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Convexity breaking of the free boundary for porous medium equations

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We investigate the preservation of convexity of the free boundary by the solutions of the porous medium equation. We prove that starting with an initial datum with some kind of suboptimal α-concavity property, the convexity of the positivity set can be lost in a short time.

1. Introduction

We consider the Cauchy problem for the porous medium equation,

$$\begin{cases} \partial_t u = \Delta u^m & \text{in} \quad \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = \varphi(x) \geqslant 0 & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $N \geqslant 2$, $m > 1$, and $\varphi \in C_0(\mathbb{R}^N)$. It is well known that the problem (1.1) has a unique strong solution $u \in C(\mathbb{R}^N \times (0, \infty))$, and that the *positivity set* of $u(\cdot, t)$,

$$P_\varphi(t) = \{x \in \mathbb{R}^N : u(x, t) > 0\},$$

is bounded for any $t \geqslant 0$ and increases with time.

This paper is concerned with the following problem.

PROBLEM (P) Let Ω be a bounded convex domain and let the initial datum φ be a nonnegative function, belonging to $C_0(\mathbb{R}^N) \cap C^\infty(\Omega)$, whose positivity set $P_\varphi(0)$ coincides with Ω. Moreover, set $f = \varphi^{m-1}$ and assume that

$$(A_\alpha) \quad \begin{cases} \text{(i)} \;\; \text{there exists a positive constant } \mu \text{ such that } f + |\nabla f| \geqslant \mu \text{ in } \Omega; \\ \text{(ii)} \; f \text{ is } \alpha\text{-concave in } \mathbb{R}^N \text{ for some } \alpha \in [-\infty, \infty]. \end{cases}$$

Is then $P_\varphi(t)$ convex for every $t > 0$? (See Section 2 for the definition of α-concavity.)

The porous medium equation provides a simple model in many physical situations, in particular, the flow of an isentropic gas through a porous medium; in such a case, u and u^{m-1} represent the *density* and the *pressure* of the gas, respectively. Due to its practical interest, regularity and geometric properties of the free boundary $\partial P_\varphi(t)$ have been extensively studied by many mathematicians; see for instance [1], [3]–[8], [14]–[16] and the monograph [17], which gives a good survey of the study of the porous medium equations and a comprehensive list of references. In connection to problem (P), in [14], Lee and Vázquez proved that $P_\varphi(t)$ is convex for all sufficiently large t, even if $P_\varphi(0)$ is not convex; in fact, they proved that the pressure u^{m-1} becomes a concave function on its support after a suitably long time, if the initial support is compact. Furthermore, under suitable regularity conditions on the initial datum φ, in [7], Daskalopoulos, Hamilton, and Lee proved that, if $f = \varphi^{m-1}$ satisfies condition (A_α) with $\alpha = 1/2$, then the pressure $u(\cdot, t)^{m-1}$ remains $1/2$-concave for $t > 0$, which again implies that the set $P_\varphi(t)$ is convex for every $t > 0$. This gives an affirmative answer to problem (P) with $\alpha \geqslant 1/2$. We may then wonder if the exponent $1/2$ has some optimality property and, similarly to [14] and [17, page 520], we ask: is the α-concavity of the initial pressure u^{m-1} preserved even if $\alpha < 1/2$? And, if not, does the α-concavity ($\alpha < 1/2$) of the initial datum implies the β-concavity of $u(\cdot, t)$ for every $t > 0$, for some suitable $\beta < \alpha$? Notice that problem (P) corresponds exactly to the latter question for $\beta = -\infty$.

In this paper we give a negative answer to problem (P) (and hence to all the stronger questions posed above) for some $\alpha > 0$, proving the following result.

THEOREM 1.1 Let Ω be a C^2 bounded convex domain in $\mathbb{R}^N$, $N \geqslant 2$, and let $t_* > 0$. Then there exists a nonnegative function $\varphi \in C_0(\mathbb{R}^N)$ satisfying the condition (A_α) for some $\alpha > 0$ and such that $P_\varphi(0) = \Omega$, while $P_\varphi(t)$ is not convex for some $t \in (0, t_*)$.

