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Transport of interfaces with surface tension by 2D viscous flows

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We consider the problem of finding a global weak solution for two-dimensional, incompressible viscous flow on a torus, containing a surface-tension bearing curve transported by the flow. This is the simplest case of a class of two-phase flows considered by Plotnikov in [16] and Abels in [1]. Our work complements Abels' analysis by examining this special case in detail. We construct a family of approximations and show that the limit of these approximations satisfies, globally in time, an incomplete set of equations in the weak sense. In addition, we examine criteria for closure of the limit system, we find conditions which imply nontrivial dependence of the limiting solution on the surface tension parameter, and we obtain a new system of evolution equations which models our flow-interface problem, in a form that may be useful for further analysis and for numerical simulations.

1. Introduction

In [16] P. I. Plotnikov considered the motion of two immiscible fluids with the same density, separated by an interface in the presence of surface tension, with distinct, strain-dependent viscosities, in two space dimensions. The interface was modelled as a *varifold* and Plotnikov proved global existence of a weak solution under a certain strain-thickening hypothesis (non-Newtonian) on the viscosities. In [1] H. Abels extended Plotnikov's work in several ways, in particular extending it to three-dimensional flows and proving the existence of *measure-valued varifold solutions* in a broader class of rheologies, which include the Newtonian case. Both varifold solutions and measure-valued varifold solutions satisfy incomplete sets of equations, and we refer to the problem of proving the existence of a global weak solution of the original system, for reasons which we will make clear, as the *closure problem*. The purpose of the present work is to complement Abels' results by considering in detail the simplest case; that of two-dimensional Newtonian flows with the same viscosity on each side of the interface. We construct a new family of approximations to this problem and we prove the existence of a weak limit satisfying an incomplete set of equations. We see that, in particular, Abels' measure-valued varifold solution actually becomes a varifold solution in this case. While we use a different approximating sequence and avoid varifolds, the proof of our main theorem has many elements in common with that of Abels. We include it basically for expository purposes,

since our simplified context and distinct approach bring the closure problem into sharper focus. We find regularity conditions under which the limiting solutions we obtain depend non-trivially on the surface tension, and we examine criteria for solving the closure problem, extending and adding further detail to the analysis performed by Abels. Finally, our method of approximation suggests a new set of evolution equations for the flow-interface dynamics.

To be more precise, we consider a pair of viscous incompressible two-dimensional fluids, which we assume to be of constant unit density, separated by a curve $\mathcal{C}_t$ on which we assume there is surface tension. The surface tension is a force that is normal to $\mathcal{C}_t$ and proportional to the curvature κ. Let $\sigma \geqslant 0$ be the coefficient of surface tension and let $\widehat{n}$ be the unit normal vector. For simplicity we will assume that the domain of the fluid flow is the two-dimensional torus, $\mathbb{T}^2$. Balance of force, together with incompressibility, in Eulerian coordinates, takes the form

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p - \nu \Delta v = \sigma \kappa \widehat{n} \delta_{\mathcal{C}_t}, \\ \nabla \cdot v = 0, \end{cases} \tag{1}$$

in $\mathbb{T}^2 \times \mathbb{R}_+$ where v is the velocity, p is the pressure, $\nu \geqslant 0$ is the coefficient of viscosity and $\delta_{\mathcal{C}_t}$ is the 1-dimensional Hausdorff measure on the curve $\mathcal{C}_t$. In addition we require that the curve $\mathcal{C}_t$ be transported by v. Throughout we assume $\nu > 0$ and $\sigma > 0$.

We are interested in the construction and approximation of solutions of (1) for all times $0 < t < \infty$, given initial conditions at $t = 0$. The difficulty is that the solution and the interface may become turbulent. The only available *a priori* estimate is given formally by the conservation of energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int |v|^2 \,\mathrm{d}x + \sigma L\right\} = -\nu \int |\nabla v|^2 \,\mathrm{d}x \tag{2}$$

where L is the length of $\mathcal{C}_t$.

Given the energy estimate above, it is natural to look for solutions $(u, \mathcal{C}_t)$ of (1) with u in $L^\infty((0,\infty); L^2) \cap L^2((0,\infty); H^1)$ (the Leray space) and $\mathcal{C}_t$ in a space of rectifiable curves. However, neither the normal vector nor the curvature are defined for general rectifiable curves. The first difficulty we must address is, therefore, that of defining the surface tension for such objects.

In Section 2, we introduce what we call *Eulerian theory of interface transport*. We describe an interface by means of a finite Radon measure, M, defined on $\mathbb{T}^2 \times \mathbb{R}^2 \times \mathbb{R}_+$, which expresses the position x of a fluid interface at each time t, together with the interface's tangent direction η. We notice that we can write the surface tension force, $\kappa \widehat{n} \delta_{\mathcal{C}_t}$, in terms of the measure M. The resulting expression for the surface tension force, which we call $\mathfrak{s}(M)$, turns out to be *linear* in M:

$$\mathfrak{s}(M) = \nabla_x \cdot \int_{\mathbb{R}^2} \widehat{\eta} \otimes \widehat{\eta} M \,\mathrm{d}\eta,$$

where $\widehat{\eta} = \eta/|\eta|$, and where we have abused notation by writing $M \,\mathrm{d}\eta$ instead of the more precise $\mathrm{d}M(\eta)$. We derive an evolution equation for M, when the interface is transported by a smooth velocity v:

$$\partial_t M + v \cdot \nabla_x M + Dv\eta \nabla_\eta M = \widehat{\eta} Dv \widehat{\eta} M. \tag{3}$$

In coordinates, this equation is

$$(\partial_t + v_i \partial_{x_i})M + (\partial_{x_i} v_j)\eta_i \partial_{\eta_j} M = \widehat{\eta}_i \widehat{\eta}_j (\partial_{x_i} v_j) M.$$

We do not know how to solve equation (3) if v is a vector field in the Leray space and the initial measure $M(t = 0)$ has support on a curve, even a smooth one. Our strategy is to mollify the velocity used to transport the interface, and to adjust the surface tension term in the Navier–Stokes system in order to retain an energy estimate. A large part of our effort is to produce a solution to this approximate problem.

With this in mind, we are ready to state our main result. We denote the space of bounded Radon measures on a domain U by $\mathcal{BM}(U)$.

THEOREM 1 Fix $\sigma > 0$ and $\nu > 0$. Let Ω_0 be a domain in $\mathbb{T}^2$ whose smooth boundary $\partial\Omega_0 \equiv \mathcal{C}_0$ is a Jordan curve. Let $v_0 \in L^2(\mathbb{T}^2)$ be a divergence-free vector field.

(i) There exist

$$v \in L^\infty((0,\infty); L^2(\mathbb{T}^2)) \cap L^2((0,\infty); H^1(\mathbb{T}^2)), \quad \widetilde{M} \in L^\infty((0,\infty); \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2))$$

such that the support of $\widetilde{M}(\cdot, t)$ is contained in $\{(x,\eta) \in \mathbb{T}^2 \times \mathbb{R}^2 : |\eta| = 1\}$ and the following equations hold in the sense of distributions:

$$\begin{cases} \partial_t v + (v \cdot \nabla_x)v + \nabla_x p - \nu \Delta_x v = \sigma \nabla_x \cdot \int_{\mathbb{R}^2} \widehat{\eta} \otimes \widehat{\eta} \widetilde{M} \, \mathrm{d}\eta, \\ \nabla_x \cdot v = 0, \\ v(x, t = 0) = v_0, \end{cases} \tag{4}$$

where $\widehat{\eta} = \eta/|\eta|$, the pressure p is a distribution, and the test functions are divergence-free.

(ii) For each $t > 0$ there exists an open set $\Omega(t) \subset \mathbb{T}^2$ with rectifiable boundary $\partial\Omega(t) \equiv \mathcal{C}_t$ such that if $b(\cdot, t) \equiv \chi_{\Omega(t)}$, then $b \in L^\infty((0,\infty); BV(\mathbb{T}^2)) \cap C([0,\infty); L^p(\mathbb{T}^2))$ for all $p < \infty$ and the following equation is satisfied in the sense of distributions:

$$\begin{cases} (\partial_t + v \cdot \nabla_x)b = 0, \\ b(x, t = 0) = \chi_{\Omega_0}. \end{cases} \tag{5}$$

Moreover, the vector function

$$\rho \equiv -\int \eta^\perp \widetilde{M} \, \mathrm{d}\eta, \tag{6}$$

where $\eta^\perp = (-\eta_2, \eta_1)$, is related to b by

$$\rho = \nabla_x b. \tag{7}$$

Note that equation (4) implies that $v \in C([0,\infty); H^{-k}(\mathbb{T}^2))$ for some k so that it makes sense to restrict v to $t = 0$. We choose to assume that the initial interface is a Jordan curve for convenience. This assumption is made simply to ensure that there are two distinct fluid regions at the initial time, without need for other conditions. The solution we obtain, however, may develop self-intersections.

Theorem 1 is a special case of Theorem 1.6 of [1], in the case where the fluids are Newtonian and viscosity-matched, and in the spatially periodic setting. This is not obvious in the statement of Theorem 1.6 because, first, Abels' result is cast in the language of geometric measure theory, while our result, given the simplified context, may be stated using the standard language of partial differential equations. Second, our notion of weak solution, which amounts to a varifold solution in the terminology of Abels, is stronger than the notion of measure-valued varifold solution whose existence is obtained in [1]. It is easy to see, however, that, in the special case of Newtonian, viscosity-matched fluids, Abels' proof of Theorem 1.6 also provides a varifold solution.

It is worth mentioning that Abels did not establish the added regularity, in time, which we obtain for b, namely, $b \in C([0, \infty); L^p(\mathbb{T}^2))$ for all $p < \infty$. This smoothness is what allows us to conclude (7) pointwise in time. The corresponding result, in Abels' work, is (1.15) of [1], which holds a.e. in time. Finally, we observe that our proof of Theorem 1 differs from Abels' proof of Theorem 1.6 of [1] in that we introduce a new sequence of approximations which transport both the approximate interface and its tangent directions.

Our approach is to obtain the solution $(v, \widetilde{M})$ of (4) as a weak limit of a sequence of suitable approximations for which the energy (2) is uniformly bounded. In particular, we obtain a sequence of smooth approximating curves with uniformly bounded length. We show that, passing to subsequences as needed, there exists a limit rectifiable curve $\mathcal{C}_t$. Now, length is weakly lower semicontinuous, so this opens the possibility that there may be a length defect in the passage to the limit. We show that, in the absence of such a length defect, v and the limit curve $\mathcal{C}_t$ are distributional solutions of the original problem (1). We also present an example which suggests that the weak limit v together with the limit curve $\mathcal{C}_t$ may satisfy (1) even if there is a length defect.

We now mention some related work. Without surface tension ($\sigma = 0$) there are the papers of Nouri and Poupaud and Giga and Takahashi for $\nu > 0$ and Delort for $\nu = 0$ [15], [9], [7]. Nouri and Poupaud studied a multifluid Navier–Stokes problem, with different densities and viscosities for each fluid, and proved the existence of global weak solutions. Giga and Takahashi similarly treated multifluid Stokes flows, proving the existence of global weak solutions when the viscosities are nearly equal. In a celebrated result, Delort showed the existence of global weak solutions for the Euler equations for vortex sheet initial data, in the case that the vortex sheet strength has a single sign. However, these papers do not provide any information at all about the nature of the interface. None of these works considered the effect of surface tension; the difficulty of handling surface tension is that one has to explicitly consider the interface.

