)^n v_{yyy}(y,s) + v(y,s)^m v_y(y,s))_y$$

where $y = \lambda(t)x$ and $s = s(t)$. To eliminate the time-dependent factors we use (4) and require $\lambda(t)s'(t) = (1/\sigma)\lambda'(t)$. Using $\lambda(t)$ given by (5) with $\lambda(0) = 1$ and integrating gives $s(t) = \ln(1+t)$. Consequently,

$$v(y,s) = (e^s - 1)^{-1/(n+4)} u((e^s - 1)^{-1/(n+4)} y, e^s - 1) \tag{6}$$

and v is a solution of

$$v_s = -\left(v^n v_{yyy} + v^{n+2} v_y - \frac{1}{n+4} yv\right)_y \tag{7}$$

with the same initial data as u. Note that $v(y,s) = \rho(y)$ is a steady state of the equation.

Focusing selfsimilar solutions. Setting $\sigma > 0$ asks for a focusing solution that blows up in finite time. In this case we normalize the solution by setting $\lambda(0) := 1$ and $\sigma := 1/(n+4)$, which leads to solution that blows up at time $t = 1$.

In contrast to selfsimilar spreading solutions, in addition to selfsimilar blowup solutions with a single maximum there exist solutions with any number of local maxima [29]. The supports of solutions are again compact and connected.

Substituting as before we obtain $s(t) = -\ln(1-t)$ and

$$v(y,s) = (1 - e^{-s})^{-1/(n+4)} u((1 - e^{-s})^{-1/(n+4)} y, 1 - e^{-s}), \tag{8}$$

which satisfies the equation

$$v_s = -\left(v^n v_{yyy} + v^{n+2} v_y + \frac{1}{n+4}\, yv\right)_y \tag{9}$$

with the same initial data as u.

2. Thin-film equations as gradient flows

The thin-film equations (2) possess the structure of a gradient flow with respect to a weighted negative norm. We first introduce the energy and present some of its properties: For $m - n \notin \{-1, -2\}$,

$$E(u) := \int_{\mathbb{R}} \left(\frac{1}{2} u_x^2 - \frac{1}{(m-n+2)(m-n+1)} u^{m-n+2} \right) \mathrm{d}x \tag{10}$$

for $u \in X$. Otherwise, if $m - n = -2$ then the second term of the energy is $-\ln u$ and if $m - n = -1$ then it is $u \ln u - u$. Since almost all of the paper is devoted to the case $m \geqslant n$ these special cases will not play a role.

Let us first note that the energy E is a dissipated quantity of the evolution with the dissipation rate $\mathrm{d}E/\mathrm{d}t = -D$:

$$D = \int_{\mathbb{R}} \frac{1}{u^n(x)} j(x)^2 \, \mathrm{d}x \tag{11}$$

where j is the flux: $j = u^n u_{xxx} + u^m u_x$.

The energy E plays a crucial role in current existence theory for the equation. Namely, as was shown in [7], energy E and mass produce a bound on the H^1-norm of the solution when $m < n + 2$ via a Gagliardo–Nirenberg inequality:

$$\|u\|_{L^p} \leqslant C \|u_x\|_{L^2}^{2(p-1)/3p} \|u\|_{L^1}^{(2+p)/3p} \qquad \text{for } p > 1.$$

In the critical case $m = n + 2$ the sharp constant in the inequality was determined by Sz.-Nagy [31]. The inequality becomes

$$\int_{\mathbb{R}} u^4 \, \mathrm{d}x \leqslant \frac{6}{M_c^2} \left(\int_{\mathbb{R}} |u| \, \mathrm{d}x \right)^2 \int_{\mathbb{R}} u_x^2 \, \mathrm{d}x.$$

For solutions of (2) with $m = n + 2$ this implies

$$\frac{1}{2} \left(1 - \frac{(\int u)^2}{M_c^2} \right) \int u_x^2 \, \mathrm{d}x \leqslant E(u(t)) \leqslant E(u_0).$$

So if the initial mass is less than the mass of a droplet steady state then the solution exists for all time.

The equations in similarity variables are gradient flows with respect to the same metric as the original equation only when $n = 1$. We should point out that if $n \neq 1$ then no gradient-flow structure is known for the rescaled equation. In the $n = 1$ case the energy for the rescaled equation has the form

$$E_{\pm}(v) := \int_{\mathbb{R}} \left(\frac{1}{2} v_y^2 - \frac{1}{12} v^4 \pm \frac{1}{10} y^2 v \right) \mathrm{d}y \tag{12}$$

where $+$ is taken for the spreading problem and $-$ for the focusing. The energy is defined for $v \in Y := X \cap \{v \mid \int_{\mathbb{R}} y^2 v \, \mathrm{d}y < \infty\}$.

2.1 *Gradient flow structure*

The geometric viewpoint of gradient flows we take was developed by Otto [27]. Consider a formal Riemannian manifold $\mathcal{M}$ whose elements are real functions on a given domain, with inner product $\langle \cdot, \cdot \rangle_u$, $u \in \mathcal{M}$. The equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F([u])$$

is a gradient flow of an energy $E : \mathcal{M} \to \mathbb{R}$ if for all $u \in \mathcal{M}$, $F([u]) \in T_u\mathcal{M}$ and

$$\langle F([u]), s \rangle_u = -\,\mathrm{d}E[s]$$

for every $s \in T_u\mathcal{M}$. By $[u]$ we denote the n-tuple of the spatial derivatives involved in the equation.

In the case of the equations (2), (7), and (9), $\mathcal{M}$ is the set of nonnegative L^1 functions with finite second moments. The tangent vectors at $u \in \mathcal{M}$ are functions whose support is a subset of the support of u, and that have zero mean on each connected component of the support of u. The inner product is, formally, defined as follows:

$$\langle s_1, s_2 \rangle_u = \int u^n f_1 f_2 \,\mathrm{d}x \tag{13}$$

where the f_i are such that $-(u^n f_i)_x = s_i$ for $i = 1, 2$ and $\lim_{x\to\infty} u(x)^n f_i(x) = 0$. Note that $\mathrm{d}E[s] = \frac{\delta E}{\delta u}[s]$, where $\frac{\delta E}{\delta u}[s]$ is the Gateaux derivative of E in the direction s. Elementary, but formal, calculations then verify that for u satisfying the equation (2) and for the energy, E, given by (10),

$$\langle u_t, s \rangle_u = -\frac{\delta E}{\delta u}[s].$$

Analogously, in the $n = 1$ case, the rescaled equations (7) and (9) are gradient flows of the energies described in (12).

The remarkable fact about the inner product with $n = 1$ is that the distance it induces on the manifold $\mathcal{M}$ is the Wasserstein metric. Various gradient flows in Wasserstein metric have been subject of a number of recent studies beginning with [18]. The reader can find further details in [27, 32].

2.2 *Linearizing a gradient flow at a steady state*

The geometric structure of gradient flows can be utilized when conducting linear-stability analysis. In particular Denzler and McCann [10], [11] have used this structure to study the linearization of the fast-diffusion equation.

The linearized dynamics near a steady state of a gradient flow on a manifold is described by the Hessian of the energy. By the definition of the Hessian, it is a symmetric operator in the metric of the gradient flow. To illustrate that in some generality, let $u(t)$ be the gradient flow of an energy E on a manifold $\mathcal{M}$ with inner product $\langle \cdot\,, \cdot \rangle$. For all $v_1, v_2 \in T_u\mathcal{M}$, $\operatorname{Hess} E(v_1, v_2) = \operatorname{Hess} E(v_2, v_1)$. The Hessian operator $\boldsymbol{H} : T_u\mathcal{M} \to T_u\mathcal{M}$ is associated to the Hessian form, $\operatorname{Hess} E$, by $\langle \boldsymbol{H} v_1, v_2 \rangle = \operatorname{Hess} E(v_1, v_2)$.

In our case the manifold structure is formal. For the gradient flows that we consider we show that at a steady state, the Hessian operator, $\boldsymbol{H}$, an object defined using the formal manifold structure, is equivalent to the standard linearization operator. More precisely, let us consider an equation

$$u_t = F([u]),$$

which we assume to be in divergence form, and hence mass preserving. Let the equation also be the gradient flow of the energy E on the manifold $\mathcal{M}$. Let η be a steady state of the equation above. Let $v \in T_\eta\mathcal{M}$. The linearized operator $\boldsymbol{L}$ at η is given by

$$\boldsymbol{L}|_\eta v = \lim_{h\to 0} \frac{F([\eta + hv]) - F([\eta])}{h}.$$

Note that $\alpha(h) = \eta + hv$ is a curve on $\mathcal{M}$. Take an arbitrary $w \in T_\eta\mathcal{M}$. By $\mathrm{D}/\mathrm{d}h$ we denote the covariant derivative along α, while we use ∇ for the Riemannian connection. Then

$$\begin{aligned}\text{Hess}\, E|_\eta(v, w) &= \langle \nabla_v \,\text{grad}\, E, w\rangle_\eta = \left\langle \frac{\mathrm{D}}{\mathrm{d}h}\bigg|_{h=0} F([\alpha(h)]), w\right\rangle_\eta \\ &= \left\langle \lim_{h\to 0} \frac{F([\alpha(h)]) - F([\eta])}{h}, w\right\rangle_\eta = \langle \boldsymbol{L}|_\eta v, w\rangle_\eta.\end{aligned}$$

To obtain the third equality we have used the fact that grad $E|_\eta = 0$. Note that the symmetry of the Hessian implies that $\boldsymbol{L}|_\eta$ is symmetric.