The above theorem essentially tells us that even starting with an initial datum with some kind of suboptimal concavity, the spatial convexity of the free boundary can be lost in an (arbitrarily chosen) short time. We remark that we are not simply saying that the α-concavity of the initial pressure (for some $\alpha < 1/2$) is not preserved, but that such a property may be completely destroyed by diffusion in a porous medium, since an α-concave initial datum can result in a solution which is not even quasi-concave at any time $t > 0$.

Notice that quasi-concavity is the weakest concavity property one can imagine: roughly speaking, a function u is quasi-concave if all its superlevel sets are convex. In fact, this corresponds to $(-\infty)$-concavity.

We finally recall that α-concavity for all $\alpha > 0$ implies log-concavity; hence Theorem 1.1 shows in particular that starting with a log-concave initial datum is not sufficient to maintain the convexity of $P_\varphi(t)$. For comparison, we recall that the heat flow, corresponding to (1.1) for $m = 1$, preserves log-concavity (see [2], [13], [10]). In this connection, the present authors proved in [9] that the mere quasi-concavity of the initial datum is not inherited, in general, by the solutions of the heat equation. Now, we are able to improve this result by showing that α-concavity, for some $\alpha < 0$, is not necessarily preserved by the heat flow. Indeed, in Theorem 4.1, we construct examples of α-concave initial data, with $\alpha < 0$, that generate a heat distribution which is not quasi-concave after a small time.

2. Preliminaries and notation

In this section we introduce some notation and recall some basic properties of α-concave functions and of solutions of the porous medium equation.

For any $x \in \mathbb{R}^N$ and $r > 0$, we put $B(x, r) = \{y \in \mathbb{R}^N : |x - y| < r\}$. Let D be a set in $\mathbb{R}^N$. We denote by χ_D the characteristic function of D, that is, $\chi_D(x) = 1$ for $x \in D$ and $\chi_D(x) = 0$ for $x \notin D$.

Let $\lambda \in (0,1)$ and $\alpha \in [-\infty, \infty]$; for $s, t \geqslant 0$ with $st > 0$ we define

$$g_\alpha(s,t:\lambda) = \begin{cases} (\lambda s^\alpha + (1-\lambda)t^\alpha)^{1/\alpha} & \text{if } \alpha \notin \{\pm\infty, 0\}, \\ \min\{s,t\} & \text{if } \alpha = -\infty, \\ \max\{s,t\} & \text{if } \alpha = \infty, \\ s^\lambda t^{1-\lambda} & \text{if } \alpha = 0, \end{cases}$$

and

$$g_\alpha(s,t:\lambda) = 0 \qquad \text{if } st = 0.$$

Let u be a nonnegative function defined in a convex set Ω. Following [2] and [11], we say that u is *α-concave* if

$$u((1-\lambda)x + \lambda y) \geqslant g_\alpha(u(x), u(y) : \lambda)$$

for all $x, y \in \Omega$ and $\lambda \in [0,1]$.

In other words, for $\alpha > 0$ [< 0], u is quasi-concave if u^α is positive and concave [convex] in a convex set P_u and vanishes outside P_u. Furthermore, u is 0-concave (or log-concave) if u is positive in a convex set P_u, vanishes outside P_u and $\log u$ is concave in P_u.

For more details regarding α-concave functions, we refer to [2] and [11]; here we recall just the following two properties, which are used in this paper:

(C1) if f is α-concave, then f is β-concave for all $\beta \leqslant \alpha$.

(C2) Let $\alpha, \beta \in [0, \infty]$, and f and g be α-concave and β-concave functions on bounded convex subsets Ω_1 and Ω_2 of $\mathbb{R}^N$, respectively. Then the convolution $f * g$, defined as usual by

$$(f * g)(x) = \int_{\Omega_1 \cap (x - \Omega_2)} f(y) g(x-y)\,\mathrm{d}y,$$

is γ-concave in $\Omega_1 + \Omega_2$, with $\gamma^{-1} = N + \alpha^{-1} + \beta^{-1}$.

