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On the two-phase Navier–Stokes equations with surface tension

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The two-phase free boundary problem for the Navier–Stokes system is considered in a situation where the initial interface is close to a halfplane. By means of L_p-maximal regularity of the underlying linear problem we show local well-posedness of the problem, and prove that the solution, in particular the interface, becomes instantaneously real analytic.

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1. Introduction and main results

In this paper we consider a free boundary problem that describes the motion of two viscous incompressible capillary Newtonian fluids. The fluids are separated by an interface that is unknown and has to be determined as part of the problem.

Let $\Omega_1(0) \subset \mathbb{R}^{n+1}$ ($n \geqslant 1$) be a region occupied by a viscous incompressible fluid, *fluid*$_1$, and let $\Omega_2(0)$ be the complement of the closure of $\Omega_1(0)$ in $\mathbb{R}^{n+1}$, corresponding to the region occupied by a second incompressible viscous fluid, *fluid*$_2$. We assume that the two fluids are immiscible. Let Γ_0 be the hypersurface that bounds $\Omega_1(0)$ (and hence also $\Omega_2(0)$) and let $\Gamma(t)$ denote the position of Γ_0 at time t. Thus, $\Gamma(t)$ is a sharp interface which separates the fluids occupying the regions $\Omega_1(t)$ and $\Omega_2(t)$, respectively, where $\Omega_2(t) := \mathbb{R}^{n+1} \setminus \overline{\Omega}_1(t)$.

We denote the normal field on $\Gamma(t)$, pointing from $\Omega_1(t)$ into $\Omega_2(t)$, by $\nu(t,\cdot)$. Moreover, we denote by $V(t,\cdot)$ and $\kappa(t,\cdot)$ the normal velocity and the mean curvature of $\Gamma(t)$ with respect to $\nu(t,\cdot)$, respectively. Here the curvature $\kappa(x,t)$ is assumed to be negative when $\Omega_1(t)$ is convex in a neighborhood of $x \in \Gamma(t)$. The motion of the fluids is governed by the following system of equations for $i = 1, 2$:

$$\begin{cases} \rho_i(\partial_t u + (u\,|\,\nabla)u) - \mu_i \Delta u + \nabla q = 0 & \text{in } \Omega_i(t), \\ \operatorname{div} u = 0 & \text{in } \Omega_i(t), \\ -[\![S(u,q)\nu]\!] = \sigma\kappa\nu & \text{on } \Gamma(t), \\ [\![u]\!] = 0 & \text{on } \Gamma(t), \\ V = (u\,|\,\nu) & \text{on } \Gamma(t), \\ u(0) = u_0 & \text{in } \Omega_i(0), \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.1}$$

Here, $S = S(u, q)$ is the stress tensor defined by

$$S(u,q) = \mu_i(\nabla u + (\nabla u)^{\mathsf{T}}) - qI \quad \text{in } \Omega_i(t),$$

and

$$[\![v]\!] = (v_{|\Omega_2(t)} - v_{|\Omega_1(t)})|_{\Gamma(t)}$$

denotes the jump of the quantity v, defined on both domains $\Omega_i(t)$, across the interface $\Gamma(t)$.

Given are the initial position Γ_0 of the interface, and the initial velocity

$$u_0 : \Omega_0 \to \mathbb{R}^{n+1}, \qquad \Omega_0 := \Omega_1(0) \cup \Omega_2(0).$$

The unknowns are the velocity field $u(t,\cdot) : \Omega(t) \to \mathbb{R}^{n+1}$, the pressure field $q(t,\cdot) : \Omega(t) \to \mathbb{R}$, and the free boundary $\Gamma(t)$, where $\Omega(t) := \Omega_1(t) \cup \Omega_2(t)$.

The constants $\rho_i > 0$ and $\mu_i > 0$ denote the densities and the viscosities of the respective fluids, and the constant σ stands for the surface tension. Hence the material parameters ρ_i and μ_i depend on the phase i, but otherwise are assumed to be constant. System (1.1) comprises the *two-phase Navier–Stokes equations with surface tension*. The first equation in (1.1) reflects the balance of momentum, while the second expresses the fact that both fluids are incompressible. If surface tension is neglected, the boundary condition on $\Gamma(t)$ would be the equality of stresses on the two sides of the surface. The effect of surface tension introduces a discontinuity in the normal component of $[\![S(u,q)]\!]$ proportional to the mean curvature of $\Gamma(t)$. The fourth equation stipulates that the velocities are continuous across $\Gamma(t)$. Finally, the fifth equation, called the kinematic boundary condition, expresses the fact that fluid particles cannot cross $\Gamma(t)$.

In order to economize our notation, we set

$$\rho = \rho_1\chi_{\Omega_1(t)} + \rho_2\chi_{\Omega_2(t)}, \qquad \mu = \mu_1\chi_{\Omega_1(t)} + \mu_2\chi_{\Omega_2(t)},$$

where χ_D denotes the indicator function of a set D. With this convention, system (1.1) can be recast as

$$\begin{cases} \rho(\partial_t u + (u\,|\,\nabla)u) - \mu \Delta u + \nabla q = 0 & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ -[\![S(u,q)\nu]\!] = \sigma\kappa\nu & \text{on } \Gamma(t), \\ [\![u]\!] = 0 & \text{on } \Gamma(t), \\ V = (u\,|\,\nu) & \text{on } \Gamma(t), \\ u(0) = u_0 & \text{in } \Omega_0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.2}$$

In this publication we consider the case where Γ_0 is the graph of a function h_0 on $\mathbb{R}^n$. We then set $\Omega_1(0) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y < h_0(x)\}$, and consequently $\Omega_2(0) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y > h_0(x)\}$. Our main result on existence, uniqueness, and regularity of solutions then reads as follows.

THEOREM 1.1 Suppose $p > n + 3$. Then given $t_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(t_0) > 0$ such that for any initial values

$$(u_0, h_0) \in W_p^{2-2/p}(\Omega_0, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n),$$

satisfying the compatibility conditions

$$[\![\mu D(u_0)\nu_0 - \mu(\nu_0 \,|\, D(u_0)\nu_0)\nu_0]\!] = 0, \quad \operatorname{div} u_0 = 0 \quad \text{on } \Omega_0, \quad [\![u_0]\!] = 0,$$

with $D(u_0) := \nabla u_0 + (\nabla u_0)^{\mathsf{T}}$, and the smallness condition

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^n)} \leqslant \varepsilon_0,$$

problem (1.2) admits a classical solution (u, q, Γ) on $(0, t_0)$. The solution is unique in the function class described in Theorem 6.3. In addition, $\Gamma(t)$ is the graph of a function $h(t)$ on $\mathbb{R}^n$, $\mathcal{M} = \bigcup_{t\in(0,t_0)}(\{t\} \times \Gamma(t))$ is a real analytic manifold, and with

$$\mathcal{O} = \{(t, x, y) : t \in (0, t_0),\ x \in \mathbb{R}^n,\ y \neq h(t, x)\},$$

the function $(u, q) : \mathcal{O} \to \mathbb{R}^{n+2}$ is real analytic.

REMARKS 1.2 (a) Theorem 1.1 shows that solutions immediately regularize and become analytic in space and time. If one thinks of the situation of oil in contact with water, this result seems plausible, as capillary forces tend to smooth out corners in the interface separating the two different fluids.

(b) More precise statements for a transformed version of problem (1.2) will be given in Section 6. Due to the restriction $p > n + 3$, we shall show that

$$h \in C(J; BUC^2(\mathbb{R}^n)) \cap C^1(J; BUC^1(\mathbb{R}^n)), \tag{1.3}$$

where $J = [0, t_0]$. In particular, the normal of $\Omega_1(t)$, the normal velocity of $\Gamma(t)$, and the mean curvature of $\Gamma(t)$ are well-defined and continuous, so that (1.2) makes sense pointwise. For u and q we obtain

$$\begin{aligned} &u(t, \cdot) \in BUC^1(\Omega(t), \mathbb{R}^{n+1}) \quad \text{for } t \in J, \qquad u \in BUC(J \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1}),\\ &q(t, \cdot) \in UC(\Omega(t)) \quad \text{for } t \in J \setminus \{0\}. \end{aligned} \tag{1.4}$$

In addition, the solution (u, q, h) depends continuously on the initial values (u_0, h_0). Also interesting is the fact that the surface pressure jump will turn out to be real analytic as well.

(c) It is possible to relax the assumption $p > n + 3$. In fact, $p > (n + 3)/2$ can be shown to be sufficient. However, to keep the arguments as simple as possible, here we impose the stronger condition $p > n + 3$.

(d) If gravity acts on the fluids then the condition on the free boundary is to be replaced by

$$-[\![S(u, q)]\!]\nu = \sigma H\nu + \gamma_a[\![\rho]\!]y\nu \quad \text{on } \Gamma(t), \tag{1.5}$$

where y denotes the vertical component of a generic point on $\Gamma(t)$, and where $\gamma_a > 0$ is the acceleration of gravity. Our approach also covers this situation, yielding a solution having the same regularity properties as stated in the theorem above. Indeed, an analysis of our proof shows that we only need to replace the symbol $s(\lambda, \tau)$, introduced in (5.9), by

$$s(\lambda, \tau) = \lambda + \sigma\tau k(z) - \frac{\gamma_a [\![\rho]\!]}{\tau} k(z)$$

(see [33, 34]). It is well-known that the case where the heavy fluid lies above the light one leads to an instability, the Rayleigh–Taylor instability (see [34] for a proof).

(e) We mention that our results also cover the one-phase Navier–Stokes equations with surface tension (1.6).

(f) The solutions we obtain exist on an interval $(0, t_0)$ with $t_0 > 0$ arbitrary, but fixed, provided the initial data are sufficiently small. It can be shown that problem (1.2) also admits unique local solutions that enjoy the same regularity properties as above, provided $\sup_{x\in\mathbb{R}} |\nabla h_0|$ is sufficiently small in relation to the horizontal component of u_0. In this case, no other smallness conditions on the data are required. The proof of this result is considerably more involved, and the analysis requires delicate estimates for the nonlinear terms. Additionally, we need a modified version of Theorem 5.1 in order to dominate some of the nonlinear terms by linear ones. The proof of this modification will involve introducing a countable partition of unity and then establishing commutator estimates for certain pseudo-differential operators. Since this paper is already rather long, we refrain from including a proof of this result here. It will be contained in the forthcoming paper [33].

(g) The case where both fluids occupy a bounded domain has recently been considered in [23], building on the approach devised here and in [33].

Let us now discuss and contrast our results with results previously obtained by other researchers. In case $\Omega_2(t) = \emptyset$ one obtains the *one-phase Navier–Stokes equations with surface tension*

$$\begin{cases} \rho(\partial_t u + (u \mid \nabla)u) - \mu\Delta u + \nabla q = 0 & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ S(u, q)\nu = \sigma\kappa\nu & \text{on } \Gamma(t), \\ V = (u \mid \nu) & \text{on } \Gamma(t), \\ u(0) = u_0 & \text{in } \Omega_0, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.6}$$

Equations (1.6) describe the motion of an isolated liquid which moves due to capillary forces acting on the free boundary.

Problem (1.6) has received wide attention in the last two decades or so. Existence and uniqueness of solutions for $\sigma > 0$, as well as for $\sigma = 0$, in the case that $\Omega(0)$ is bounded (corresponding to an isolated fluid drop) has been extensively studied in a long series of papers by Solonnikov (see for instance [41]–[47] and [28] for the case $\sigma > 0$). Solonnikov proves existence and uniqueness results in various function spaces, including anisotropic Hölder and Sobolev–Slobodetskiĭ spaces. Moreover, it is shown in [42] that if Ω_0 is sufficiently close to a ball and the initial velocity u_0 is sufficiently small, then the solution exists globally, and converges to a uniform rigid rotation of the liquid about a certain axis which is moving uniformly with a constant speed (see also [29]). More recently, local existence and uniqueness of solutions for (1.6) (in the case that Ω is a bounded domain, a perturbed infinite layer, or a perturbed halfspace) in anisotropic Sobolev spaces $W^{2,1}_{p,q}$ with $2 < p < \infty$ and $n < q < \infty$ has been established by Shibata and Shimizu in [39, 40]. For results concerning (1.6) with $\sigma = 0$ we refer to the recent contributions [37, 38] and the references therein.

The motion of a layer of viscous, incompressible fluid in an ocean of infinite extent, bounded below by a solid surface and above by a free surface which includes the effects of surface tension and gravity (in which case Ω_0 is a strip, bounded above by Γ_0 and below by a fixed surface Γ_b) is considered by Allain [1], Beale [6], Beale and Nishida [7], Tani [49], and by Tani and Tanaka [50]. If the initial state and the initial velocity are close to equilibrium, global existence of solutions is proved in [6] for $\sigma > 0$, and in [50] for $\sigma \geqslant 0$, and the asymptotic decay rate for $t \to \infty$ is studied in [7].

Results concerning the *two-phase problem* (1.2) are more recent. Existence and uniqueness of local solutions is studied in [10, 11, 12, 48]. In more detail, Denisova [11] establishes existence and uniqueness of solutions (of the transformed problem in Lagrangian coordinates) with $v \in W_2^{r,r/2}$ for $r \in (5/2, 3)$ in the case that one of the domains is bounded. Tanaka [48] considers the two-phase Navier–Stokes equations with thermo-capillary convection in bounded domains, and he obtains existence and uniqueness of solutions with $(v, \theta) \in W_2^{r,r/2}$ for $r \in (7/2, 4)$, with θ denoting the temperature.

The approach used by Solonnikov, and also in [10–12, 37–40, 48–50], relies on a formulation in Lagrangian coordinates. In this formulation one obtains a transformed problem for the velocity and the pressure on a fixed domain, where the free boundary does not occur explicitly. The free boundary is then given by

$$\Gamma(t) = \left\{ \xi + \int_0^t v(\tau, \xi)\, \mathrm{d}\tau : \xi \in \Gamma_0 \right\},$$

where v is the velocity field in Lagrangian coordinates. It is not clear whether this formulation allows one to obtain smoothing results for the free boundary, as the regularity of $\Gamma(t)$ seems to be restricted by the regularity of Γ_0. To the best of our knowledge, the regularity of the free boundary for the Navier–Stokes equations with surface tension (1.1) or (1.6) has not been addressed in the literature before, with the notable exception of [6]. Beale considers the ocean problem with $\Omega(t) = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} : -b(x) < y < h(t, x)\}$ and he shows by a boot-strapping argument that solutions are C^k for any given fixed $k \in \mathbb{N}$, where the size of the initial data must be adjusted in dependence on k. As in our case, his approach does not rely on a formulation in Lagrangian coordinates.

In order to prove our main result we transform problem (1.2) into a problem on a fixed domain. The transformation is expressed in terms of the unknown height function h describing the free boundary. Our analysis proceeds by studying solvability properties of some associated linear problems. It is important to point out that we succeed in establishing optimal solvability results (also referred to as maximal regularity): see Theorem 3.1, Proposition 3.3, Theorem 4.1, Corollary 4.2 and Theorem 5.1. In other words, we show that the linear problems define isomorphisms between properly chosen function spaces. This property, in turn, allows us to resort to the implicit function theorem to establish the analyticity of solutions to the nonlinear problem, as will be pointed out below. All our results for the associated linear problems mentioned above seem to be new, as they give sufficient as well as necessary conditions for solvability. Our analysis is greatly facilitated by studying the Dirichlet-to-Neumann operator for the Stokes equations (see Section 4). It is interesting, and maybe even surprising, to observe the mapping properties of this operator (see Theorem 4.1). Our approach to establishing solvability results relies on the powerful theory of maximal regularity, in particular on the H^∞-calculus for sectorial operators, the Dore–Venni theorem, and the Kalton–Weis theorem (see for instance [13, 25, 26, 31]).