2.3 *Local coordinates*

In the description above we were loose in describing the space of functions that form the tangent space. For the gradient flows with the inner product defined as in (13) the description is easier after a change of coordinates. The particular coordinates when $n = 1$ were suggested by work of Otto, and were used by Denzler and McCann [10], [11].

The definition of the inner product suggests identifying the tangent plane at $u \in \mathcal{M}$ with the set of functions

$$L^2_{u^n} = \left\{ f \;\middle|\; \int u(x)^n f(x)^2 \,\mathrm{d}x < \infty \right\}.$$

The inner product is the weighted L^2 inner product, $\langle f, g\rangle_u = \int u(x)^n f(x)g(x)\,\mathrm{d}x$. The coordinate change that transforms from this description to the old one is $s = -(u^n f)_x$. When $n = 1$ this is the transformation from Lagrangian description, f, to Eulerian description, s. That is, the s describes the infinitesimal change in the height of fluid, while f is the vector field the fluid is perturbed by, with all particles located above the same spot moving by the same amount.

As it turns out, Lagrangian coordinates for the tangent plane can be useful even when $n \neq 1$. Although it is possible to use the coordinates suggested by the inner product directly, for our particular problem the Lagrangian coordinates yield a slightly simpler form of the operator. In this case, the tangent plane is identified with the weighted L^2-space, $L^2_{u^{2-n}}$. The inner product is $\langle f, g\rangle_u = \int_{\{u>0\}} u(x)^{2-n} f(x)g(x)\,\mathrm{d}x$. The coordinate transformation to Eulerian coordinates is $s = -(uf)_x$.

3. Stability of steady states

We now study the stability of steady states of the equations

$$u_t = -(u^n u_{xxx} + u^m u_x)_x$$

with $0 < n < 3$ and $n \leqslant m$. The steady states have been studied by Laugesen and Pugh [20–23]. There are two classes of the steady states. The first are the positive, periodic steady states. Their stability was studied in [20]. Constant steady states are long-wave unstable. Positive periodic steady states were shown to be unstable to zero-mean perturbations of the same period if $m \geqslant n + 1$ or $m < n$. For $n \leqslant m < n + 1$ evidence was presented that periodic steady states can be stable. The stability in [20] is characterized in terms of time and area maps of a related nonlinear oscillator.

The techniques of [20] use the fact that the linearized operator is nondegenerate, which is not the case in the problems we consider. We should also point out the difference in the definitions of stability in [20] and here. Steady states in [20] were defined to be stable if the spectrum of the linearized operator is nonnegative, while we distinguish between positivity (stability) and nonnegativity (marginal stability) of the spectrum.

The second class of steady states are the ones with compact support. We study the stability of such states here. Recall that if the set where a compactly supported steady is positive is connected, we call it a *droplet steady state*. Otherwise the steady state is a *droplet configuration*. That is, any compactly supported steady state is a sum of droplet steady states with disjoint positivity sets.

We distinguish between two types of droplet configurations: the ones where the supports of droplets (closures of positivity sets) are pairwise disjoint, and the ones where there exist two droplets that have nonempty intersection of supports (the intersection is a point where they "touch"). In the former case, the stability of the droplet configuration follows from the stability of each of the droplet steady states that form it, when considered alone. In the latter case, our analysis shows that the stability of droplet steady states that touch only implies that their configuration is marginally stable. It is an open problem to determine if it is stable. In either case if there exists an unstable droplet in the droplet configuration, then the configuration is unstable.

Let us remark that it turns out that the stability of droplet steady states depends only on the powers of nonlinearities. So in a droplet configuration, either all droplets are stable or all droplets are unstable.

From now on, we concentrate on droplet steady states. Let η be such a state. We know from [21] that η is symmetric and hence, by translating it if necessary, we can assume that η is centered at 0, and that the support of η is the interval $[-L, L]$. From [21] it also follows that η is a C^1 function, η restricted to $[-L, L]$ is a smooth function, but is not a C^2 function on $\mathbb{R}$. Furthermore, $\lim_{x\to L-} \eta''(x) > 0$.

The linearized equation can be obtained in a classical way, by perturbing the steady state in Eulerian variables. For this we refer the reader to the work of Witelski, Bernoff, and Bertozzi [33, Sec. 5.2] who carried it out for the $n = 1$ case. The delicate part of this procedure is handling the contact line, that is, the boundary of the support of η.

Following Otto [27] and Denzler and McCann [10], [11], we consider the linearization using the geometry of the equation. This approach handles the contact line in a natural and straightforward manner. We will first compute the Hessian form Hess E in Lagrangian local coordinates, mentioned above.

Given a tangent vector f and a function G, we denote by $f[G]$ the action of the vector on the function, that is, $f[G]$ is the derivative of G in the direction f. To compute Hess $E(f, f)$ at the steady state η, given a tangent vector $f \in L^2_{\eta^{2-n}} \cap C^2([-L, L])$ we use the fact that

$$\operatorname{Hess} E(f, f) = \langle \nabla_f \operatorname{grad} E, f\rangle \overset{\operatorname{grad} E|_\eta = 0}{=} f[\langle \operatorname{grad} E, f\rangle] = f[f[E]],$$

which is equal to the second derivative of E along a curve whose tangent vector is f. When $n = 1$ the geodesic in the direction f has a simple expression. Even for $n \neq 1$ this geodesic is a curve with tangent vector f at η. Hence we use these curves to compute the Hessian for any n.

The geodesics were used in the works of McCann [24], Otto [27] and Denzler and McCann [10], [11], and we refer to these works or the book by Villani [32] for the details. Here we just state what the geodesics are. Let $\rho \in \mathcal{M}$ and $f \in T_\rho\mathcal{M}$ be a bounded function. Then the geodesic γ is for

$|s| < 1/\|f\|_{L^\infty}$ given by

$$\gamma(s) = (\mathrm{Id} + sf)_{\#}\rho.$$

Here $F_{\#}\rho$ represents the push forward of the measure with density ρ via the function F. In the case above, that represents translating each particle beneath the graph of ρ by the vector sf. So the new location of the particle originally at x is $\Phi_s(x) = x + sf(x)$. If f is differentiable then

$$\gamma(s)(y) = \frac{\rho(\Phi_s^{-1}(y))}{\Phi_s'(\Phi_s^{-1}(y))}.$$

The Hessian quadratic form of the energy E at a steady state η is

$$\mathrm{Hess}\, E(f, f) = f[f[E]] = \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E(\gamma(s)).$$

The energies we study involve the following terms:

$$E_1(u) = \int u_x^2 \,\mathrm{d}x, \qquad E_2(u) = \int u^\beta \,\mathrm{d}x, \quad \text{and} \quad E_3(u) = \int x^2 u \,\mathrm{d}x.$$

Let us compute Hess E_1 at a steady state η. We write $x = \Phi_s^{-1}(y)$. Let $f \in L^2_{\eta^{2-n}} \cap C^2([-L, L])$. Then

$$\begin{aligned}
\mathrm{Hess}\, E_1(f, f) &= \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} E_1(\gamma(s)) = \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} \int \left(\partial_y \frac{\eta(\Phi_s^{-1}(y))}{\Phi_s'(\Phi_s^{-1}(y))}\right)^2 \mathrm{d}y \\
&= \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} \int \left(\frac{\eta'(x)}{(\Phi_s'(x))^2} - \frac{\eta(x)\Phi_s''(x)}{(\Phi_s'(x))^3}\right) \mathrm{d}y \\
&= \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} \int \left(\frac{\eta'(x)}{(1 + sf'(x))^2} - \frac{\eta(x) s f''(x)}{(1 + sf'(x))^3}\right)^2 (1 + sf'(x)) \,\mathrm{d}x \\
&= \int [12(\eta'(x))^2 (f'(x))^2 + 16\eta(x)\eta'(x) f'(x) f''(x) + 2\eta(x)^2 (f''(x))^2] \,\mathrm{d}x.
\end{aligned}$$

The Hessians of the energies E_2 and E_3 are computed similarly (this was already done by Otto [27]):

$$\begin{aligned}
\mathrm{Hess}\, E_2(f, f) &= \beta(\beta - 1) \int \eta(x)^\beta (f'(x))^2 \,\mathrm{d}x, \\
\mathrm{Hess}\, E_3(f, f) &= 2 \int \eta(x) f(x)^2 \,\mathrm{d}x.
\end{aligned}$$

The Hessian of the energy $E(u)$ given by (10) at η is thus equal to

$$\begin{aligned}
\mathrm{Hess}\, E(f, f) &= \frac{1}{2} \mathrm{Hess}\, E_1(f, f) + \frac{1}{(m - n + 2)(m - n + 1)} \mathrm{Hess}\, E_2(f, f) \\
&= \int_{-L}^{L} \eta(x)^2 (f''(x))^2 - \frac{m - n - 2}{m - n + 2} \eta(x)^{m-n+2} (f'(x))^2 \,\mathrm{d}x. \qquad (14)
\end{aligned}$$

Here E_2 is taken with $\beta = m - n + 2$. In computing the expression above we have used the fact that η is a steady state, that it has a compact support and that $\eta' = 0$ at the edge of the support.