Next we recall some properties of solutions of (1.1). For any nonnegative function $\varphi \in L^1(\mathbb{R}^N)$, there exists a unique (strong) solution $u = S(t)\varphi$ of (1.1), and the following statements hold (see Section 9 in [17]):

(S1) $(S(t)\varphi)(x)$ is a continuous function in $\mathbb{R}^N \times (0, \infty)$.

(S2) If $0 \leqslant \varphi_1 \leqslant \varphi_2$ almost everywhere in $\mathbb{R}^N$ and $\varphi_2 \in L^1(\mathbb{R}^N)$, then

$$0 \leqslant (S(t)\varphi_1)(x) \leqslant (S(t)\varphi_2)(x) \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

(S3) If $P_\varphi(0)$ is bounded, then $P_\varphi(t)$ is bounded for all $t > 0$ and

$$P_\varphi(t_1) \subset P_\varphi(t_2) \quad \text{if } 0 < t_1 < t_2.$$

It is well known that the equation (1.1) has a family of self-similar solutions called *Barenblatt–Pattle solutions* given by

$$U_M(x,t) = t^{-kN}\left[c_M - \frac{(m-1)k}{2m}|x|^2 t^{-2k}\right]_+^{1/(m-1)},$$

where $[v]_+ = \max\{v, 0\}$, $k = (N(m-1)+2)^{-1}$ and c_M is given by mass conservation

$$\int_{\mathbb{R}^N} U_M(x,t)\,\mathrm{d}x = M \geqslant 0.$$

Then, for any $t > 0$, we have

$$\{x \in \mathbb{R}^N : U_M(x,t) > 0\} = \{x \in \mathbb{R}^N : |x| < c'_M t^k\}, \tag{2.1}$$

where $c'_M = \sqrt{2mc_M/(m-1)k}$.

The following proposition, taken from [4], gives a lower bound of the speed of propagation of the support of solutions for porous medium equations.

PROPOSITION 2.1 (See Proposition 3.1 in [4]) Let $\varphi \in L^1(\mathbb{R}^N)$ be a nonnegative function and set

$$E(x : \varphi) = \sup_{R>0} R^{-(N+\frac{2}{m-1})} \int_{|y-x|<R} \varphi(y)\, \mathrm{d}y. \tag{2.2}$$

There exists a constant $c_* = c_*(N, m)$ such that $(S(t)\varphi)(x) = 0$ for every $(x,t) \in \mathbb{R}^N \times (0,\infty)$ such that

$$0 < t < c_* E(x : \varphi)^{1-m}. \tag{2.3}$$

3. Proof of Theorem 1.1

Let Ω be a C^2 bounded convex domain in $\mathbb{R}^N$. Here we can assume, without loss of generality, that

$$\Omega \subset \{x_N < 0\}, \quad 0 \in \partial\Omega. \tag{3.1}$$

For any $r > 0$, put

$$\Omega(r) = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > r\}.$$

Then there exists a positive constant δ such that

$$\bigcup_{x\in\Omega(r)} B(x,r) = \Omega, \quad 0 < r < \delta. \tag{3.2}$$

For $n = 1, 2, \ldots$, define

$$\Omega_n = n\Omega, \quad \Omega'_n = n\Omega(n^{-2}), \quad \Omega''_n = n\Omega(2n^{-2})$$

By (3.1), the convexity and the regularity of Ω, we have

$$\Omega_n \subset \Omega_{n+1} \subset \{x_N < 0\}, \quad \bigcup_{n=1}^{\infty} \Omega_n = \bigcup_{n=1}^{\infty} \Omega'_n = \bigcup_{n=1}^{\infty} \Omega''_n = \{x_N < 0\}. \tag{3.3}$$

Set

$$\rho(x) = \begin{cases} 1 - |x|^2 & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1. \end{cases} \tag{3.4}$$

