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Long-time behaviour of two-phase solutions to a class of forward-backward parabolic equations

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We consider two-phase solutions to the Neumann initial-boundary value problem for the parabolic equation $u_t = [\phi(u)]_{xx}$, where ϕ is a nonmonotone cubic-like function. First, we prove global existence for a restricted class of initial data u_0, showing that two-phase solutions can be obtained as limiting points of the family of solutions to the Neumann initial-boundary value problem for the regularized equation $u_t^\varepsilon = [\phi(u^\varepsilon)]_{xx} + \varepsilon u_{txx}^\varepsilon$ ($\varepsilon > 0$). Then, assuming global existence, we study the long-time behaviour of two-phase solutions for any initial datum u_0.

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1. Introduction

In this paper we consider the Neumann initial-boundary value problem for the equation

$$u_t = [\phi(u)]_{xx} \quad \text{in } Q := \Omega \times (0, \infty), \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded interval (ω_1, ω_2) and the function $\phi \in C^2(\mathbb{R})$ satisfies the following assumption:

$$(H_1) \quad \begin{cases} \phi'(u) > 0 & \text{if } u \in (-\infty, b) \cup (c, \infty),\ b < 0 < c, \\ \phi'(u) < 0 & \text{if } u \in (b, c), \\ B := \phi(b) > \phi(c) =: A, & \phi(u) \to \pm\infty \text{ as } u \to \pm\infty, \\ \phi''(b) \neq 0, & \phi''(c) \neq 0. \end{cases} \tag{1.2}$$

We also denote by $a \in (-\infty, b)$ and $d \in (c, \infty)$ the roots of the equation $\phi(u) = A$, respectively $\phi(u) = B$ (see Figure 1).

In view of the nonmonotone character of ϕ, (1.1) is a *forward-backward* parabolic equation, since it is well-posed forward in time at the points where $\phi' > 0$ and it is ill-posed where $\phi' < 0$. In this connection, we denote by

$$S_1 := \{(u, \phi(u)) \mid u \in (-\infty, b]\} \equiv \{(s_1(v), v) \mid v \in (-\infty, B]\} \tag{1.3}$$

and

$$S_2 := \{(u, \phi(u)) \mid u \in [c, \infty)\} \equiv \{(s_2(v), v) \mid v \in [A, \infty)\} \tag{1.4}$$

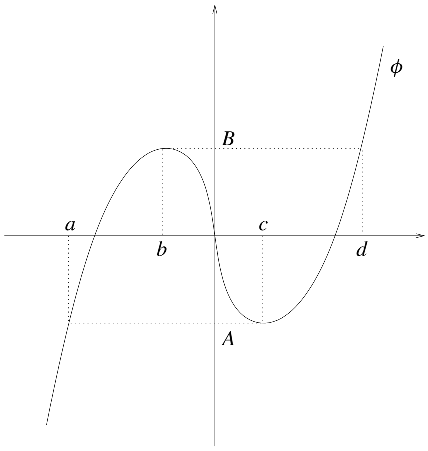

FIG. 1. Assumption (H_1).

the *stable branches* of the equation $v = \phi(u)$, whereas

$$S_0 := \{(u, \phi(u)) \mid u \in (b, c)\} \equiv \{(s_0(v), v)) \mid v \in (A, B)\} \tag{1.5}$$

is referred to as the *unstable branch*.

Equation (1.1) with a function ϕ satisfying assumption (H_1) naturally arises in the theory of phase transitions. In this context, u represents the phase field and equation (1.1) describes the evolution between stable phases. With a nonlinearity ϕ of a different shape, whose main feature is *degeneracy at infinity*, equation (1.1) describes models in population dynamics ([Pa]), oceanography ([BBDU]), image processing ([PM]) and gradient systems associated with nonconvex functionals ([BFG]). In these cases equation (1.1) can be obtained by differentiating the one-dimensional *Perona–Malik equation*

$$z_t = [\phi(z_x)]_x \tag{1.6}$$

with respect to the space variable x. The transformation $u := z_x$ gives a relation between equations (1.6) and (1.1). Here typical choices of the function ϕ are either $\phi(s) = s(1 + s^2)^{-1}$ or $\phi(s) = s \exp(-s)$. Observe that in both cases ϕ degenerates to zero as s diverges to infinity.