Specifically we have used the equation $\eta'''(x) = -\eta(x)^{m-n}\eta'(x)$, as well as the integrated forms of the equation:

$$\eta'' = \frac{\eta(0)^{m-n+1}}{(m-n+1)(m-n+2)} - \frac{\eta^{m-n+1}}{m-n+1} \quad \text{and} \quad (\eta')^2 = \frac{2\eta(\eta(0)^{m-n+1} - \eta^{m-n+1})}{(m-n+1)(m-n+2)}.$$

It is clear that the form Hess E is nonnegative when $m \leqslant n+2$ (recall that we always assume that $m \geqslant n$). Note that

$$\eta''(L) = \frac{\eta(0)^{m-n+1}}{(m-n+1)(m-n+2)} > 0$$

and hence there exist positive constants C_1 and $C_2 > 1$ such that $C_1(L-|x|)^2 < \eta(x) < C_2(L-|x|)^2$ for all $x \in (-L, L)$. Thus the interpolation inequality (21) implies that the form is semibounded for $m > n+2$. That is, there exists A such that Hess $E(f, f) \geqslant A\langle f, f\rangle$ for all $f \in L^2_{\eta^{2-n}} \cap C^2([-L, L])$. The form domain is the weighted Sobolev space $Y = W^{2,2}((-L, L), 4-2n, 2m-2n+4, 4)$ as defined by (17) in the Appendix. The linearized operator itself has the form

$$\boldsymbol{L} f = \eta^{n-2}\left((\eta^2 f'')'' + \frac{m-n-2}{m-n+2}(\eta^{m-n+2} f')'\right).$$

Note that it is symmetric on $L^2_{\eta^{2-n}} \cap C^4([-L, L])$, with no additional boundary conditions at $-L$ and L. The form Hess E determines the Friedrichs extension (see [28]) of the operator $\boldsymbol{L}$. The extended operator $\boldsymbol{L}$ is selfadjoint.

Perturbing in the direction $f = 1$ corresponds to translations. Formally Hess $E(1, 1) = 0$. However, only when $0 < n < 5/2$ does 1 belong to $L^2_{\eta^{2-n}}$ and hence only then is $f = 1$ an eigenvector of the operator $\boldsymbol{L}$ that corresponds to the eigenvalue 0. The fact that $f = 1$ is a neutral direction is not surprising; it is a consequence of translation invariance of the equation (2).

Perturbing the solution in direction $f = x$ corresponds to dilations. Note that if $m > n+2$ and $0 < n < 5/2$ then Hess $E(x, x) < 0$ and $x \in Y$. Hence these steady states are linearly unstable and dilations represent an unstable direction. If $m = n+2$ then $f = x$ is an eigenvector corresponding to the eigenvalue 0. When $m \leqslant n+2$ the droplet steady states are linearly stable, which we prove in the next theorem.

THEOREM 1 (subcritical case) Assume $0 < n < 3$ and $n \leqslant m < n+2$. Let η be a droplet steady state of the equation (2) supported on $[-L, L]$. Let Hess E be the Hessian at η of the energy E given by (14). Let Y denote the weighted Sobolev space $W^{2,2}((-L, L), 4-2n, 2m-2n+4, 4)$, defined in (17).

(i) If $0 < n \leqslant 2$ then η is linearly stable modulo translations, that is, there exists $\lambda > 0$ such that

$$\text{Hess}\, E(f, f) > \lambda\langle f, f\rangle_\eta$$

for all functions $f \in Y$ such that $\langle f, 1\rangle_\eta = 0$.

(ii) If $2 < n < 5/2$ then η is marginally stable modulo translations, that is, Hess $E(f, f) > 0$ for all $f \in Y \setminus \{0\}$ such that $\langle f, 1\rangle_\eta = 0$. However,

$$\inf_{f \in Y\setminus\{0\},\, \langle f,1\rangle_\eta = 0} \frac{\text{Hess}\, E(f, f)}{\langle f, f\rangle_\eta} = 0.$$

(iii) If $5/2 \leqslant n < 3$ then η is marginally stable. That is, $\operatorname{Hess} E(f, f) > 0$ for all $f \in Y \setminus \{0\}$, but

$$\inf_{f \in Y \setminus \{0\}} \frac{\operatorname{Hess} E(f, f)}{\langle f, f \rangle_\eta} = 0.$$

Since $\boldsymbol{L}$ is a selfadjoint operator, the claims above imply lower bounds on the spectrum of the operator $\boldsymbol{L}$ (restricted to the orthogonal complement of 1 when $n < 5/2$).

Proof. Assume that $n \leqslant m \leqslant n+2$. If $0 < n \leqslant 2$ then applying Corollary 10 establishes that there exists $\lambda > 0$ such that $\operatorname{Hess} E(f, f) \geqslant \int_{-L}^{L} f(x)^2 \, \mathrm{d}x \geqslant \|\eta\|_{L^\infty}^{n-2} \int_{-L}^{L} \eta(x)^{2-n} f(x)^2 \, \mathrm{d}x$ for all $f \in C^2([-L, L])$ such that $\langle f, 1 \rangle_\eta = 0$. We claim that C^2 functions in the orthogonal complement of the vector 1 are dense in the orthogonal complement of 1 in Y. Let $\varepsilon > 0$ and $g \in Y$ be such that $\langle g, 1 \rangle_\eta = 0$. By Lemma 6, $C^2([-L, L])$ is dense in Y. Thus there exists $g_\varepsilon \in C^2([-L, L])$ such that $\|g_\varepsilon - g\|_Y < \varepsilon$. Note that the projection of g_ε on the orthogonal complement of vector 1, $\tilde{g}_\varepsilon = g_\varepsilon - \langle g_\varepsilon, 1 \rangle_\eta / \langle 1, 1 \rangle_\eta$, is also in $C^2([-L, L])$. Furthermore, $\|\tilde{g}_\varepsilon - g\|_Y \leqslant \|g_\varepsilon - g\|_Y < \varepsilon$, which establishes the density claim.

Since all functionals involved are continuous with respect to the norm on Y the claim of (i) follows.

If $2 < n < 5/2$ then it is clear from the form of $\operatorname{Hess} E$ that $\operatorname{Hess} E(f, f) > 0$ for all $f \in Y \setminus \{0\}$ such that $\langle f, 1 \rangle_\eta = 0$. To establish claim (ii) let $g \not\equiv 0$ be a smooth function supported on a subset of $(0, 1)$. Let the function g_β be given by $g_\beta(x) = g(\beta(L - |x|))$. Let $f_\beta = g_\beta - \langle g_\beta, 1 \rangle_\eta / \langle 1, 1 \rangle_\eta$. Then for $\beta > 1/L$, f_β is smooth and

$$\operatorname{Hess} E(f_\beta, f_\beta) \leqslant 2 \int_0^L \eta(x)^2 (g_\beta''(x))^2 + \eta(x)^{m-n+2} (g_\beta'(x))^2 \, \mathrm{d}x.$$

An elementary calculation that uses the estimate $\eta(x) \leqslant C_2 (L - |x|)^2$ shows how the relevant quantities scale with β:

$$\int_0^L \eta(x)^2 (g_\beta''(x))^2 \, \mathrm{d}x \leqslant \frac{C_2^2}{\beta} \int_0^1 z^4 (g''(z))^2 \, \mathrm{d}z,$$

$$\int_0^L \eta(x)^{m-n+2} (g_\beta'(x))^2 \, \mathrm{d}x \leqslant \frac{C_2^{m-n+2}}{\beta^{2m-2n+3}} \int_0^1 z^{2m-2n+4} (g'(z))^2 \, \mathrm{d}z,$$

$$\int_{-L}^{L} \eta^{2-n}(x) f_\beta(x)^2 \, \mathrm{d}x \geqslant \int_{-L}^{L} \eta^{2-n} g_\beta^2 \, \mathrm{d}x - 2 \left(\int_{-L}^{L} \eta^{2-n} g_\beta \, \mathrm{d}x \right)^2 \Big/ \langle 1, 1 \rangle_\eta$$

$$\geqslant 2 C_2^{2-n} \beta^{2n-5} \int_0^1 z^{4-2n} g(z)^2 \, \mathrm{d}z - 4 C_2^{4-2n} \beta^{2(2n-5)} \frac{(\int_0^1 z^{4-2n} g(z) dz)^2}{\langle 1, 1 \rangle_\eta}.$$

Since $0 > 2n - 5 > -1$ and $2m - 2n + 3 \geqslant 1$ the scalings above imply

$$\lim_{\beta \to \infty} \frac{\operatorname{Hess} E(f_\beta, f_\beta)}{\langle f_\beta, f_\beta \rangle_\eta} = 0,$$

which establishes (ii).