Based on the linear estimates we can solve the nonlinear problem by the contraction mapping principle. Analyticity of the solution is obtained in a rather short and elegant way by the implicit function theorem in conjunction with a scaling argument, relying on an idea that goes back to Angenent [3, 4] and Masuda [27]; see also [17, 18, 20]. More precisely, by introducing parameters which represent scaling in time, and translation into space, the implicit function theorem yields analytic dependence of the solution of a parameter dependent-problem on the parameters, and this can be translated into a smoothness result in space and time for the original problem.

The plan of this paper is as follows. Section 2 contains the transformation of the problem to a half-space and the determination of the proper underlying linear problem. In Sections 3, 4 and 5 we study this linearization and prove in particular the crucial maximal regularity results in an L_p-setting. Section 6 is then devoted to the nonlinear problem and contains the proof of our main result.

2. Reduction to a flat interface

In this section we first transform the free boundary problem (1.2) to a fixed domain, and we then introduce some function spaces that will be used throughout the paper. Suppose that $\Gamma(t)$ is a graph over $\mathbb{R}^n$, parametrized as

$$\Gamma(t) = \{(x, h(t,x)) : x \in \mathbb{R}^n\}, \qquad t \in J,$$

with $\Omega_2(t)$ lying "above" $\Gamma(t)$, i.e. $\Omega_2(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y > h(t,x)\}$ for $t \in J := [0,a]$. Reduction from deformed into true halfspaces is achieved by means of the transformations

$$\begin{aligned} v(t,x,y) &= \begin{bmatrix} u_1(t,x,h(t,x)+y) \\ \vdots \\ u_n(t,x,h(t,x)+y) \end{bmatrix}, \\ w(t,x,y) &= u_{n+1}(t,x,h(t,x)+y), \\ \pi(t,x,y) &= q(t,x,h(t,x)+y), \end{aligned}$$

where $t \in J$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $y \neq 0$. Since for $j,k = 1,\dots,n$ we have

$$\begin{aligned} \partial_j u_k &= \partial_j v_k - \partial_j h \partial_y v_k, & \partial_{n+1} u_k &= \partial_y v_k, \\ \partial_j u_{n+1} &= \partial_j w - \partial_j h \partial_y w, & \partial_{n+1} u_{n+1} &= \partial_y w, \\ \partial_j q &= \partial_j \pi - \partial_j h \partial_y \pi, & \partial_{n+1} q &= \partial_y \pi, \\ \partial_t u_k &= \partial_t v_k - \partial_t h \partial_y v_k, & \partial_t u_{n+1} &= \partial_t w - \partial_t h \partial_y w, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \Delta u_k &= \Delta_x v_k - 2(\nabla h \mid \nabla_x)\partial_y v_k + (1 + |\nabla h|^2)\partial_y^2 v_k - \Delta h \partial_y v_k, \\ \Delta u_{n+1} &= \Delta_x w - 2(\nabla h \mid \nabla_x)\partial_y w + (1 + |\nabla h|^2)\partial_y^2 w - \Delta h \partial_y w, \end{aligned}$$

we obtain from (1.2) the following quasilinear system with initial conditions:

$$\begin{cases} \rho\partial_t v - \mu\Delta_x v - \mu\partial_y^2 v + \nabla_x\pi = F_v(v,w,\pi,h) & \text{in } (0,\infty)\times\dot{\mathbb{R}}^{n+1}, \\ \rho\partial_t w - \mu\Delta_x w - \mu\partial_y^2 w + \partial_y\pi = F_w(v,w,h) & \text{in } (0,\infty)\times\dot{\mathbb{R}}^{n+1}, \\ \operatorname{div}_x v + \partial_y w = F_d(v,h) & \text{in } (0,\infty)\times\dot{\mathbb{R}}^{n+1}, \\ v(0,x,y) = v_0(x,y), \quad w(0,x,y) = w_0(x,y) & \text{in } \dot{\mathbb{R}}^{n+1}, \end{cases} \tag{2.2}$$

where $\dot{\mathbb{R}}^{n+1} = \{(x,y)\in\mathbb{R}^n\times\mathbb{R} : y\neq 0\}$. Here and in the following, ∇h and Δh always denote the gradient and the Laplacian of h with respect to $x\in\mathbb{R}^n$. Note that ρ and μ in general have jumps at $y=0$, i.e. $\rho=\rho_2$ for $y>0$, $\rho=\rho_1$ for $y<0$, and similarly for μ. The nonlinearities are given by

$$\begin{aligned} F_v(v,w,\pi,h) &= \mu\{-2(\nabla h\mid\nabla_x)\partial_y v + |\nabla h|^2\partial_y^2 v - \Delta h\partial_y v\} + \partial_y\pi\nabla h \\ &\quad + \rho\{-(v\mid\nabla_x)v + (\nabla h\mid v)\partial_y v - w\partial_y v\} + \rho\partial_t h\partial_y v, \\ F_w(v,w,h) &= \mu\{-2(\nabla h\mid\nabla_x)\partial_y w + |\nabla h|^2\partial_y^2 w - \Delta h\partial_y w\} \\ &\quad + \rho\{-(v\mid\nabla_x)w + (\nabla h\mid v)\partial_y w - w\partial_y w\} + \rho\partial_t h\partial_y w, \\ F_d(v,h) &= (\nabla h\mid\partial_y v). \end{aligned} \tag{2.3}$$

Note that these functions are polynomials in the derivatives of (v,w,π,h), hence analytic, and linear with respect to second derivatives, with coefficients of first order. This exhibits the quasilinear character of the problem.

To obtain the transformed interface conditions we observe that the outer normal ν of $\Omega_1(t)$ is given by

$$\nu(t,x) = \frac{1}{\sqrt{1+|\nabla h(t,x)|^2}}\begin{bmatrix} -\nabla h(t,x) \\ 1 \end{bmatrix},$$

where, as above, $\nabla h(t,x)$ denotes the gradient vector of h with respect to $x\in\mathbb{R}^n$. The normal velocity V of $\Gamma(\cdot)$ is

$$V(t,x) = \partial_t h(t,x)/\sqrt{1+|\nabla h(t,x)|^2}.$$

The kinematic condition $V=(u\mid\nu)$ on $\Gamma(\cdot)$ now reads

$$\partial_t h - \gamma w = H(v,h), \qquad H(v,h) := -(\gamma v\mid\nabla h). \tag{2.4}$$

Here $(\gamma w)(x) := w(x,0)$ denotes the trace of the function $w:\dot{\mathbb{R}}^{n+1}\to\mathbb{R}$ and, correspondingly, γv is the trace of $v:\dot{\mathbb{R}}^{n+1}\to\mathbb{R}^n$. Since u is continuous across $\Gamma(t)$, γv and γw are unambiguously defined. It is also noteworthy that the tangential derivatives of v and w are continuous across $\mathbb{R}^n$. The curvature of $\Gamma(t)$ is given by

$$\kappa(t,x) = \operatorname{div}_x\left(\frac{\nabla h(t,x)}{\sqrt{1+|\nabla h(t,x)|^2}}\right) = \Delta h - G_\kappa(h)$$

(see for instance equation (24) in [8, Appendix]) with

$$G_\kappa(h) = \frac{|\nabla h|^2\Delta h}{(1+\sqrt{1+|\nabla h|^2})\sqrt{1+|\nabla h|^2}} + \frac{(\nabla h\mid\nabla^2 h\nabla h)}{(1+|\nabla h|^2)^{3/2}}, \tag{2.5}$$

where $\nabla^2 h$ denotes the Hessian matrix of all second order derivatives of h. The components of $\mathcal{D}(v, w, h)$, the transformed version of the deformation tensor $D(u) = \nabla u + (\nabla u)^{\mathsf{T}}$, are given by

$$\begin{aligned} \mathcal{D}_{ij}(v, w, h) &= \partial_i v_j + \partial_j v_i - (\partial_i h \partial_y v_j + \partial_j h \partial_y v_i), \\ \mathcal{D}_{n+1,j}(v, w, h) &= \mathcal{D}_{j,n+1}(v, w, h) = \partial_y v_j + \partial_j w - \partial_j h \partial_y w, \\ \mathcal{D}_{n+1,n+1}(v, w, h) &= 2\partial_y w, \end{aligned} \tag{2.6}$$

for $i, j = 1, \dots, n$, where δ_{ij} denotes the Kronecker symbol. For the jumps of the components of the deformation tensor this yields

$$\begin{aligned} [\![\mu \mathcal{D}_{ij}(v, w, h)]\!] &= [\![\mu(\partial_i v_j + \partial_j v_i)]\!] - \partial_i h [\![\mu \partial_y v_j]\!] - \partial_j h [\![\mu \partial_y v_i]\!], \\ [\![\mu \mathcal{D}_{n+1,j}(v, w, h)]\!] &= [\![\mu \mathcal{D}_{j,n+1}(v, w, h)]\!] = [\![\mu \partial_j w]\!] + [\![\mu \partial_y v_j]\!] - \partial_j h [\![\mu \partial_y w]\!], \\ [\![\mu \mathcal{D}_{n+1,n+1}(v, w, h)]\!] &= 2[\![\mu \partial_y w]\!]. \end{aligned}$$

Therefore, the jump condition for the normal stress at the interface yields the following boundary conditions:

$$\begin{aligned} -[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] &= G_v(v, w, [\![\pi]\!], h), \\ -2[\![\mu \partial_y w]\!] + [\![\pi]\!] - \sigma \Delta h &= G_w(v, w, h), \end{aligned} \tag{2.7}$$

where the nonlinearities (G_v, G_w) have the form

$$\begin{aligned} G_v(v, w, [\![\pi]\!], h) = &- [\![\mu(\nabla_x v + (\nabla_x v)^{\mathsf{T}})]\!] \nabla h + |\nabla h|^2 [\![\mu \partial_y v]\!] + (\nabla h \mid [\![\mu \partial_y v]\!]) \nabla h \\ &- [\![\mu \partial_y w]\!] \nabla h + \{[\![\pi]\!] - \sigma(\Delta h - G_\kappa(h))\} \nabla h, \\ G_w(v, w, h) = &- (\nabla h \mid [\![\mu \partial_y v]\!]) - (\nabla h \mid [\![\mu \nabla_x w]\!]) + |\nabla h|^2 [\![\mu \partial_y w]\!] - \sigma G_\kappa(h). \end{aligned} \tag{2.8}$$

We note that $G = (G_v, G_w)$ is analytic in $(v, w, [\![\pi]\!], h)$. Moreover, G is linear in $(v, w, [\![\pi]\!])$, and in the second derivatives of h. Thus the boundary conditions are quasilinear as well.

Summarizing, we arrive at the following problem for $u = (v, w)$, π, and h:

$$\begin{cases} \partial_t u - \mu \Delta u + \nabla \pi = F(u, \pi, h) & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = F_d(u, h) & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = G_v(u, [\![\pi]\!], h) & \text{on } \mathbb{R}^n, \\ -2[\![\mu \partial_y w]\!] + [\![\pi]\!] - \sigma \Delta h = G_w(u, h) & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ \partial_t h - \gamma w = H(u, h) & \text{on } \mathbb{R}^n, \\ u(0) = u_0, \quad h(0) = h_0, \end{cases} \tag{2.9}$$

for $t > 0$. This is problem (1.2) transformed to the halfspaces $\mathbb{R}^{n+1}_{\pm} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \pm y > 0\}$.

Before studying solvability results for problem (2.9) let us first introduce suitable function spaces. Let $\Omega \subseteq \mathbb{R}^m$ be open and X be an arbitrary Banach space. By $L_p(\Omega; X)$ and $H^s_p(\Omega; X)$, for $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, we denote the X-valued Lebesgue and Bessel potential spaces of order s, respectively. We will also frequently make use of the fractional Sobolev–Slobodetskiĭ spaces $W^s_p(\Omega; X)$, $1 \leqslant p < \infty$, $s \in \mathbb{R} \setminus \mathbb{Z}$, with norm

$$\|g\|_{W^s_p(\Omega;X)} = \|g\|_{W^{[s]}_p(\Omega;X)} + \sum_{|\alpha|=[s]} \left(\int_\Omega \int_\Omega \frac{\|\partial^\alpha g(x) - \partial^\alpha g(y)\|^p_X}{|x - y|^{m+(s-[s])p}} \, dx \, dy \right)^{1/p}, \tag{2.10}$$

where $[s]$ denotes the largest integer smaller than s. Let $a \in (0, \infty]$ and $J = [0, a]$. We set

$$ {}_0W_p^s(J; X) := \begin{cases} \{g \in W_p^s(J; X) : g(0) = g'(0) = \cdots = g^{(k)}(0) = 0\}, \\ \qquad\qquad\qquad \text{if } k + 1/p < s < k + 1 + 1/p,\ k \in \mathbb{N} \cup \{0\}, \\ W_p^s(J; X) \quad \text{if } s < 1/p. \end{cases} $$

The spaces ${}_0H_p^s(J; X)$ are defined analogously. Here we remind the reader that $H_p^k = W_p^k$ for $k \in \mathbb{Z}$ and $1 < p < \infty$, and that $W_p^s = B_{pp}^s$ for $s \in \mathbb{R} \setminus \mathbb{Z}$.

For $\Omega \subset \mathbb{R}^m$ open and $1 \leqslant p < \infty$, the homogeneous Sobolev spaces $\dot{H}_p^1(\Omega)$ of order 1 are defined as

$$ \begin{aligned} &\dot{H}_p^1(\Omega) := (\{g \in L_{1,\mathrm{loc}}(\Omega) : \|\nabla g\|_{L_p(\Omega)} < \infty\}, \|\cdot\|_{\dot{H}_p^1(\Omega)}), \\ &\|g\|_{\dot{H}_p^1(\Omega)} := \Big(\sum_{j=1}^m \|\partial_j g\|_{L_p(\Omega)}^p\Big)^{1/p}. \end{aligned} \tag{2.11} $$

Then $\dot{H}_p^1(\Omega)$ is a Banach space, provided we factor out the constant functions and equip the resulting space with the corresponding quotient norm (see for instance [21, Lemma II.5.1]). We will always consider this quotient space topology without change of notation. In the case that Ω is locally Lipschitz, it is known that $\dot{H}_p^1(\Omega) \subset H_{p,\mathrm{loc}}^1(\overline{\Omega})$ (see [21, Remark II.5.1]), and consequently, any function in $\dot{H}_p^1(\Omega)$ has a well-defined trace on $\partial\Omega$.