Other articles which do treat the effect of surface tension either treat only smooth solutions or are numerical works. Beale proved the global existence of small, smooth solutions in a one-fluid viscous case (see [5]). Hou, Lowengrub, and Shelley (HLS) developed an efficient numerical method for the inviscid, irrotational case [11], [12]. Ambrose used ideas of HLS to prove short time well-posedness of smooth solutions of any size, still in the irrotational inviscid setting [3]. The works of Ambrose and HLS used in a fundamental way that the free surface was non-self-intersecting, so that topological transitions could not be treated. Smooth solutions for short times with surface tension have also been studied in three dimensions in the inviscid case by Ambrose and Masmoudi [4]; Cheng, Coutand, and Shkoller [6]; and Shatah and Zeng [17]. Tauber, Unverdi, and Tryggvason have a numerical method for viscous fluids with surface tension, but again, the interface is explicitly tracked, so that topological transitions cannot be studied [18]. Herrmann has introduced a numerical method (using level sets) in the inviscid case; by means of level sets, topological transitions can be handled [10]. Herrmann's method, however, fundamentally uses the vortex sheet structure, which would not be possible with viscosity.

The plan of the paper is as follows: In Section 2 we discuss the classical problem and our Eulerian model for interface transport. In Section 3 we introduce the approximate system, discuss its properties, and solve the approximate system. In Section 4 we take the weak limits, proving Theorem 1. In Section 5 we discuss possible defects. This includes a discussion as to the proper definition of such defects, a sufficient condition which excludes such defects, and the presentation of an illuminating example. In Section 6 we conditionally prove the nontriviality of solutions in

certain ways. Finally, in Section 7, we give a new approximate system which is more amenable to numerical simulation.

2. Eulerian interface dynamics

Our objective in this section is to formulate an Eulerian model for the evolution of interfacial Navier–Stokes flows with surface tension. To this end, we introduce an equation designed to propagate a curve (the interface) and its tangent vector. Let us assume that $v = v(x,t) \in C^\infty(\mathbb{T}^2 \times \mathbb{R}_+)$ is a smooth, divergence-free, vector field and consider the initial-value problem

$$\begin{cases} \partial_t M + v \cdot \nabla_x M + Dv(x,t)\eta \cdot \nabla_\eta M = \widehat{\eta} Dv(x,t)\widehat{\eta} M, \\ M(x,0) = M_0(x). \end{cases} \tag{8}$$

It is elementary to verify that $\nabla_\eta \cdot (Dv\,\eta) = \nabla_x \cdot v = 0$.

Let $X = X(\theta) \in C^1([a,b];\mathbb{T}^2)$ be any regular parametrization (i.e., $X'(\theta) \neq 0$) of a smooth Jordan curve in $\mathbb{T}^2$. We denote by $\mathcal{M}[X]$ the measure on $\mathbb{T}^2 \times \mathbb{R}^2$ defined by duality as

$$\langle \phi, \mathcal{M}[X] \rangle = \int_a^b \phi\left(X, \frac{\partial X}{\partial \theta}\right) \left|\frac{\partial X}{\partial \theta}\right| \mathrm{d}\theta \tag{9}$$

for all $\phi \in C_0(\mathbb{T}^2 \times \mathbb{R}^2)$.

We use the notation w^*-$\mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2)$ to mean $\mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2)$ endowed with the weak* topology.

PROPOSITION 1 Let $\mathcal{C}_0$ be a smooth Jordan curve on the torus, with a regular parametrization $x_0 = x_0(\theta)$, $a \leqslant \theta \leqslant b$. Then there is a unique distributional solution $M \in C([0,\infty); w^*\text{-}\mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2))$ of (8) with $M(t=0) = \mathcal{M}[x_0]$. Let $x(\theta,t)$ satisfy

$$\begin{cases} \partial x/\partial t = v(x,t), \\ x(\theta,0) = x_0(\theta). \end{cases}$$

The solution $M = M(\cdot,t)$ of (8) is given explicitly by $M = \mathcal{M}[x(\cdot,t)]$.

Proof. We use the method of characteristics. We introduce the characteristic curves $x = x(\theta,t)$, $\eta = \eta(\theta,t)$ and $\zeta = \zeta(\theta,t)$ as the unique solutions of the system of ordinary differential equations below:

$$\begin{cases} \partial x/\partial t = v(x,t), \\ \partial \eta/\partial t = Dv(x,t)\eta, \\ \partial \zeta/\partial t = \widehat{\eta} Dv(x,t)\widehat{\eta}\zeta, \\ x(\theta,0) = x_0(\theta), \\ \eta(\theta,0) = x_0'(\theta), \\ \zeta(\theta,0) = |x_0'(\theta)|, \end{cases} \tag{10}$$

The solution exists for all time because v is globally Lipschitz continuous.

Let $M \in L^\infty((0,\infty); \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2))$ denote the time-dependent measure defined through the expression

$$\langle \phi, M(\cdot,t) \rangle = \int_a^b \phi(x(\theta,t), \eta(\theta,t))\zeta(\theta,t)\,\mathrm{d}\theta$$

for all $\phi \in C_c(\mathbb{T}^2 \times \mathbb{R}^2)$. Actually, M is continuous in t with values in the space of bounded measures, endowed with the weak* topology. Note that, since $\theta \mapsto x_0(\theta)$ is a regular parametrization, $\eta(\theta, 0)$ does not vanish. Therefore, since the equation for η in (10) is linear with a smooth coefficient Dv, $\eta(\theta, t)$ does not vanish, and the support of $M(\cdot, t)$ stays bounded away from $\mathbb{T}^2 \times \{0\}$. It is a straightforward calculation to check that for all $\phi \in C_c^\infty((0, \infty) \times \mathbb{T}^2 \times \mathbb{R}^2)$,

$$\begin{aligned}\langle \phi, \partial_t M\rangle = -\langle \partial_t \phi, M\rangle &= -\int_0^\infty \int_a^b \partial_t \phi(x(\theta, t), \eta(\theta, t), t)\zeta(\theta, t)\,\mathrm{d}\theta\,\mathrm{d}t \\ &= \langle v \cdot \nabla_x \phi + Dv(x, t)\eta \cdot \nabla_\eta \phi + \widehat{\eta} Dv(x, t)\widehat{\eta}\phi, M\rangle.\end{aligned}$$

Hence, since v is divergence-free with respect to x and $Dv\,\eta$ is divergence-free with respect to η, M is a distributional solution of equation (8).

Next, we observe that, if x, η, ζ is a solution of (10) then x, $\partial_\theta x$, $|\partial_\theta x|$ is also a solution of (10) with the same initial data. The calculations that show this are elementary and straightforward. It follows by uniqueness for systems of ODEs that x, $\partial_\theta x$, $|\partial_\theta x|$ is the only solution of (10). Furthermore, since the equation for ζ is linear with respect to ζ and since the initial parametrization was assumed to be regular, it follows that $x = x(\cdot, t)$ is a regular parametrization for all $t \geqslant 0$. Therefore $M = \mathcal{M}[x(\cdot, t)]$.

Finally, uniqueness of the distributional solution follows by a Holmgren-type argument in a straightforward manner. Indeed, by duality the uniqueness is equivalent to the existence of a smooth solution ϕ to the adjoint problem in $[0, T]$ for arbitrary T with the right-hand side being a test function ψ and with final condition $\phi(x, T) = 0$. The existence is proved as usual by the method of characteristics. □

Next, we introduce a distribution which incorporates the surface tension term, written in terms of the measure M. If $M \in \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2)$ has support bounded away from the set $\{\eta = 0\}$ then we define the $\mathbb{R}^2$-valued distribution

$$\mathfrak{s}[M](x) \equiv \nabla_x \cdot \int_{\mathbb{R}^2} \widehat{\eta} \otimes \widehat{\eta}\, M(x, \eta)\,\mathrm{d}\eta. \tag{11}$$

We reiterate that above and throughout the remainder of this paper we will abuse notation, writing for instance $M\,\mathrm{d}\eta$ instead of $\mathrm{d}M(\eta)$, and $\mathfrak{s}[M]\,\mathrm{d}x$ instead of $\mathrm{d}\mathfrak{s}[M]$.

PROPOSITION 2 Let $\mathcal{C}$ be a smooth closed curve on the torus $\mathbb{T}^2$ with a regular parametrization $x = x(\theta)$, $a \leqslant \theta \leqslant b$. Let $M = \mathcal{M}[x]$. Then

$$\mathfrak{s}[M] = \kappa \widehat{n} \delta_{\mathcal{C}},$$

where κ is the curvature of $\mathcal{C}$ and $\widehat{n}$ is the unit normal vector to $\mathcal{C}$ chosen so that $\{x', \widehat{n}\}$ is a positive basis.

Proof. Let $\varphi = \varphi(x) = (\varphi^1(x), \varphi^2(x))$ be a smooth, compactly supported test vector field. We write

$$\begin{aligned}\langle \varphi, \mathfrak{s}[M]\rangle &= \sum_{i,j=1}^{2} \left\langle \varphi^i \frac{\eta^i}{|\eta|}\frac{\eta^j}{|\eta|}, \partial_{x_j} M \right\rangle \\ &= -\sum_{i=1}^{2}\int_a^b \sum_{j=1}^{2} \partial_{x_j}\varphi^i(x(\theta))x_j'(\theta)\frac{x_i'(\theta)}{|x'(\theta)|}\,\mathrm{d}\theta = -\sum_{i=1}^{2}\int_a^b (\varphi^i(x(\cdot)))'(\theta)\frac{x_i'(\theta)}{|x'(\theta)|}\,\mathrm{d}\theta \\ &= \sum_{i=1}^{2}\left(\int_a^b \varphi^i(x(\theta))\kappa(x(\theta))\widehat{n}^i(x(\theta))|x'(\theta)|\,\mathrm{d}\theta - \varphi^i(x(b))\frac{x_i'(b)}{|x'(b)|} + \varphi^i(x(a))\frac{x_i'(a)}{|x'(a)|}\right),\end{aligned}$$

as $(x_i'/|x'|)'(\theta) = \kappa(x(\theta))\widehat{n}^i(x(\theta))|x'(\theta)|$. Hence we obtain

$$\langle \varphi, \mathfrak{s}[M]\rangle = \langle \varphi, \kappa\widehat{n}\delta_{\mathcal{C}}\rangle - \langle \varphi, \widehat{x'(b)}\delta_{x(b)} - \widehat{x'(a)}\delta_{x(a)}\rangle.$$

The boundary terms disappear since the curve $\mathcal{C}$ is closed, which concludes the proof. □

Coupling the equations for transport of the interface with a viscous flow v, using the surface tension term introduced in (11), we write the following system of equations:

$$\begin{cases} \partial_t M + v(x,t)\cdot\nabla_x M + Dv(x,t)\eta\cdot\nabla_\eta M = \widehat{\eta}Dv(x,t)\widehat{\eta}M, \\ \partial_t v + (v\cdot\nabla_x)v = -\nabla_x p + \nu\Delta_x v + \sigma\mathfrak{s}[M], \\ \nabla_x\cdot v(x,t) = 0, \\ M(x,\eta,0) = M_0(x), \\ v(x,0) = v_0(x). \end{cases} \tag{12}$$

This set of equations is what we propose as an Eulerian model for the evolution of viscous interfacial dynamics in the presence of surface tension. (Although we do not study inviscid models in the present work, notice that Propositions 1 and 2 do not depend in any way on the presence of viscosity. Thus, setting $\nu = 0$ in (12) would also provide us with an Eulerian model for the evolution of inviscid interfacial dynamics in the presence of surface tension.) We conclude this section with energy estimates for solutions of the system (12).