The initial-boundary value problem for equation (1.1) (either under Dirichlet or Neumann boundary conditions) has been widely addressed in the literature. Most techniques consist in *modifying* the (possibly) ill-posed equation (hence the boundary conditions) by some regularization which leads to a well-posed problem. Then a natural question is whether the approximating solutions define a solution (in some suitable sense, depending on the regularization itself) of (1.1) as the regularization parameter goes to zero. Many regularizations of equation (1.1) have been proposed and investigated ([BBDU, NP, Sl]). Among them, let us mention the *pseudoparabolic* regularization, described by the Sobolev equation

$$u_t = [\phi(u)]_{xx} + \varepsilon[u_t]_{xx}. \tag{1.7}$$

In particular, (1.7) has been studied in [NP] for the corresponding Neumann initial-boundary value problem in $Q_T := \Omega \times (0, T)$ ($T > 0$) and for cubic-like response functions ϕ

satisfying assumption (H_1) (analogous results have been given in [Pa] in the case of "Perona–Malik" ϕ). Moreover, in [Pl1] it is shown that the limiting points of the family of the approximating solutions $(u^\varepsilon,\ \phi(u^\varepsilon))$ define a class of solutions (u, v)—precisely the *weak entropy measure-valued solutions*—to the Neumann initial-boundary value problem in Q_T for the original unperturbed equation (1.1). The main properties of such solutions (u, v) obtained in the limiting process $\varepsilon \to 0$ can be summarized as follows:

- $u \in L^\infty(Q_T),\ v \in L^\infty(Q_T) \cap L^2((0, T); H^1(\Omega))$ and there exist $\lambda_i \in L^\infty(Q_T)$ $(i = 0, 1, 2)$, $0 \leqslant \lambda_i \leqslant 1$ and $\sum_{i=0}^{2} \lambda_i = 1$ such that

$$u = \sum_{i=0}^{2} \lambda_i s_i(v);$$

- the couple (u, v) solves in the weak sense the equation

$$u_t = v_{xx} \quad \text{in } \mathcal{D}'(Q_T);$$

- the couple (u, v) satisfies the following class of *entropy inequalities*:

$$\iint_{Q_T} [G^*\psi_t - g(v)v_x\psi_x - g'(v)\, v_x^2\psi]\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega G(u_0)\psi(x, 0)\, \mathrm{d}x \geqslant 0$$

  for any $\psi \in C^1(\overline{Q}_T)$ with $\psi \geqslant 0$, $\psi(\cdot, T) \equiv 0$. Here, for any $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$, the function G^* is defined by setting

$$G^* = \sum_{i=0}^{2} \lambda_i G(s_i(v)), \quad \text{where} \quad G(\lambda) := \int_0^\lambda g(\phi(s))\, \mathrm{d}s + k \quad (k \in \mathbb{R}).$$

Let us also mention that both in [NP] and [Pl1] the general case $\Omega \subseteq \mathbb{R}^N,\ N > 1$, is considered as well.

As already pointed out, the main results in [Pl1] show that weak entropy measure-valued solutions to both the Neumann and Dirichlet initial-boundary value problems for equation (1.1) exist for any initial datum $u_0 \in L^\infty(\Omega)$ and for any regular cubic-like ϕ satisfying assumption (H_1). As a matter of fact, actually the uniqueness of such solutions is unknown, albeit this class seems a natural candidate in view of the entropy inequalities (see also [H, Z] for general nonuniqueness results). On the other hand, a natural question is whether uniqueness can be recovered by introducing some additional constraints. For this purpose, *two-phase solutions* have been introduced in [EP] and investigated in [MTT1, MTT2, T]. Roughly speaking, a two-phase solution to the Neumann initial-boundary value problem associated to equation (1.1) in $Q_T = \Omega \times (0, T)$ is a weak entropy measure-valued solution (u, v) (in the sense of [Pl1]) which describes transitions only between stable phases. Such solutions exhibit a smooth interface $\xi : [0, T] \to \overline{\Omega}$ such that

$$u = s_1(v) \quad \text{in } \{(x, t) \in Q_T \mid \omega_1 \leqslant x < \xi(t)\},$$
$$u = s_2(v) \quad \text{in } \{(x, t) \in Q_T \mid \xi(t) < x \leqslant \omega_2\}.$$

Here s_1 and s_2 are defined in (1.3) and (1.4) (recall also that $\Omega = (\omega_1, \omega_2)$). It is worth observing that the interface $\xi(t)$ evolves obeying admissibility conditions which follow from the entropy inequalities (see Remark 2.2).