For $s \in \mathbb{R}$ and $1 < p < \infty$ we also consider the homogeneous Bessel potential spaces $\dot{H}_p^s(\mathbb{R}^n)$ of order s, defined by

$$ \begin{aligned} &\dot{H}_p^s(\mathbb{R}^n) := (\{g \in \mathcal{S}'(\mathbb{R}^n) : \dot{I}^s g \in L_p(\mathbb{R}^n)\}, \|\cdot\|_{\dot{H}_p^s(\mathbb{R}^n)}), \\ &\|g\|_{\dot{H}_p^s(\mathbb{R}^n)} := \|\dot{I}^s g\|_{L_p(\mathbb{R}^n)}, \end{aligned} \tag{2.12} $$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of all tempered distributions, and $\dot{I}^s$ is the Riesz potential given by

$$ \dot{I}^s g := (-\Delta)^{s/2} g := \mathcal{F}^{-1}(|\xi|^s \mathcal{F} g), \qquad g \in \mathcal{S}'(\mathbb{R}^n). $$

By factoring out all polynomials, $\dot{H}_p^s(\mathbb{R}^n)$ becomes a Banach space with the natural quotient norm. For $s \in \mathbb{R} \setminus \mathbb{Z}$, the homogeneous Sobolev–Slobodetskiĭ spaces $\dot{W}_p^s(\mathbb{R}^n)$ of fractional order can be obtained by real interpolation as

$$ \dot{W}_p^s(\mathbb{R}^n) := (\dot{H}_p^k(\mathbb{R}^n), \dot{H}_p^{k+1}(\mathbb{R}^n))_{s-k,p}, \qquad k < s < k + 1, $$

where $(\cdot, \cdot)_{\theta,p}$ is the real interpolation method. It follows that

$$ \dot{I}^s \in \mathrm{Isom}(\dot{H}_p^{t+s}(\mathbb{R}^n), \dot{H}_p^t(\mathbb{R}^n)) \cap \mathrm{Isom}(\dot{W}_p^{t+s}(\mathbb{R}^n), \dot{W}_p^t(\mathbb{R}^n)), \qquad s, t \in \mathbb{R}, \tag{2.13} $$

with $\dot{W}_p^k = \dot{H}_p^k$ for $k \in \mathbb{Z}$. We refer to [5, Section 6.3] and [52, Section 5] for more information on homogeneous function spaces. In particular, it follows from parts (ii) and (iii) of [52, Theorem 5.2.3.1] that the definitions (2.11) and (2.12) are consistent if $\Omega = \mathbb{R}^n$, $s = 1$, and $1 < p < \infty$. We note in passing that

$$ \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|g(x) - g(y)|^p}{|x - y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}, \qquad \Big(\int_0^\infty t^{(1-s)p} \Big\|\frac{\mathrm{d}}{\mathrm{d}t} P(t) g\Big\|_{L_p(\mathbb{R}^n)}^p \frac{\mathrm{d}t}{t}\Big)^{1/p} \tag{2.14} $$

define equivalent norms on $\dot{W}_p^s(\mathbb{R}^n)$ for $0 < s < 1$, where $P(\cdot)$ denotes the Poisson semigroup (see [52, Theorem 5.2.3.2 and Remark 5.2.3.4]). Moreover,

$$\gamma_\pm \in \mathcal{L}(\dot{W}_p^1(\mathbb{R}_\pm^{n+1}), \dot{W}_p^{1-1/p}(\mathbb{R}^n)), \tag{2.15}$$

where $\gamma_\pm$ denotes the trace operators (see for instance [21, Theorem II.8.2]).

3. The linearized two-phase Stokes problem

In this section we consider the linear two-phase (inhomogeneous) Stokes problem

$$\begin{cases} \partial_t u - \mu \Delta u + \nabla \pi = f & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = f_d & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = g_v & \text{on } \mathbb{R}^n, \\ -2[\![\mu \partial_y w]\!] + [\![\pi]\!] = g_w & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ u(0) = u_0 & \text{in } \dot{\mathbb{R}}^{n+1}. \end{cases} \tag{3.1}$$

Here the initial value u_0 as well as the inhomogeneities (f, f_d, g_v, g_w) are given. We want to establish maximal regularity for this problem in the framework of L_p-spaces. Thus we are interested in solutions (u, π) in the class

$$u \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \qquad \pi \in L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1})).$$

We recall that $J = [0, a]$ and $\dot{\mathbb{R}}^{n+1} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \neq 0\}$. If (u, π) is a solution of (3.1) in this class we necessarily have $f \in L_p(J; L_p(\mathbb{R}^{n+1}))$, and additionally $u_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$ by trace theory. Moreover,

$$f_d \in H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^{n+1})),$$

as the operator div maps $L_p(\mathbb{R}^{n+1})$ onto $\dot{H}_p^{-1}(\mathbb{R}^{n+1})$. Taking traces at the interface $y = 0$ results in $g_v \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{1-1/2p}(\mathbb{R}^n, \mathbb{R}^n))$, and $g_w \in L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$. If, in addition,

$$[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/2p}(\mathbb{R}^n))$$

then g_w shares this regularity.

Here and in the following, the notation $1/2p$ means $1/(2p)$. The main result of this section states the converse of these assertions, i.e. maximal L_p-regularity for (3.1).

THEOREM 3.1 Let $1 < p < \infty$ be fixed, $p \neq 3/2, 3$, and assume that ρ_j and μ_j are positive constants for $j = 1, 2$, and set $J = [0, a]$. Then the Stokes problem (3.1) admits a unique solution (u, π) with regularity

$$u \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \qquad \pi \in L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1}))$$

if and only if the data (f, f_d, g_v, g_w, u_0) satisfy the following regularity and compatibility conditions:

(a) $f \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$,

(b) $f_d \in H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^{n+1}))$,

(c) $g_v \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^n))$, $g_w \in L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$,

(d) $u_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$,

(e) $\operatorname{div} u_0 = f_d(0)$ in $\dot{\mathbb{R}}^{n+1}$ and $[\![u_0]\!] = 0$ on $\mathbb{R}^n$ if $p > 3/2$,

(f) $-[\![\mu \partial_y v_0]\!] - [\![\mu \nabla_x w_0]\!] = g_v(0)$ on $\mathbb{R}^n$ if $p > 3$.

In addition, $[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n))$ if and only if

$$g_w \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)).$$

The solution map $(f, f_d, g_v, g_w, g_h, u_0, h_0) \mapsto (u, \pi, [\![\pi]\!])$ is continuous between the corresponding spaces.

Proof. The basic idea of the proof is to reduce system (3.1) to the case where $(f, f_d, u_0) = (0, 0, 0)$ and $g_v(0) = 0$, and then to solve the resulting problem by means of the *Dirichlet-to-Neumann operator* for the Stokes problem. We can achieve this goal in four steps, as follows.

STEP 1 For given data (f, g_v, u_0) subject to the conditions of the theorem we first solve the parabolic problem without pressure and divergence, i.e. we solve

$$\begin{cases} \partial_t u - \mu \Delta u = f & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = g_v & \text{on } \mathbb{R}^n, \\ -2[\![\mu \partial_y w]\!] = \tilde{g}_w & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ u(0) = u_0 & \text{in } \dot{\mathbb{R}}^{n+1}. \end{cases} \tag{3.2}$$

Here we set $\tilde{g}_w = -2e^{-D_n t}[\![\mu \partial_y w_0]\!]$ with $D_n := -\Delta$ in $L_p(\mathbb{R}^n)$. The function $\tilde{g}_w$ has the same regularity as g_v, and the necessary compatibility conditions are satisfied. By reflection of the $\{y < 0\}$-part of this problem to the upper halfplane, we obtain a parabolic system on a halfspace with boundary conditions satisfying the Lopatinskiĭ–Shapiro conditions. Therefore, the theory of parabolic boundary value problems yields a unique solution u_1 for (3.2) with regularity

$$u_1 \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})).$$

We refer to Denk, Hieber and Prüss [13, 14] for this.

STEP 2 In this step we solve the Stokes equations

$$\begin{cases} \rho \partial_t u - \mu \Delta u + \nabla \pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = f_d - \operatorname{div} u_1 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ u(0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \end{cases} \tag{3.3}$$

where u_1 is the solution obtained in Step 1. It follows from assumption (e) that system (3.3) satisfies the compatibility condition $\operatorname{div} u(0) = f_d(0) - \operatorname{div} u_1(0) = 0$. We recall that $\rho = \rho_2 \chi_{\mathbb{R}_+^{n+1}} + \rho_1 \chi_{\mathbb{R}_-^{n+1}}$ and $\mu = \mu_2 \chi_{\mathbb{R}_+^{n+1}} + \mu_1 \chi_{\mathbb{R}_-^{n+1}}$. Concentrating on the upper halfplane, we extend the function $f_d -$

$\operatorname{div} u_1$ evenly in y to all of $\mathbb{R}^{n+1}$ and solve the Stokes problem with coefficients ρ_2, μ_2 in the whole space (see [9, Theorem 5.1]). This gives a solution which has the property that the normal velocity w vanishes at the interface; the latter is due to the symmetries of the equations. We restrict this solution to $\mathbb{R}^{n+1}_+$. We then do the same on the lower halfplane. This results in a solution (u_2, π_2) for system (3.3) that satisfies

$$\begin{aligned} &u_2 \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \\ &\pi_2 \in L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})), \qquad w_2 = 0 \quad \text{on } \mathbb{R}^n, \end{aligned}$$

where, as before, $u_2 = (v_2, w_2)$. We remark that the tangential part of the velocity, i.e. v_2, may now have a jump at the boundary $y = 0$.

STEP 3 To remove the jump in the tangential velocity, we solve the homogeneous Stokes problem in the lower halfplane with this jump as Dirichlet datum, that is, we solve

$$\begin{cases} \rho_1 \partial_t u - \mu_1 \Delta u + \nabla \pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1}_-, \\ \operatorname{div} u = 0 & \text{in } \dot{\mathbb{R}}^{n+1}_-, \\ v = [\![v_2]\!], \quad w = 0 & \text{on } \mathbb{R}^n, \\ u_0(0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1}_-, \end{cases} \tag{3.4}$$

where $u_2 = (v_2, w_2)$ is the solution obtained in Step 2. It follows from Proposition 3.3 below that system (3.4) has a unique solution with the regularity properties of Theorem 3.1. Let (u_3, π_3) be defined by

$$(u_3, \pi_3) := \begin{cases} (0, 0) & \text{in } \dot{\mathbb{R}}^{n+1}_+, \\ \text{the solution of (3.4)} & \text{in } \dot{\mathbb{R}}^{n+1}_-. \end{cases}$$

Then (u_3, π_3) also satisfies the regularity properties stated in Theorem 3.1 and we have $[\![v_3]\!] = -[\![v_2]\!]$ and $[\![w_3]\!] = 0$ on $\mathbb{R}^n$.

STEP 4. In this step we consider the problem

$$\begin{cases} \rho \partial_t u - \mu \Delta u + \nabla \pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = [\![\mu \partial_y (v_2 + v_3)]\!] + [\![\mu \nabla_x (w_2 + w_3)]\!] & \text{on } \mathbb{R}^n, \\ -2[\![\mu \partial_y w]\!] + [\![\pi]\!] = g_w - \tilde{g}_w + 2[\![\mu \partial_y (w_2 + w_3)]\!] - [\![\pi_2 + \pi_3]\!] & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ u(0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \end{cases} \tag{3.5}$$

with (v_2, w_2, π_2) and (v_3, w_3, π_3) the solutions obtained in Steps 2 and 3. Here it should be observed that the function on the right hand side of line 3 appearing as boundary condition has zero time trace. Problem (3.5), which is also of independent interest, will be studied in detail in the next section. It will be shown in Corollary 4.2 that it admits a unique solution, denoted here by (u_4, π_4), which satisfies the regularity properties stated in Theorem 3.1.

To finish the proof of Theorem 3.1 we set $(u, \pi) = (\sum_{i=1}^4 u_i, \sum_{i=1}^4 \pi_i)$, where (u_i, π_i) are the solutions obtained in Step i, with $\pi_1 := 0$. Then (u, π) satisfies the regularity properties stated in the theorem and it is the unique solution of (3.1). □

REMARK 3.2 We refer to the recent paper by Bothe and Prüss [9] for results related to Theorem 3.1 for the more general and involved situation of a generalized Newtonian fluid.

Let us now consider the problem

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ u = u_b & \text{on } \mathbb{R}^n, \\ u(0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \end{cases} \tag{3.6}$$

and prove the result that was used in Step 3 above.

PROPOSITION 3.3 Let $1 < p < \infty$ and assume that ρ_j and μ_j are positive constants, $j = 1, 2$, and set $J = [0, a]$. Then problem (3.6) admits a unique solution (u, π) with

$$u \in {}_0H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \qquad \pi \in L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1}))$$

if and only if the data $u_b = (v_b, w_b)$ satisfy the following regularity assumptions:

(a) $v_b \in {}_0W^{1-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n, \mathbb{R}^n))$,
(b) $w_b \in {}_0H^1_p(J; \dot{W}^{-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n))$.

Proof. (i) Assume for the moment that we have a solution in the proper regularity class even on the halfline $J = \mathbb{R}_+$. Then we may employ the Laplace transform in t and the Fourier transform in the tangential variables $x \in \mathbb{R}^n$, to obtain the following boundary value problem for a system of ordinary differential equations on $\dot{\mathbb{R}}$:

$$\begin{cases} \omega^2\hat{v} - \mu\partial_y^2\hat{v} + i\xi\hat{\pi} = 0, & y \neq 0, \\ \omega^2\hat{w} - \mu\partial_y^2\hat{w} + \partial_y\hat{\pi} = 0, & y \neq 0, \\ (i\xi \mid \hat{v}) + \partial_y\hat{w} = 0, & y \neq 0, \\ \hat{v}(0) = \hat{v}_b, \ \hat{w}(0) = \hat{w}_b. \end{cases}$$

Here we have set $\omega_j^2 = \rho_j\lambda + \mu_j|\xi|^2$, $j = 1, 2$, and

$$\hat{v}_j(\lambda, \xi, y) = (2\pi)^{-n/2}\int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda t}e^{-i(x\mid\xi)}v(t, x, y)\,\mathrm{d}x\,\mathrm{d}t, \qquad (-1)^j y > 0.$$

This system of equations is easily solved to yield

$$\begin{bmatrix} \hat{v}_2 \\ \hat{w}_2 \\ \hat{\pi}_2 \end{bmatrix} = e^{-\omega_2 y/\sqrt{\mu_2}} \begin{bmatrix} a_2 \\ \frac{\sqrt{\mu_2}}{\omega_2}(i\xi \mid a_2) \\ 0 \end{bmatrix} + \alpha_2 e^{-|\xi|y} \begin{bmatrix} -i\xi \\ |\xi| \\ \rho_2\lambda \end{bmatrix} \tag{3.7}$$

for $y > 0$, and

$$\begin{bmatrix} \hat{v}_1 \\ \hat{w}_1 \\ \hat{\pi}_1 \end{bmatrix} = e^{\omega_1 y/\sqrt{\mu_1}} \begin{bmatrix} a_1 \\ -\frac{\sqrt{\mu_1}}{\omega_1}(i\xi \mid a_1) \\ 0 \end{bmatrix} + \alpha_1 e^{|\xi|y} \begin{bmatrix} -i\xi \\ -|\xi| \\ \rho_1\lambda \end{bmatrix} \tag{3.8}$$

for $y < 0$. Here $a_i \in \mathbb{R}^n$ and α_i have to be determined by the boundary conditions $\hat{v}(0) = \hat{v}_b$ and $\hat{w}(0) = \hat{w}_b$. We have

$$a_2 - i\xi\alpha_2 = \hat{v}_b = a_1 - i\xi\alpha_1,$$

and

$$\frac{\sqrt{\mu_2}}{\omega_2}(i\xi \mid a_2) + |\xi|\alpha_2 = \hat{w}_b = -\frac{\sqrt{\mu_1}}{\omega_1}(i\xi \mid a_1) - |\xi|\alpha_1$$

where $(a \mid b) := \sum a^j b^j$ for $a, b \in \mathbb{C}^n$. This yields

$$\begin{aligned} a_j &= \hat{v}_b + i\xi\alpha_j, \qquad j = 1, 2,\\ \alpha_2 &= -\frac{\omega_2 + \sqrt{\mu_2}|\xi|}{\rho_2\lambda|\xi|}(\sqrt{\mu_2}(i\xi \mid \hat{v}_b) - \omega_2\hat{w}_b),\\ \alpha_1 &= -\frac{\omega_1 + \sqrt{\mu_1}|\xi|}{\rho_1\lambda|\xi|}(\sqrt{\mu_1}(i\xi \mid \hat{v}_b) + \omega_1\hat{w}_b). \end{aligned} \tag{3.9}$$

(ii) By parabolic theory, the velocity u has the correct regularity provided the pressure gradient is in L_p, and provided

$$u_b \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1}))$$

(see for instance Denk, Hieber and Prüss [14]). In particular this regularity of u_b is necessary. Note that the embedding

$${}_0H_p^1(J; \dot{W}_p^{-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)) \hookrightarrow {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \tag{3.10}$$

is valid. This follows from the fact that $\dot{W}_p^{-1/p}(\mathbb{R}^n) \hookrightarrow W_p^{-1/p}(\mathbb{R}^n)$ by a similar argument to the proof of [30, Lemma 6.3] where we set $Au := (1 - \Delta)u$.