PROPOSITION 3 (i) Let (M, v) be a solution of (12) with $M \in C^1([0,\infty); w^*\text{-}\mathcal{BM}(\mathbb{T}^2\times\mathbb{R}^2))$ and $v \in C^1(\mathbb{T}^2\times[0,\infty))$ and with initial data (M_0, v_0). If, additionally, M decays fast enough at infinity with respect to η and has support bounded away from $\{\eta = 0\}$, then

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^2}|v(x,t)|^2\,\mathrm{d}x + \sigma\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} M(x,\eta,t)\,\mathrm{d}\eta\,\mathrm{d}x + \nu\int_0^t\int_{\mathbb{T}^2}|\nabla v|^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \\ = \frac{1}{2}\int_{\mathbb{T}^2}|v_0(x)|^2\,\mathrm{d}x + \sigma\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} M_0(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}x.\end{aligned}$$

(ii) If M is of the form $\mathcal{M}[x]$ with x a regular parametrization of a smooth Jordan curve $\mathcal{C}$ on $\mathbb{T}^2$ then the quantity

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} M(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}x$$

is the length of the curve $\mathcal{C} = \{x = x(\theta)\}$.

Proof. We begin by multiplying the evolution equation for v in (12) by v itself and integrating over $\mathbb{T}^2$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\frac{1}{2}|v(x,t)|^2\,\mathrm{d}x = \sigma\int_{\mathbb{T}^2} v(x,t)\cdot\mathfrak{s}[M](x,t)\,\mathrm{d}x - \nu\sum_{i=1}^{2}\int_{\mathbb{T}^2}|\nabla v^i(x,t)|^2\,\mathrm{d}x. \tag{13}$$

Next note that

$$\begin{aligned}
\sigma\int_{\mathbb{T}^2} v(x,t)\cdot\mathfrak{s}[M](x,t)\,\mathrm{d}x &= \sigma\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} v^i\frac{\eta^i}{|\eta|}\partial_{x_j}M\frac{\eta^j}{|\eta|}\,\mathrm{d}\eta\,\mathrm{d}x\\
&= -\sigma\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\frac{\eta^i}{|\eta|}\partial_{x_j}v^i\frac{\eta^j}{|\eta|}M\,\mathrm{d}\eta\,\mathrm{d}x = -\sigma\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\widehat{\eta}Dv\widehat{\eta}M\,\mathrm{d}\eta\,\mathrm{d}x\\
&= -\sigma\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}(\partial_t M + v\cdot\nabla_x M + Dv\,\eta\cdot\nabla_\eta M)\,\mathrm{d}\eta\,\mathrm{d}x = -\sigma\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}M(x,\eta,t)\,\mathrm{d}\eta\,\mathrm{d}x.
\end{aligned}$$

We have used here the facts that the x-divergence of v and the η-divergence of $Dv\,\eta$ vanish and that M decays sufficiently fast as $|\eta|\to\infty$. Using this observation in (13) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{T}^2}\frac{1}{2}|v(x,t)|^2\,\mathrm{d}x + \sigma\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}M(x,\eta,t)\,\mathrm{d}\eta\,\mathrm{d}x\right) = -\nu\sum_{i=1}^{2}\int_{\mathbb{T}^2}|\nabla v^i(x,t)|^2\,\mathrm{d}x.$$

which yields (i), upon integration in time.

Next, to obtain (ii) we note that, if M is of the form $\mathcal{M}[x]$, then

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}M(x,\eta,t)\,\mathrm{d}\eta\,\mathrm{d}x \equiv \langle 1, M\rangle = \int_a^b\left|\frac{\partial x}{\partial\theta}(\theta,t)\right|\mathrm{d}\theta,$$

which is precisely the length of $\mathcal{C}$. This concludes the proof. □

3. The approximate system

Let $\mathcal{C}_0$ be a smooth Jordan curve on the torus, i.e. a simple curve which divides the torus into two connected components, with the regular parametrization $x_0 = x_0(\theta)$, $a\leqslant\theta\leqslant b$. For each $\varepsilon>0$, let J^ε be a standard Friedrichs mollifier in x. We introduce the following approximate system:

$$\begin{cases}
\partial_t v_\varepsilon + v_\varepsilon\cdot\nabla_x v_\varepsilon = -\nabla_x p_\varepsilon + \nu\Delta_x v_\varepsilon + J^\varepsilon\sigma\mathfrak{s}(M_\varepsilon),\\
\partial_t M_\varepsilon + (J^\varepsilon v_\varepsilon)\cdot\nabla_x M_\varepsilon + (J^\varepsilon Dv_\varepsilon)\eta\cdot\nabla_\eta M_\varepsilon = \widehat{\eta}(DJ^\varepsilon v_\varepsilon)\widehat{\eta}M_\varepsilon,\\
\nabla_x\cdot v_\varepsilon = 0,\\
v_\varepsilon(x,0) = v_0(x),\\
M(x,\eta,0) = \mathcal{M}[x_0].
\end{cases} \tag{14}$$

Note that, once we establish existence for (14), we will infer, by virtue of Proposition 1, and because $J^\varepsilon v_\varepsilon$ is smooth and divergence-free, that M^ε is the measure $\mathcal{M}[x_\varepsilon(\cdot,t)]$, where $x_\varepsilon(\cdot,t)$ is a regular parametrization of the image of $\mathcal{C}_0$ by the flow induced by $J^\varepsilon v_\varepsilon$.

LEMMA 1 The mollified system satisfies the basic energy conservation:

$$\frac{1}{2}\int_{\mathbb{T}^2} |v_\varepsilon|^2 \,\mathrm{d}x + \nu \int_0^t \int_{\mathbb{T}^2} |\nabla v_\varepsilon|^2 \,\mathrm{d}x + \sigma \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} M_\varepsilon \,\mathrm{d}x \,\mathrm{d}\eta = E_0,$$

where

$$E_0 = \frac{1}{2}\int_{\mathbb{T}^2} |v_0|^2 \,\mathrm{d}x + \sigma \int_{\mathbb{R}^2}\int_{\mathbb{T}^2} \mathcal{M}[x_0] \,\mathrm{d}x \,\mathrm{d}\eta.$$

We do not include a proof, since the proof is the same as in Proposition 3(i).

We denote by $\mathbb{P}$ the Leray projector on the torus, given by $\mathbb{P}v \equiv v - \nabla \Delta_0^{-1} \operatorname{div} v$, where Δ_0^{-1} is the solution operator for the Laplacian with zero average. We now establish existence for the approximate problem (14). The proof is surprisingly delicate. The strategy is based on Picard's existence theorem for ODEs on Banach spaces; we use several levels of mollification in order to produce approximations which satisfy an energy estimate, which preserve nonnegativity of the transported scalar and which keep the support of the transported scalar bounded away from $\eta = 0$.

THEOREM 2 Let $\varepsilon > 0$ and $T > 0$. Then there are $v_\varepsilon \in L^\infty((0,T); L^2(\mathbb{T}^2)) \cap L^2((0,T); H^1(\mathbb{T}^2))$ and $M_\varepsilon \in L^\infty((0,T); \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2))$ which satisfy (14) in the sense of distributions.

Proof. We wish to find a global solution to the approximate problem (14). To begin, we introduce a system with three further regularization parameters, $\delta' > 0$, $\delta > 0$ and $\gamma > 0$. These parameters are all mollification parameters. We will apply mollifiers $\mathcal{J}_\gamma^x$ and $\mathcal{J}_\gamma^\eta$ to the initial data. We also will include mollifiers which have $\delta' > 0$ and $\delta > 0$ as the mollification parameter in the evolution equations. We denote these new mollifiers by $\mathcal{J}_{\delta'}^x$, $\mathcal{J}_\delta^x$ and $\mathcal{J}_\delta^\eta$; in each case, the mollification occurs with respect to the variable indicated in the superscript.

Before introducing our mollified system, we make an aside on transport equations. Consider the class of transport equations

$$\partial_t f + J^\varepsilon g \cdot \nabla_x f + k_1 J^\varepsilon Dg\eta \cdot \nabla_\eta f = k_2 f. \tag{15}$$

We take $g(x,t)$, $k_1(\eta)$, and $k_2(x,\eta,t)$ to be given functions, and we take the initial data $f(x,\eta,0) = f_0$, with $\operatorname{supp}(f_0) \subseteq \{(x,\eta) : 1/2 \leqslant |\eta| \leqslant 3/2\}$. We furthermore assume that $\|g\|_{L^2} \leqslant E_0$ for all t and $|k_1(\eta)| \leqslant 1$ for all η. This equation can be solved along characteristics; much as in Proposition 1, we have the characteristic equation $\dot\eta = k_1 J^\varepsilon Dg\eta$. Together with the assumption on the L^2-norm of g, this implies a bound on the growth of the η-support of f, and in particular, there exists a closed annulus $Y \subseteq \mathbb{R}^2$ such that for all $t \in (0,T)$, $\operatorname{supp}(f) \subseteq \{(x,\eta,t) : \eta \in Y\}$. It is important to note that Y can be taken to be independent of f_0, g, k_1, and k_2, as long as the above conditions are satisfied. Naturally, given the above condition on the initial support of f, we have $\{\eta : 1/2 \leqslant |\eta| \leqslant 3/2\} \subseteq Y$. We note that Y can be taken to be bounded and to exclude the origin. We introduce a smooth cutoff function $\Upsilon = \Upsilon(\eta)$. We take Υ to be identically equal to 1 on the set Y, and $0 \leqslant \Upsilon(\eta) \leqslant 1$ for all η, and we take Υ to be compactly supported such that the origin is not included in the support of Υ. We reiterate that it is important that we are only solving until time T, and that Υ is independent of time.

Using this cutoff Υ, the new mollifiers, and the Leray projector, and replacing M with m^2, we write the new system

$$\partial_t v + \mathbb{P}\mathcal{J}_\delta^x (v \cdot \nabla_x \mathcal{J}_\delta^x v) - \nu (\mathcal{J}_{\delta'}^x)^2 \Delta_x v = \mathbb{P}\sigma J^\varepsilon \mathfrak{s}(m^2 \Upsilon), \tag{16}$$

$$\partial_t m + \mathcal{J}_\delta^x(J^\varepsilon v \cdot \nabla_x \mathcal{J}_\delta^x m) + \mathcal{J}_\delta^\eta(\Upsilon J^\varepsilon Dv\eta \cdot \nabla_\eta((\mathcal{J}_\delta^\eta m)\Upsilon)) = \frac{1}{2}\widehat{\eta}(J^\varepsilon Dv)\widehat{\eta} m \Upsilon. \tag{17}$$

We take the initial data

$$v(x,0) = \mathcal{J}_\gamma^x v_0(x), \qquad m(x,\eta,0) = \sqrt{\mathcal{J}_\gamma^x \mathcal{J}_\gamma^\eta \mathcal{M}[x_0]}, \tag{18}$$

where of course we assume $\nabla_x \cdot v_0 = 0$. We also place a requirement on the operator $\mathcal{J}_\gamma^\eta$: we require that $\operatorname{supp}(m(x,\eta,0)) \subseteq \{(x,\eta) : 1/2 \leqslant |\eta| \leqslant 3/2\}$ for all γ.

With the presence of the cutoff function Υ and the smoothing operators $\mathcal{J}_{\delta'}^x$, $\mathcal{J}_\delta^x$, $\mathcal{J}_\delta^\eta$ and J^ε, all of the terms except the time derivatives in (16), (17) are now bounded (i.e. Lipschitz), with respect to v and m, where the space $L^2(\mathbb{T}^2)$ is used for v and $L^2(\mathbb{T}^2 \times \mathbb{R}^2)$ is used for m. The Picard Theorem for ODEs on a Banach space guarantees existence of a unique solution until a time $T_{\varepsilon,\gamma,\delta,\delta'}$. We give the names $v_{\varepsilon,\gamma,\delta,\delta'}$, $m_{\varepsilon,\gamma,\delta,\delta'}$ to solutions of (16), (17); these solutions are in $C^1([0, T_{\varepsilon,\gamma,\delta,\delta'}]; L^2)$ in each case. (The reader might consult [14] to see a similar approach used to prove the existence of solutions for just the Navier–Stokes equations.)