Local existence and *uniqueness* of *smooth* two-phase solutions to the Cauchy problem associated to equation (1.1) was studied in [MTT2] for *piecewise* response functions ϕ. Actually, *global* existence of such solutions is proven to hold for initial data functions u_0 satisfying the condition $a \leqslant u_0 \leqslant d$ (see Figure 1), whereas it is still unknown in the general case.

In this paper we obtain global existence of two-phase solutions to the Neumann problem associated to equation (1.1) in the case of cubic-like response functions ϕ and initial data functions u_0 subject to the constraint $a \leqslant u_0 \leqslant b$ in $(\omega_1, 0)$ and $c \leqslant u_0 \leqslant d$ in $(0, \omega_2)$; such a result can be regarded as the counterpart of the one obtained in [MTT2] for the Cauchy problem associated to equation (1.1) in the case of piecewise nonlinearities ϕ. In particular, we will prove that global two-phase solutions (in the sense of Definition 2.1 below) can be obtained as limiting points of the solutions $(u^\varepsilon, \phi(u^\varepsilon))$ to the Neumann initial-boundary value problems associated to the pseudoparabolic regularization (1.7) of equation (1.1).

We also study the long-time behaviour of two-phase solutions in the general case of arbitrary cubic-like response functions ϕ and for arbitrary initial data u_0.

The paper is organized as follows. In Section 2 we describe the mathematical framework and give the main results, while Sections 3–6 are essentially devoted to the proofs.

2. Mathematical framework and results

2.1 *Basic properties*

Let us consider the initial-boundary value problem

$$\begin{cases} u_t = [\phi(u)]_{xx} & \text{in } \Omega \times (0, \infty) =: Q, \\ [\phi(u)]_x = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.1}$$

where $\phi \in C^2(\mathbb{R})$ satisfies assumption (H_1). For simplicity, in what follows we will always assume that $0 \in \Omega$. Concerning the initial datum $u_0 \in L^\infty(\Omega)$ we formulate the following, quite natural, assumption:

$$(H_2) \qquad \begin{cases} u_0 \leqslant b & \text{almost everywhere in } (\omega_1, 0), \\ u_0 \geqslant c & \text{almost everywhere in } (0, \omega_2) \end{cases}$$

(see Figure 1). Following [EP, MTT1, MTT2], we give the definition of two-phase solution to problem (2.1).

DEFINITION 2.1 By a *two-phase solution* of problem (2.1) we mean any triple (u, v, ξ) such that:

(i) $u \in L^\infty(Q)$, $v \in L^\infty(Q) \cap L^2((0, T); H^1(\Omega))$ for any $T > 0$, and $\xi : [0, \infty) \to \overline{\Omega}$ is Lipschitz continuous, $\xi(0) = 0$;

(ii) we have

$$u = s_i(v) \quad \text{in } V_i \quad (i = 1, 2), \tag{2.2}$$

where

$$V_1 := \{(x, t) \in Q \mid \omega_1 < x < \xi(t),\ t \in (0, \infty)\}, \tag{2.3}$$

$$V_2 := \{(x, t) \in Q \mid \xi(t) < x < \omega_2,\ t \in (0, \infty)\}, \tag{2.4}$$

$$\gamma := \partial V_1 \cap \partial V_2 = \{(\xi(t), t) \mid t \in [0, \infty)\}; \tag{2.5}$$