If $5/2 \leqslant n < 3$ then $1 \notin Y$. Hence $\operatorname{Hess} F(f, f) > 0$ for all $f \in Y$. Let $f_\beta = g_\beta$ where g_β was defined in the case above. The scalings above then show that

$$\lim_{\beta \to \infty} \frac{\operatorname{Hess} E(f_\beta, f_\beta)}{\langle f_\beta, f_\beta \rangle_\eta} = 0. \qquad \square$$

THEOREM 2 (critical case) Assume $0 < n < 3$ and $m = n + 2$. Let η be a droplet steady state of the equation (2) supported on $[-L, L]$. Let Hess E be the Hessian at η of the energy E given by (14) and let $Y = W^{2,2}((-L, L), 4 - 2n, 2m - 2n + 4, 4)$.

(i) If $0 < n < 2$ then η is linearly stable modulo translations and dilations, that is, there exists $\lambda > 0$ such that

$$\text{Hess}\, E(f, f) > \lambda \langle f, f \rangle_\eta$$

for all functions $f \in Y$ such that $\langle f, 1 \rangle_\eta = 0$ and $\langle f, x \rangle_\eta = 0$.

(ii) If $2 < n < 5/2$ then η is marginally stable modulo translations and dilations. That is, Hess $E(f, f) > 0$ for all $f \in Y \setminus \{0\}$ such that $\langle f, 1 \rangle_\eta = 0$ and $\langle f, x \rangle_\eta = 0$. However,

$$\inf_{f \in Y \setminus \{0\},\, \langle f,1 \rangle_\eta = 0,\, \langle f,x \rangle_\eta = 0} \frac{\text{Hess}\, E(f, f)}{\langle f, f \rangle_\eta} = 0.$$

(iii) If $5/2 \leqslant n < 3$ then η is marginally stable. That is, Hess $E(f, f) > 0$ for all $f \in Y \setminus \{0\}$. However,

$$\inf_{f \in Y \setminus \{0\}} \frac{\text{Hess}\, E(f, f)}{\langle f, f \rangle_\eta} = 0.$$

Proof. The proofs are analogous to the proofs in the subcritical case. The only significant difference is that the existence of the desired $\lambda > 0$ in the case $0 < n < 2$ follows from the Hardy-type inequality established in claim (i) of Lemma 9. □

THEOREM 3 (supercritical case) Assume $0 < n < 3$ and $m > n + 2$. Let η be a droplet steady state of the equation (2) supported on $[-L, L]$. Let Hess E be the Hessian at η of the energy E given by (14) and let $Y = W^{2,2}((-L, L), 4 - 2n, 2m - 2n + 4, 4)$. The steady state η is linearly unstable. In particular, when $0 < n < 5/2$, $f = x$ belongs to the space Y and represents an unstable direction: Hess $E(x, x) < 0$. When $5/2 \leqslant n < 3$ there exists $f \in C_0^\infty(-L, L) \cap Y$ such that Hess $E(f, f) < 0$.

Proof. Since $\eta \geqslant C_1(L - |x|)^2$ near $\pm L$ it readily follows that the function $f(x) = x$ is in Y precisely when $n < 5/2$. The form of Hess E then shows that Hess $E(x, x) < 0$.

We now consider the case $5/2 \leqslant n < 3$. Let κ be a smooth, nondecreasing cut-off function such that $\kappa = 0$ on $(-\infty, 0]$, and $\kappa = 1$ on $[1, \infty)$. For $\beta > 1$ let $f_\beta(x) = x\kappa(\beta(L - |x|))$ for $x \in (-L, L)$. Note that f_β is smooth and odd. Thus $\langle f_\beta, 1 \rangle_\eta = 0$.

Consider how the terms of Hess $E(f_\beta, f_\beta)$ scale with β. Since $\eta \leqslant C_2(L - |x|)^2$,

$$\begin{aligned}\int_{-L}^{L} \eta(x)^2 (f_\beta''(x))^2\, dx &\leqslant 2C_2^2 \int_0^L (L - x)^4 \big(\beta^2 x \kappa''(\beta(L - x)) - 2\beta\kappa'(\beta(L - x))\big)^2\, dx \\ &\leqslant 8C_2^2 \int_0^1 \frac{y^4}{\beta} L^2 (\kappa''(y))^2 + \frac{y^4}{\beta^3} (\kappa'(y))^2\, dy,\end{aligned}$$

which converges to 0 as $\beta \to \infty$. On the other hand,

$$\int_{-L}^{L} \eta(x)^{m-n+2} (f_\beta'(x))^2\, dx \geqslant 2 \int_0^{L(1-1/\beta)} \eta(x)^{m-n+2}\, dx$$

is bounded from below. Therefore, for β large enough, Hess $E(f_\beta, f_\beta) < 0$. □

4. Stability of blowup and source type selfsimilar solutions

The equation (2) has dynamical selfsimilar solutions only when $m = n + 2$. We study the stability of these solutions via the stability analysis of steady states of the equations in similarity variables: (7) and (9). As already mentioned, only when $n = 1$ is the gradient-flow structure of (7) and (9) known. Thus all the considerations in this section are for the case $n = 1$ and $m = 3$.

4.1 *Selfsimilar blowup solutions*

The equation in similarity variables (9) is a gradient flow of the energy $E = \frac{1}{2}E_1 - \frac{1}{12}E_2 - \frac{1}{10}E_3$. Using the computations of the Hessians of E_1, E_2 and E_3 given in Section 3 we obtain the Hessian of E at similarity profile ρ:

$$\operatorname{Hess} E(f, f) = \int \left(\rho^2 (f'')^2 + 8\rho\rho' f' f'' + 6(\rho')^2 (f')^2 - \rho^4 (f')^2 - \frac{1}{5}\rho f^2 \right) \mathrm{d}x$$

for $f \in C^2([-L, L])$.

The profile, ρ, of a symmetric selfsimilar blowup solution satisfies the equation

$$\rho'''(x) = -\frac{x}{5} - \rho(x)^2 \rho'(x)$$

with $\rho'(0) = 0$ and zero contact angle: $\rho'(L) = 0$. Using the identities obtained by integrating the equation:

$$\rho''(x) = \rho''(0) + \frac{\rho(0)^3}{3} - \frac{x^2}{10} - \frac{\rho(x)^3}{3}$$

and

$$\rho'(x)^2 = 2\left(\rho''(0) + \frac{\rho(0)^3}{3}\right)\rho(x) - \frac{\rho(x)^4}{6} - \frac{x^2 \rho(x)}{5} - \frac{2}{5}\int_x^L s\rho(s)\,\mathrm{d}s$$

we obtain

$$\operatorname{Hess} E(f, f) = \int \left(\rho(x)^2 (f''(x))^2 - \frac{4}{5}\varphi(x)(f'(x))^2 - \frac{1}{5}\rho(x) f(x)^2 \right) \mathrm{d}x \tag{15}$$

where

$$\varphi(x) = \int_x^L s\rho(s)\,\mathrm{d}s.$$

Since ρ is an even and positive function on $(-L, L)$, so is φ. Furthermore, $\varphi(L) = \varphi'(L) = \varphi''(L) = 0$. The form domain is the weighted Sobolev space $Y = W^{2,2}((-L, L), \rho, \varphi, \rho^2)$.

The linearized operator $\boldsymbol{L}$ has the form

$$\boldsymbol{L}f = \frac{1}{\rho}\left((\rho^2 f'')'' + \frac{4}{5}(\varphi f')' - \frac{1}{5}\rho f \right).$$

It is symmetric on $C^4([-L, L])$ with no boundary conditions.

Note that $\operatorname{Hess} E(1, 1) < 0$ and $\operatorname{Hess} E(x, x) < 0$. Furthermore, $f = 1$ is an eigenvector corresponding to the eigenvalue $-1/5$ and $f = x$ is an eigenvector corresponding to the eigenvalue

-1. The functions $f = 1$, which corresponds to translations, and $f = x$, which corresponds to dilations, represent unstable directions for the operator. This is a consequence of the invariances of the original equation and the rescaling to similarity variables. However, this does not mean that selfsimilar blowup solutions are structurally unstable; it just means that a small perturbation of initial data may result in shift in the location or the time of the blowup. If we want to investigate whether ρ describes the asymptotic shape of the blowup solution near a point, we need to find out if there are other eigenvectors corresponding to a negative eigenvalue.

Hence we say that a selfsimilar blowup solution is *linearly stable* if there exists a positive constant λ such that at ρ,

$$\operatorname{Hess} E(f, f) \geqslant \lambda \langle f, f \rangle_\rho$$

for all functions $f \in Y$ such that $\langle f, 1 \rangle_\rho = 0$ and $\langle f, x \rangle_\rho = 0$.

4.1.1 *Stability of single-bump selfsimilar blowup solutions.* To formulate and prove the result about stability we need to recall several facts about existence and properties of both steady states and selfsimilar blowup profiles.