These are in fact autonomous ODEs on a Banach space, and the continuation theorem for such ODEs guarantees existence of a solution as long as the L^2-norm of the solution does not blow up. We multiply (16) by v and (17) by $2m$, and we find the following energy conservation:

$$\frac{1}{2}\int_{\mathbb{T}^2} v_{\varepsilon,\gamma,\delta,\delta'}^2 \,\mathrm{d}x + \int_0^T \int_{\mathbb{T}^2} \nu |\nabla \mathcal{J}_{\delta'}^x v_{\varepsilon,\gamma,\delta,\delta'}|^2 \,\mathrm{d}x + \sigma \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} m_{\varepsilon,\gamma,\delta,\delta'}^2 \,\mathrm{d}\eta\,\mathrm{d}x = \tilde{E}_0. \tag{19}$$

In particular, we have $v_{\varepsilon,\gamma,\delta,\delta'} \in L^\infty((0,T); L^2(\mathbb{T}^2))$ and $m_{\varepsilon,\gamma,\delta,\delta'} \in L^\infty((0,T); L^2(\mathbb{T}^2 \times \mathbb{R}^2))$, for any $T > 0$. Thus, there is a solution to the $(\varepsilon, \gamma, \delta, \delta')$-IVP on $[0, T)$ for arbitrary $T > 0$.

At this point, we also have higher regularity of $m_{\varepsilon,\gamma,\delta,\delta'}$. That is, $m_{\varepsilon,\gamma,\delta,\delta'}$ is in any Sobolev space, and is bounded in these spaces uniformly in δ' and uniformly in δ. That $m_{\varepsilon,\gamma,\delta,\delta'}(t = 0)$ is in any Sobolev space is clear, because of the presence of the $\mathcal{J}_\gamma$ mollifiers. Now, we let $P \in \mathbb{Z}^+$ and we let p be a multi-index of order at most P. Applying the derivative operator ∂^p to equation (17), we have

$$\partial_t \partial^p m = -\mathcal{J}_\delta^x \partial^p(J^\varepsilon v \cdot \nabla_x \mathcal{J}_\delta^x m) - \mathcal{J}_\delta^\eta \partial^p(\Upsilon J^\varepsilon Dv\eta \cdot \nabla_\eta((\mathcal{J}_\delta^\eta m)\Upsilon)) + \partial^p\left(\frac{1}{2}\widehat{\eta} J^\varepsilon Dv \widehat{\eta} m \Upsilon\right). \tag{20}$$

We multiply by $\partial^p m$ and integrate with respect to η and x. We get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p m)^2 \,\mathrm{d}\eta\,\mathrm{d}x = &- \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p m)(\mathcal{J}_\delta^x \partial^p(J^\varepsilon v \cdot \nabla_x \mathcal{J}_\delta^x m))\,\mathrm{d}\eta\,\mathrm{d}x \\
&- \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p m)(\mathcal{J}_\delta^\eta \partial^p(\Upsilon J^\varepsilon Dv\eta \cdot \nabla_\eta((\mathcal{J}_\delta^\eta m)\Upsilon)))\,\mathrm{d}\eta\,\mathrm{d}x \\
&+ \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p m)\left(\partial^p\left(\frac{1}{2}\widehat{\eta} J^\varepsilon Dv \widehat{\eta} m \Upsilon\right)\right)\mathrm{d}\eta\,\mathrm{d}x \\
= &\ I + II + III.
\end{aligned} \tag{21}$$

We now perform estimates for I; the estimates for II and III are entirely similar. Using the fact that $\mathcal{J}_\delta^x$ is self-adjoint and commutes with derivatives, we have

$$I = -\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p \mathcal{J}_\delta^x m)(\partial^p(J^\varepsilon v \cdot \nabla_x \mathcal{J}_\delta^x m))\,\mathrm{d}\eta\,\mathrm{d}x.$$

We add and subtract to rewrite the second factor:

$$I = -\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p \mathcal{J}_\delta^x m)(\{J^\varepsilon v\}\cdot\nabla_x \partial^p \mathcal{J}_\delta^x m)\,\mathrm{d}\eta\,\mathrm{d}x$$
$$-\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} (\partial^p \mathcal{J}_\delta^x m)(\partial^p[\{J^\varepsilon v\}\cdot\nabla_x \mathcal{J}_\delta^x m] - [\{J^\varepsilon v\}\cdot\nabla_x \partial^p \mathcal{J}_\delta^x m])\,\mathrm{d}\eta\,\mathrm{d}x = I_A + I_B. \quad (22)$$

We can integrate I_A by parts, and we can use the divergence-free condition of v:

$$I_A = -\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{1}{2}\{J^\varepsilon v\}\cdot\nabla(\partial^p \mathcal{J}_\delta^x m)^2\,\mathrm{d}\eta\,\mathrm{d}x = \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{1}{2}(\mathrm{div}\{J^\varepsilon v\})(\partial^p \mathcal{J}_\delta^x m)^2\,\mathrm{d}\eta\,\mathrm{d}x = 0.$$

Taking P sufficiently large, and using standard commutator estimates, we have the following estimate for I_B:

$$|I_B| \leqslant c\|m\|_{H^P}\|J^\varepsilon v\|_{H^P}\|\nabla_x \mathcal{J}_\delta^x m\|_{H^{P-1}} \leqslant c\|m\|_{H^P}^2.$$

In the above estimate, the constant c depends on ε and on E_0, since we have used Lemma 4 below. Summing over all multi-indices of order at most P yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m\|_{H^P}^2 \leqslant c\|m\|_{H^P}^2.$$

Thus, we see that the H^P norm of m grows at most exponentially, for any sufficiently large P. Furthermore, the growth rate depends only on ε, P, and E_0, and the initial H^P norm depends on γ. Thus, the norm of m in any Sobolev space is bounded independently of δ' and δ, for all $t \in [0, T)$, as claimed.

We will now in turn send each of δ', δ, and γ to zero, taking weak* limits, passing to subsequences as needed. The argument in each case is essentially the same. We begin with δ'. Using the conserved energy (19), $m_{\varepsilon,\gamma,\delta,\delta'}$ is bounded, uniformly in δ', in $L^\infty((0,T);L^2)$. Also, because of the presence of the cutoff and the abundance of mollifiers, (17) together with the conserved energy implies that $\partial_t m_{\varepsilon,\gamma,\delta,\delta'}$ is bounded (again, uniformly in δ') in $L^\infty((0,T);H^{-1})$. This implies that (up to subsequences) $m_{\varepsilon,\gamma,\delta,\delta'}$ converges in $C([0,T];w\text{-}L^2)$ to an element $m_{\varepsilon,\gamma,\delta} \in L^\infty((0,T);L^2)$, where $w\text{-}L^2$ denotes L^2 with the weak topology (cf. Appendix C of [13]). Next we note that, using the energy estimate, it follows that $\partial_t v_{\varepsilon,\gamma,\delta,\delta'}$ is bounded in $L^\infty((0,T);H^{-2})$, and hence, (up to subsequences) $v_{\varepsilon,\gamma,\delta,\delta'}$ also converges in $C([0,T];w\text{-}L^2)$ to an element $v_{\varepsilon,\gamma,\delta}$, with $v_{\varepsilon,\gamma,\delta} \in L^\infty((0,T);L^2)$.

We now verify that the limit as $\delta' \to 0$ is a solution of the system

$$\begin{cases} \partial_t v + \mathcal{J}_\delta^x \mathbb{P}(v\cdot\nabla_x \mathcal{J}_\delta^x v) - \nu\Delta_x v = \mathbb{P}\sigma J^\varepsilon \mathfrak{s}(m^2\Upsilon), \\ \partial_t m + \mathcal{J}_\delta^x(J^\varepsilon v\cdot\nabla_x \mathcal{J}_\delta^x m) + \mathcal{J}_\delta^\eta(\Upsilon J^\varepsilon Dv\eta\cdot\nabla_\eta((\mathcal{J}_\delta^\eta m)\Upsilon)) = \frac{1}{2}\widehat{\eta}J^\varepsilon Dv\widehat{\eta}m\Upsilon, \end{cases} \quad (23)$$

with initial data (18). First we discuss convergence in the $m_{\varepsilon,\gamma,\delta,\delta'}$ evolution equation; for the linear term this is straightforward. The nonlinear terms each involve a product of $J^\varepsilon v_{\varepsilon,\gamma,\delta,\delta'}$ and $m_{\varepsilon,\gamma,\delta,\delta'}$. By Lemma 4 below, we know that $J^\varepsilon v_{\varepsilon,\gamma,\delta,\delta'}$ is bounded, uniformly with respect to δ' and t, in any Sobolev space. Furthermore, as we have already seen, using equation (16) together with the bounds for $v_{\varepsilon,\gamma,\delta,\delta'}$ and $m_{\varepsilon,\gamma,\delta,\delta'}$, we can establish bounds in low-regularity spaces for $\partial_t v_{\varepsilon,\gamma,\delta,\delta'}$ and $\partial_t m_{\varepsilon,\gamma,\delta,\delta'}$, uniformly in δ'. Thus, using the Aubin–Lions lemma, we find that $J^\varepsilon v_{\varepsilon,\gamma,\delta,\delta'}$ is precompact, with respect to δ', in $C([0,T];H^s)$ for arbitrarily large s and hence the nonlinear

terms in the $m_{\varepsilon,\gamma,\delta,\delta'}$ equation each form a weak-strong pair. (That is, we get strong convergence in the $J^\varepsilon v_{\varepsilon,\gamma,\delta,\delta'}$ term because of the compactness afforded us by its smoothness.) Similarly, to get convergence in the $v_{\varepsilon,\gamma,\delta,\delta'}$ evolution equation, we only need to pay careful attention to the nonlinear terms, which are quadratic in $v_{\varepsilon,\gamma,\delta,\delta'}$ or in $m_{\varepsilon,\gamma,\delta,\delta'}$. For the first of these, we have a uniform bound, in any Sobolev space, on $\mathcal{J}_\delta^x v_{\varepsilon,\gamma,\delta,\delta'}$ from the conserved energy (19) and properties of mollifiers. Furthermore, using (16) and (19), we can establish a uniform-in-δ' estimate for $\partial_t \mathcal{J}_\delta^x v$ in H^{-2}. Thus, by the Aubin–Lions lemma, we find that $\mathcal{J}_\delta^x v_{\varepsilon,\gamma,\delta,\delta'}$ are contained in a compact subset of $C([0,T]; H^1)$. This implies that $v_{\varepsilon,\gamma,\delta,\delta'}$ and $\nabla_x \mathcal{J}_\delta^x v_{\varepsilon,\gamma,\delta,\delta'}$ form a weak-strong pair. We treat the second nonlinear term (i.e., the surface tension term) in the same way, this time using the uniform bounds for $m_{\varepsilon,\gamma,\delta,\delta'}$ in any Sobolev space which we established above. Of course, using this regularity for $m_{\varepsilon,\gamma,\delta,\delta'}$, Lemma 4 below, and (17), we can bound $\partial_t m_{\varepsilon,\gamma,\delta,\delta'}$ uniformly in δ' in H^{-1}.

Notice that by Lemma 2 below, we have additional regularity for solutions of (23): $v_{\varepsilon,\gamma,\delta} \in L^\infty((0,T); L^2) \cap L^2((0,T); H^1)$.