(iii) for any $T > 0$ set $Q_T := \Omega \times (0, T)$; then

$$\iint_{Q_T} [u\psi_t - v_x\psi_x]\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x)\psi(x,0)\,\mathrm{d}x = 0 \tag{2.6}$$

for any $\psi \in C^1(\overline{Q}_T)$, $\psi(\cdot, T) \equiv 0$ in $\overline{\Omega}$;

(iv) for any $g \in C^1(\mathbb{R})$, set

$$G(\lambda) := \int_0^\lambda g(\phi(s))\,\mathrm{d}s + k \quad (k \in \mathbb{R}), \tag{2.7}$$

where k is arbitrary, so that G is *any* primitive of the function $g \circ \phi$; then, for any $T > 0$ and under the assumption $g' \geqslant 0$, the *entropy inequalities*

$$\iint_{Q_T} [G(u)\psi_t - g(v)v_x\psi_x - g'(v)v_x^2\psi]\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega G(u_0(x))\psi(x,0)\,\mathrm{d}x \geqslant 0 \tag{2.8}$$

hold for any $\psi \in C^1(\overline{Q}_T)$ with $\psi \geqslant 0$ and $\psi(\cdot, T) \equiv 0$ in $\overline{\Omega}$.

REMARK 2.1 Let us denote by $C^{2,1}(Q)$ the set of functions $f \in C(Q)$ such that $f_t, f_x, f_{xx} \in C(Q)$. Let (u, v, ξ) be any two-phase solution of problem (2.1) (in the sense of Definition 2.1). Then u is a weak solution of the equation

$$u_t = [\phi(u)]_{xx} \quad \text{in } V_i$$

($i = 1, 2$). This implies that $u \in C(V_i)$ ([AdB]). Moreover

$$u \in C^{2,1}(V_1 \setminus V_b), \qquad V_b := \{(x,t) \in V_1 \mid u(x,t) = b\},$$
$$u \in C^{2,1}(V_2 \setminus V_c), \qquad V_c := \{(x,t) \in V_2 \mid u(x,t) = c\}$$

([LSU, Va]).

REMARK 2.2 Let (u, v, ξ) be any two-phase solution of problem (2.1) and *assume* that $\xi \in C^1([0,\infty))$, $u \in C^{2,1}(\overline{V_i})$ ($i = 1, 2$). Moreover, denote by

$$[f(\xi(t), t)] := \lim_{h \to 0^+} \{f(\xi(t) + h, t) - f(\xi(t) - h, t)\} = f(\xi(t)^+, t) - f(\xi(t)^-, t)$$

the jump across the interface γ of any piecewise continuous function f. Then:

- the Rankine–Hugoniot condition

$$\xi'(t) = -\frac{[v_x(\xi(t), t)]}{[u(\xi(t), t)]} \tag{2.9}$$

holds for any $t > 0$ ([EP, MTT1]);
- by the entropy inequalities (2.8) we obtain

$$\xi'(t)[G(u)(\xi(t), t)] \geqslant -g(v(\xi(t), t))[v_x(\xi(t), t)] \tag{2.10}$$

for any $t > 0$ and for any G defined by (2.7) in terms of $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$. Observe that the above condition implies that for any $t > 0$,

$$\begin{cases} \xi'(t) \geqslant 0 & \text{if } v(\xi(t), t) = A, \\ \xi'(t) \leqslant 0 & \text{if } v(\xi(t), t) = B, \\ \xi'(t) = 0 & \text{if } v(\xi(t), t) \neq A, \ v(\xi(t), t) \neq B \end{cases} \tag{2.11}$$

([EP, MTT1]). That is, jumps between the stable phases S_1 and S_2 occur only at the points (x, t) where the function $v(x, t)$ takes the value A (jumps from S_2 to S_1) or B (jumps from S_1 to S_2).

Finally, if we weaken the conditions $\xi \in C^1([0, \infty))$, $u \in C^{2,1}(\overline{V}_i)$ $(i = 1, 2)$ to the less restrictive assumptions $\xi \in \mathrm{Lip}([0, \infty))$, $u \in C^{2,1}(V_i)$ and $v_x(\cdot, t) \in BV(\Omega)$ for almost every $t > 0$, then the Rankine–Hugoniot conditions (2.9) and the jump conditions (2.10)–(2.11) continue to hold almost everywhere in t.