(iii) We will now introduce some operators that will play a crucial role in our analysis. We set $G := \partial_t$ in $X := L_p(J; L_p(\mathbb{R}^n))$ with domain

$$\mathsf{D}(G) = {}_0H_p^1(J; L_p(\mathbb{R}^n)).$$

Then it is well-known that G is closed, invertible and sectorial with angle $\pi/2$, and $-G$ is the generator of a C_0-semigroup of contractions in $L_p(\mathbb{R}^n)$. Moreover, G admits an H^∞-calculus in X with H^∞-angle $\pi/2$ as well; see e.g. [24]. The symbol of G is λ, the time covariable.

Next we set $D_n := -\Delta$, the Laplacian in $L_p(\mathbb{R}^n)$ with domain $\mathsf{D}(D_n) = H_p^2(\mathbb{R}^n)$. It is also well-known that D_n is closed and sectorial with angle 0, and it admits a bounded H^∞-calculus which is even $\mathcal{R}$-bounded with $\mathcal{R}H^\infty$-angle 0; see e.g. [15]. These results also hold for the canonical extension of D_n to X, and also for the fractional power $D_n^{1/2}$ of D_n. Note that the domain of $D_n^{1/2}$ is

$$\mathsf{D}(D_n^{1/2}) = L_p(J; H_p^1(\mathbb{R}^n)).$$

The symbol of D_n is $|\xi|^2$, and that of $D_n^{1/2}$ is $|\xi|$, where ξ is the covariable of x. By the Dore–Venni theorem for sums of commuting sectorial operators (cf. [16, 35]), we see that the parabolic operators $L_j := \rho_j G + \mu_j D_n$ with natural domain

$$\mathsf{D}(L_j) = \mathsf{D}(G) \cap \mathsf{D}(D_n) = {}_0H_p^1(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^2(\mathbb{R}^n))$$

are closed, invertible and sectorial with angle $\pi/2$. Moreover, L_j also admits a bounded H^∞-calculus in X with H^∞-angle $\pi/2$; cf. e.g. [31]. The same results are valid for the operators $F_j = L_j^{1/2}$, their H^∞-angle is $\pi/4$, and their domains are

$$\mathsf{D}(F_j) = \mathsf{D}(G^{1/2}) \cap \mathsf{D}(D_n^{1/2}) = {}_0H_p^{1/2}(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^1(\mathbb{R}^n)).$$

The symbol of L_j is $\rho_j\lambda + \mu_j|\xi|^2$ and that of F_j is given by $\sqrt{\rho_j\lambda + \mu_j|\xi|^2}$.

Let R denote the Riesz operator with symbol $\zeta = \xi/|\xi|$. It follows from the Mikhlin–Hörmander theorem that R is a bounded linear operator on $W_p^s(\mathbb{R}^n)$, and hence also on $L_p(J; W_p^s(\mathbb{R}^n))$ by canonical extension.

(iv) Let $\beta_2 = \rho_2\lambda\alpha_2$. Then the transform of the pressure π_2 in $\mathbb{R}_+^{n+1}$ is given by $e^{-|\xi|y}\beta_2$. The pressure gradient will be in L_p provided the inverse transform of β_2 is in $L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$. In fact, $e^{-|\xi|y}$ is the symbol of the Poisson semigroup $P(\cdot)$ in $L_p(\mathbb{R}^n)$, and the negative generator of $P(\cdot)$ is $D_n^{1/2}$. Then the second part of (2.14) shows that $D_n^{1/2}P(\cdot)\beta_2 \in L_p(\mathbb{R}_+; L_p(\mathbb{R}^n))$ if and only $\beta_2 \in \dot{W}_p^{1-1/p}(\mathbb{R}^n)$. This result extends canonically to $L_p(J; L_p(\mathbb{R}_+^{n+1}))$.

Therefore, let us look more closely at β_2. We easily obtain

$$\beta_2 = \rho_2\frac{\lambda}{|\xi|}\hat{w}_b + (\sqrt{\mu}_2\omega_2 + \mu_2|\xi|)(\hat{w}_b - (i\zeta \mid \hat{v}_b)),$$

where $\zeta = \xi/|\xi|$. We recall that $\dot{D}_n^{1/2} := \mathcal{F}^{-1}(|\xi|\mathcal{F}\cdot) : \dot{W}_p^s(\mathbb{R}^n) \to \dot{W}_p^{s-1}(\mathbb{R}^n)$ is an isomorphism.

With the operators introduced above, b_2, the inverse transform of β_2, can be represented by

$$b_2 = \rho_2 G\dot{D}_n^{-1/2}w_b + (\sqrt{\mu_2}F_2 + \mu_2 D_n^{1/2})(w_b - i(R \mid v_b)) =: b_{21} + b_{22}.$$

Due to (3.10) and ${}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)) = D_{F_j}(2 - 1/p, p)$, the second term b_{22} is in

$$D_{F_j}(1 - 1/p, p) = {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)),$$

which embeds into $L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$. Here we use the notation

$$D_{F_j}(\theta, p) = (X, \mathsf{D}(F_j))_{\theta,p}, \qquad D_{F_j}(1+\theta, p) = (\mathsf{D}(F_j), \mathsf{D}(F_j^2))_{\theta,p}, \qquad \theta \in (0, 1).$$

Thus it remains to look at the first term $b_{21} = \rho_2 G D_n^{-1/2} w_b$. Since

$$G\dot{D}_n^{-1/2} : {}_0H_p^1(J; \dot{W}_p^{-1/p}(\mathbb{R}^n)) \to L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$$

is bounded and invertible, we see that the condition $w_b \in {}_0H_p^1(J; \dot{W}_p^{-1/p}(\mathbb{R}^n))$ is necessary and sufficient for $b_{21} \in L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$. Of course, similar arguments apply for the lower half-plane. □

4. The Dirichlet-to-Neumann operator for the Stokes equation

The main ingredient in analyzing problem (3.1) with $(f, f_d, u_0) = (0, 0, 0)$ and $g_v(0) = 0$ is the Dirichlet-to-Neumann operator. It is defined as follows. Let (u, π) be the solution of the Stokes problem (3.6) with Dirichlet boundary condition u_b on $\mathbb{R}^n$ (see Proposition 3.3). We then define the *Dirichlet-to-Neumann operator* by means of

$$(\mathcal{DN})u_b = -[\![S(u,\pi)]\!]e_{n+1} = -[\![\mu(\nabla u + (\nabla u)^{\mathsf{T}})]\!]e_{n+1} + [\![\pi]\!]e_{n+1}. \tag{4.1}$$

For this purpose it is convenient to split u into $u = (v, w)$ as before, and u_b into $u_b = (v_b, w_b)$. Then we obtain

$$(\mathcal{DN})u_b = (-[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!], -2[\![\mu\partial_y w]\!] + [\![\pi]\!]). \tag{4.2}$$

We will now formulate and prove the main result of this section.

THEOREM 4.1 The Dirichlet-to-Neumann operator $\mathcal{DN}$ for the Stokes problem is an isomorphism from the *Dirichlet space* of $u_b = (v_b, w_b)$ with

$$v_b \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^n)),$$
$$w_b \in {}_0H_p^1(J; \dot{W}_p^{-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))$$

onto the *Neumann space* of $g = (g_v, g_w)$ with

$$g_v \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^n)),$$
$$g_w \in L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n)).$$

Proof. (i) Let $(\hat{v}_1, \hat{w}_1, \hat{\pi}_1)$ and $(\hat{v}_2, \hat{w}_2, \hat{\pi}_2)$ be as in (3.7)–(3.8). We may now compute the symbol of the Dirichlet-to-Neumann operator. We have

$$(\mathcal{DN})\hat{u}_b = \begin{bmatrix} \omega_1\sqrt{\mu_1}a_1 + \omega_2\sqrt{\mu_2}a_2 - (\alpha_1\mu_1 + \alpha_2\mu_2)|\xi|i\xi - [\![\mu]\!]i\xi\hat{w}_b \\ 2i(\mu_2 a_2 - \mu_1 a_1 \mid \xi) + 2(\alpha_2\mu_2 - \alpha_1\mu_1)|\xi|^2 + \lambda(\alpha_2\rho_2 - \alpha_1\rho_1) \end{bmatrix},$$

where the functions α_j and a_j are given in (3.9). Simple algebraic manipulations then yield the following symbol:

$$(\mathcal{DN})(\lambda, \xi) = \begin{bmatrix} \alpha + \beta\zeta\otimes\zeta & i\gamma\zeta \\ -i\gamma\zeta^T & \alpha + \delta \end{bmatrix}, \tag{4.3}$$

where $\zeta = \xi/|\xi|$ and

$$\begin{aligned} \alpha &= \sqrt{\mu_1}\omega_1 + \sqrt{\mu_2}\omega_2, & \beta &= (\mu_1 + \mu_2)|\xi|, \\ \gamma &= (\sqrt{\mu_2}\omega_2 - \sqrt{\mu_1}\omega_1) - [\![\mu]\!]|\xi|, & \delta &= (\omega_1^2 + \omega_2^2)/|\xi| = \beta + (\rho_1 + \rho_2)\lambda/|\xi|. \end{aligned} \tag{4.4}$$

Next we want to compute the inverse of the Dirichlet-to-Neumann operator. Thus we have to solve the equation $(\mathcal{DN})u_b = g$. As before we use the decomposition $u_b = (v_b, w_b)$ and $g = (g_v, g_w)$. Then in transformed variables we have the system

$$\alpha\hat{v}_b + \beta\zeta(\zeta \mid \hat{v}_b) + i\gamma\zeta\hat{w}_b = \hat{g}_v,$$
$$-i\gamma(\zeta \mid \hat{v}_b) + (\alpha + \delta)\hat{w}_b = \hat{g}_w.$$

This yields

$$\hat{v}_b = \alpha^{-1}[\hat{g}_v - \zeta\beta(\zeta \mid \hat{v}_b) + i\gamma\hat{w}_b)]. \tag{4.5}$$

(ii) This last equation shows that it is sufficient to determine $(\hat{v}_b \mid \zeta)$ and $\hat{w}_b$. If the inverses of $\beta(\hat{v}_b \mid \zeta)$ and $\gamma \hat{w}_b$ belong to the class of g_v, then v_b is uniquely determined and has the claimed regularity. Indeed, α is the symbol of

$$F := \sqrt{\mu_1} F_1 + \sqrt{\mu_2} F_2, \qquad \mathsf{D}(F) = {}_0H_p^{1/2}(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^1(\mathbb{R}^n)),$$

which is a bounded invertible operator from its domain into $L_p(J; L_p(\mathbb{R}^n))$, and hence also from $D_F(2 - 1/p, p)$ into $D_F(1 - 1/p, p)$. Here we note that

$$D_F(\theta, p) = D_{F_j}(\theta, p) = {}_0W_p^{\theta/2}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{\theta}(\mathbb{R}^n))$$

for $\theta \in (0, 2)$, $\theta \neq 1$. Therefore, $F^{-1} g_v$ belongs to $D_F(2 - 1/p, p)$ if and only if $g_v \in D_F(1 - 1/p, p)$. Next we note that γ is the symbol of $\sqrt{\mu_2} F_2 - \sqrt{\mu_1} F_1 - [\![\mu]\!] D_n^{1/2}$ which is bounded from $D_F(2 - 1/p, p)$ to $D_F(1 - 1/p, p)$, and β is the symbol of $(\mu_1 + \mu_2) D_n^{1/2}$ which has the same mapping properties.

(iii) It remains to show that w_b and $(R \mid v_b)$ belong to $D_F(2 - 1/p, p)$. For $\hat{w}_b$ and $(\zeta \mid \hat{v}_b)$ we have the equations

$$\begin{aligned} (\alpha + \beta)(\zeta \mid \hat{v}_b) + i\gamma \hat{w}_b &= (\zeta \mid \hat{g}_v), \\ -i\gamma(\zeta \mid \hat{v}_b) + (\alpha + \delta)\hat{w}_b &= \hat{g}_w \end{aligned}$$

since $|\zeta| = 1$. Solving this 2-D system we obtain

$$\begin{aligned} \hat{w}_b &= m^{-1}[i\gamma(\zeta \mid \hat{g}_v) + (\alpha + \beta)\hat{g}_w], \\ (\zeta \mid \hat{v}_b) &= m^{-1}[(\alpha + \delta)(\zeta \mid \hat{g}_v) - i\gamma \hat{g}_w], \end{aligned} \tag{4.6}$$

where

$$m = (\alpha + \beta)(\alpha + \delta) - \gamma^2.$$

Since $\delta = \beta + (\rho_1 + \rho_2)\lambda/|\xi|$ we obtain the following relation for m:

$$m = (\alpha + \beta)\left[(\rho_1 + \rho_2)\frac{\lambda}{|\xi|} + 4\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^{-1}\right] =: (\alpha + \beta)n,$$

where $\eta_1 = \sqrt{\mu_1}\omega_1 + \mu_2|\xi|$ and $\eta_2 = \sqrt{\mu_2}\omega_2 + \mu_1|\xi|$. This yields

$$\begin{aligned} \hat{w}_b &= \frac{i\gamma}{(\alpha + \beta)n}(\zeta \mid \hat{g}_v) + \frac{\hat{g}_w}{n}, \\ (\zeta \mid \hat{v}_b) &= \frac{(\rho_1 + \rho_2)\lambda/|\xi|}{(\alpha + \beta)n}(\zeta \mid \hat{g}_v) + \frac{1}{n}\left[(\zeta \mid \hat{g}_v) - \frac{i\gamma}{\alpha + \beta}\hat{g}_w\right]. \end{aligned} \tag{4.7}$$

We define the operators T_j by means of their symbols η_j, i.e.

$$T_1 := \sqrt{\mu_1} F_1 + \mu_2 D_n^{1/2}, \qquad T_2 := \sqrt{\mu_2} F_2 + \mu_1 D_n^{1/2}, \qquad \mathsf{D}(T_j) = \mathsf{D}(F_j) = \mathsf{D}(F).$$

Then by the Dore–Venni theorem, the operators T_j with domains $\mathsf{D}(T_j) = \mathsf{D}(F_j) = \mathsf{D}(F)$ are invertible, sectorial with angle $\pi/4$. Moreover, they admit an H^∞-calculus with H^∞-angle $\pi/4$ (see for instance [31]). The harmonic mean T of T_1 and T_2, i.e.

$$T := 2T_1T_2(T_1 + T_2)^{-1} = 2(T_1^{-1} + T_2^{-1})^{-1}$$

enjoys the same properties, as another application of the Dore–Venni theorem shows. The symbol of T is given by $\eta := 2\eta_1\eta_2/(\eta_1+\eta_2)$.