We now take the limit as $\delta \to 0$. By the same argument as before, $m_{\varepsilon,\gamma,\delta}$ converges, in $C([0,T]; w\text{-}L^2)$, to an element $m_{\varepsilon,\gamma} \in L^\infty((0,T); L^2)$; furthermore, using the new energy estimate in Lemma 2, we find (up to subsequences) that $v_{\varepsilon,\gamma,\delta}$ converges weak* to $v_{\varepsilon,\gamma} \in L^\infty((0,T); L^2) \cap L^2((0,T); H^1)$. In the same way as before, we can verify that $m_{\varepsilon,\gamma}$ and $v_{\varepsilon,\gamma}$ satisfy the following system:

$$\begin{cases} \partial_t v + \mathbb{P}(v \cdot \nabla_x v) - \nu \Delta_x v = \mathbb{P}\sigma J^\varepsilon \mathfrak{s}(m^2 \Upsilon), \\ \partial_t m + J^\varepsilon v \cdot \nabla_x m + \Upsilon J^\varepsilon Dv\eta \cdot \nabla_\eta (m\Upsilon) = \frac{1}{2}\widehat{\eta} J^\varepsilon Dv \widehat{\eta} m \Upsilon, \end{cases} \tag{24}$$

with initial data (18). Indeed, the main difference from the previous argument comes when handling the mollified convective term in the velocity equation; previously, for this term, we relied on the presence of the operator $\mathcal{J}_\delta^x$, while now we need estimates independent of δ. The estimate implied by Lemma 2 includes the basic energy estimate for solutions of the Navier–Stokes equations and hence the same argument used to pass to the limit in the convection term in order to obtain Leray weak solutions for Navier–Stokes can be applied in our case as well, i.e., compactness in $L^2((0,T); L^2)$ for velocity. The nonlinear surface tension term is dealt with analogously to the limit $\delta' \to 0$, using the compactness afforded by the higher order energy estimates for m.

The evolution equation for m in (24) is a transport equation for m, and in particular, this is a transport equation of the type (15) with $g = v$, $k_1 = \Upsilon^2$, and $k_2 = \frac{1}{2}\widehat{\eta}(J^\varepsilon Dv)\widehat{\eta}\Upsilon - \Upsilon(J^\varepsilon Dv)\eta \cdot \nabla_\eta \Upsilon$. This implies that $\operatorname{supp}(m_{\varepsilon,\gamma}) \subseteq \{(x,\eta,t) : \eta \in Y\}$, and thus $m\Upsilon = m$ and $m\nabla_\eta \Upsilon = 0$. This implies that the system satisfied by $v_{\varepsilon,\gamma}$, $m_{\varepsilon,\gamma}$ is in fact

$$\begin{cases} \partial_t v_{\varepsilon,\gamma} + \mathbb{P}(v_{\varepsilon,\gamma} \cdot \nabla_x v_{\varepsilon,\gamma}) - \nu \Delta_x v_{\varepsilon,\gamma} = \mathbb{P}\sigma J^\varepsilon \mathfrak{s}(m_{\varepsilon,\gamma}^2), \\ \partial_t m_{\varepsilon,\gamma} + J^\varepsilon v_{\varepsilon,\gamma} \cdot \nabla_x m_{\varepsilon,\gamma} + J^\varepsilon Dv_{\varepsilon,\gamma}\eta \cdot \nabla_\eta m_{\varepsilon,\gamma} = \frac{1}{2}\widehat{\eta} J^\varepsilon Dv_{\varepsilon,\gamma} \widehat{\eta} m_{\varepsilon,\gamma}, \end{cases} \tag{25}$$

with initial data (18).

We introduce now $M_{\varepsilon,\gamma} = m_{\varepsilon,\gamma}^2$. We multiply the $m_{\varepsilon,\gamma}$ evolution equation in (25) by $m_{\varepsilon,\gamma}$ to get the system satisfied by $v_{\varepsilon,\gamma}$, $M_{\varepsilon,\gamma}$:

$$\begin{cases} \partial_t v_{\varepsilon,\gamma} + \mathbb{P}(v_{\varepsilon,\gamma} \cdot \nabla_x v_{\varepsilon,\gamma}) - \nu \Delta_x v_{\varepsilon,\gamma} = \mathbb{P}\sigma J^\varepsilon \mathfrak{s}(M_{\varepsilon,\gamma}), \\ \partial_t M_{\varepsilon,\gamma} + J^\varepsilon v_{\varepsilon,\gamma} \cdot \nabla_x M_{\varepsilon,\gamma} + J^\varepsilon Dv_{\varepsilon,\gamma}\eta \cdot \nabla_\eta M_{\varepsilon,\gamma} = \widehat{\eta} J^\varepsilon Dv_{\varepsilon,\gamma} \widehat{\eta} M_{\varepsilon,\gamma}, \end{cases} \tag{26}$$

now with initial data

$$v_{\varepsilon,\gamma}(x,0) = \mathcal{J}_\gamma^x v_0(x), \qquad M_{\varepsilon,\gamma}(x,\eta,0) = \mathcal{J}_\gamma^x \mathcal{J}_\gamma^\eta \mathcal{M}[x_0]. \tag{27}$$

Since $M_{\varepsilon,\gamma}$ and $v_{\varepsilon,\gamma}$ are uniformly bounded in their spaces (the space for $M_{\varepsilon,\gamma}$ is $L^\infty((0,T);L^1)$), we obtain weak* limits M_ε and v_ε as $\gamma \to 0$ in the same manner as before. Furthermore, we check that v_ε and M_ε satisfy the approximate system (14). This is straightforward given the above remark about the support, and in fact is just the same as the prior proof that $v_{\varepsilon,\gamma}$ and $m_{\varepsilon,\gamma}$ satisfy (24), but with $M_{\varepsilon,\gamma}$ now taken in L^1 and the limit M_ε taken in $\mathcal{BM}$, and the surface tension term now being linear. □

In the preceding proof, we relied on the following lemmas.

LEMMA 2 The mollified systems (i.e., with $\varepsilon > 0$, $\gamma \geqslant 0$, and $\delta \geqslant 0$, but without δ') satisfy the basic energy conservation:

$$\frac{1}{2}\int_{\mathbb{T}^2} v_{\varepsilon,\gamma,\delta}^2 \,\mathrm{d}x + \int_0^T \int_{\mathbb{T}^2} \nu|\nabla v_{\varepsilon,\gamma,\delta}|^2 \,\mathrm{d}x + \sigma \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} m_{\varepsilon,\gamma,\delta}^2 \,\mathrm{d}\eta\,\mathrm{d}x = \tilde{E}_0.$$

Again, the proof of this is omitted, as it is essentially the same as the proof of energy conservation in the non-mollified case.

LEMMA 3 If Q is a bounded positive measure such that the support of Q does not include $\eta = 0$ and if there exists $K > 0$ such that $\int\int Q\,\mathrm{d}\eta\,\mathrm{d}x \leqslant K$ then for any nonnegative γ, p_1 and p_2, $\partial_{x_1}^{p_1}\partial_{x_2}^{p_2} J^\varepsilon \int_{\mathbb{R}^2}(\widehat{\eta}\otimes\widehat{\eta})Q(x,\eta)\,\mathrm{d}\eta$ is in $L^1\cap L^\infty$, and the L^1 and L^∞ norms are bounded by a constant depending only on K and ε.

Proof. This is the standard theory of mollifiers, together with the bound for Q. We denote by f_ε the function such that for all g, $J^\varepsilon g = f_\varepsilon * g$. We start by looking at the absolute value of the quantity in question:

$$\begin{aligned}\left|\partial_{x_1}^{p_1}\partial_{x_2}^{p_2} J^\varepsilon \int_{\mathbb{R}^2}(\widehat{\eta}\otimes\widehat{\eta})Q(x,\eta)\,\mathrm{d}\eta\right| &= \left|\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}(\partial_{x_1}^{p_1}\partial_{x_2}^{p_2} f_\varepsilon(x-y))(\widehat{\eta}\otimes\widehat{\eta})Q(y,\eta)\,\mathrm{d}\eta\,\mathrm{d}y\right| \\ &\leqslant C_\varepsilon\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} Q(y,\eta)\,\mathrm{d}\eta\,\mathrm{d}y \leqslant C_\varepsilon K. \end{aligned} \tag{28}$$

This proves the L^∞ estimate. The L^1 estimate follows just by integrating in x. □

LEMMA 4 If there exists $K > 0$ such that $\|v\|_{L^2} \leqslant K$, then $J^\varepsilon Dv \in H^s$ for any s, and the norm is bounded by a constant depending only on K and ε.

We exclude the proof, since this is exactly the standard theory of mollifiers. See, for example, [2] or [14].

4. Proof of Theorem 1

In this section we give the proof of our main result.

First, for any smooth Jordan curve $\mathcal{C}$ with regular parametrization $X = X(\theta)$, $a \leqslant \theta \leqslant b$, we introduce another measure $\widetilde{M} = \widetilde{\mathcal{M}}_{\mathcal{C}}$, supported on $\{|\eta| = 1\}$, induced by $M = \mathcal{M}[X]$. Define

$$\langle \phi, \widetilde{\mathcal{M}}_{\mathcal{C}}\rangle \equiv \int_a^b \phi\left(X(\theta), \widehat{\frac{\partial X}{\partial\theta}}\right)\left|\frac{\partial X}{\partial\theta}\right|\mathrm{d}\theta, \qquad \phi \in C_0(\mathbb{T}^2\times\mathbb{R}^2). \tag{29}$$

Here $\widehat{V}$ means $V/|V|$. Note that $\widetilde{\mathcal{M}}_{\mathcal{C}}$ is indeed *independent* of parametrization. Notice also that, from the definition of $\mathfrak{s}(M)$ in (11), if $M = \mathcal{M}[X]$ then, for any test vector field φ,

$$\langle \varphi, \mathfrak{s}(M)\rangle = -\int_a^b \widehat{\frac{\partial X}{\partial \theta}} D\varphi(X) \widehat{\frac{\partial X}{\partial \theta}} \left|\frac{\partial X}{\partial \theta}\right| \mathrm{d}\theta = \langle \varphi, \mathfrak{s}(\widetilde{M})\rangle. \tag{30}$$

Next, fix $T > 0$. Let $\mathcal{C}_0$ be a smooth Jordan curve on the torus with a regular parametrization $x_0 = x_0(\theta)$, $a \leqslant \theta \leqslant b$. Let $v_0 \in L^2(\mathbb{T}^2)$ be a divergence-free vector field and set $M(x, \eta, 0) = \mathcal{M}[x_0]$, defined by (9). Let us start by observing that Theorem 2 gives the existence of a family M_ε of measures uniformly bounded in $L^\infty((0, T); \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2))$, and a family of smooth divergence-free vector fields v_ε, uniformly bounded in $L^\infty((0, T); L^2(\mathbb{T}^2)) \cap L^2((0, T); H^1(\mathbb{T}^2))$, which satisfy (14) with initial data v_0, $M(x, \eta, 0)$. Let $\mathcal{C}_t^\varepsilon$ denote the image of the curve $\mathcal{C}_0$ under the flow induced by v_ε. As stated below (14), $M_\varepsilon = \mathcal{M}[x_\varepsilon(\cdot, t)]$, where $x_\varepsilon(\cdot, t)$ is the regular parametrization of $\mathcal{C}_t^\varepsilon$ induced by $x_0(\cdot)$ and v_ε. Consider also the induced measures $\widetilde{M}_\varepsilon = \widetilde{\mathcal{M}}_{\mathcal{C}_t^\varepsilon}$, which are measures with support on $\{|\eta| = 1\}$ that are uniformly bounded in $L^\infty((0, T); \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2))$.