Let us denote by η the droplet steady state with support $[-1, 1]$. Let $H_1 = \eta(0)$. It follows from [29, eq. (11)] that $5 < H_1 < 6$. Let $l(x) = 1 - |x|$. Using the equation $\eta'' = H_1^3/4 - \eta^3/3$ it is easy to show that $4l(x)^2 < \eta(x) < 30l(x)^2$ for $x \in (-1, 1)$. The facts listed below follow from Theorem 11 and Lemma 13 in [29].

- For all H large enough there exists a symmetric single-bump selfsimilar blowup profile ρ_H with $\rho_H(0) = H$ and zero contact angle at $x = \pm L_H$. Furthermore, $5/H < L_H < 7/H$.
- Let $\sigma_H(z) := \rho(L_H z)/H$. For all H large enough, $\|\sigma_H - \eta\|_{C^2([-1,1])} < 1$.

Therefore $3l(x)^2 < \sigma_H(x) < 31l(x)^2$ for all H large enough. A consequence is that $W^{2,2}((-L_H, L_H), \rho, \varphi, \rho^2) = W^{2,2}((-L_H, L_H), 2, 3, 4)$. The density of $C^\infty([-L_H, L_H])$ follows from Lemma 6.

THEOREM 4 There exist positive C and λ such that for all $H > C$, and all functions $f \in Y = W^{2,2}((-L_H, L_H), 2, 3, 4)$ such that $\langle f, 1 \rangle_{\rho_H} = 0$ and $\langle f, x \rangle_{\rho_H} = 0$,

$$\operatorname{Hess} E(f, f) > \lambda \langle f, f \rangle_{\rho_H}.$$

Therefore the form Hess E is semibounded and hence the Friedrichs extension of $\boldsymbol{L}$ is defined and selfadjoint. In conclusion for $H > C$ single-bump similarity profiles ρ_H are linearly stable and λ is a lower bound on the spectrum of $\boldsymbol{L}$ restricted to the orthogonal complement of 1.

Proof. Let H be large enough that the properties of ρ_H listed above hold. Let f be a C^2 function on $[-1, 1]$ such that $\langle f, 1 \rangle_{\rho_H} = 0$ and $\langle f, x \rangle_{\rho_H} = 0$. The inequality (i) of Lemma 9 then yields

$$\begin{aligned}\int_{-L_H}^{L_H} \rho_H(x)^2 (f''(x))^2 \, \mathrm{d}x &= \int_{-1}^{1} H^2 L_H^{-3} \sigma_H(z)^2 \left(\frac{\mathrm{d}^2 f(L_H z)}{\mathrm{d}z^2} \right)^2 \mathrm{d}z \\ &\geqslant c_1 H^2 L_H^{-3} \int_{-1}^{1} l(z)^2 f(L_H z)^2 \, \mathrm{d}z \geqslant c_2 H^5 \int_{-L_H}^{L_H} \rho_H(x)^2 f(x)^2 \, \mathrm{d}x.\end{aligned}$$

The constants c_1 and c_2 above are positive and independent of H.

Let $\phi_H(x) = \int_x^1 s\sigma_H(s)\,\mathrm{d}s$. Since σ_H is an even function, so is ϕ_H. Furthermore, since $3l^2 < \sigma_H < 31l^2$ on $(-1,1)$ an elementary calculation gives $1/16l^3 < \phi_H < 12l^3$ on $(-1,1)$. Using the inequality (ii) of Lemma 9 we obtain

$$\begin{aligned}\int_{-L_H}^{L_H} \rho_H(x)^2 (f''(x))^2\,\mathrm{d}x &= \int_{-1}^{1} H^2 L_H^{-3} \sigma_H(z)^2 \left(\frac{\mathrm{d}^2 f(L_H z)}{\mathrm{d}z^2}\right)^2 \mathrm{d}z \\ &\geqslant c_3 H^2 L_H^{-3} \int_{-1}^{1} l(z)^3 \left(\frac{\mathrm{d}f(L_H z)}{\mathrm{d}z}\right)^2 \mathrm{d}z \\ &\geqslant c_4 H^2 L_H^{-1} \int_{-1}^{1} \phi_H(z)(f'(L_H z))^2\,\mathrm{d}z \\ &\geqslant c_5 H^5 \int_{-L_H}^{L_H} \int_x^{L_H} s\rho_H(s)\,\mathrm{d}s\,(f'(x))^2\,\mathrm{d}x.\end{aligned}$$

The constants c_i for $i = 3, 4, 5$ are again positive and independent of H.

Combining the inequalities above shows that $\operatorname{Hess} E(f, f) > \langle f, f\rangle$ for all H large enough ($> \max\{2/\sqrt[5]{c_2}, 1/\sqrt[5]{c_5}\}$). Arguing as in Theorem 1 one can show the density of the orthogonal complement of $\{1, x\}$ in $C^2([-L, L])$ in the orthogonal complement of $\{1, x\}$ in Y. The continuity of the functionals involved with respect to the topology of Y implies that $\operatorname{Hess} E(f, f) > \langle f, f\rangle_\rho$ for all $f \in Y$ orthogonal to 1 and x. □

4.1.2 *Instability of the multi-bump selfsimilar blowup solutions.* We show the instability of the multi-bump selfsimilar blowup solutions by constructing an unstable direction, f, orthogonal to translations and dilations (see Figure 1). Perturbing the profile ρ in the direction f is effectively dilating out the solution from α to the right, while dilating in the solution to the left of $-\alpha$. The dynamical effect of this perturbation is that the bumps on the left blow up sooner than the ones on the right. Thus as the solution is blowing up, it is attaining a shape rather different from ρ.

We present the details for the solution with an odd number of bumps. The construction for a solution with even number of bumps is similar, so we only comment on differences. We first recall some facts about existence and properties of the selfsimilar solutions from [29].

- Let k be an odd integer and $k \geqslant 3$. For all H large enough there exists a symmetric selfsimilar blowup profile $\rho_{H,k}$ with $\rho_{H,k}(0) = H$, zero contact angle at $x = \pm L_{H,k}$, and exactly k local maxima. Furthermore, $5k/H < L_{H,k} < 7k/H$.
- Let k be an even integer and $k \geqslant 2$. For all θ large enough there exists a symmetric selfsimilar blowup profile $\rho_{\theta,k}$ with $\rho''_{\theta,k}(0) = \theta$, zero contact angle at $x = \pm L_{\theta,k}$, and exactly k local maxima.
- For $H > 0$ let η_H be the steady state centered at 0 with $\eta_H(0) = H$ and zero contact angles at $\pm\overline{L}/H$ (the constant $\overline{L} \approx 6$ is known). Let $\overline{\eta}_H(x) = \eta_H(x - \lfloor xH/\overline{L}\rfloor \overline{L}/H)$ for $x \in \mathbb{R}$. For $k \geqslant 3$ odd and all H large enough,

$$\|\rho_{H,k} - \overline{\eta}_H\|_{L^\infty([-L_{H,k}, L_{H,k}])} < H^{-7/2} \quad \text{and} \quad \|\rho''_{H,k} - \overline{\eta}''_H\|_{L^\infty([-L_{H,k}, L_{H,k}])} < H^{-3/2}.$$

The facts listed follow from [29, Theorem 29, Theorem 30, Lemma 23, Lemma 26, and the argument of Corollary 21].

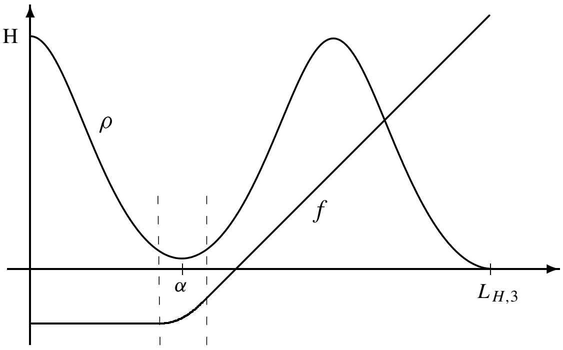

FIG. 1. Illustration of the right half of the unstable direction f in $k = 3$ case.

THEOREM 5 For all odd integers $k \geqslant 3$ and all H large enough there exists a function $f \in C^\infty([-L_{H,k}, L_{H,k}])$ such that

$$\langle f, 1\rangle_{\rho_{H,k}} = 0, \quad \langle f, x\rangle_{\rho_{H,k}} = 0, \quad \text{and} \quad \operatorname{Hess} E(f, f) < 0.$$

The statement also holds for $k \geqslant 2$ even, with H replaced by θ.