Furthermore, for any $n = 1, 2, \ldots$, we put

$$\rho_n(x) = n^N \rho(nx), \quad d_n(x) = \int_{\Omega'_n} \rho_n(x-y)\, \mathrm{d}y. \tag{3.5}$$

Then, by (3.2), we have

$$\{x \in \mathbb{R}^N : d_n(x) > 0\} = \Omega_n$$

for all $n = 1, 2, \dots$ with $n > \delta^{-1}$. Furthermore, since ρ is 1-concave, by (C2) the function d_n is $(N+1)^{-1}$-concave in $\mathbb{R}^N$. This implies that $d_n^{1/(N+1)}$ is concave in $\mathbb{R}^N$. Therefore, since $d_n(x) = 1$ in Ω_n'' and $d_n(x) = 0$ on $\partial\Omega_n$, for any sufficiently large n there exists a constant $c_{n,1}$ such that

$$d_n(x)^{1/(N+1)} \geqslant c_{n,1}\mathrm{dist}(x, \partial\Omega_n) \quad \text{in } \Omega_n \setminus \Omega_n''. \tag{3.6}$$

Let A be a positive constant to be chosen later. Put

$$\varphi_n(x) = \left(A\int_{\Omega_n'} e^{-|y|^2}\rho_n(x-y)\,\mathrm{d}y\right)^{1/(N+1)(m-1)} \tag{3.7}$$

and $u_n(x,t) = (S(t)\varphi_n)(x)$. For any sufficiently large n, since $e^{-|x|^2}\chi_{\Omega_n'}$ is α_n-concave for some $\alpha_n > 0$, by (C2) there exists a positive constant β_n such that

$$\text{the function } \varphi_n^{m-1} \text{ is } \beta_n\text{-concave in } \mathbb{R}^N. \tag{3.8}$$

On the other hand, by (3.2)–(3.5), for $n > \delta^{1/2}$, we have

$$\{\varphi_n(x) > 0\} = \bigcup_{x\in\Omega_n'} B(x, n^{-1}) = n \bigcup_{x\in\Omega(n^{-2})} B(x, n^{-2}) = n\Omega. \tag{3.9}$$

Furthermore, by (3.5)–(3.7), for any sufficiently large n, there exists a constant $c_{n,2}$ such that

$$\varphi_n(x)^{m-1} \geqslant (Ac_{n,2}d_n(x))^{1/(N+1)} \geqslant c_{n,1}(Ac_{n,2})^{1/(N+1)}\mathrm{dist}(x, \partial\Omega_n)$$

in $\Omega_n \setminus \Omega_n''$. This implies that there exist positive constants $c_{n,3}$ and δ_n such that

$$|\nabla(\varphi_n(x)^{m-1})| \geqslant c_{n,3} \tag{3.10}$$

for all $x \in \Omega_n$ with $\mathrm{dist}(x, \partial\Omega_n) < \delta_n$.

Let $t_* > 0$. By (2.2), there exists a positive constant L such that

$$0 < t_* < c_* E(Le_N : \chi_{\{x_N<0\}})^{1-m}, \tag{3.11}$$

where c_* is the constant given in Proposition 2.1. Since $c'_M \to \infty$ as $M \to \infty$ we can take a sufficiently large constant M such that

$$c'_M + L + 2 < c'_M(t_* + 1)^k. \tag{3.12}$$

Since

$$\operatorname{supp} U_M(\cdot + (c'_M + 2)e_N, 1) \subset B(-(c'_M + 2)e_N, c'_M) \subset \{x_N < -1\},$$

by (3.3) and (3.9) we can take a sufficiently large constant A so that

$$\varphi_n(x) \geqslant U_M(x + (c'_M + 2)e_N, 1) \quad \text{in } \mathbb{R}^N \tag{3.13}$$

for all sufficiently large n. Then, by (S2) and (3.13), we have

$$u_n(x,t) \geqslant U_M(x + (c'_M + 2)e_N, t+1) \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

and by (2.1) and (3.12) there exists a positive constant $L' > L$ such that

$$u_n(le_N, t_*) \geqslant U_M((l + c'_M + 2)e_N, t_* + 1) > 0 \tag{3.14}$$

for all $0 < l \leqslant L'$ and sufficiently large n.