Let us conclude the section with the following refinement of the entropy inequalities (2.8).

PROPOSITION 2.1 Let (u, v, ξ) be a two-phase solution of problem (2.1). For any $g \in C^1(\mathbb{R})$, let G be the function defined by (2.7). Then, for any $0 \leqslant t_1 < t_2$ and for any $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geqslant 0$,

$$\int_\Omega G(u(x, t_1))\varphi(x)\,\mathrm{d}x - \int_\Omega G(u(x, t_2))\varphi(x)\,\mathrm{d}x \geqslant \int_{t_1}^{t_2} \int_\Omega [g(v)v_x\varphi_x + g'(v)v_x^2\varphi]\,\mathrm{d}x\,\mathrm{d}t \tag{2.12}$$

for any $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$.

REMARK 2.3 Observe that inequalities (2.12) can be regarded as a *pointwise version* of the entropy inequalities (2.8).

2.2 *The case $a \leqslant u_0 \leqslant d$: smoothness and uniqueness*

This subsection is devoted to the study of problem (2.1) in the case of initial data u_0 satisfying the following reinforcement of assumption (H_2):

$$(H_3) \qquad \begin{cases} a \leqslant u_0 \leqslant b & \text{almost everywhere in } (\omega_1, 0), \\ c \leqslant u_0 \leqslant d & \text{almost everywhere in } (0, \omega_2). \end{cases}$$

The following proposition shows that the interval $[a, d]$ is *positively invariant* for problem (2.1) and that under assumption (H_3) the interface $\gamma = \{(\xi(t), t)\}$ does not move.

PROPOSITION 2.2 Assume that $u_0 \in L^\infty(\Omega)$ satisfies assumption (H_3) and let (u, v, ξ) be any two-phase solution of problem (2.1) with initial datum u_0. Then

$$a \leqslant u(x, t) \leqslant d \tag{2.13}$$

for almost every $(x, t) \in Q$. Moreover,

$$\xi(t) \equiv 0 \quad \text{for any } t \geqslant 0. \tag{2.14}$$

REMARK 2.4 As a consequence of the above result, for initial data u_0 subject to assumption (H_3) we have $V_1 = (\omega_1, 0) \times (0, \infty)$, $V_2 = (0, \omega_2) \times (0, \infty)$. Therefore, following the terminology in [MTT2], problem (2.1) can be regarded as a *steady boundary* problem, since $\xi'(t) = 0$ for any $t \geqslant 0$.

The following proposition shows that the set where $\phi'(u) = 0$ is nonincreasing in time.

PROPOSITION 2.3 Let $u_0 \in L^\infty(\Omega)$ satisfy assumption (H_3), assume $\phi(u_0) \in C(\overline{\Omega})$ and let $(u, v, 0)$ be a two-phase solution of problem (2.1) with initial datum u_0. Then:

(i) for any $x \in (\omega_1, 0)$ such that $u_0(x) < b$, we have $u(x,t) < b$ for any $t > 0$;
(ii) for any $x \in (0, \omega_2)$ such that $u_0(x) > c$, we have $u(x,t) > c$ for any $t > 0$.

In view of the above proposition and Remark 2.1 the following smoothness result holds.

COROLLARY 2.4 Let $u_0 \in L^\infty(\Omega)$ satisfy the following assumption:

$$(A_1) \qquad \begin{cases} \phi(u_0) \in C(\overline{\Omega}), \\ u_0 < b \quad \text{in } (\omega_1, 0), \\ u_0 > c \quad \text{in } (0, \omega_2). \end{cases}$$

Let $(u, v, 0)$ be a two-phase solution of problem (2.1) with initial datum u_0. Then $u \in C^{2,1}(V_i)$ $(i = 1, 2)$.

Let us denote by $BV(\Omega)$ the set of functions with bounded total variation on Ω. The following uniqueness result is the counterpart of the one proven in [MTT2].