Next we consider the operator $GD_n^{-1/2}$ with domain

$$\begin{aligned}\mathsf{D}(GD_n^{-1/2}) &= \{h \in \mathcal{R}(D_n^{1/2}) : D_n^{-1/2}h \in \mathsf{D}(G)\}\\ &= {}_0H^1_p(J; \dot H^{-1}_p(\mathbb{R}^n)) \cap L_p(J; L_p(\mathbb{R}^n))\end{aligned}$$

The inclusion from left to right in the last equality is obvious. The converse can be seen as follows. Let $h \in {}_0H^1_p(J; \dot H^{-1}_p(\mathbb{R}^n)) \cap L_p(J; L_p(\mathbb{R}^n))$ and define $g := \dot D_n^{-1/2}h$. Then

$$g \in {}_0H^1_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; \dot H^1_p(\mathbb{R}^n)) \hookrightarrow L_p(J; H^1_p(\mathbb{R}^n)),$$

and $D_n^{1/2}g = \dot D_n^{1/2}g = h \in L_p(J; L_p(\mathbb{R}^n))$, which implies that $h \in \mathcal{R}(D_n^{1/2})$ and $g = \dot D_n^{-1/2}h = D_n^{-1/2}h \in \mathsf{D}(G)$. The operator $GD_n^{-1/2}$ is closed, sectorial and admits a bounded H^∞-calculus with H^∞-angle $\pi/2$ on $X = L_p(J; L_p(\mathbb{R}^n))$; see for instance [22, Corollary 2.2]. Its symbol is given by $\lambda/|\xi|$.

Finally, we consider the operator

$$N := (\rho_1+\rho_2)GD_n^{-1/2} + 2T, \tag{4.8}$$

with domain

$$\mathsf{D}(N) = \mathsf{D}(GD_n^{-1/2}) \cap \mathsf{D}(T) = {}_0H^1_p(J; \dot H^{-1}_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\mathbb{R}^n));$$

recall (3.10). By the Dore–Venni theorem N is closed, invertible, and by [31] it admits a bounded H^∞-calculus as well, with H^∞-angle $\pi/2$. Its symbol is n.

The operator with symbol γ is then given by $T_2 - T_1$, and the operator with symbol $\alpha+\beta$ by $T_1 + T_2$. For the inverse transforms w_b and $(R\,|\,v_b)$ of $\hat w_b$ and $(\zeta\,|\,\hat v_b)$ we then obtain the representations

$$\begin{aligned} w_b &= N^{-1}[(T_2-T_1)(T_1+T_2)^{-1}i(R\,|\,g_v) + g_w],\\ (R\,|\,v_b) &= (T_1+T_2)^{-1}(\rho_1+\rho_2)GD_n^{-1/2}N^{-1}(R\,|\,g_v)\\ &\quad + N^{-1}(R\,|\,g_v) - (T_2-T_1)(T_1+T_2)^{-1}N^{-1}ig_w.\end{aligned} \tag{4.9}$$

We note that N^{-1} has the following mapping properties:

$$\begin{aligned} N^{-1} &: L_p(J; L_p(\mathbb{R}^n)) \to {}_0H^1_p(J; \dot H^{-1}_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\mathbb{R}^n)) \hookrightarrow L_p(J; L_p(\mathbb{R}^n)),\\ N^{-1} &: L_p(J; \dot H^1_p(\mathbb{R}^n)) \to {}_0H^1_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; \dot H^2_p(\mathbb{R}^n)) \hookrightarrow L_p(J; L_p(\mathbb{R}^n)).\end{aligned}$$

Therefore by three-fold real interpolation

$$N^{-1} : L_p(J; \dot W^{1-1/p}_p(\mathbb{R}^n)) \to {}_0H^1_p(J; \dot W^{-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)). \tag{4.10}$$

Moreover, N^{-1} maps ${}_0W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n))$ into

$${}_0W^{3/2-1/2p}_p(J; \dot H^{-1}_p(\mathbb{R}^n)) \cap {}_0W^{1/2-1/2p}_p(J; H^1_p(\mathbb{R}^n)). \tag{4.11}$$

Next we note that the operators $T_j(T_1 + T_2)^{-1}$ are bounded in $D_F(1 - 1/p, p)$, as is the Riesz transform R, and the assertion for w_b follows now from (4.9)–(4.10) and

$$ {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)) \hookrightarrow L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n)). $$

The assertions for $(R \mid v_b)$ follow readily from (3.10) and (4.9)–(4.11). □

We can now formulate our second main result of this section concerning the solvability of the problem

$$ \begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = g_v & \text{on } \mathbb{R}^n, \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] = g_w & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ u(0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1}. \end{cases} \tag{4.12} $$

COROLLARY 4.2 Let $1 < p < \infty$ and assume that ρ_j and μ_j are positive constants, $j = 1, 2$, and set $J = [0, a]$. Then (4.12) admits a unique solution (u, π) with

$$ u \in {}_0H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \qquad \pi \in L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1})) $$

if and only if $g = (g_v, g_w)$ satisfies the following regularity assumptions:

(a) $g_v \in {}_0W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^n))$,
(b) $g_w \in L_p(J; \dot{W}_p^{1-1/p}(\mathbb{R}^n))$.

Proof. Let $u_b := (v_b, w_b) := (\mathcal{DN})^{-1}(g_v, g_w)$, and let (u, π) be the solution of (3.6). Thanks to Theorem 4.1 and Proposition 3.3, (u, π) satisfies the regularity assertion of the corollary, and it is the unique solution of (4.12) due to the definition of $\mathcal{DN}$. □

REMARK 4.3 The representation formulas in (3.7)–(3.8) have also been derived and used by other authors (see for instance [11, 36]). However, the optimal regularity results in Theorem 3.1, Proposition 3.3, Theorem 4.1, and Corollary 4.2 are new. Moreover, the computations and arguments leading to these results are shorter than in [11] (which only deals with the case $p = 2$) and in [36]. We should mention, however, that these authors consider more general domains.

5. The linearized two-phase Stokes problem with free boundary

In this section we consider the full linearized problem

$$ \begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = f & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = f_d & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = g_v & \text{on } \mathbb{R}^n, \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h = g_w & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ \partial_t h - \gamma w = g_h & \text{on } \mathbb{R}^n, \\ u(0) = u_0, \quad h(0) = h_0. & \end{cases} \tag{5.1} $$

We are interested in the same regularity classes for u and π as before. Then the equation for the height function h lives in the trace space $W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))$, hence the natural space for h is given by

$$h \in W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)).$$

Our next theorem states that problem (5.1) admits maximal regularity, in particular defines an isomorphism between the solution space and the space of data.

THEOREM 5.1 Let $1 < p < \infty$ be fixed, $p \neq 3/2, 3$, and assume that ρ_j and μ_j are positive constants for $j = 1, 2$, and set $J = [0, a]$. Then the Stokes problem with free boundary (5.1) admits a unique solution (u, π, h) with regularity

$$\begin{aligned} &u \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \\ &\pi \in L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1})), \\ &[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)), \\ &h \in W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)) \end{aligned} \tag{5.2}$$

if and only if the data $(f, f_d, g, g_h, u_0, h_0)$ satisfy the following regularity and compatibility conditions:

(a) $f \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$,
(b) $f_d \in H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^{n+1}))$,
(c) $g = (g_v, g_w) \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1}))$,
(d) $g_h \in W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))$,
(e) $u_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$, $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$,
(f) $\operatorname{div} u_0 = f_d(0)$ in $\dot{\mathbb{R}}^{n+1}$ and $[\![u_0]\!] = 0$ on $\mathbb{R}^n$ if $p > 3/2$,
(g) $-[\![\mu \partial_y v_0]\!] - [\![\mu \nabla_x w_0]\!] = g_v(0)$ on $\mathbb{R}^n$ if $p > 3$.

The solution map $(f, f_d, g, g_h, u_0, h_0) \mapsto (u, \pi, [\![\pi]\!], h)$ is continuous between the corresponding spaces.

Proof. Similarly to the proof of Thereom 3.1 we will reduce system (5.1) to the case where $(f, f_d, g, u_0, h_0) = (0, 0, 0, 0, 0)$ and $g_h(0) = 0$. The Neumann-to-Dirichlet operator will once again play an essential role in the resulting reduced problem.

(i) Let

$$h_1(t) := [2e^{-D_n^{1/2}t} - e^{-2D_n^{1/2}t}]h_0 + (1 + D_n)^{-1}[e^{-(1+D_n)t} - e^{-2(1+D_n)t}](g_h(0) + \gamma w_0),$$

where $u_0 = (v_0, w_0)$ and $\gamma : \mathbb{R}_{\pm}^{n+1} \to \mathbb{R}^n$ is the trace operator. The function h_1 has the following properties:

$$\begin{aligned} h_1 \in\ &W_p^{1/2-1/2p}(J; H_p^2(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)) \\ &\cap W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)), \\ h_1(0) = h_0,&\quad \partial_t h_1(0) = g_h(0) + \gamma w_0 \end{aligned} \tag{5.3}$$

(see [30, Lemma 6.4] for a proof of a similar result). Let then (u_1, π_1) be the solution of problem (3.1) with g_w replaced by $g_w + \sigma\Delta h_1$. It follows from Theorem 3.1, the assumptions on $g = (g_v, g_w)$, and from the first line in (5.3) that (u_1, π_1) satisfies the regularity properties stated in Theorem 5.1.

(ii) Next we consider the reduced problem

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \operatorname{div} u = 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = 0 & \text{on } \mathbb{R}^n, \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h = 0 & \text{on } \mathbb{R}^n, \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n, \\ \partial_t h - \gamma w = \tilde{g}_h & \text{on } \mathbb{R}^n, \\ u(0) = 0, \quad h(0) = 0, \end{cases} \tag{5.4}$$

with $\tilde{g}_h := g_h - (\partial_t h_1 - \gamma w_1)$, where $u_1 = (v_1, w_1)$ is the solution obtained in step (i). We conclude from (5.3) and the regularity properties of γw_1 that

$$\tilde{g}_h \in {}_0W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)). \tag{5.5}$$

Suppose that problem (5.4) admits a solution (u_2, π_2, h_2) with the regularity properties stated in (5.2). One readily verifies that $(u, \pi, h) := (u_1 + u_2, \pi_1 + \pi_2, h_1 + h_2)$ is a solution of problem (5.1) in the regularity class of (5.2).

(iii) It thus remains to show that the reduced problem (5.4) admits a unique solution (u, π, h) in the regularity class stated in Theorem 5.1. We note that once h has been determined, Corollary 4.2 yields the corresponding pair (u, π) in problem (5.4).

To determine h we extract the *boundary symbol* for this problem as follows. Applying the Neumann-to-Dirichlet operator $(\mathcal{DN})^{-1}$ to $(g_v, g_w) = (0, \sigma D_n h)$ yields $\gamma u = u_b$, the trace of u. According to (4.7), the transform of the normal component $\gamma w = w_b$ of u_b is given by

$$\hat{w}_b = \frac{-\sigma|\xi|^2}{(\rho_1 + \rho_2)\lambda/|\xi| + 4\eta_1\eta_2/(\eta_1 + \eta_2)}\hat{h}.$$

Let us now consider the equation $\partial_t h - \gamma w = \tilde{g}_h$. Inserting this expression for $\hat{w}_b$ into the transformed equation $\lambda\hat{h} - \hat{w}_b = \hat{\tilde{g}}_h$ results in $s(\lambda, |\xi|)\hat{h} = \hat{\tilde{g}}_h$ where the boundary symbol $s(\lambda, |\xi|)$ is given by

$$s(\lambda, |\xi|) = \lambda + \frac{\sigma|\xi|^2}{(\rho_1 + \rho_2)\lambda/|\xi| + 4\eta_1\eta_2/(\eta_1 + \eta_2)}. \tag{5.6}$$

The operator corresponding to this symbol is

$$S = G + \sigma D_n N^{-1}, \tag{5.7}$$

where the meaning of the operators G, D_n and N is as in Section 4. The operator S has the following mapping properties:

$$S : {}_0H_p^{r+1}(J; K_p^s(\mathbb{R}^n)) \cap {}_0H_p^r(J; K_p^{s+1}(\mathbb{R}^n)) \to {}_0H^r(J; K_p^s(\mathbb{R}^n)), \tag{5.8}$$

where $K \in \{H, W\}$. In order to find h we need to solve the equation $Sh = \tilde{g}_h$, that is, we need to show that S is invertible in appropriate function spaces.

All operators in the definition of S commute, and admit an H^∞-calculus. The H^∞-angle of D_n is zero, that of N is $\pi/2$, and that of G is $\pi/2$ as well. Thus we cannot a priori guarantee that the sum of the power-angles of the single operators in S is strictly less than π, and the Dore–Venni approach is therefore not directly applicable. We will instead apply a result of Kalton and Weis [25, Theorem 4.4].

For this purpose note that for complex numbers w_j with $\arg w_j \in [0, \pi/2)$, we have $\arg{(w_1 w_2)/(w_1 + w_2)} = \arg{(1/w_1 + 1/w_2)^{-1}} \in [0, \pi/2)$ as well. This implies that $s(\lambda, |\xi|)$ has strictly positive real part for each λ in the closed right halfplane and for each $\xi \in \mathbb{R}^n$, $(\lambda, \xi) \neq (0, 0)$, hence $s(\lambda, |\xi|)$ does not vanish for such λ and ξ.