Using these uniform bounds only and passing to subsequences as necessary, we obtain weak limits $\widetilde{M}$ and v such that

$$v_\varepsilon \rightharpoonup v \quad \text{weak}^* \text{ in } L^\infty((0, T); L^2(\mathbb{T}^2)) \text{ and weakly in } L^2((0, T); H^1(\mathbb{T}^2)), \tag{31}$$

$$\widetilde{M}_\varepsilon \rightharpoonup \widetilde{M} \quad \text{weak}^* \text{ in } L^\infty((0, T); \mathcal{BM}(\mathbb{T}^2 \times \mathbb{R}^2)), \tag{32}$$

and such that the support of $\widetilde{M}$ is contained in $\{(x, \eta, t) : |\eta| = 1\}$. Using additionally that the surface tension term appearing in the equation for v is linear, together with the fact that $\widehat{\eta}$ is a legitimate test function for $\widetilde{M}$, we can pass to the limit in all the linear terms of the evolution equation for v_ε. Additionally, from the uniform bounds we have for v_ε and for $\widetilde{M}_\varepsilon$, together with the system (14), we can find estimates for $\partial_t v_\varepsilon$ which are enough to ensure that v_ε is precompact in $C((0, T); L^2)$ by the Aubin–Lions lemma. This allows us to pass to the limit in the nonlinear term of the Navier–Stokes equations, thereby obtaining that v and $\widetilde{M}$ satisfy, in the sense of distributions,

$$\partial_t v + v \cdot \nabla v + \nabla p - \nu \Delta v = \sigma \mathfrak{s}(\widetilde{M}), \qquad \nabla \cdot v = 0. \tag{33}$$

Since the initial velocity for the approximate problem (14) is $v_\varepsilon(x, t = 0) = v_0$ and given that $v \in C((0, T); H^{-k}(\mathbb{T}))$ for some k, it follows that $v(x, t = 0) = v_0$ as well. This establishes the first part of Theorem 1, namely that v and $\widetilde{M}$ are distributional solutions of (4).

We are not able to pass to the limit in the evolution equation for M due to insufficient regularity of v but will come back to this issue in Section 5. Instead, we observe that there exists a "limiting curve" $\mathcal{C}_t$ for a.e. $t \geq 0$, and we deduce an equation for its evolution. We have not succeeded in proving that $\widetilde{M} = \widetilde{\mathcal{M}}_{\mathcal{C}_t}$, i.e., that the limiting measure is given by the limiting curve, but we will establish a weak link between $\mathcal{C}_t$ and the measure $\widetilde{M}$.

As observed, $\mathcal{C}_t^\varepsilon$ is a smooth Jordan curve on $\mathbb{T}^2$ for each $\varepsilon > 0$. Let Ω_t^ε denote the domain bounded by $\mathcal{C}_t^\varepsilon$ consisting of the image of Ω_0 under the flow induced by $J^\varepsilon v_\varepsilon$. Set $b_\varepsilon \equiv \chi_{\Omega_t^\varepsilon}$. The evolution equation for b_ε is simply

$$\partial_t b_\varepsilon + J^\varepsilon v_\varepsilon \cdot \nabla b_\varepsilon = 0, \tag{34}$$

i.e., b_ε is a distributional solution of this transport equation. Moreover, $b_\varepsilon(x, 0) = \chi_{\Omega_0}$ for all $\varepsilon > 0$. Now the energy estimate of Lemma 1 gives a bound on the total length of the boundary $\mathcal{C}_t^\varepsilon$ of Ω_t^ε.

Since $\mathcal{C}_t^\varepsilon$ is smooth, it follows from Theorem 5.8.1 in [19] that $\nabla b_\varepsilon(\cdot,t) = \delta_{\mathcal{C}_t^\varepsilon}\widehat{n}_t^\varepsilon$, where $\widehat{n}_t^\varepsilon$ is the unit normal to C_t^ε chosen so that $(\partial x^\varepsilon/\partial t, \widehat{n}_t^\varepsilon)$ is a positive basis. Therefore $|\nabla b_\varepsilon(\cdot,t)| = \delta_{\mathcal{C}_t^\varepsilon}$ so that $\|\nabla b_\varepsilon(\cdot,t)\|_{\mathcal{BM}}$ is the length of $\mathcal{C}_t^\varepsilon$. This establishes the uniform boundedness of $\{b_\varepsilon\}$ in $L^\infty((0,T);BV(\mathbb{T}^2))$ for any $T>0$. Furthermore $b_\varepsilon(x,t)\in\{0,1\}$ for all (x,t), so that b_ε is also uniformly bounded in $L^\infty(\mathbb{R}_+;L^\infty(\mathbb{T}^2))$.

We use DiPerna–Lions transport theory to characterize the limit as $\varepsilon\to 0$. Using Theorem II.3, part 1, of [8] we find that $\{b_\varepsilon\}$ are *renormalized solutions* of (34), with transporting divergence-free vector fields $\{J^\varepsilon v_\varepsilon\}$ converging in $L^2((0,T);L^2(\mathbb{T}^2))$ to v. In particular, these vector fields satisfy the hypotheses of Theorem II.4, part 2, in [8], as does the initial data for (34). It follows immediately from this result (passing to subsequences as needed) not only that there exists the strong limit $b_\varepsilon\to b$ in $C((0,T);L^p(\mathbb{T}^2))$ for every $1\leqslant p<\infty$, but also that b is a renormalized solution of (5). Given the extra regularity of b and of v we find by consistency (cf. Theorem II.3, part 1, in [8]) that b is a distributional solution of (5), as desired. As $b_\varepsilon\to b$ strongly in $C((0,T);L^p(\mathbb{T}^2))$ for every $1\leqslant p<\infty$, it follows that $b_\varepsilon\to b$ *almost everywhere* as well. Thus, we conclude that $b(x,t)\in\{0,1\}$ a.e. $\mathbb{T}^2\times(0,T)$. Setting $\Omega_t\equiv\{x : b(x,t)=1\}$, we have $b=\chi_{\Omega_t}$ a.e. Since $b\in L^\infty((0,T);BV(\mathbb{T}^2))$, we deduce that the set Ω_t has bounded perimeter, that is, the boundary is a rectifiable curve. This concludes the proof of the second part of Theorem 1.

For fixed ε, let $\rho_\varepsilon = -\int_{\mathbb{R}^2}\eta^\perp\widetilde{M}_\varepsilon\,\mathrm{d}\eta$, where $\eta^\perp=(-\eta_2,\eta_1)$. This is the measure given by

$$\langle\varphi,\rho_\varepsilon\rangle = -\int_a^b\left(\widehat{\frac{\partial x_\varepsilon}{\partial\theta}}(\theta,t)\right)^\perp\varphi(x_\varepsilon(\theta,t))\left|\frac{\partial x_\varepsilon}{\partial\theta}(\theta,t)\right|\mathrm{d}\theta$$

for any $\varphi=\varphi(x)\in C(\mathbb{T}^2)$. It is immediate that $\rho_\varepsilon\equiv\delta_{\mathcal{C}_t^\varepsilon}\widehat{n}_t^\varepsilon$. Hence we find

$$\rho_\varepsilon = \nabla b_\varepsilon = \delta_{\mathcal{C}_t^\varepsilon}\widehat{n}_t^\varepsilon. \tag{35}$$

We have already shown that $\widetilde{M}_\varepsilon\rightharpoonup\widetilde{M}$ weak* in $L^\infty((0,T);\mathcal{BM}(\mathbb{T}^2\times\mathbb{R}^2))$ and that these measures are all supported on the unit circle with respect to η. It follows that there exists the limit

$$\rho_\varepsilon\rightharpoonup\rho \tag{36}$$

weak* in $L^\infty((0,T);\mathcal{BM}(\mathbb{T}^2))$. We also know that $b_\varepsilon\to b$ strongly in $C((0,T);L^p(\mathbb{T}^2))$ for every $p\geqslant 1$. Now, since $\{b_\varepsilon\}$ is bounded in $L^\infty((0,T);BV(\mathbb{T}^2))$, we have $\nabla b_\varepsilon\rightharpoonup\mu$ weak* in $L^\infty((0,T);\mathcal{BM}(\mathbb{T}^2))$. By linearity of the gradient we get $\mu=\nabla b$. Putting this together with (35) and (36), we obtain $\rho=\nabla b$, which concludes the proof of Theorem 1.

5. Remarks on surface tension defects

In this section, we introduce the notion of a surface tension defect and we discuss surface tension defects and length defects. The present section is divided into two subsections. In the first, we give the precise definition of a surface tension defect. In the second, we prove a sufficient condition for the absence of such defects.

5.1 *Surface tension defects*

In this subsection we do not consider fluid dynamics; instead, we only look at questions of convergence of sequences of curves.

We begin by observing that, although we defined $\widetilde{\mathcal{M}}_{\mathcal{C}}$ in (29) only for smooth curves, the definition can be extended to all rectifiable curves since, for such curves, there exists, a.e., a tangent vector.

Let us now consider the following problem. Assume that $\widetilde{M}_\varepsilon = \widetilde{\mathcal{M}}_{\mathcal{C}_\varepsilon}$ (defined in (29)) for some smooth curves $\mathcal{C}_\varepsilon$. Suppose, as usual, that $\mathcal{C}_\varepsilon$ are Jordan curves, so that $\mathcal{C}_\varepsilon = \partial\Omega_\varepsilon$ for some Ω_ε. Suppose also that the curves $\mathcal{C}_\varepsilon$ have uniformly bounded lengths. It follows that $\chi_{\Omega_\varepsilon}$, which are bounded in $BV(\mathbb{T}^2)$, converge, up to subsequences, strongly in $L^p(\mathbb{T}^2)$, $1 \leqslant p < \infty$, to χ_Ω for some domain Ω with rectifiable boundary $\mathcal{C}$, as in the preceding proof. We ask about conditions under which

$$\mathfrak{s}[\widetilde{M}_\varepsilon] \rightharpoonup \mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}}] \quad \text{in } \mathcal{D}'(\mathbb{T}^2).$$

Recall that, if $x^\varepsilon = x^\varepsilon(\theta)$ denotes a regular parametrization of $\mathcal{C}^\varepsilon$ then by definition

$$\langle \varphi, \mathfrak{s}[\widetilde{M}_\varepsilon]\rangle = -\sum_{i,j} \int_{a_\varepsilon}^{b_\varepsilon} \partial_{x_j}\varphi_i \, \widehat{\frac{\partial x_i^\varepsilon}{\partial\theta}} \widehat{\frac{\partial x_j^\varepsilon}{\partial\theta}} \left|\frac{\partial x^\varepsilon}{\partial\theta}\right| \mathrm{d}\theta$$

for $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathbb{T}^2)$. If we choose the test function $\varphi(x) = (x_1, x_2)$, which has a nonzero divergence, then

$$\langle \varphi, \mathfrak{s}[\widetilde{M}_\varepsilon]\rangle = -\int_{a_\varepsilon}^{b_\varepsilon} \left|\frac{\partial x^\varepsilon}{\partial\theta}\right| \mathrm{d}\theta,$$

which is the length of the curve $\mathcal{C}^\varepsilon$. Therefore if the lengths of $\mathcal{C}^\varepsilon$ do not converge to the length of $\mathcal{C}$, i.e., if there is a *length defect* in the approximation process, then $\mathfrak{s}[\widetilde{M}_\varepsilon]$ cannot converge to $\mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}}]$ in $\mathcal{D}'$ (note that the length is lower semicontinuous, so that the length of $\mathcal{C}$ is not greater than the limit of the lengths of $\mathcal{C}_\varepsilon$). However, given that the flow of interest is incompressible we will see that we need only concern ourselves with convergence of the divergence-free part of the surface tension term. Recall the notation for the Leray projector, $\mathbb{P}$, introduced in Section 3.