THEOREM 2.5 Let u_0 satisfy assumption (A_1) and let $(u_1, v_1, 0)$, $(u_2, v_2, 0)$ be two two-phase solutions of problem (2.1) with initial datum u_0. Moreover, assume that $v_{1x}(\cdot, t), v_{2x}(\cdot, t) \in BV(\Omega)$ for almost every $t \in (0, \infty)$. Then $(u_1, v_1) = (u_2, v_2)$.

2.3 *The case $a \leqslant u_0 \leqslant d$: global existence*

STEP 1: *The approximating problems.* For any $\varepsilon > 0$ let us consider the pseudoparabolic regularization of (2.1), described by the problem

$$\begin{cases} u_t = [\phi(u) + \varepsilon u_t]_{xx} \equiv v_{xx} & \text{in } Q, \\ [\phi(u) + \varepsilon u_t]_x \equiv v_x = 0 & \text{in } \{\omega_1, \omega_2\} \times (0, \infty), \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.15}$$

where the *chemical potential* v is defined by setting

$$v := \phi(u) + \varepsilon u_t. \tag{2.16}$$

DEFINITION 2.2 Let $u_0 \in L^\infty(\Omega)$. For any $\varepsilon > 0$ a solution to problem (2.15)–(2.16) is a couple $(u^\varepsilon, v^\varepsilon)$, where $u^\varepsilon \in C^1([0, \infty); L^\infty(\Omega))$ and $v^\varepsilon \in C([0, \infty); W^{2,\infty}(\Omega))$, which satisfies (2.15)–(2.16) in the strong sense.

The following well-posedness result was proven in [NP].

THEOREM 2.6 For any $u_0 \in L^\infty(\Omega)$ and $\varepsilon > 0$ there exists a unique solution $(u^\varepsilon, v^\varepsilon)$ of (2.15)–(2.16). Moreover, for any $t > 0$ the function $v^\varepsilon(\cdot, t)$ solves the problem

$$\begin{cases} -\varepsilon v^\varepsilon_{xx}(\cdot, t) + v^\varepsilon(\cdot, t) = \phi(u^\varepsilon)(\cdot, t) & \text{in } \Omega, \\ v^\varepsilon_x(\cdot, t) = 0 & \text{in } \{\omega_1, \omega_2\}. \end{cases} \tag{2.17}$$

For any $\varepsilon > 0$ the solution $(u^\varepsilon, v^\varepsilon)$ to problem (2.15)–(2.16) satisfies a family of *viscous entropy inequalities*, this terminology being suggested by a formal analogy with the entropy inequalities for *viscous conservation laws* ([EP, MTT2, NP, Pl1]).

PROPOSITION 2.7 For any $u_0 \in L^\infty(\Omega)$ and $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) given by Theorem 2.6. For any $g \in C^1(\mathbb{R})$ let G be the function defined by (2.7). Then, for any $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$, the entropy inequalities

$$\int_\Omega G(u^\varepsilon(x,t_2))\psi(x,t_2)\,\mathrm{d}x - \int_\Omega G(u^\varepsilon(x,t_1))\psi(x,t_1)\,\mathrm{d}x \leqslant \int_{t_1}^{t_2}\int_\Omega \{G(u^\varepsilon)\psi_t - g(v^\varepsilon)v_x^\varepsilon\psi_x - g'(v^\varepsilon)(v_x^\varepsilon)^2\psi\}\,\mathrm{d}x\,\mathrm{d}t \tag{2.18}$$

hold for any $\psi \in C^1(\overline{Q})$ with $\psi \geqslant 0$, and for any $0 \leqslant t_1 < t_2$.

One of the main consequences of the viscous entropy inequalities (2.18) is the existence of *positively invariant regions* for the regularized problems (2.15)–(2.16) ([NP]), therefore a priori estimates for both the families $\{u^\varepsilon\}$, $\{v^\varepsilon\}$. In particular, the following theorem holds.

THEOREM 2.8 For any $u_0 \in L^\infty(\Omega)$ and $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution to problem (2.15)–(2.16). Assume that the initial datum u_0 satisfies assumption (H_3). Then

$$a \leqslant u^\varepsilon \leqslant d \tag{2.19}$$

almost everywhere in Q. Moreover, for any $t > 0$ we have:

(i) $a \leqslant u^\varepsilon(x,t) \leqslant b$ for almost every $x \in (\omega_1, 0)$;
(ii) $c \leqslant u^\varepsilon(x,t) \leqslant d$ for almost every $x \in (0, \omega_2)$.