We write $s(\lambda, |\xi|)$ in the following way:

$$s(\lambda, \tau) = \lambda + \sigma\tau k(z), \qquad z = \lambda/\tau^2,\ \lambda \in \mathbb{C},\ \tau \in \mathbb{C} \setminus \{0\}, \tag{5.9}$$

where

$$k(z) = \left[(\rho_1 + \rho_2)z + 4\left(\frac{1}{\sqrt{\mu_1}\sqrt{\rho_1 z + \mu_1} + \mu_2} + \frac{1}{\sqrt{\mu_2}\sqrt{\rho_2 z + \mu_2} + \mu_1}\right)^{-1}\right]^{-1}.$$

The asymptotics of $k(z)$ are given by

$$k(0) = \frac{1}{2(\mu_1 + \mu_2)}, \qquad zk(z) \to \frac{1}{\rho_1 + \rho_2} \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}_- \text{ with } |z| \to \infty.$$

This shows that for any $\vartheta \in [0, \pi)$ there is a constant $C = C(\vartheta) > 0$ such that

$$|k(z)| \leqslant \frac{C}{1 + |z|}, \qquad z \in \bar{\Sigma}_\vartheta.$$

Hence we see that

$$|s(\lambda, |\xi|)| \leqslant C(|\lambda| + |\xi|), \qquad \operatorname{Re}\lambda \geqslant 0,\ \xi \in \mathbb{R}^n,$$

for some constant $C > 0$. Next we are going to prove that for each $\lambda_0 > 0$ there are $\eta, c > 0$ such that

$$|s(\lambda, \tau)| \geqslant c[|\lambda| + |\tau|] \qquad \text{for all } \lambda \in \Sigma_{\pi/2+\eta},\ |\lambda| \geqslant \lambda_0,\ \tau \in \Sigma_\eta. \tag{5.10}$$

This can be seen as follows: since $\operatorname{Re} k(z) > 0$ for $\operatorname{Re} z \geqslant 0$, by continuity of the modulus and argument we obtain an estimate of the form

$$|s(\lambda, \tau)| \geqslant c_0[|\lambda| + |\tau|\,|k(z)|] \geqslant c[|\lambda| + |\tau|], \qquad \lambda \in \Sigma_{\pi/2+\eta},\ \tau \in \Sigma_\eta,$$

provided $|z| \leqslant M$, with some $\eta, c > 0$ depending on M, but not on λ and τ. On the other hand, for $m > 0$ fixed we consider the case with $|\lambda| \geqslant m|\tau|$, $|z| \geqslant M$. We then have

$$|s(\lambda, \tau)| \geqslant |\lambda| - \sigma|\tau|\,|k(z)| \geqslant \frac{1}{2}[|\lambda| + m|\tau|] - \sigma C|\tau|/(1 + M) \geqslant c[|\lambda| + |\tau|]$$

provided $m > 2\sigma C/(1 + M)$, and then by extension

$$|s(\lambda, \tau)| \geqslant c[|\lambda| + |\tau|], \qquad \lambda \in \Sigma_{\pi/2+\eta},\ \tau \in \Sigma_\eta,\ |\lambda| \geqslant m|\tau|,\ |z| \geqslant M,$$

provided $\eta, c > 0$ are sufficiently small. One easily sees that the intersection point of the curves $y = Mx^2$ and $y = mx$ in $\mathbb{R}^2$ has distance $d = (m/M)\sqrt{1+m^2}$ from the origin. By choosing M large enough so that $d \leqslant \lambda_0$, (5.10) follows by combining the two estimates.

By means of the $\mathcal{R}$-boundedness of the functional calculus for D_n in $K_p^s(\mathbb{R}^n)$ (cf. Desch, Hieber and Prüss [15]) we see that

$$(\lambda + D_n^{1/2})s^{-1}(\lambda, D_n^{1/2})$$

is of class H^∞ and $\mathcal{R}$-bounded on $\Sigma_{\pi/2+\eta} \setminus B_{\lambda_0}(0)$. The operator-valued H^∞-calculus for $G = \partial_t$ on ${}_0H_p^r(J; K_p^s(\mathbb{R}^n))$ (cf. Hieber and Prüss [24]) implies boundedness of

$$(G + D_n^{1/2})s^{-1}(G, D_n^{1/2}) \quad \text{in } {}_0H_p^r(J; K_p^s(\mathbb{R}^n)).$$

This shows that $s^{-1}(G, D_n^{1/2})$ has the following mapping properties:

$$s^{-1}(G, D_n^{1/2}) : {}_0H_p^r(J; K_p^s(\mathbb{R}^n)) \to {}_0H_p^{r+1}(J; K_p^s(\mathbb{R}^n)) \cap {}_0H_p^r(J; K_p^{s+1}(\mathbb{R}^n)). \tag{5.11}$$

We conclude that S is invertible and that $S^{-1} = s^{-1}(G, D_n^{1/2})$. Choosing $r = 0$ and $s = 2 - 1/p$ and $K = W$ in (5.11) yields

$$S^{-1} : L_p(J; W_p^{2-1/p}(\mathbb{R}^n)) \to {}_0H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)). \tag{5.12}$$

Moreover, we also deduce from (5.11) that

$$S^{-1} : L_p(J; L_p(\mathbb{R}^n)) \to {}_0H_p^1(J; L_p(\mathbb{R}^n)),$$
$$S^{-1} : {}_0H_p^1(J; L_p(\mathbb{R}^n)) \to {}_0H_p^2(J; L_p(\mathbb{R}^n)).$$

Interpolating with the real method $(\cdot,\cdot)_{1-1/p,p}$ then yields

$$S^{-1} : {}_0W_p^{1-1/p}(J; L_p(\mathbb{R}^n)) \to {}_0W_p^{2-1/p}(J; L_p(\mathbb{R}^n)). \tag{5.13}$$

(5.12)–(5.13) shows that the equation $Sh = \tilde{g}_h$ has for each $\tilde{g}_h$ satisfying (5.5) a unique solution h in the regularity class (5.2).

(iv) Since the function h is now known we can use Corollary 4.2 to determine the pair (u, π) in problem (5.4). For this we note that

$${}_0H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)) \hookrightarrow {}_0W_p^{1-1/p}(J; H_p^2(\mathbb{R}^n)) \tag{5.14}$$

(see [30, Lemma 6.2] for a proof). This shows that the function h determined in step (iii) satisfies

$$\Delta h \in {}_0W^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n))$$

and Corollary 4.2 yields a solution (u, π) in the regularity class (5.2).

(v) Steps (i)–(iv) render a solution (u, π, h) for problem (5.1) that satisfies the regularity properties asserted in the theorem. It follows from step (iv) and from Theorem 3.1 that problem (5.4) with $(f, f_d, g, g_h, u_0, h_0) = (0, 0, 0, 0, 0, 0)$ has only the trivial solution, and this gives uniqueness. The proof of Theorem 5.1 is now complete. □

REMARK 5.2 Further mapping properties of the symbol $s(\lambda, \tau)$ and the associated operator S have been derived in [32]. In particular, we have investigated the singularities and zeros of the boundary symbol s, and we have studied the mapping properties of S in the case of low and high frequencies, respectively.

6. The nonlinear problem

In this section we derive estimates for the nonlinear mappings occurring on the right hand side of (2.9). In order to facilitate this task, we first introduce some notation, and then study the mapping properties of the nonlinear functions appearing on the right hand side of (2.9). In the following we set

$$\begin{aligned}
\mathbb{E}_1(a) &:= \{u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})) : [\![u]\!] = 0\},\\
\mathbb{E}_2(a) &:= L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{E}_3(a) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n)),\\
\mathbb{E}_4(a) &:= W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n))\\
&\quad \cap W^{1/2-1/2p}_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n)),\\
\mathbb{E}(a) &:= \{(u, \pi, q, h) \in \mathbb{E}_1(a) \times \mathbb{E}_2(a) \times \mathbb{E}_3(a) \times \mathbb{E}_4(a) : [\![\pi]\!] = q\}.
\end{aligned} \tag{6.1}$$

The space $\mathbb{E}(a)$ is given the natural norm

$$\|(u, \pi, q, h)\|_{\mathbb{E}(a)} = \|u\|_{\mathbb{E}_1(a)} + \|\pi\|_{\mathbb{E}_2(a)} + \|q\|_{\mathbb{E}_3(a)} + \|h\|_{\mathbb{E}_4(a)},$$

which turns it into a Banach space. We recall that $\mathbb{E}_2(a)$ is equipped with the norm $\|\pi\|_{\mathbb{E}_2(a)} = (\sum_{j=1}^{n+1} \|\partial_j \pi\|^p_{L_p(J; L_p(\dot{\mathbb{R}}^{n+1}))})^{1/p}$ for $\pi : \dot{\mathbb{R}}^{n+1} \to \mathbb{R}$.

In addition, we define

$$\begin{aligned}
\mathbb{F}_1(a) &:= L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})),\\
\mathbb{F}_2(a) &:= H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{F}_3(a) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1})),\\
\mathbb{F}_4(a) &:= W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),\\
\mathbb{F}(a) &:= \mathbb{F}_1(a) \times \mathbb{F}_2(a) \times \mathbb{F}_3(a) \times \mathbb{F}_4(a).
\end{aligned} \tag{6.2}$$

The generic elements of $\mathbb{F}(a)$ are the functions (f, f_d, g, g_h).

We list some properties of the function spaces introduced above that will be used below. In the following we say that a function space is a *multiplication algebra* if it is a Banach algebra under the operation of multiplication.

LEMMA 6.1 Suppose $p > n + 3$ and let $J = [0, a]$. Then

(a) $\mathbb{E}_3(a)$ and $\mathbb{F}_4(a)$ are multiplication algebras.
(b) $\mathbb{E}_1(a) \hookrightarrow C(J; BUC^1(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})) \cap C(J; BUC(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$.
(c) $\mathbb{E}_3(a) \hookrightarrow C(J; BUC(\mathbb{R}^n))$.
(d) $\mathbb{E}_4(a) \hookrightarrow BC^1(J; BC^1(\mathbb{R}^n)) \cap BC(J; BC^2(\mathbb{R}^n))$.
(e) $W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n)) \hookrightarrow \mathbb{E}_4(a)$.

Proof. (a) The assertion that $\mathbb{E}_3(a)$ and $\mathbb{F}_4(a)$ are multiplication algebras can be shown as in the proof of [30, Lemma 6.6(ii)].

(b) It follows from [2, Theorem III.4.10.2] that $\mathbb{E}_1(a) \hookrightarrow C(J; W^{2-2/p}_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}))$ and this implies the first inclusion, thanks to Sobolev's embedding theorem. The second assertion follows from the fact that u is continuous across $y = 0$.

(c) This follows from [19, Remark 5.3(d)] and Sobolev's embedding theorem.

(d) We infer from [2, Theorem III.4.10.2] that

$$H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)) \hookrightarrow C(J; W_p^{3-2/p}(\mathbb{R}^n)),$$

and the inclusion $\mathbb{E}_4(a) \hookrightarrow C(J; BC^2(\mathbb{R}^n))$ then follows from Sobolev's embedding theorem. In addition, we conclude from [30, Remark 5.3(d)] and Sobolev's embedding theorem that

$$W_p^{1-2/p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)) \hookrightarrow BC(J; BC^1(\mathbb{R}^n)),$$

and this implies that $\mathbb{E}_4(a) \hookrightarrow BC^1(J; BC^1(\mathbb{R}^n))$.

(e) This follows from (5.14). □

Let

$$N(u, \pi, q, h) := (F(u, \pi, h), F_d(u, h), G(u, q, h), H(u, h)) \tag{6.3}$$

for $(u, \pi, q, h) \in \mathbb{E}(a)$, where as before $u = (v, w)$, $F = (F_v, F_w)$ and $G = (G_v, G_w)$. We show that the mapping N is real analytic.

PROPOSITION 6.2 Suppose $p > n + 3$. Then

$$N \in C^\omega(\mathbb{E}(a), \mathbb{F}(a)) \quad \text{and} \quad N(0) = 0,\ DN(0) = 0, \tag{6.4}$$

where DN denotes the Fréchet derivative of N. In addition we have

$$DN(u, \pi, q, h) \in \mathcal{L}({}_0\mathbb{E}(a), {}_0\mathbb{F}(a)) \quad \text{for any } (u, \pi, q, h) \in \mathbb{E}(a).$$

Proof. We first note that the mapping $(u, \pi, q, h) \mapsto N(u, \pi, q, h)$ is polynomial. It thus suffices to verify that $N : \mathbb{E}(a) \to \mathbb{F}(a)$ is well-defined and continuous.

(i) We first consider the term $F(u, \pi, h)$, and observe that it contains the expressions ∇h, Δh and $\partial_t h$. Without changing notation we here consider the extension of h from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ defined by $h(t, x, y) = h(t, x)$ for $t \in J$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. With this interpretation we clearly have

$$\|\partial h\|_{\infty, J\times\mathbb{R}^{n+1}} = \|\partial h\|_{\infty, J\times\mathbb{R}^n}, \quad h \in \mathbb{E}(a),\ \partial \in \{\partial_j, \Delta, \partial_t\}, \tag{6.5}$$

where $\|\cdot\|_{\infty,U}$ denotes the sup-norm for the set $U \subset J \times \mathbb{R}^{n+1}$. Next we note that

$$\begin{aligned} &BC(J; BC(\mathbb{R}^{n+1})) \cdot L_p(J; L_p(\mathbb{R}^{n+1})) \hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})),\\ &BC(J; BC(\mathbb{R}^{n+1})) \cdot BC(J; BC(\mathbb{R}^{n+1})) \hookrightarrow BC(J; BC(\mathbb{R}^{n+1})), \end{aligned} \tag{6.6}$$

that is, multiplication is continuous and bilinear in the indicated function spaces. We can now conclude from (6.5)–(6.6) and Lemma 6.1 that

$$F \in C^\omega(\mathbb{E}_1(a) \times \mathbb{E}_2(a) \times \mathbb{E}_4(a), \mathbb{F}_1(a)), \qquad F(0) = 0,\ DF(0) = 0.$$

(ii) We will now consider the nonlinear function $F_d(u, h) = (\nabla h|\partial_y v)$. Since h does not depend on y we have

$$F_d(u, h) = (\nabla h \,|\, \partial_y u) = \partial_y(\nabla h \,|\, u). \tag{6.7}$$

Observing that

$$BC^1(J;BC(\mathbb{R}^{n+1}))\cdot H^1_p(J;L_p(\mathbb{R}^{n+1}))\hookrightarrow H^1_p(J;L_p(\mathbb{R}^{n+1})),$$
$$BC(J;BC^1(\dot{\mathbb{R}}^{n+1}))\cdot L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1}))\hookrightarrow L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1})),$$

and

$$\partial_y\in\mathcal{L}\big(H^1_p(J;L_p(\mathbb{R}^{n+1})),H^1_p(J;H^{-1}_p(\mathbb{R}^{n+1}))\big)$$
$$\cap\,\mathcal{L}\big(L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1})),L_p(J;L_p(\mathbb{R}^{n+1}))\big),$$

we infer from Lemma 6.1(d) that

$$F_d\in C^\omega(\mathbb{E}_1(a)\times\mathbb{E}_4(a),\mathbb{F}_2(a)),\qquad F_d(0)=0,\ DF_d(0)=0.$$

(iii) We recall that

$$[\![\mu\partial_i\ \cdot]\!]\in\mathcal{L}\big(H^1_p(J;L_p(\mathbb{R}^{n+1}))\cap L_p(J;H^2_p(\dot{\mathbb{R}}^{n+1})),\mathbb{E}_3(a)\big),\tag{6.8}$$

where $[\![\mu\partial_i u]\!]$ denotes the jump of the quantity $\mu\partial_i u$ with u a generic function $\dot{\mathbb{R}}^{n+1}\to\mathbb{R}$, and where $\partial_i=\partial_{x_i}$ for $i=1,\dots,n$ and $\partial_{n+1}=\partial_y$.