DEFINITION 1 Let $\{\mathcal{C}_\varepsilon\}$ be smooth Jordan curves on $\mathbb{T}^2$, $\mathcal{C}_\varepsilon = \partial\Omega_\varepsilon$. Assume that $\chi_{\Omega_\varepsilon}$ converges, strongly in $L^p(\mathbb{T}^2)$, $1 \leqslant p < \infty$, to χ_Ω for some domain Ω with rectifiable boundary. Let $\mathcal{C} = \partial\Omega$. Let $\widetilde{M}_\varepsilon = \widetilde{\mathcal{M}}_{\mathcal{C}_\varepsilon}$ be as defined in (29). We say that there is a *surface tension defect* in $\widetilde{M}_\varepsilon$ if $\mathbb{P}(\mathfrak{s}[\widetilde{M}_\varepsilon])$ does not converge to $\mathbb{P}(\mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}}])$ in $\mathcal{D}'(\mathbb{T}^2)$.

We now show that the occurrence of a length defect does not imply occurrence of a surface tension defect, i.e. we consider the question of whether the Leray projector of the surface tension term is weakly continuous with respect to ε.

More precisely, let us consider divergence-free test functions $\varphi = (\varphi^1, \varphi^2) \in C^\infty(\mathbb{T}^2)$ and let $y = y(\theta) = (y_1(\theta), y_2(\theta))$ be a regular parametrization of some curve $\mathcal{C}$. By (30), we have

$$\begin{aligned}\langle \varphi, \mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}}]\rangle &= -\int_a^b \sum_{i,j=1}^{2} \frac{\mathrm{d}y_i}{\mathrm{d}\theta}\frac{\mathrm{d}y_j}{\mathrm{d}\theta}\, \partial_{x_i}\varphi^j(y(\theta)) \frac{\mathrm{d}\theta}{|y'(\theta)|} \\ &= -\int_a^b \{[(y_1')^2 - (y_2')^2]\partial_{x_1}\varphi^1 + y_1'y_2'[\partial_{x_1}\varphi^2 + \partial_{x_2}\varphi^1]\} \frac{\mathrm{d}\theta}{|y'(\theta)|}.\end{aligned}$$

The issue of whether $\mathbb{P}(\mathfrak{s}[\widetilde{M}_\varepsilon]) \rightharpoonup \mathbb{P}(\mathfrak{s}[\mathcal{M}_{\mathcal{C}}])$ in $\mathcal{D}'$ is equivalent to whether

$$\int_{a_\varepsilon}^{b_\varepsilon} (\partial_{x_2}\varphi^1 + \partial_{x_1}\varphi^2)(y^\varepsilon(\theta)) \frac{(\partial y_1^\varepsilon/\partial\theta)(\partial y_2^\varepsilon/\partial\theta)}{|\partial y^\varepsilon/\partial\theta|}\, \mathrm{d}\theta$$

and

$$\int_{a_\varepsilon}^{b_\varepsilon} (\partial_{x_1}\varphi^1)(y^\varepsilon(\theta))\left[\frac{(\partial y_1^\varepsilon/\partial\theta)^2}{|\partial y^\varepsilon/\partial\theta|} - \frac{(\partial y_2^\varepsilon/\partial\theta)^2}{|\partial y^\varepsilon/\partial\theta|}\right]\mathrm{d}\theta$$

converge to the corresponding expressions without ε. To see this, let $\varphi \in C^\infty(\mathbb{T}^2)$ and consider the Hodge–Kodaira decomposition $\varphi = \nabla\Delta_0^{-1}\operatorname{div}\varphi + \mathbb{P}\varphi$. Then $\langle\varphi, \mathbb{P}(\mathfrak{s}[\widetilde{M}_\varepsilon])\rangle = \langle\mathbb{P}\varphi, \mathbb{P}(\mathfrak{s}[\widetilde{M}_\varepsilon])\rangle$, whence it is enough to test against a divergence-free vector field.

EXAMPLE Consider the straight line segment $(\theta, 0)$, $0 \leqslant \theta \leqslant 1$, being approximated by a "staircase function"

$$y^n(\theta) = \begin{cases} \left(\theta, \theta - \frac{i}{n}\right) & \text{if } \frac{i}{n} \leqslant \theta < \frac{2i+1}{2n}, \\ \left(\theta, \frac{i+1}{n} - \theta\right) & \text{if } \frac{2i+1}{2n} \leqslant \theta < \frac{i+1}{n}, \end{cases}$$

for $i = 0, \ldots, n-1$. Although these are not Jordan curves, they can be easily closed. We will not concern ourselves with this issue, however, concentrating instead on the local behavior of the tangent vectors, which are highly oscillatory.

The approximating curves $\mathcal{C}_n : y^n = y^n(\theta)$, $0 \leqslant \theta \leqslant 1$, have length equal to $\sqrt{2}$ for all n, whereas the limiting curve $\mathcal{C}$ has length 1, so that there is a length defect.

We see easily that $\partial_\theta y_1^n = 1$ and

$$\partial_\theta y_2^n = (-1)^j \quad \text{if } \frac{j}{2n} \leqslant \theta < \frac{j+1}{2n} \quad \text{for } j = 0, \ldots, 2n-1.$$

Therefore $|\partial_\theta y^n(\theta)| = \sqrt{2}$ for all $0 \leqslant \theta \leqslant 1$ and

$$\frac{\partial_\theta y_1^n \partial_\theta y_2^n}{|\partial_\theta y^n|} = \frac{1}{\sqrt{2}}\partial_\theta y_2^n \rightharpoonup 0 \quad \text{as } n \to \infty.$$

Moreover,

$$\frac{(\partial_\theta y_1^n)^2 - (\partial_\theta y_2^n)^2}{|\partial_\theta y^n|} = 0.$$

On the other hand, the limiting curve is $\mathcal{C} : y = y(\theta) = (\theta, 0)$, $0 \leqslant \theta \leqslant 1$, for which

$$\frac{\partial_\theta y_1 \partial_\theta y_2}{|\partial_\theta y|} = 0$$

and

$$\frac{(\partial_\theta y_1)^2 - (\partial_\theta y_2)^2}{|\partial_\theta y|} = 1. \tag{37}$$

However, when the expression (which is identically equal to 1) in (37) is tested against $\partial_{x_1}\varphi^1$, we get zero. Thus we see that $\langle\varphi, \mathbb{P}(\mathfrak{s}[\widetilde{\mathcal{M}}_\mathcal{C}])\rangle = 0$.

In this example, there is a length defect; as discussed above, this obviously implies that $\mathfrak{s}[\widetilde{M}_n]$ does not converge to $\mathfrak{s}[\widetilde{\mathcal{M}}_\mathcal{C}]$, where $\widetilde{M}_n = \widetilde{\mathcal{M}}_{\mathcal{C}_n}$. Nevertheless, $\mathbb{P}(\mathfrak{s}[\widetilde{M}_n])$ does converge weakly to $\mathbb{P}(\mathfrak{s}[\widetilde{M}_\mathcal{C}])$ in $\mathcal{D}'(\mathbb{T}^2)$.

5.2 *A sufficient condition*

In this subsection, we prove a sufficient condition for the absence of a surface tension defect, namely that, in the absence of length defect there is no surface tension defect. Again, this is a purely analytical result, irrespective of fluid dynamics.

THEOREM 3 Consider a family of rectifiable closed curves $\mathcal{C}_n \subseteq \mathbb{R}^2$ of length L^n and let y^n be their arclength parametrizations. Assume that $\{y^n\}$ is bounded in $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and that $L^n \leqslant K$ for all n. Then there exist y and $\bar{L}$ (and a subsequence) such that $y^n \rightharpoonup y$, weakly* in $W^{1,\infty}_{\mathrm{loc}}$ and $L^n \to \bar{L}$. Let $\mathcal{C}$ be the curve with parametrization y and let L be the length of $\mathcal{C}$. If $\bar{L} = L$, then $\mathbb{P}(\mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}_n}])$ converges to $\mathbb{P}(\mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}}])$ in $\mathcal{D}'$.

Consequently, the limit y above is, in fact, an arc length parametrization of $\mathcal{C}$.

Proof. By weak compactness, together with a diagonal argument, we can extract a single subsequence, which we do not relabel, such that

$$y^n \rightharpoonup y \quad \text{weak}^* \text{ in } W^{1,\infty}_{\mathrm{loc}}.$$

As $W^{1,\infty}$ is compactly imbedded in L^∞ on bounded sets, we can assume that $y^n \to y$ uniformly on compact subsets of $\mathbb{R}$. Also, without loss of generality we may assume that $L^n \to \bar{L}$ by passing to a further subsequence if necessary. We observe that y is $\bar{L}$-periodic. Indeed, we have

$$y^n(L^n + s) = y^n(s), \quad \text{and} \quad y^n(L^n + s) \to y(\bar{L} + s), \quad y^n(s) \to y(s),$$

using the uniform convergence. Note that, since $\mathrm{d}y^n/\mathrm{d}s \rightharpoonup \mathrm{d}y/\mathrm{d}s$ weak* in L^∞, we have $|\mathrm{d}y/\mathrm{d}s| \leqslant 1$. The length of the limiting curve is

$$L = \int_0^{\bar{L}} \left| \frac{\mathrm{d}y}{\mathrm{d}s} \right| \mathrm{d}s.$$

We now impose the hypothesis that there is no length defect, i.e., $\bar{L} = L$. From this it follows that

$$\left| \frac{\mathrm{d}y}{\mathrm{d}s} \right| = 1 \quad \text{a.e. on } \mathbb{R},$$

since $|\mathrm{d}y/\mathrm{d}s| \leqslant 1$. Therefore the limiting curve is parametrized by arclength.

Now for any interval I in the real line, observe that

$$\int_I \left| \frac{\mathrm{d}y^n}{\mathrm{d}s} \right|^2 \mathrm{d}s = \int_I \left| \frac{\mathrm{d}y}{\mathrm{d}s} \right|^2 \mathrm{d}s = |I|.$$

Consider

$$\int_I \left| \frac{\mathrm{d}y^n}{\mathrm{d}s} - \frac{\mathrm{d}y}{\mathrm{d}s} \right|^2 \mathrm{d}s = \int_I \left(2 - 2 \frac{\mathrm{d}y^n}{\mathrm{d}s} \cdot \frac{\mathrm{d}y}{\mathrm{d}s} \right) \mathrm{d}s.$$

Since $\mathrm{d}y^n/\mathrm{d}s \rightharpoonup \mathrm{d}y/\mathrm{d}s$ weak* in L^∞ and since $\mathrm{d}y/\mathrm{d}s \in L^1_{\mathrm{loc}}$ it follows that $\mathrm{d}y^n/\mathrm{d}s \to \mathrm{d}y/\mathrm{d}s$ strongly in L^2_{loc}. Passing to a further subsequence if needed, this implies the almost everywhere convergence of each component of $\mathrm{d}y^n/\mathrm{d}s$. Hence

$$\frac{\mathrm{d}y_1^n}{\mathrm{d}s} \frac{\mathrm{d}y_2^n}{\mathrm{d}s} \to \frac{\mathrm{d}y_1}{\mathrm{d}s} \frac{\mathrm{d}y_2}{\mathrm{d}s}$$

and

$$\left(\frac{\mathrm{d}y_1^n}{\mathrm{d}s}\right)^2 - \left(\frac{\mathrm{d}y_2^n}{\mathrm{d}s}\right)^2 \to \left(\frac{\mathrm{d}y_1}{\mathrm{d}s}\right)^2 - \left(\frac{\mathrm{d}y_2}{\mathrm{d}s}\right)^2$$

a.e. and strongly in L^1_{loc}. This is all we need with regard to the Leray projection of the surface tension term, which concludes the proof. □

Note that, in fact, we proved above that $\mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}_n}]$ converges to $\mathfrak{s}[\widetilde{\mathcal{M}}_{\mathcal{C}}]$ in $\mathcal{D}'$ in the absence of length defects, and not only their Leray projections.