In view of Theorem 2.8, for any initial datum $u_0 \in L^\infty(\Omega)$ subject to assumption (H_3) the solution u^ε of problem (2.15)–(2.16) satisfies

$$u^\varepsilon = \begin{cases} s_1(\phi(u^\varepsilon)) & \text{almost everywhere in } (\omega_1, 0) \times (0,\infty), \\ s_2(\phi(u^\varepsilon)) & \text{almost everywhere in } (0, \omega_2) \times (0,\infty), \end{cases} \tag{2.20}$$

where s_1 and s_2 are defined in (1.3) and (1.4), respectively.

STEP 2: *Vanishing viscosity limit.* Fix any $u_0 \in L^\infty(\Omega)$ and for any $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) with initial datum u_0. As already remarked, crucial consequences of the viscous entropy inequalities (2.18) are a priori estimates for both the families $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$. In particular in [Pl1] (see also [EP, MTT1, ST]) the following estimates are proved:

$$\|u^\varepsilon\|_{L^\infty(Q)} \leqslant C, \tag{2.21}$$

$$\|v^\varepsilon\|_{L^\infty(Q)} \leqslant C, \tag{2.22}$$

$$\|v_x^\varepsilon\|_{L^2(Q)} + \|\sqrt{\varepsilon}u_t^\varepsilon\|_{L^2(Q)} \leqslant C, \tag{2.23}$$

for some constant $C > 0$ independent of ε. In view of (2.21)–(2.23), we obtain the following convergence results.

THEOREM 2.9 Let $u_0 \in L^\infty(\Omega)$ and for any $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) with initial datum u_0. Then:

(i) there exist a sequence $\{\varepsilon_k\}$, $\varepsilon_k \to 0$, and two functions $u \in L^\infty(Q)$, $v \in L^\infty(Q) \cap L^2((0,T); H^1(\Omega))$ for any $T > 0$ with $v_x \in L^2(Q)$, such that

$$u^{\varepsilon_k} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q), \tag{2.24}$$

$$\phi(u^{\varepsilon_k}), v^{\varepsilon_k} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(Q), \tag{2.25}$$

$$v_x^{\varepsilon_k} \rightharpoonup v_x \quad \text{in } L^2(Q) \tag{2.26}$$

as $k \to \infty$;

(ii) for any $T > 0$ let $Q_T = \Omega \times (0,T)$; then for any $T > 0$ the couple (u, v) satisfies the equality

$$\iint_{Q_T} \{u\psi_t - v_x\psi_x\}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x)\psi(x,0)\,\mathrm{d}x = 0 \tag{2.27}$$

for any $\psi \in C^1(\overline{Q}_T)$ with $\psi(\cdot, T) \equiv 0$ in $\overline{\Omega}$.

If we had $v = \phi(u)$, the couple (u, v) given by Theorem 2.9 would be a weak solution of problem (2.1). However, in view of the nonmonotone character of ϕ, we cannot guarantee such a conclusion in the general case of arbitrary initial data $u_0 \in L^\infty(\Omega)$. In particular, in [Pl1] it is shown that there exist $\lambda_i \in L^\infty(Q)$ $(i = 0, 1, 2)$ with $0 \leqslant \lambda_i \leqslant 1$ and $\sum_{i=0}^2 \lambda_i = 1$ almost everywhere in Q such that

$$u(x,t) = \sum_{i=0}^{2} \lambda_i(x,t) s_i(v(x,t)) \tag{2.28}$$

for almost every $(x,t) \in Q$, where $s_i(v)$ denote the three roots of the equation $v = \phi(u)$ $(i = 0, 1, 2)$. In other words, for arbitrary initial data $u_0 \in L^\infty(\Omega)$, the limiting function u is a *superposition of different phases* and the coefficients λ_i $(i = 0, 1, 2)$ can be regarded as *phase fractions* (see also [EP, MTT1, Sm, ST]). In the light of the above characterization, a natural question is whether all the coefficients λ_i play a role in (2.28) or whether under suitable assumptions on the initial datum u_0 we can arrange that $v = \phi(u)$ almost everywhere in Q. Observe that in this last case the couple (u, v) would be a weak solution of problem (2.1). In this connection, using an alternative proof of the results given in [Pl1], in the following theorem we show that under the more restrictive assumption (H_3) the characterization (2.20) carries over to the limiting functions u, v.