Proof. Let $k \geqslant 3$ be an odd integer. Let $H > 5$ be large enough that the properties above hold. As H and k are set, from now on we omit the H, k indices. Let α be the location of the first local minimum of ρ. Let $a = H^{-3}$ and g be an even C^∞ function on $[-L, L]$ such that

$$g(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \alpha - a, \\ x - \alpha & \text{if } \alpha + a \leqslant x. \end{cases}$$

Furthermore, g is required to be nondecreasing on $(0, \infty)$ and to satisfy $|g''| < 5/a$. Let f be the projection of g to the orthogonal complement of the vector 1, that is, let

$$f = g - \frac{\langle g, 1\rangle_\rho}{\langle 1, 1\rangle_\rho}.$$

Then $\langle f, 1\rangle_\rho = 0$ and since f is even, $\langle f, x\rangle = 0$. From the estimates on $\rho - \overline{\eta}$ listed above, the fact that $\overline{\eta}$ has minimum 0 and $\overline{\eta}'' < H^3/12$ it follows that $\rho(\alpha) < H^{-7/2}$ and $\rho'' < H^3$ on $[-L, L]$. Thus $\rho(x) \leqslant \rho(\alpha) + H^3(x - \alpha)^2$. Therefore

$$\int_{-L}^{L} \rho(x)^2 (f''(x))^2 \,\mathrm{d}x \leqslant 2\int_{\alpha - a}^{\alpha + a} (H^{-7/2} + H^3 a^2)^2 \left(\frac{5}{a}\right)^2 \mathrm{d}x \leqslant 400 H^{-3}.$$

On the other hand, using that $L > 15/H > 2\overline{L}$, $\alpha + a < 4/3\overline{L}$, and $\rho > \overline{\eta}_H - 1 > H/3$ on $[5/3\overline{L}, 2\overline{L}]$ we obtain

$$\int_{-L}^{L} \int_{x}^{L} s\rho(s)\,\mathrm{d}s\,(f'(x))^2\,\mathrm{d}x \geqslant 2\int_{\frac{4}{3}\overline{L}}^{\frac{5}{3}\overline{L}} \int_{\frac{5}{3}\overline{L}}^{2\overline{L}} s\rho(s)\,\mathrm{d}s\,\mathrm{d}x \geqslant 2\frac{1}{H}\frac{1}{H}\frac{5}{H}\frac{H}{4} \geqslant H^{-2}.$$

These inequalities imply that for H large enough,

$$\operatorname{Hess} E(f, f) \leqslant 400 H^{-3} - \frac{4}{5} H^{-2} < 0.$$

In the case that k is even one can construct the test function by shifting g to the left by $\alpha - a$. This gives a V-shaped test function. □

Let $\tilde{\boldsymbol{L}}$ be the restriction of $\boldsymbol{L}$ to the orthogonal complement of $\{1, x\}$. As a symmetric and real operator, $\tilde{\boldsymbol{L}}$ has a selfadjoint extension [28]. In terms of the spectrum of any such extension, also dentored by $\tilde{\boldsymbol{L}}$, the above lemma implies that the spectrum of $\tilde{\boldsymbol{L}}$ contains a negative number.

4.2 *Source-type selfsimilar solutions*

The stability analysis of these solutions is straightforward. We say that a source-type selfsimilar solution of (2) is stable if the associated steady state (i.e. the similarity profile) of the equation (7) is stable, that is, if the Hessian of the energy (12) is a uniformly positive-definite quadratic form.

Using $E = \frac{1}{2} E_1 - \frac{1}{12} E_2 + \frac{1}{10} E_3$ and the computations of the Hessians in Section 3 we find that for $f \in L^2_\rho \cap C^2([-L, L])$ the Hessian of E at ρ is

$$\operatorname{Hess} E(f, f) = \int \left(\rho^2 (f'')^2 + 8\rho\rho' f' f'' + 6(\rho')^2 (f')^2 - \rho^4 (f')^2 + \frac{1}{5} \rho f^2 \right) \mathrm{d}x.$$

The profile ρ of a symmetric spreading selfsimilar solution satisfies the equation

$$\rho'''(x) = \frac{x}{5} - \rho(x)^2 \rho'(x)$$

with $\rho'(0) = 0$ and zero contact angle: $\rho'(L) = 0$. We also use the following identities that follow by integrating the equation:

$$\rho''(x) = \rho''(0) + \frac{\rho(0)^3}{2} + \frac{x^2}{10} - \frac{\rho(x)^3}{3}$$

and

$$\rho'(x)^2 = 2\left(\rho''(0) + \frac{\rho(0)^3}{3} \right) \rho(x) - \frac{\rho(x)^4}{6} + \frac{x^2 \rho(x)}{5} + \frac{2}{5} \int_x^L s\rho(s)\, \mathrm{d}s.$$

An elementary calculation yields

$$\operatorname{Hess} E(f, f) = \int \left(\rho(x) (f''(x))^2 + \frac{4}{5} \int_x^L s\rho(s)\, \mathrm{d}s\, (f'(x))^2 + \frac{1}{5} \rho(x) f(x)^2 \right) \mathrm{d}x. \tag{16}$$

It is obvious that $\operatorname{Hess} E(f, f) \geqslant \frac{1}{5} \langle f, f \rangle_\rho$ for all f in the form domain and hence the selfsimilar spreading solutions are linearly stable.

5. Appendix

Here we establish some properties of the weighted Sobolev spaces relevant for the stability analysis. In particular, the weights that appear in our considerations are equivalent to powers of the distance to the boundary of the domain.

Given an interval I let us denote by $d(x)$ the distance of a point $x \in I$ to the boundary. We denote by $W^{2,2}(I, a, c, b)$ the space of functions f on I whose distributional derivatives satisfy

$$\|f\|^2_{W^{2,2}(I,a,b,c)} = \int_I [d(x)^c (f''(x))^2 + d(x)^b (f'(x))^2 + d(x)^a f(x)^2]\,\mathrm{d}x < \infty. \tag{17}$$

The particular weights of interest will be $c = 4$, $b \geqslant 4$, while $-2 < a < 4$.

5.1 *Density of smooth functions*

LEMMA 6 Consider the weighted Sobolev space $W(a, b) = W^{2,2}((0, 1), a, b, 4)$.

(i) The set $C^\infty([0, 1])$ is dense in $W(a, b)$ if $b \geqslant 2$ and $a > -1$.
(ii) The set $C_0^\infty(0, 1)$ is dense in $W(a, b)$ if $b \geqslant 2$ and $a < 0$.

In the statement above the functions defined on $[0, 1]$ are restricted to $(0, 1)$ to be considered elements of $W(a, b)$. This convention holds throughout the paper.

Proof. If $a \geqslant 0$ claim (i) follows by using standard arguments; see Kufner [19, Sec. 7]. In the case $0 > a > -1$ the claim follows from claim (ii); we listed it above just to point out that $C^\infty([0, 1]) \subset W(a, b)$ as long as $a > -1$.

To show (ii), consider $f \in W(a, b)$ with $b \geqslant 2$ and $a < 0$. Let κ be a smooth, nondecreasing cut-off function: $\kappa = 0$ on $(-\infty, 0]$, $\kappa = 1$ on $[1, \infty)$. It suffices to approximate κf and $(1 - \kappa) f$ by smooth functions. As the two cases are analogous, we only consider $\tilde{f} = (1 - \kappa) f$, that is, we consider $\tilde{f}$ such that $\tilde{f} = 0$ in some neighborhood of $x = 1$. From here on, let $f = \tilde{f}$.

Let $\kappa_\alpha(x) = \kappa(\alpha x)$ and let $f_\alpha = \kappa_\alpha f$. We claim that $f_\alpha \to f$ in $W(a, b)$ as $\alpha \to \infty$. It suffices to show that

$$\int_0^1 x^a (f_\alpha(x) - f(x))^2\,\mathrm{d}x \to 0, \tag{18}$$

$$\int_0^1 x^b (f'_\alpha(x) - f'(x))^2\,\mathrm{d}x \to 0, \tag{19}$$

$$\int_0^1 x^4 (f''_\alpha(x) - f''(x))^2\,\mathrm{d}x \to 0, \tag{20}$$

as $\alpha \to \infty$. The claim in (18) follows immediately, since

$$\int_0^1 x^a (f_\alpha(x) - f(x))^2\,\mathrm{d}x = \int_0^{1/\alpha} x^a (\kappa(\alpha x) - 1)^2 f(x)^2\,\mathrm{d}x \leqslant \int_0^{1/\alpha} x^a f(x)^2\,\mathrm{d}x \to 0$$

as $\alpha \to \infty$, since $\int_0^1 x^a f(x)^2\,\mathrm{d}x < \infty$. To show (19) we estimate

$$\begin{aligned}\int_0^1 & x^b (\alpha\kappa'(\alpha x) f(x) + \kappa(\alpha x) f'(x) - f'(x))^2\,\mathrm{d}x \\ &\leqslant 2\|\kappa'\|^2_{L^\infty} \alpha^2 \int_0^{1/\alpha} x^{b-a} x^a f(x)^2\,\mathrm{d}x + 2\int_0^{1/\alpha} x^b (f'(x))^2\,\mathrm{d}x \\ &\leqslant 2\|\kappa'\|^2_{L^\infty} \alpha^2 \alpha^{-b+a} \int_0^{1/\alpha} x^a f(x)^2\,\mathrm{d}x + o(1) \to 0\end{aligned}$$

as $\alpha \to \infty$ since $-b + a + 2 < 0$. To show (20) we estimate

$$\int_0^1 x^4(\alpha^2\kappa''(\alpha x) f(x) + 2\alpha\kappa'(\alpha x) f'(x) + (\kappa(\alpha x) - 1) f''(x))^2 \,\mathrm{d}x$$
$$\leqslant 3\|\kappa''\|^2_{L^\infty} \int_0^{1/\alpha} f(x)^2 \,\mathrm{d}x + 12 \int_0^{1/\alpha} (\kappa'(\alpha x))^2 x^2 (f'(x))^2 \,\mathrm{d}x + 3 \int_0^{1/\alpha} x^4 (f''(x))^2 \,\mathrm{d}x.$$