The mapping $G(u,q,h)$ is made up of terms of the form

$$[\![\mu\partial_i u_k]\!]\partial_j h,\quad [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h,\quad q\partial_j h,\quad \Delta h\partial_j h,\quad G_\kappa(h),\quad G_\kappa(h)\partial_j h,$$

where u_k denotes the k-th component of a function $u\in\mathbb{E}_1(a)$. From (6.8) and the fact that $\mathbb{E}_3(a)$ is a multiplication algebra it follows that the mappings

$$(u,h)\mapsto[\![\mu\partial_i u_k]\!]\partial_j h,\ [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h:\mathbb{E}_1(a)\times\mathbb{E}_4(a)\to\mathbb{E}_3(a),$$
$$(q,h)\mapsto q\partial_j h:\mathbb{E}_3(a)\times\mathbb{E}_4(a)\to\mathbb{E}_3(a),\qquad h\mapsto\Delta h\partial_j h:\mathbb{E}_4(a)\to\mathbb{E}_3(a)$$

are multilinear and continuous, and hence real analytic. The fact that $\mathbb{E}_3(a)$ is an algebra additionally implies that the mapping $[h\mapsto G_\kappa(h)]:\mathbb{E}_4(a)\to\mathbb{E}_3(a)$ is analytic. In summary we conclude that

$$G\in C^\omega(\mathbb{E}_1(a)\times\mathbb{E}_3(a)\times\mathbb{E}_4(a),\mathbb{E}_3(a)),\qquad G(0)=0,\ DG(0)=0.$$

(iv) We infer from $\gamma\in\mathcal{L}\big(H^1_p(J;L_p(\mathbb{R}^{n+1}))\cap L_p(J;H^2_p(\dot{\mathbb{R}}^{n+1})),\mathbb{F}_a(a)\big)$ and the fact that $\mathbb{F}_4(a)$ is an algebra that the mapping $[(u,h)\mapsto(\nabla h\,|\,\gamma u)]:\mathbb{E}_1(a)\times\mathbb{E}_4(a)\to\mathbb{F}_4(a)$ is bilinear and continuous. This immediately yields

$$H\in C^\omega(\mathbb{E}_1(a)\times\mathbb{E}_4(a),\mathbb{F}_4(a)),\qquad H(0)=0,\ DH(0)=0.$$

(v) As the terms of N are made up of products of u,π,q,h and derivatives thereof, one easily verifies that

$$DN(u,\pi,q,h)[\bar u,\bar\pi,\bar q,\bar h]\in{}_0\mathbb{F}(a)\quad\text{whenever}\quad(u,\pi,q,h)\in\mathbb{E}(a),\ (\bar u,\bar\pi,\bar q,\bar h)\in{}_0\mathbb{E}(a).$$

Combining the results obtained in steps (i)–(v) yields the assertions of the proposition. □

We are now ready to prove our main result of this section, yielding existence and uniqueness of solutions for the nonlinear problem (2.9).

THEOREM 6.3 (Existence of solutions for the nonlinear problem (2.9)) (a) For every $t_0 > 0$ there exists a number $\varepsilon = \varepsilon(t_0) > 0$ such that for all initial values

$$(u_0, h_0) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n), \qquad [\![u_0]\!] = 0,$$

satisfying the compatibility conditions

$$[\![\mu \mathcal{D}(u_0, h_0)\nu_0 - \mu(\nu_0 \,|\, \mathcal{D}(u_0, h_0)\nu_0)\nu_0]\!] = 0, \qquad \operatorname{div} u_0 = F_d(u_0, h_0), \qquad [\![u_0]\!] = 0 \quad (6.9)$$

and the smallness condition

$$\|u_0\|_{W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^n)} \leqslant \varepsilon, \tag{6.10}$$

where $\mathcal{D}(u, h)$ is defined in (2.6), the nonlinear problem (2.9) admits a unique solution $(u, \pi, [\![\pi]\!], h) \in \mathbb{E}(t_0)$.

(b) The solution has the additional regularity properties

$$(u, \pi) \in C^\omega((0, t_0) \times \dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+2}), \qquad [\![\pi]\!], h \in C^\omega((0, t_0) \times \mathbb{R}^n).$$

In particular, $\mathcal{M} = \bigcup_{t \in (0, t_0)} (\{t\} \times \Gamma(t))$ is a real analytic manifold.

Proof. In order to economize our notation we set $z := (u, \pi, q, h)$ for $(u, \pi, q, h) \in \mathbb{E}(a)$. With this notation, the nonlinear problem (2.9) can be restated as

$$Lz = N(z), \qquad (u(0), h(0)) = (u_0, h_0), \tag{6.11}$$

where L denotes the linear operator on the left hand side of (2.9), and where N is defined in (6.3).

It is convenient to first introduce an auxiliary function $z^* \in \mathbb{E}(a)$ which resolves the compatibility conditions (6.9) and the initial conditions in (6.11), and then to solve the resulting reduced problem

$$Lz = N(z + z^*) - Lz^* =: K_0(z), \qquad z \in {}_0\mathbb{E}(a), \tag{6.12}$$

by means of a fixed point argument.

(i) Suppose that the initial values (u_0, h_0) satisfy the (first) compatibility condition in (6.9), and set

$$[\![\pi_0]\!] := [\![\mu(\nu_0 \,|\, \mathcal{D}(u_0, h_0)\nu_0)]\!] + \sigma(\Delta h_0 - G_\kappa(h_0)).$$

It is then clear that the following compatibility conditions hold:

$$\begin{aligned} -[\![\mu \partial_y v_0]\!] - [\![\mu \nabla_x w_0]\!] &= G_v(u_0, [\![\pi_0]\!], h_0) && \text{on } \mathbb{R}^n, \\ -2[\![\mu \partial_y w_0]\!] + [\![\pi_0]\!] - \sigma \Delta h_0 &= G_w(u_0, h_0) && \text{on } \mathbb{R}^n, \end{aligned} \tag{6.13}$$

where $u_0 = (v_0, w_0)$. Next we introduce special functions $(0, f_d^*, g^*, g_h^*) \in \mathbb{F}(a)$ which resolve the necessary compatibility conditions. First we set

$$c^*(t) := \begin{cases} \mathcal{R}_+ e^{-tD_{n+1}} \mathcal{E}_+(v_0 \,|\, \nabla h_0) & \text{in } \mathbb{R}^{n+1}_+, \\ \mathcal{R}_- e^{-tD_{n+1}} \mathcal{E}_-(v_0 \,|\, \nabla h_0) & \text{in } \mathbb{R}^{n+1}_-, \end{cases} \tag{6.14}$$

where $\mathcal{E}_\pm \in \mathcal{L}(W_p^{2-2/p}(\mathbb{R}^{n+1}_\pm), W_p^{2-2/p}(\mathbb{R}^{n+1}))$ is an appropriate extension operator and $\mathcal{R}_\pm$ is the restriction operator. Since $(v_0 \mid \nabla h_0) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})$ we obtain

$$c^* \in H^1_p(J; L_p(\mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1})).$$

Consequently,

$$f_d^* := \partial_y c^* \in \mathbb{F}_2(a) \quad \text{and} \quad f_d^*(0) = F_d(v_0, h_0). \tag{6.15}$$

Next we set

$$g^*(t) := e^{-D_n t} G(u_0, [\![\pi_0]\!], h_0), \qquad g_h^*(t) := e^{-D_n t} H(u_0, h_0). \tag{6.16}$$

It then follows from (6.15) and [19, Lemma 8.2] that $(0, f_d^*, g^*, g_h^*) \in \mathbb{F}(a)$. (6.13) and the second and third conditions in (6.9) show that the necessary compatibility conditions of Theorem 5.1 are satisfied and we can conclude that the linear problem

$$Lz^* = (0, f_d^*, g^*, g_h^*), \qquad (u^*(0), h^*(0)) = (u_0, h_0), \tag{6.17}$$

has a unique solution $z^* \in \mathbb{E}(a)$. With the auxiliary function z^* now determined, we can focus on the reduced equation (6.12), which can be converted into the fixed point equation

$$z = L_0^{-1} K_0(z), \qquad z \in {}_0\mathbb{E}(a), \tag{6.18}$$

where L_0 denotes the restriction of L to ${}_0\mathbb{E}(a)$. Due to the choice of (f_d^*, g^*, g_h^*) we have $K_0(z) \in {}_0\mathbb{F}(a)$ for any $z \in {}_0\mathbb{E}(a)$, and it follows from Proposition 6.2 that

$$K_0 \in C^\omega({}_0\mathbb{E}(a), {}_0\mathbb{F}(a)).$$

Consequently, $L_0^{-1} K_0 : {}_0\mathbb{E}(a) \to {}_0\mathbb{E}(a)$ is well-defined and smooth.

(ii) In the following, $t_0 > 0$ is a fixed number. We set

$$E_1 := \{(u_0, h_0) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n) : [\![u]\!] = 0\},$$

and observe that E_1 is a Banach space. Given $(u_0, h_0) \in E_1$ let (f_d^*, g^*, g_h^*) be defined as in (6.15)–(6.16). It is not difficult to see that the mapping

$$F^* : E_1 \to \mathbb{F}(t_0), \qquad F^*(u_0, h_0) := (0, f_d^*, g^*, g_h^*),$$

is C^1 (in fact real analytic), and that $F^*(0) = 0$ and $DF^*(0) = 0$. Hence given $\delta \in (0, 1)$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\|F^*(u_0, h_0)\|_{\mathbb{F}(t_0)} \leqslant \delta \|(u_0, h_0)\|_{E_1}, \qquad (u_0, h_0) \in \varepsilon\overline{\mathbb{B}}_{E_1}. \tag{6.19}$$

Let $\mathbb{G}(t_0)$ denote the closed subspace of $\mathbb{F}(t_0) \times E_1$ consisting of all functions $(f, f_d, g, g_h, u_0, h_0)$ satisfying the compatibility conditions of Theorem 5.1.

Suppose that $(u_0, h_0) \in \varepsilon\overline{\mathbb{B}}_{E_1}$ satisfies the compatibility conditions (6.9). Then, due to (6.13) and the definition of F^*, the mapping

$$G^* : E_1 \to \mathbb{G}(t_0), \qquad G^*(u_0, h_0) := (F^*(u_0, h_0), u_0, h_0),$$

is well-defined and $\|G^*(u_0,h_0)\|_{\mathbb{G}(t_0)} \leqslant 2\|(u_0,h_0)\|_{E_1}$. It then follows from Theorem 5.1 that (6.17) has a unique solution $z^* = z^*(u_0,h_0)$ which satisfies

$$\|z^*\|_{\mathbb{E}(t_0)} \leqslant C_0\|(u_0,h_0)\|_{E_1}, \qquad (u_0,h_0) \in \varepsilon\overline{\mathbb{B}}_{E_1}, \tag{6.20}$$

where the constant C_0 does not depend on (u_0,h_0).

(iii) Theorem 5.1 also implies that $L_0 : {}_0\mathbb{E}(t_0) \to {}_0\mathbb{F}(t_0)$ is an isomorphism. Let then

$$M := \|L_0^{-1}\|_{\mathcal{L}({}_0\mathbb{F}(t_0),{}_0\mathbb{E}(t_0))}. \tag{6.21}$$

We can assume that the number δ in step (ii) was already chosen so small that

$$\delta < \min\left(1, \frac{1}{M(2+C_0)}\right). \tag{6.22}$$

(iv) We shall show that the fixed point equation (6.18) has for each initial value (u_0,h_0) satisfying (6.9)–(6.10) a unique fixed point $\hat{z} = \hat{z}(u_0,h_0) \in \varepsilon\overline{\mathbb{B}}_{{}_0\mathbb{E}(t_0)}$. It follows from Proposition 6.2 and (6.20) that

$$\|DN(z+z^*)\|_{\mathcal{L}(\mathbb{E}(t_0),\mathbb{F}(t_0))}, \|DK_0(z)\|_{\mathcal{L}({}_0\mathbb{E}(t_0),{}_0\mathbb{F}(t_0))} \leqslant \delta \tag{6.23}$$

for all (u_0,h_0) satisfying (6.9)–(6.10) and all $z \in \varepsilon\overline{\mathbb{B}}_{{}_0\mathbb{E}(t_0)}$, provided ε is chosen small enough. From (6.19)–(6.23) it follows for $z, z_j \in \varepsilon\overline{\mathbb{B}}_{{}_0\mathbb{E}(t_0)}$ that

$$\|L_0^{-1}(K_0(z_1) - K_0(z_2))\|_{{}_0\mathbb{E}(t_0)} \leqslant M\delta\|z_1 - z_2\|_{{}_0\mathbb{E}(t_0)} \leqslant \tfrac{1}{2}\|z_1 - z_2\|_{{}_0\mathbb{E}(t_0)}$$

and

$$\|L_0^{-1}K_0(z)\|_{{}_0\mathbb{E}(t_0)} \leqslant M(\|N(z+z^*)\|_{\mathbb{F}(t_0)} + \|F^*(u_0,h_0)\|_{\mathbb{F}(t_0)}) \leqslant M\delta(2+C_0)\varepsilon \leqslant \varepsilon.$$

This shows that the mapping $L_0^{-1}K_0 : \varepsilon\overline{\mathbb{B}}_{{}_0\mathbb{E}(t_0)} \to \varepsilon\overline{\mathbb{B}}_{{}_0\mathbb{E}(t_0)}$ is a contraction for any initial value (u_0,h_0) satisfying (6.9)–(6.10).

(v) By the contraction mapping principle $L_0^{-1}K_0$ has a unique fixed point $\hat{z} \in \varepsilon\overline{\mathbb{B}}_{{}_0\mathbb{E}(t_0)} \subset {}_0\mathbb{E}(t_0)$ and it follows from (6.11)–(6.12) that $\hat{z} + z^*$ is the (unique) solution of the nonlinear problem (2.9) in $\mathbb{E}(t_0)$, proving the assertion in part (a) of the theorem.

(vi) In order to show that (u,π,q,h) is analytic in space and time we can use the same strategy as in [19, Section 8]. Since the proof is similar we will refrain from giving all the details, and will rather point out the underlying ideas.