On the other hand, since

$$\sup_{n\in\mathbb{N}} \|\varphi_n\|_{L^\infty(\mathbb{R}^N)} < \infty, \qquad \lim_{|x|\to\infty} \varphi_n(x) = 0 \quad \text{uniformly for all } n,$$

by (3.3) and (3.9) there exists a constant R_* such that

$$E(x : \varphi_n) = R^{-(N+\frac{2}{m-1})} \sup_{x\in\mathbb{R}^N} \int_{|y-x|<R} \varphi_n(y)\,\mathrm{d}y \leqslant E(Le_N : \chi_{\{x_N<0\}}) \tag{3.15}$$

for all $x = (x', x_N)$ with $x' \geqslant R_*$ and $x_N \geqslant L$ and for all $n \in \mathbb{N}$. By Proposition 2.1, (3.11) and (3.15), we have

$$u_n(x, t_*) = 0 \tag{3.16}$$

for all $x = (x', x_N)$ with $x' \geqslant R_*$ and $x_N \geqslant L$ and all $n \in \mathbb{N}$.

Put

$$\lambda = \frac{L + L' + 2}{2(L' + 1)}.$$

Then $\lambda < 1$ and

$$1 - \lambda = \frac{L' - L}{2(L' + 1)}.$$

By (3.3), we can take a large integer n so that (3.14) holds and

$$\bar{x} = \left(\frac{R_*}{1-\lambda}, 0, \ldots, 0, -1\right) \in n\Omega, \tag{3.17}$$

and fix n. Then, by (3.9) and (3.17), we have

$$u_n(\bar{x}, t) > 0 \quad \text{for all } t > 0. \tag{3.18}$$

Furthermore, the set $P_{\varphi_n}(t_*)$ is not convex. Indeed, if $P_{\varphi_n}(t_*)$ is convex, then, by (3.14), (3.17), and (3.18), we have

$$(1-\lambda)\bar{x} + \lambda\,(0, \ldots, 0, L') = (R_*, 0, \ldots, 0, (L + L')/2) \in P_{\varphi_n}(t_*).$$

Since $(L + L')/2 > L$, this contradicts (3.16).

Finally, we put

$$\varphi(x) = n^N \varphi_n(nx), \qquad U(x, t) = n^N u_n(nx, n^{1/k}t), \qquad \bar{t} = n^{-1/k} t_* \in (0, t_*),$$

where $k = (N(m-1) + 2)^{-1}$. Then $U(t) = S(t)\varphi$, and by (3.8), φ^{m-1} is β_n-concave in $\mathbb{R}^N$. Furthermore, by (3.9) and (3.10), we have $P_\varphi(0) = \Omega$ and

$$\varphi(x)^{m-1} + |\nabla\varphi(x)^{m-1}| \geqslant C \quad \text{in } \Omega$$

for some constant C. Therefore, since $P_\varphi(\bar{t}) = n^{-1} P_{\varphi_n}(t_*)$ is not convex, the proof of Theorem 1.1 is complete. □

4. Heat equation

In this section we apply the argument of Section 3 to the heat equation, and improve a result of our previous paper [9]. Let u be a nonnegative solution of

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \text{ if } \partial\Omega \neq \emptyset, \\ u(x,0) = \varphi(x) \geqslant 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where Ω is a convex smooth domain in $\mathbb{R}^N$ and $N \geqslant 2$. Then, for any nonnegative solution u of the Cauchy–Dirichlet problem (4.1), if the initial datum $u(\cdot,0)$ is 0-concave, then for any $t > 0$, the solution $u(\cdot,t)$ is 0-concave, in particular, $u(\cdot,t)$ is quasi-concave. In our previous paper [9], we discussed the preservation of the quasi-concavity by the heat flow, and gave an example of a quasi-concave initial datum for which the solution of (4.1) is not quasi-concave at some time. By the arguments in the previous sections and [9], we can now prove the following theorem.