The first and third terms converge to zero as $\alpha \to \infty$ since $a < 0$ and $\alpha x \leqslant 1$ on the support of the integrands. To analyze the second term note that the integrand is supported on a compact subset of $(0, 1)$. Since κ is smooth and $f \in W^{2,2}$ on the support of the integrand, we can integrate by parts to obtain

$$\begin{aligned}
\int_0^{1/\alpha} (\kappa'(\alpha x))^2 x^2 (f'(x))^2 \,\mathrm{d}x &= -\int_0^{1/\alpha} 2\alpha\kappa'(\alpha x)\kappa''(\alpha x) x^2 f'(x) f(x) \,\mathrm{d}x \\
&\quad - \int_0^{1/\alpha} 2(\kappa'(\alpha x))^2 x f'(x) f(x) \,\mathrm{d}x - \int_0^{1/\alpha} (\kappa'(\alpha x))^2 x^2 f(x) f''(x) \,\mathrm{d}x \\
&\leqslant \frac{1}{4} \int_0^{1/\alpha} (\kappa'(\alpha x))^2 x^2 (f'(x))^2 \,\mathrm{d}x + 16\|\kappa''\|^2_{L^\infty(\mathbb{R})} \int_0^{1/\alpha} f(x)^2 \,\mathrm{d}x \\
&\quad + \frac{1}{4} \int_0^{1/\alpha} (\kappa'(\alpha x))^2 x^2 (f'(x))^2 \,\mathrm{d}x + 16\|\kappa'\|^2_{L^\infty(\mathbb{R})} \int_0^{1/\alpha} f(x)^2 \,\mathrm{d}x \\
&\quad + \|\kappa'\|^2_{L^\infty(\mathbb{R})} \left(\int_0^{1/\alpha} x^4 (f''(x))^2 \,\mathrm{d}x + \int_0^{1/\alpha} f(x)^2 \,\mathrm{d}x \right).
\end{aligned}$$

Above we have used $x\alpha \leqslant 1$ and $yz \leqslant \varepsilon y^2 + \varepsilon^{-1} z^2$ with appropriate $\varepsilon > 0$. Therefore

$$\int_0^{1/\alpha} (\kappa'(\alpha x))^2 x^2 (f'(x))^2 \,\mathrm{d}x \leqslant \mathrm{const} \cdot \left(\int_0^{1/\alpha} x^4 (f''(x))^2 \,\mathrm{d}x + \int_0^{1/\alpha} f(x)^2 \,\mathrm{d}x \right) = o(1).$$

Hence $f_\alpha \to f$ in $W(a, b)$ as $\alpha \to \infty$. The functions f_α are supported on compact subsets of $(0, 1)$. The fact that any f_α can be approximated by a function in $C_0^\infty(0, 1)$ follows by a standard use of mollifiers, as can be found in [19]. □

5.2 *A weighted interpolation inequality*

LEMMA 7 Let $n > 0$, $\beta \geqslant \max\{4 - n, 6 - 2n, 2\}$, and $l(x) = 1 - |x|$. There exists $C > 0$ such that for all $1 > \varepsilon > 0$ and all $f \in Y = W^{2,2}((-1, 1), 4 - 2n, \beta, 4)$,

$$\int_{-1}^1 l(x)^\beta (f'(x))^2 \,\mathrm{d}x \leqslant \varepsilon \int_{-1}^1 l(x)^4 (f''(x))^2 \,\mathrm{d}x + \frac{C}{\varepsilon} \int_{-1}^1 l(x)^{4-2n} f(x)^2 \,\mathrm{d}x. \tag{21}$$

Proof. Since all of the expressions involved in the inequality are continuous with respect to norm on Y, and smooth functions (that is, $C^\infty([-1, 1])$ when $n < 5/2$ and $C_0^\infty(-1, 1)$ when $n \geqslant 5/2$) are dense in Y, it suffices to show the inequality for smooth functions. Let $f \in C^\infty([-1, 1])$ if $n < 5/2$

and $f \in C_0^\infty(-1,1)$ when $n \geqslant 5/2$. Recall that in either case $f \in L^2_{l^{4-2n}}$. For any $1 > \varepsilon > 0$,

$$\begin{aligned}\int_{-1}^1 l(x)^\beta (f'(x))^2\,\mathrm{d}x &\leqslant \int_{-1}^1 l(x)^\beta |f''(x) f(x)|\,\mathrm{d}x + \beta \int_{-1}^1 l(x)^{\beta-1} |l'(x)|\,|f'(x) f(x)|\,\mathrm{d}x \\ &\leqslant \left(\int_{-1}^1 l(x)^4 (f''(x))^2\,\mathrm{d}x\right)^{1/2} \left(\int_{-1}^1 l(x)^{2\beta-4} f(x)^2\,\mathrm{d}x\right)^{1/2} \\ &\quad + \beta \left(\int_{-1}^1 l(x)^\beta (f'(x))^2\,\mathrm{d}x\right)^{1/2} \left(\int_{-1}^1 l(x)^{\beta-2} f(x)^2\,\mathrm{d}x\right)^{1/2} \\ &\leqslant \frac{\varepsilon}{2} \int_{-1}^1 l(x)^4 (f''(x))^2\,\mathrm{d}x + \frac{1}{2\varepsilon} \int_{-1}^1 l(x)^{4-2n} f(x)^2\,\mathrm{d}x \\ &\quad + \frac{1}{2} \int_{-1}^1 l(x)^\beta (f'(x))^2\,\mathrm{d}x + \frac{\beta^2}{2} \int_{-1}^1 l(x)^{4-2n} f(x)^2\,\mathrm{d}x.\end{aligned}$$

The claim with $C = (1+\beta^2)/2$ then follows. □

5.3 *Hardy-type inequalities*

LEMMA 8 Let $g \in C^1([0,a])$ be such that for some $\kappa > 0$,

$$\int_0^a z^\kappa g(z)^2\,\mathrm{d}z > c > 0.$$

Assume that $|g(a)| < \varepsilon$ for $\varepsilon > 0$ such that $2a^{\kappa+1}\varepsilon^2 < (\kappa+1)c$. Then

$$\int_0^a z^{\kappa+2} (g'(z))^2\,\mathrm{d}z \geqslant \frac{(\kappa+1)^2}{16} \int_0^a z^\kappa g(z)^2\,\mathrm{d}z.$$

Proof. Using integration by parts and the assumptions above we obtain

$$\begin{aligned}\int_0^a z^\kappa g(z)^2\,\mathrm{d}z &= \frac{a^{\kappa+1}}{\kappa+1}\varepsilon^2 - 2\int_0^a \frac{z^{\kappa+1}}{\kappa+1} g(z) g'(z)\,\mathrm{d}z \\ &\leqslant \frac{a^{\kappa+1}}{\kappa+1}\varepsilon^2 + \frac{2}{\kappa+1}\left(\int_0^a z^\kappa g(z)^2\,\mathrm{d}z\right)^{1/2} \left(\int_0^a z^{\kappa+2} (g'(z))^2\,\mathrm{d}z\right)^{1/2}\end{aligned}$$

Therefore

$$\begin{aligned}\left(\int_0^a z^\kappa g(z)^2\,\mathrm{d}z\right)^{1/2} &< \frac{a^{\kappa+1}}{\kappa+1}\frac{\varepsilon^2}{\sqrt{c}} + \frac{2}{\kappa+1}\left(\int_0^a z^{\kappa+2} (g'(z))^2\,\mathrm{d}z\right)^{1/2} \\ &\leqslant \frac{\sqrt{c}}{2} + \frac{2}{\kappa+1}\left(\int_0^a z^{\kappa+2} (g'(z))^2\,\mathrm{d}z\right)^{1/2}\end{aligned}$$

Thus

$$\frac{1}{2}\left(\int_0^a z^\kappa g(z)^2\,\mathrm{d}z\right)^{1/2} \leqslant \frac{2}{\kappa+1}\left(\int_0^a z^{\kappa+2} (g'(z))^2\,\mathrm{d}z\right)^{1/2}.$$

□

LEMMA 9 Let $l(x) = 1 - |x|$ and $M > 0$. There exists $\lambda > 0$ such that for any even measurable function ρ on $[-1, 1]$ such that $Ml^2 > \rho > l^2$ and any $f \in Y = W^{2,2}((-1, 1), 0, 2, 4)$ such that

$$\int_{-1}^{1} \rho(x) f(x)\, dx = 0 \quad \text{and} \quad \int_{-1}^{1} \rho(x) x f(x)\, dx = 0$$

the following hold:

(i) $\int_{-1}^{1} \rho(x)^2 (f''(x))^2\, dx \geqslant \lambda \int_{-1}^{1} f(x)^2\, dx,$

(ii) $\int_{-1}^{1} \rho(x)^2 (f''(x))^2\, dx \geqslant \lambda \int_{-1}^{1} l(x)^2 (f'(x))^2\, dx.$

Proof. Assume that the claim (i) is false. We know that $C^2([-1, 1])$ is dense in Y and that the functionals above are continuous with respect to the topology of Y. Arguing as in Theorem 1 one can show that the set of functions in $f \in C^2([-1, 1])$ such that $\int_{-1}^{1} \rho(x) f(x)\, dx = 0$ and $\int_{-1}^{1} \rho(x) x f(x)\, dx = 0$ is dense in the set of functions in Y satisfying the two equalities. Hence there exists a sequence of functions ρ_i and $f_i \in C^2([-1, 1])$ satisfying the assumptions above, such that $\int_{-1}^{1} f_i(x)^2\, dx = 1$ and $\int_{-1}^{1} \rho_i(x)^2 (f_i''(x))^2\, dx \to 0$ as $i \to \infty$.