Let $(u,\pi,q,h) \in \mathbb{E}(t_0)$ be the solution of (2.9) with initial value (u_0,h_0). Let $a \in (0,t_0)$ be fixed and choose $\delta > 0$ so that $(1+\delta)a \leqslant t_0$. Moreover, let φ be a smooth cut-off function with $\varphi \equiv 1$ on $[-R,R]$ for some $R > 0$ and suppose that $\delta > 0$ is chosen small enough so that

$$1 + \varphi(y)\tau t > 0, \qquad 1 + (y\varphi(y))'\tau t > 0, \qquad t \in [0,a],\ \tau \in (-\delta,\delta),\ y \in \mathbb{R}.$$

For given parameters $(\lambda,\nu,\tau) \in (1-\delta,1+\delta) \times \mathbb{R}^n \times (-\delta,\delta)$ we set

$$\begin{aligned}(u_{\lambda,\nu,\tau},\pi_{\lambda,\nu,\tau})(t,x,y) &:= (u,\pi)(\lambda t, x+t\nu, y(1+\varphi(y)\tau t)),\\ (q_{\lambda,\nu},h_{\lambda,\nu})(t,x) &:= (q,h)(\lambda t, x+t\nu),\\ z_{\lambda,\nu,\tau} &:= (u_{\lambda,\nu,\tau},\pi_{\lambda,\nu,\tau},q_{\lambda,\nu},h_{\lambda,\nu}),\end{aligned} \tag{6.24}$$

where $(t, x, y) \in [0, a] \times \mathbb{R}^n \times \dot{\mathbb{R}}$. Suppose we know that

$$[(\lambda, \nu, \tau) \mapsto z_{\lambda,\nu,\tau}] \in C^\omega(\Lambda, \mathbb{E}(a)) \tag{6.25}$$

with $\Lambda \subset (1-\delta, 1+\delta) \times \mathbb{R}^n \times (-\delta, \delta)$ a neighborhood of $(\lambda, \nu, \tau) = (1, 0, 0)$. Pick $(s_0, x_0, y_0) \in (0, t_0) \times \dot{\mathbb{R}}^{n+1}$ and choose $a \in (s_0, t_0)$. Without loss of generality we can assume that $y_0 \in [-R, R]$. Thanks to the embeddings

$$\mathbb{E}_1(a) \hookrightarrow C(I; BC(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})), \qquad \mathbb{E}_3(a), \mathbb{E}_4(a) \hookrightarrow C(I; BC(\mathbb{R}^n)),$$

(see Lemma 6.1) we conclude that

$$\begin{aligned}
&[(\lambda, \nu, \tau) \mapsto u_{\lambda,\nu,\tau}] \in C^\omega\big(\Lambda, C(I; BC(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))\big),\\
&[(\lambda, \nu, \tau) \mapsto (q_{\lambda,\nu}, h_{\lambda,\nu})] \in C^\omega(\Lambda, C(I; BC(\mathbb{R}^n))) \times C(I; BC(\mathbb{R}^n))
\end{aligned}$$

for $I = [0, a]$. Thus

$$\begin{aligned}
&[(\lambda, \nu, \tau) \mapsto u(\lambda s_0, x_0 + s_0\nu, y_0(1 + \tau s_0)] \in C^\omega(\Lambda, \mathbb{R}^{n+1}),\\
&[(\lambda, \nu, \tau) \mapsto (q, h)(\lambda s_0, x_0 + s_0\nu)] \in C^\omega(\Lambda, \mathbb{R}^2),
\end{aligned}$$

and this implies that

$$u \in C^\omega((0, t_0) \times \dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}), \qquad q, h \in C^\omega((0, t_0) \times \mathbb{R}^n). \tag{6.26}$$

This in turn together with (2.2)–(2.3) shows that $\nabla\pi \in C^\omega((0, t_0) \times \dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$ as well, and we can now conclude that

$$\pi \in C^\omega((0, t_0) \times \dot{\mathbb{R}}^{n+1}), \tag{6.27}$$

where the pressure π is normalized by $\pi(t, 0, 0-) \equiv 0$, i.e.

$$\pi(t, x, y) = \begin{cases} q(t, 0) + \displaystyle\int_0^1 [(\nabla_x \pi(t, sx, sy) \,|\, x) + \partial_y \pi(t, sx, sy) y]\, \mathrm{d}s, & y > 0,\\ \displaystyle\int_0^1 [(\nabla_x \pi(t, sx, sy) \,|\, x) + \partial_y \pi(t, sx, sy) y]\, \mathrm{d}s, & y < 0. \end{cases}$$

(vii) We will now explain the steps needed to establish the crucial property (6.25). First we note that there exists a neighborhood $\Lambda \subset (1-\delta, 1+\delta) \times \mathbb{R}^n \times (-\delta, \delta)$ of $(1, 0, 0)$ such that

$$[(\lambda, \nu, \tau) \mapsto (0, f^*_{d,\lambda,\nu,\tau}, g^*_{\lambda,\nu}, g^*_{h,\lambda,\nu})] \in C^\omega(\Lambda, \mathbb{F}(a)), \tag{6.28}$$

where the functions (f^*_d, g^*, g^*_h) are defined in (6.15)–(6.16). In fact, the assertion follows immediately from [19, Lemma 8.2] for the functions (g^*, g^*_h). Let us then consider the function c^* defined in (6.14). Let $w(t) := e^{-tD_{n+1}} w_0$ for some function $w_0 \in W_p^{2-2/p}(\mathbb{R}^{n+1})$ and define $w_{\lambda,\nu,\tau}(t, x, y)$ for $(t, x, y) \in I \times \mathbb{R}^{n+1}$ as above, with $I = [0, a]$. Then one verifies as in the proof of [19, Lemma 8.2] that

$$w_{\lambda,\nu,\tau} \in H^1_p(I; L_p(\mathbb{R}^{n+1})) \cap L_p(I; H^2_p(\mathbb{R}^{n+1})) =: \mathbb{X}_1(I)$$

for $(\lambda, \nu, \tau) \in (1-\delta, 1+\delta) \times \mathbb{R}^n \times (-\delta, \delta)$, and that $w_{\lambda,\nu,\tau}$ solves the parameter-dependent parabolic equation

$$\partial_t u - \mathcal{A}_{\lambda,\nu,\tau} u = 0, \quad u(0) = w_0,$$

in $\mathbb{R}^{n+1}$, where $\mathcal{A}_{\lambda,\nu,\tau}$ is a parameter-dependent differential operator given by

$$\mathcal{A}_{\lambda,\nu,\tau} = \lambda \Delta_x + \frac{\lambda}{(1+\alpha'(y)\tau t)^2}\partial_y^2 + \tau\left(\frac{\alpha(y)}{1+\alpha'(y)\tau t} - \frac{\lambda\alpha''(y)t}{(1+\alpha'(y)\tau t)^3}\right)\partial_y + (\nu \mid \nabla_x)$$

for $t \in [0, a]$ and $y \in \dot{\mathbb{R}}$, where $\alpha(y) := y\varphi(y)$. Here we observe that

$$\mathcal{A}_{1,0,0} = \Delta, \quad [(\lambda, \nu, \tau) \mapsto \mathcal{A}_{\lambda,\nu,\tau}] \in C^\omega(\Lambda, \mathcal{L}(\mathbb{X}_1(I), \mathbb{X}_0(I))),$$

with $\mathbb{X}_0(I) := L_p(I, L_p(\mathbb{R}^{n+1}))$. As in the proof of [19, Lemma 8.2] it follows from the implicit function theorem that there exists a neighborhood $\Lambda \subset (1-\delta, 1+\delta) \times \mathbb{R}^n \times (-\delta, \delta)$ of $(1, 0, 0)$ such that

$$[(\lambda, \nu, \tau) \mapsto w_{\lambda,\nu,\tau}] \in C^\omega(\Lambda, \mathbb{X}_1(I)). \tag{6.29}$$

Applying (6.29) separately to $w_0 = \mathcal{E}_\pm(v_0 \nabla h_0)$, and then applying $\mathcal{R}_\pm$ yields

$$[(\lambda, \nu, \tau) \mapsto c^*_{\lambda,\nu,\tau}] \in C^\omega\big(\Lambda, H^1_p(I; L_p(\mathbb{R}^{n+1})) \cap L_p(I; H^2_p(\dot{\mathbb{R}}^{n+1}))\big).$$

It then follows from the definition of f^*_d that $[(\lambda, \nu, \tau) \mapsto f^*_{d,\lambda,\nu,\tau}] \in C^\omega(\Lambda, \mathbb{F}_2(a))$. In a next step one verifies that the function $z^*_{\lambda,\nu,\tau}$ solves the linear parameter-dependent problem

$$\left\{\begin{aligned}
\rho\partial_t u - \mathcal{A}_{\lambda,\nu,\tau} u + \mathcal{B}_{\lambda,\tau}\pi &= 0 && \text{in } \dot{\mathbb{R}}^{n+1},\\
\mathcal{C}_\tau u &= f^*_{d,\lambda,\nu,\tau} && \text{in } \dot{\mathbb{R}}^{n+1},\\
-\frac{1}{1+\tau t}[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] &= g^*_{v,\lambda,\nu} && \text{on } \mathbb{R}^n,\\
-\frac{2}{1+\tau t}[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h &= g^*_{w,\lambda,\nu} && \text{on } \mathbb{R}^n,\\
[\![u]\!] &= 0 && \text{on } \mathbb{R}^n,\\
\partial_t h - \lambda\gamma w + \mathcal{D}_\nu h &= \lambda g^*_{h,\lambda,\nu} && \text{on } \mathbb{R}^n,\\
u(0) = u_0, \quad h(0) &= h_0
\end{aligned}\right. \tag{6.30}$$

where

$$\begin{aligned}
\mathcal{A}_{\lambda,\nu,\tau} &:= \lambda\mu\Delta_x + \frac{\lambda\mu}{(1+\alpha'(y)\tau t)^2}\partial_y^2 + \tau\left(\frac{\rho\alpha(y)}{1+\alpha'(y)\tau t} - \frac{\lambda\mu\alpha''(y)t}{(1+\alpha'(y)\tau t)^3}\right)\partial_y + \rho(\nu \mid \nabla_x),\\
\mathcal{B}_{\lambda,\tau}\pi &:= \lambda\left(\nabla_x \pi, \frac{1}{1+\alpha'(y)\tau t}\partial_y \pi\right), \qquad \mathcal{C}_\tau u := \operatorname{div}_x v + \frac{1}{1+\alpha'(y)\tau t}\partial_y w,\\
\mathcal{D}_\nu h &:= -(\nu \mid \nabla h).
\end{aligned}$$

We note that

$$\mathcal{A}_{1,0,0} = \mu\Delta, \quad \mathcal{B}_{1,0} = \nabla, \quad \mathcal{C}_1 = \operatorname{div}, \quad \mathcal{D}_0 = 0.$$

It is easy to see that the differential operators $\mathcal{A}_{\lambda,\nu,\tau}$, $\mathcal{B}_{\lambda,\tau}$, $\mathcal{C}_\tau$ and $\mathcal{D}_\nu$ depend analytically on the parameters (λ, ν, τ) in the appropriate function spaces. Using Thereom 5.1 and the implicit function theorem one shows similarly to [19, Lemma 8.3] that there is a neighborhood $\Lambda \subset (1-\delta, 1+\delta) \times \mathbb{R}^n \times (-\delta, \delta)$ of $(1, 0, 0)$ such that

$$[(\lambda, \nu, \tau) \mapsto z^*_{\lambda,\nu,\tau}] \in C^\omega(\Lambda, \mathbb{E}(a)). \tag{6.31}$$

Let $\hat{z}$ be the solution of (6.12) obtained in step (v) above. Then one verifies that $\hat{z}_{\lambda,\nu,\tau} \in 2\varepsilon \mathbb{B}_{{}_0\mathbb{E}(t_0)}$ for $(\lambda, \nu, \tau) \in \Lambda$, with Λ a sufficiently small neighborhood of $(1, 0, 0)$. Moreover, $\hat{z}_{\lambda,\nu,\tau}$ solves the nonlinear parameter-dependent problem

$$L_{\lambda,\nu,\tau} z = K_{\lambda,\nu,\tau}(z), \qquad z \in {}_0\mathbb{E}(a), \tag{6.32}$$

for $(\lambda, \nu, \tau) \in \Lambda$, where $L_{\lambda,\nu,\tau} z$ is defined by the left hand side of (6.30) and where

$$K_{\lambda,\nu,\tau}(z) := \begin{pmatrix} \lambda F_\tau(u + u^*_{\lambda,\nu,\tau}, \pi + \pi^*_{\lambda,\nu,\tau}, h + h^*_{\lambda,\nu}) \\ F_{d,\tau}(u + u^*_{\lambda,\nu,\tau}, h + h^*_{\lambda,\nu}) - f^*_{d,\lambda,\nu,\tau} \\ G_\tau(u + u^*_{\lambda,\nu,\tau}, q + q^*_{\lambda,\nu}, h + h^*_{\lambda,\nu}) - g^*_{\lambda,\nu} \\ \lambda H(u + u^*_{\lambda,\nu,\tau}, h + h^*_{\lambda,\nu}) - g^*_{h,\lambda,\nu} \end{pmatrix}. \tag{6.33}$$

The functions F_τ, $F_{d,\tau}$ and G_τ are obtained from F, F_d and G, respectively, by replacing terms containing partial derivatives ∂_y and ∂_y^2 in the following way:

$$\partial_y \omega \mapsto \frac{1}{1+\alpha'(y)\tau t}\partial_y \omega, \qquad \partial_y^2 \omega \mapsto \frac{1}{(1+\alpha'(y)\tau t)^2}\partial_y^2 \omega - \frac{\alpha''(y)\tau t}{(1+\alpha'(y)\tau t)^3}\partial_y \omega$$

for $\omega \in \{v, w, \pi\}$. Equation (6.32) can be reformulated as

$$\Psi(z, (\lambda, \nu, \tau)) := z - (L_{\lambda,\nu,\tau})^{-1} K_{\lambda,\nu,\tau}(z) = 0, \qquad z \in {}_0\mathbb{E}(a). \tag{6.34}$$

Here we observe that $\Psi(\hat{z}, (1, 0, 0)) = 0$ for the solution $\hat{z}$ of the fixed point equation (6.18). It follows from (6.28), (6.31) and Proposition 6.2 that

$$[(z, (\lambda, \nu, \tau)) \mapsto \Psi(z, (\lambda, \nu, \tau))] \in C^\omega({}_0\mathbb{E}(a) \times \Lambda, {}_0\mathbb{E}(a)).$$

Moreover, it follows from (6.21)–(6.23) that

$$D_1\Psi(\hat{z}, (1, 0, 0)) = I - D(L^{-1}K_0)(\hat{z}) \in \mathrm{Isom}({}_0\mathbb{E}(a), {}_0\mathbb{E}(a)).$$

By the implicit function theorem there exists a neighborhood $\Lambda \subset (1-\delta, 1+\delta) \times \mathbb{R}^n \times (-\delta, \delta)$ of $(\lambda, \nu, \tau) = (1, 0, 0)$ such that

$$[(\lambda, \nu, \tau) \mapsto \hat{z}_{\lambda,\nu,\tau}] \in C^\omega(\Lambda, {}_0\mathbb{E}(a)). \tag{6.35}$$

Combining (6.31) and (6.35) yields (6.25). This completes the proof of Theorem 6.3. $\square$

Proof of Theorem 1.1. We first observe that the compatibility conditions of Theorem 1.1 are satisfied if and only if (6.9) is satisfied. Next we note that the mapping Θ_{h_0} given by $\Theta_{h_0}(x, y) := (x, y + h_0(x))$ defines for each $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$ a C^2-diffeomorphism from $\mathbb{R}^{n+1}_\pm$ onto $\Omega_i(0)$ with $\det[D\Theta_{h_0}(x, y)] = 1$. Its inverse is given by $\Theta^{-1}_{h_0}(x, y) := (x, y - h_0(x))$. It then follows from the chain rule and the transformation rule for integrals that

$$\frac{1}{C(h_0)}\|u_0\|_{W_p^{2-2/p}(\Omega_0)} \leqslant \|(v_0, w_0)\|_{W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})} \leqslant C(h_0)\|u_0\|_{W_p^{2-2/p}(\Omega_0)},$$

where $C(h_0) := M[1 + \|\nabla h_0\|_{BC^1(\mathbb{R}^n)}]$, with M an appropriate constant. Consequently, there exists $\varepsilon_0 > 0$ such that $\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^n)} \leqslant \varepsilon_0$ implies the smallness condition (6.10). Theorem 6.3 then yields a unique solution $(v, w, \pi, [\pi], h) \in \mathbb{E}(t_0)$ which satisfies the additional regularity properties listed in part (b) of the theorem. Setting

$$(u, q)(t, x, y) = (v, w, \pi)(t, x, y - h(t, x)), \qquad (t, x, y) \in \mathcal{O},$$

we then conclude that $(u, q) \in C^\omega(\mathcal{O}, \mathbb{R}^{n+2})$ and $[\![q]\!] \in C^\omega(\mathcal{M})$. The regularity properties listed in (1.3)–(1.4) are implied by Lemma 6.1(b)–(c). Finally, since $q(t, x, y)$ is defined for every $(t, x, y) \in \mathcal{O}$, we can conclude that $q(t, \cdot) \in \dot{H}_p^1(\Omega(t)) \subset UC(\Omega(t))$ for every $t \in (0, t_0)$. □

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