THEOREM 4.1 Let Ω be any smooth convex domain in $\mathbb{R}^N$ (possibly $\Omega = \mathbb{R}^N$) and $t_* > 0$. Then there exists a nonnegative function $\varphi \in C_0(\Omega)$ such that

(1) φ is α-concave in Ω for some $\alpha \in (-\infty, 0)$,
(2) the solution of (4.1) is not quasi-concave in Ω for some $t \leqslant t_*$.

Proof. Let

$$\psi(x) = (1 + e^{-|x|^2})\chi_{\{x_N<0\}},$$
$$v(x,t) = (e^{t\Delta}\psi)(x) \equiv (4\pi t)^{-N/2} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right)\psi(y)\,dy.$$

Let $t_* > 0$. Then, by the same argument as in Lemma 3.1 of [9], there exist two points X and Y in $\mathbb{R}^N$ such that

$$v(X,t_*) > \frac{1}{2}, \quad v(Y,t_*) > \frac{1}{2}, \quad v\left(\frac{X+Y}{2}, t_*\right) < \frac{1}{2}. \tag{4.2}$$

Next, for any $n = 1, 2, \ldots$, we put

$$\psi_n(x) = \psi(x)\chi_{B(x_n,n)}(x), \quad x_n = -(n + n^{-1})e_N, \quad u_n(x,t) = (e^{t\Delta}\psi_n)(x).$$

Then we have

$$\lim_{n\to\infty} \sup_{x\in B(0,R)} |u_n(x,t) - v(x,t)| = 0$$

for any $R > 0$ and $t > 0$, and by (4.2), for any sufficiently large n,

$$u_n(X,t_*) > \frac{1}{2}, \quad u_n(Y,t_*) > \frac{1}{2}, \quad u_n\left(\frac{X+Y}{2}, t_*\right) < \frac{1}{2}. \tag{4.3}$$

We take a sufficiently large n so that (4.3) holds, and fix n. Put

$$\eta_m(x) = \begin{cases} 0 & \text{if } |x - x_n| < n - m^{-1}, \\ (n - |x - x_n|)^{-1} - m & \text{if } n - m^{-1} \leqslant |x - x_n| < n, \end{cases}$$