6. Dependence on the surface tension

In this section we examine the dependence of our solutions on the surface tension coefficient σ. We establish a conditional nontriviality theorem.

LEMMA 5 Let $\mathcal{C}_0$ be a smooth Jordan curve and let $v_0 \in L^2(\mathbb{T}^2)$. Set $\widetilde{M}_0 \equiv \widetilde{\mathcal{M}}_{\mathcal{C}_0}$. Suppose there exists a solution $(v, \widetilde{M})$ to (4) with initial data v_0. Assume that $\mathbb{P}(\mathfrak{s}[M]) \in C([0, T); H^{-k}(\mathbb{T}^2))$ for some $k > 0$. If v is also a weak solution of the Navier–Stokes equations without surface tension, then $\mathbb{P}(\mathfrak{s}[\widetilde{M}_0]) = 0$.

Proof. Assume that v is a weak solution of both the Navier–Stokes equations with zero surface tension and of our system (4). Then $\mathbb{P}(\mathfrak{s}[M]) = 0$ for all $t > 0$. From the hypothesis that $\mathbb{P}(\mathfrak{s}[M]) \in C([0, T); H^{-k}(\mathbb{T}^2))$ for some $k > 0$ we find that $\mathbb{P}(\mathfrak{s}[M]) = 0$ at $t = 0$ as well. To conclude the proof recall that $\mathfrak{s}[M] = \mathfrak{s}[\widetilde{M}]$. □

Thus, with this assumption of regularity, if initially there is some surface tension, then there is also some surface tension at some later time.

THEOREM 4 Let $\mathcal{C}_0$ be a smooth Jordan curve, let $v_0 = 0$, $\sigma > 0$, and let $\widetilde{M}_0 = \widetilde{\mathcal{M}}_{\mathcal{C}_0}$ with $\mathbb{P}(\mathfrak{s}(\widetilde{M}_0)) \neq 0$. Assume that there exists a solution $(v, \widetilde{M})$ to (4), corresponding to the initial data v_0, with parameter σ. Suppose, additionally, that $\mathbb{P}(\mathfrak{s}[M]) \in C([0, T); H^{-k}(\mathbb{T}^2))$ for some $k > 0$. Then there is a number $\sigma' > 0$ such that, if $0 < \sigma'' < \sigma'$, then all solutions $(v', \widetilde{M}')$ of (4), corresponding to the initial data $v_0 = 0$ and $\mathcal{C}_0$ but with surface tension coefficient σ'', satisfy $v \neq v'$.

Proof. First, we prove that v is not identically zero. We prove this by contradiction. Assume that there exists a solution $(v, \widetilde{M})$ of (4) with data $v_0 = 0$ which vanishes identically. In this case the evolution equation for v reduces to the statement that $\mathfrak{s}(M)$ must always be a gradient. But this cannot be the case, by Lemma 5 (since we have a nontrivial $\mathbb{P}(\mathfrak{s}[\widetilde{M}_0])$). Thus v is not identically zero.

Second, since v is not identically zero, there must be a time $0 < T' < T$ at which there is a positive amount of kinetic energy. Let K be the kinetic energy for the solution $(v, \widetilde{M})$ at time T'. Let the length of the initial curve be L_0. Take $\sigma' = K/L_0$, and σ'' to be a positive number smaller than σ'. Then, for any solution (v', M') corresponding to data $v_0 = 0$ and surface tension coefficient σ'', the kinetic energy K is unattainable, since the total energy at any time can never be greater than the initial energy, and since the initial energy is $\sigma'' L_0 < K$. Thus $v \neq v'$. This completes the proof. □

7. Moment closure

In this section, we propose another formulation of the problem.

We define the following moments of M:

$$g_1 = \int_{\mathbb{R}^2} \frac{\eta_1 \eta_2}{|\eta|^2} M \, \mathrm{d}\eta, \tag{38}$$

$$g_2 = \int_{\mathbb{R}^2} \frac{\eta_1^2 - \eta_2^2}{|\eta|^2} M \, \mathrm{d}\eta. \tag{39}$$

Let $\mathcal{C}$ be a smooth curve and consider an arclength parametrization $y = (y_1, y_2)$. Note that, if $M = \mathcal{M}_{\mathcal{C}}$, then

$$\langle \varphi, \mathbb{P}(\mathfrak{s}(M)) \rangle = -\int_{\mathcal{C}} \frac{\partial y}{\partial s} D\varphi \frac{\partial y}{\partial s} \, \mathrm{d}s \tag{40}$$

for all test functions $\varphi = (\varphi^1, \varphi^2)$ such that $\varphi^1_{x_1} + \varphi^2_{x_2} = 0$. Writing $\partial y / \partial s = (y_{1,s}, y_{2,s})$, and using the divergence-free property of the test functions, we see that this is the same as

$$\langle \varphi, \mathfrak{s}(M) \rangle = -\int_0^T \int_{\mathcal{C}} ((y^2_{1,s} - y^2_{2,s})(\varphi^1_{x_1}) \circ y + y_{1,s} y_{2,s} (\varphi^1_{x_2} + \varphi^2_{x_2}) \circ y) \, \mathrm{d}s \, \mathrm{d}t. \tag{41}$$

Interpreting this in terms of the g_i, we have

$$\langle \varphi, \mathfrak{s}(M) \rangle = -\langle \varphi^1_{x_1}, g_2 \rangle - \langle \varphi^1_{x_2} + \varphi^2_{x_1}, g_1 \rangle.$$

Moving derivatives from the φ^i to the g_j, this is

$$\langle \varphi, \mathfrak{s}(M) \rangle = \langle \varphi^1, g_{2,x_1} \rangle + \langle \varphi^1, g_{1,x_2} \rangle + \langle \varphi^2, g_{1,x_1} \rangle = \langle \varphi, (g_{2,x_1} + g_{1,x_2}, g_{1,x_1}) \rangle.$$

We can hence express $\mathbb{P}(\mathfrak{s}(M))$ as

$$\mathbb{P}(\mathfrak{s}(M)) = \begin{pmatrix} \partial_{x_1} g_2 + \partial_{x_2} g_1 \\ \partial_{x_1} g_1 \end{pmatrix}.$$

We now turn to the question of finding evolution equations for the g_i. If we multiply the M evolution equation (3) by $\eta_1 \eta_2 / |\eta|^2$ and integrate with respect to η, we can find an evolution equation for g_1. Similarly, if we multiply instead by $(\eta_1^2 - \eta_2^2)/|\eta|^2$ and integrate, we find an evolution equation for g_2. Unfortunately, this does not lead to a closed system of equations (with a given velocity), as the equations for g_{1t} and g_{2t} would include terms of the form $\int \frac{p_4(\eta)}{|\eta|^4} M \, \mathrm{d}\eta$, where p_4 is a fourth degree polynomial.

There are, however, a large number of other moments of M which, given the velocity, do form a finite, closed system of evolution equations (in x and t). Let P_n be the set of all polynomials with domain $\mathbb{R}^2$ of degree at most n, and let $Q_n = \{\int \frac{p(\eta)}{|\eta|} M \, \mathrm{d}\eta : p \in P_n\}$. Then, given the velocity, the evolution equations for the elements of Q_n involve only elements of Q_n. To see this, first write the equation for M in the form

$$\partial_t M + v \cdot \nabla_x M + |\eta| \operatorname{div}_\eta \left(Dv \frac{\eta}{|\eta|} M \right) = 0. \tag{42}$$

Multiplication of (42) by $p/|\eta|$, for some $p \in P_n$, yields

$$\partial_t\left(\frac{p}{|\eta|}M\right) + v \cdot \nabla_x\left(\frac{p}{|\eta|}M\right) + p\,\mathrm{div}_\eta\left(Dv\frac{\eta}{|\eta|}M\right) = 0. \tag{43}$$

After integrating (43) in the η variable and integrating by parts we find

$$\partial_t\left(\int \frac{p}{|\eta|}M\,\mathrm{d}\eta\right) + v \cdot \nabla_x\left(\int_\eta \frac{p}{|\eta|}M\,\mathrm{d}\eta\right) - \int (\nabla_\eta p) Dv\frac{\eta}{|\eta|}M\,\mathrm{d}\eta = 0, \tag{44}$$

which is the same as

$$\partial_t\left(\int \frac{p}{|\eta|}M\,\mathrm{d}\eta\right) + v \cdot \nabla_x\left(\int_\eta \frac{p}{|\eta|}M\,\mathrm{d}\eta\right) - \mathrm{tr}\left[Dv\int (\nabla_\eta p)\otimes\frac{\eta}{|\eta|}M\,\mathrm{d}\eta\right] = 0. \tag{45}$$

Clearly equation (45) involves only elements of Q_n. Since Q_n is a finite-dimensional vector space it follows that there is a finite closed system of evolution equations for the basis elements.

Going back to the evolution of g_1 and g_2, which come from the surface tension term, we can introduce an approximation based on the Taylor polynomials of $\eta_1\eta_2/|\eta|$ and $(\eta_1^2 - \eta_2^2)/|\eta|$ about some unit vector η_0. We write

$$\frac{\eta_1\eta_2}{|\eta|^2} = \frac{1}{|\eta|}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} a_\alpha(\eta-\eta_0)^\alpha \tag{46}$$

and

$$\frac{\eta_1^2 - \eta_2^2}{|\eta|^2} = \frac{1}{|\eta|}\sum_{k=0}^{\infty}\sum_{|\alpha|=k} b_\alpha(\eta-\eta_0)^\alpha, \tag{47}$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ is a multi-index. We now propose to truncate these series at some $k_0 < \infty$ and we consider the evolution equations for $g_1^{k_0}$ and $g_2^{k_0}$, where these are the moments of M with respect to the truncated series above. Since $g_1^{k_0}$ and $g_2^{k_0}$ are elements of Q_{k_0}, and since there is a finite, closed system of evolution equations for a basis of Q_{k_0} (given a velocity field), we have an approximation to our system if we take the Q_{k_0}-system together with the following equations:

$$\begin{cases} \partial_t v^{k_0} + v^{k_0}\cdot\nabla v^{k_0} + \nabla p^{k_0} - \nu\Delta v^{k_0} = \sigma(\partial_{x_1} g_2^{k_0} + \partial_{x_2} g_1^{k_0}, \partial_{x_1} g_1^{k_0}), \\ \nabla \cdot v^{k_0} = 0. \end{cases} \tag{48}$$

It is a subject of further work to determine how well a solution of this approximate system approximates a solution of the original system.

We now have a new system of evolution equations in (48) with the following features: the surface tension term is linear, and the spatial domain is two-dimensional (as compared to the four-dimensional domain of M). Because of these features, this system might allow for efficient numerical simulations of interfacial Navier–Stokes flows with surface tension. It is also possible that the formulation in this section may be applicable to numerical simulations in the Euler equation case, i.e. $\nu = 0$. Although analysis (of global weak solutions) in this case would seem beyond our reach at present because of the lack of regularity of the velocity, this problem should be numerically tractable. Again, the formulation of this section would allow for the study of topological transitions in the free-surface problem with surface tension, by an alternative method to level-set and phase-field methods.

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