Therefore $f_i'' \to 0$ in $L^2([-a, a])$ for any $0 < a < 1$. Let us now show that $f_i(0)$ and $f_i'(0)$ are bounded sequences. By taking the mirror images of f_i about the x and/or the y-axis, we can assume that $f_i(0) \geqslant 0$ and $f_i'(0) \geqslant 0$. Since $\rho_i \geqslant 1/4$ on $[0, 1/2]$ we can also assume that $1 > \int_0^{1/2} (f_i''(x))^2\, dx \geqslant (\int_0^{1/2} |f_i''(x)|\, dx)^2$ for all i. In the following computations we make use of the estimate $(b + c)^2 \geqslant 3b^2/4 - 3c^2$. There exists C large such that for all i,

$$\begin{aligned} C &> \int_0^{1/2} f_i(x)^2\, dx = \int_0^{1/2} \left(f_i(0) + f_i'(0)x + \int_0^x \int_0^r f''(s)\, ds\, dr \right)^2 dx \\ &\geqslant \int_0^{1/2} \left(\frac{3}{4}(f_i(0) + f_i'(0)x)^2 - 3 \right) dx \geqslant \frac{3}{4} \int_0^{1/2} [f_i(0)^2 + (f_i'(0))^2 x^2]\, dx - 2 \\ &\geqslant \frac{1}{32}(f_i(0)^2 + (f_i'(0))^2) - 2. \end{aligned}$$

Thus there exists a subsequence along which $f_i(0)$ and $f_i'(0)$ converge. For notational simplicity we assume that the entire sequence converges: $f_i(0) \to \alpha$ and $f_i'(0) \to \beta$ as $i \to \infty$. By expanding $f(x)$ as above in estimating the H^2-norm of $|f_i - \alpha x - \beta|$ it is elementary to verify that

$$f_i \to \alpha x + \beta \quad \text{in } H^2([-a, a])$$

for any $a \in (0, 1)$. The Sobolev inequality implies that the convergence is also in $C^{1,1/2}$.

Let us show that $\beta = 0$. Assume that $\beta \neq 0$. Let $0 < \varepsilon < |\beta|/2$. There exists $i(\varepsilon)$ such that for all $i > i(\varepsilon)$, $\| f_i(x) - \alpha x - \beta \|_{C^1([-1+\varepsilon, 1-\varepsilon])} < \varepsilon$. Let $I_\varepsilon = [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]$. From $\int_{-1}^{1} \rho_i(x) f_i(x)\, dx = 0$ it follows that

$$\left| \int_{I_\varepsilon} \rho_i(x) f_i(x)\, dx \right| \geqslant \left| \int_{-1+\varepsilon}^{1-\varepsilon} \rho_i(x)(\alpha x + \beta)\, dx \right| - \left| \int_{-1+\varepsilon}^{1-\varepsilon} \rho_i(x)(f_i(x) - \alpha x - \beta)\, dx \right|.$$

Note that $\int_{-1}^{1} l(x)^2 f_i(x)^2 \,\mathrm{d}x \leqslant \int_{-1}^{1} f_i(x)^2 \,\mathrm{d}x = 1$. For $i > i(\varepsilon)$, it then follows that

$$\begin{aligned}\sqrt{\frac{2M^2\varepsilon^3}{3}} &\geqslant \sqrt{\int_{I_\varepsilon} Ml(x)^2 \,\mathrm{d}x}\sqrt{\int_{I_\varepsilon} Ml(x)^2 f_i(x)^2 \,\mathrm{d}x} \\ &\geqslant \int_{I_\varepsilon} \sqrt{\rho_i(x)}(\sqrt{\rho_i(x)} f_i(x)) \,\mathrm{d}x \\ &\geqslant \left|\int_{-1+\varepsilon}^{1-\varepsilon} \rho_i(x)\beta \,\mathrm{d}x\right| - \varepsilon\left|\int_{-1+\varepsilon}^{1-\varepsilon} \rho(x) \,\mathrm{d}x\right| \geqslant \frac{|\beta|}{2}\int_{-1+\varepsilon}^{1-\varepsilon} l(x)^2 \,\mathrm{d}x.\end{aligned}$$

Choosing ε small enough leads to a contradiction. Thus $\beta = 0$.

The proof that $\alpha = 0$ is similar so we omit it. Hence $f_i \to 0$ in $C^{1,1/2}$ on compact subsets of $(-1, 1)$. There exists i_0 such that for all $i > i_0$, $|f_i(0)| + |f_i'(0)| < 1/8$. Let us consider the case that $\int_0^1 f_i(x)^2 \,\mathrm{d}x \geqslant 1/2$. The case $\int_{-1}^0 f_i(x)^2 \,\mathrm{d}x \geqslant 1/2$ is considered analogously.

Lemma 8, applied with $\kappa = 0$, $g = f_i$, $z = 1 - x$, $a = 1$, $\varepsilon = 1/8$, and $c = 1/2$, implies

$$\int_0^1 (1-x)^2 (f_i'(x))^2 \,\mathrm{d}x > \frac{1}{16}\int_0^1 f_i(x)^2 \,\mathrm{d}x > \frac{1}{32}.$$

Applying the lemma once more, this time to $g = f_i'$ with $\kappa = 2$, yields

$$\int_0^1 \rho(x)^2 (f_i''(x))^2 \,\mathrm{d}x > \int_0^1 (1-x)^4 (f_i''(x))^2 \,\mathrm{d}x > \frac{9}{16}\int_0^1 (1-x)^2 (f_i'(x))^2 \,\mathrm{d}x > \frac{1}{64}.$$

This contradicts the assumption $\int_{-1}^1 \rho(x)^2 (f_i''(x))^2 \,\mathrm{d}x \to 0$ as $i \to \infty$.

Let us now prove claim (ii). Assume that the claim is false. Let $g_i = f_i'$. Arguing as above one finds that $g_i \to \alpha$ in $C^{1/2}$ on compact subsets on $(-1, 1)$.

Let $\phi_i(x) = \int_x^1 s\rho_i(s) \,\mathrm{d}s$. Since ρ_i is an even, positive function on $(-1, 1)$, so is ϕ_i. Also $\phi_i(1) = \phi_i'(1) = \phi_i''(1) = 0$. Furthermore, $\phi_i(x) \leqslant \int_x^1 \rho_i(s) \,\mathrm{d}s \leqslant M(1-x)^3$. Since ϕ_i is even, $\phi_i(x) \leqslant Ml(x)^3$. The condition $0 = \int_{-1}^1 \rho_i(x) x f_i(x) \,\mathrm{d}x = -\int_{-1}^1 \phi'(x) f_i(x) \,\mathrm{d}x$ implies, after integration by parts, that

$$\int_{-1}^1 \phi_i(x) g_i(x) \,\mathrm{d}x = 0.$$

This condition can now be used to show that $\alpha = 0$. The argument is analogous to the way we used $\int_{-1}^1 \rho_i(x) f_i(x) \,\mathrm{d}x = 0$ to show that $\beta = 0$ so we leave the details to the reader. Lemma 8, applied with $g = g_i$ and $k = 2$, now leads to a contradiction as above. □

COROLLARY 10 Let $M > 0$, $l(x) = 1 - |x|$, and ψ an integrable function, positive on $(-1, 1)$. There exists $\lambda > 0$ such that for any even measurable function ρ on $(-1, 1)$ such that $Ml^2 > \rho > l^2$ and any $f \in C^2([-1, 1])$ such that $\int_{-1}^1 \rho(x) f(x) \,\mathrm{d}x = 0$ the following holds:

$$\int_{-1}^1 [\rho(x)^2 (f''(x))^2 + \psi(x)(f'(x))^2] \,\mathrm{d}x \geqslant \lambda \int_{-1}^1 f(x)^2 \,\mathrm{d}x.$$

The proof of the corollary closely follows the proof of claim (i) of the lemma. The only difference is that the fact that $\alpha = 0$ now follows from the assumption that $\int [\rho(x)^2 (f_i''(x))^2 + \psi(x)(f_i'(x))^2] \,\mathrm{d}x \to 0$ as $i \to \infty$.

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