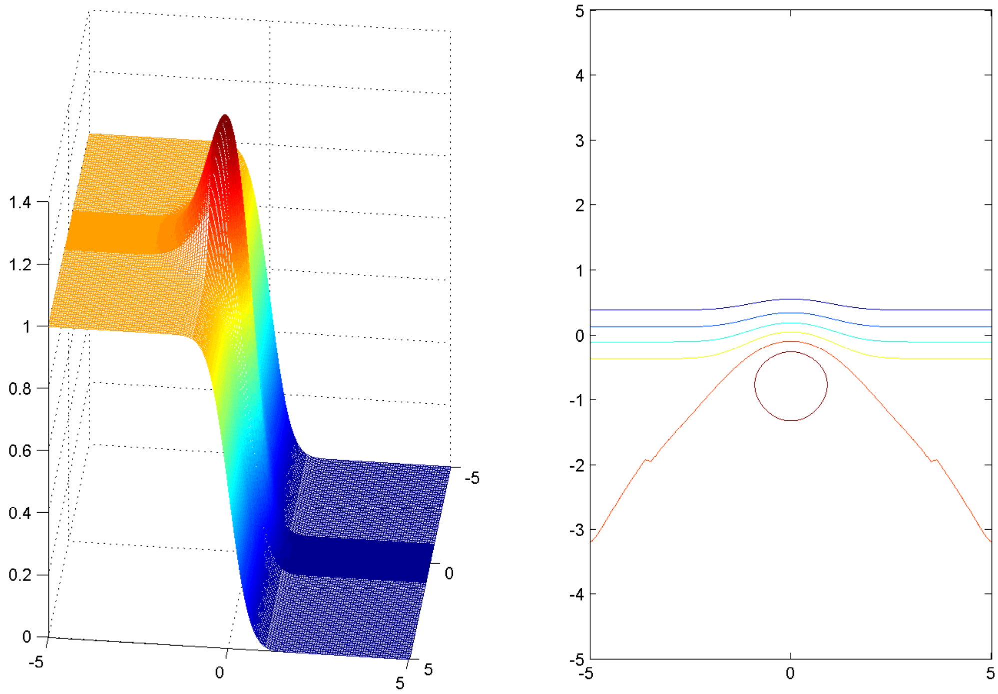

FIG. 1. The function v and some of its level sets at time $t = 0.1$.

where $m = 1, 2, \ldots$. Then $\eta_m(x)$ is a convex function in $B(x_n, n)$. On the other hand, since ψ_n is α_n-concave for some $\alpha_n < 0$, $\psi_n^{\alpha_n}$ is convex in $B(x_n, n)$, and the function $\psi_n^{\alpha_n} + \eta_m$ is also convex in $B(x_n, n)$. Put

$$\varphi_m(x) = \begin{cases} (\psi_n(x)^{\alpha_n} + \eta_m(x))^{1/\alpha_n} & \text{if } x \in B(x_n, n), \\ 0 & \text{otherwise,} \end{cases}$$

$$u_m(x,t) = (e^{t\Delta}\varphi_m)(x).$$

Then $\varphi_m(x)$ is an α_n-concave function such that

$$\begin{aligned} &\varphi_m \in C_0(\mathbb{R}^N), \qquad \operatorname{supp} \varphi_m = \overline{B(x_n, n)} \subset \{x_N < -n^{-1}\}, \\ &\lim_{m\to\infty} \sup_{x \in B(x_n, n)} \|\varphi_m(x) - \psi_n(x)\| = 0. \end{aligned} \tag{4.4}$$

Hence

$$\lim_{m\to\infty} \sup_{B(0,R)} |u_m(x,t) - u_n(x,t)| = 0$$

for any $R > 0$ and $t > 0$. This together with (4.3) implies that, for any sufficiently large m,

$$u_m(X, t_*) > \frac{1}{2}, \qquad u_m(Y, t_*) > \frac{1}{2}, \qquad u_m\left(\frac{X+Y}{2}, t_*\right) < \frac{1}{2}; \tag{4.5}$$

thus $u_m(\cdot, t_*)$ is not quasi-concave in $\mathbb{R}^N$.

If $\Omega = \mathbb{R}^N$, we put $\varphi(x) = \varphi_m(x)$, and the proof of Theorem 4.1 is complete.

For $\Omega \neq \mathbb{R}^N$, without loss of generality, we can assume (3.1). We pick a sufficiently large m so that (4.5) holds. By (3.3) and (4.4), we can take an integer l such that $\varphi_m \in C_0(\Omega_l)$. Let U_l be the solution of

$$\begin{cases} \partial_t U = \Delta U & \text{in } \Omega_l \times (0, \infty), \\ U(x,t) = 0 & \text{on } \partial\Omega_l \times (0, \infty), \\ U(x,0) = \varphi_m(x) & \text{in } \Omega_l. \end{cases} \tag{4.6}$$

Then

$$\lim_{m\to\infty} \sup_{B(0,R)} |U_l(x,t) - u_m(x,t)| = 0$$

for any $R > 0$ and $t > 0$. By (4.5), there exists a sufficiently large L such that

$$U_L(X, t_*) > \frac{1}{2}, \quad U_L(Y, t_*) > \frac{1}{2}, \quad U_L\left(\frac{X+Y}{2}, t_*\right) < \frac{1}{2}; \tag{4.7}$$

thus $U_L(\cdot, t_*)$ is not quasi-concave in Ω_L. Finally, we put

$$u(x,t) = U_L(Lx, L^2 t), \qquad \varphi(x) = \varphi_m(Lx) \in C_0(\Omega)$$

for all $x \in \Omega$ and $t > 0$. Then u is a solution of (1.1) and φ is α_n-concave in Ω. Furthermore, by (4.7), $u(\cdot, \bar{t})$ is not quasi-concave in Ω, for $\bar{t} = L^{-2} t_* \leqslant t_*$; thus the proof of Theorem 4.1 is complete. □

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REFERENCES

1. ARONSON, D. G., & VÁZQUEZ, J. L. Eventual C^∞-regularity and concavity for flows in one-dimensional porous media. *Arch. Ration. Mech. Anal.* **99** (1987), 329–348. Zbl 0642.76108 MR 0898714
2. BRASCAMP, H. J., & LIEB, E. H. On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** (1976), 366–389. Zbl 0334.26009 MR 0450480
3. CAFFARELLI, L. A., & FRIEDMAN, A. Regularity of the free boundary of a gas flow in an n-dimensional porous medium. *Indiana Univ. Math. J.* **29** (1980), 361–391. Zbl 0439.76085 MR 0570687
4. CAFFARELLI, L. A., VÁZQUEZ, J. L., & WOLANSKI, N. I. Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation. *Indiana Univ. Math. J.* **36** (1987), 373–401. Zbl 0644.35058 MR 0891781
5. DASKALOPOULOS, P., & HAMILTON, R. The free boundary for the n-dimensional porous medium equation. *Int. Math. Res. Notices* **1997**, no. 17, 817–831. Zbl 0886.35114 MR 1474840
6. DASKALOPOULOS, P., & HAMILTON, R. Regularity of the free boundary for the porous medium equation. *J. Amer. Math. Soc.* **11** (1998), 899–965. Zbl 0910.35145 MR 1623198
7. DASKALOPOULOS, P., HAMILTON, R., & LEE, K. All time C^∞-regularity of the interface in degenerate diffusion: a geometric approach. *Duke Math. J.* **108** (2001), 295–327. Zbl 1017.35052 MR 1833393

8. GALAKTIONOV, V. A., & VÁZQUEZ, J. L. Geometrical properties of the solutions of one-dimensional nonlinear parabolic equations. *Math. Ann.* **303** (1995), 741–769. Zbl 0842.35006 MR 1359958
9. ISHIGE, K., & SALANI, P. Is quasi-concavity preserved by heat flow? *Arch. Math. (Basel)* **90** (2008), 450–460. Zbl 1176.35012 MR 2414248
10. GRECO, A., & KAWOHL, B. Log-concavity in some parabolic problems. *Electron. J. Differential Equations* **1999**, no. 19, 12 pp. Zbl 0921.35030 MR 1692565
11. KENNINGTON, A. U. Power concavity and boundary value problems. *Indiana Univ. Math J.* **34** (1985), 687–704. Zbl 0549.35025 MR 0794582
12. KENNINGTON, A. U. Convexity of level curves for an initial value problem. *J. Math. Anal. Appl.* **133** (1988), 324–330. Zbl 0696.35080 MR 0954709
13. KOREVAAR, N. J. Convex solutions to nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.* **32** (1983), 603–614. Zbl 0481.35024 MR 0703287
14. LEE, K.-A., & VÁZQUEZ, J. L. Geometrical properties of solutions of the porous medium equation for large times. *Indiana Univ. Math. J.* **52** (2003), 991–1016. Zbl 1080.35037 MR 2001942
15. LEE, K.-A., & VÁZQUEZ, J. L. Parabolic approach to nonlinear elliptic eigenvalue problems. *Adv. Math.* **219** (2008), 2006–2028. Zbl 1152.35060 MR 2456273
16. NICKEL, K. Gestaltaussagen über Lösungen parabolischer Differentialgleichungen. *J. Reine Angew. Math.* **211** (1962), 78–94. Zbl 0127.31801 MR 0146534
17. VÁZQUEZ, J. L. *The Porous Medium Equation. Mathematical Theory.* Oxford Math. Monogr., Oxford Univ. Press, Oxford (2007). Zbl 1107.35003 MR 2286292