THEOREM 2.10 Let $u_0 \in L^\infty(\Omega)$ satisfy assumption (H_3) and let (u, v) be the limiting couple given by Theorem 2.9, corresponding to u_0. Then:

(i) the following characterization holds:

$$u = \begin{cases} s_1(v) & \text{in } (\omega_1, 0) \times (0, \infty), \\ s_2(v) & \text{in } (0, \omega_2) \times (0, \infty); \end{cases} \tag{2.29}$$

(ii) there exists a subsequence $\{\varepsilon_j\} \subseteq \{\varepsilon_k\}$ such that

$$\phi(u^{\varepsilon_j}), v^{\varepsilon_j} \to v, \tag{2.30}$$

$$u^{\varepsilon_j} \to u \tag{2.31}$$

almost everywhere in Q;

(iii) for any $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$ let G be the function defined by (2.7), corresponding to g; then the entropy inequalities (2.8) hold.

In other words, if assumption (H_3) holds, Theorems 2.9–2.10 give a triple $(u, v, 0)$ that is a *natural candidate* for a two-phase solution to problem (2.1) with initial datum u_0. Such a *global existence* result is the content of the following corollary.

COROLLARY 2.11 Let $u_0 \in L^\infty(\Omega)$ and let assumption (H_3) hold. Then there exists a two-phase solution $(u, v, 0)$ to problem (2.1).

REMARK 2.5 (i) The *global existence* result of Corollary 2.11 is analogous to the one obtained—with very different methods—in [MTT2] for the Cauchy problem associated to equation (1.1).

(ii) As already remarked, when $u_0 \in L^\infty(\Omega)$ is an arbitrary initial datum to problem (2.1) subject to the only assumption (H_2), passing to the limit as $\varepsilon \to 0$ in the regularized problems (2.15) need not give a two-phase solution to problem (2.1) ([EP, Pl1, ST]). In other words, *global existence* of two-phase solutions to both the Neumann initial-boundary value problem (2.1) and the Cauchy problem for equation (1.1) is proven to hold under assumption (H_3), but if we consider arbitrary initial data u_0 subject to the less restrictive assumption (H_2), the situation is more complicated and global existence actually remains an open problem. However, under the weaker condition (H_2), in [MTT2] *local existence* of two-phase solutions to the Cauchy problem associated to equation (1.1) is established for piecewise response functions ϕ.

2.4 *Long-time behaviour*

In what follows, *assuming global existence*, we investigate the asymptotic behaviour in time of two-phase solutions to problem (2.1) for any initial datum $u_0 \in L^\infty(\Omega)$ satisfying assumption (H_2). The techniques and the results we obtain are quite similar to those proven in [ST] where the long-time behaviour of general weak entropy measure-valued solutions is studied. However, in this case some specific novel features arise, in particular the characterization of the asymptotic behaviour of the interface $\xi(t)$ (see Theorem 2.16 below).

STEP 1: *A priori estimates*. Let us begin with some basic properties of two-phase solutions to problem (2.1). For any initial datum $u_0 \in L^\infty(\Omega)$ set

$$M_{u_0} := \frac{1}{|\Omega|} \int_\Omega u_0(x)\,\mathrm{d}x. \tag{2.32}$$

By the homogeneous Neumann boundary conditions in (2.1), we deduce the following conservation law.

PROPOSITION 2.12 Let $u_0 \in L^\infty(\Omega)$ and let (u, v, ξ) be a two-phase solution of problem (2.1) with initial datum u_0. Then for any $t \geqslant 0$,

$$\frac{1}{|\Omega|} \int_\Omega u(x,t)\,\mathrm{d}x = M_{u_0}. \tag{2.33}$$

Moreover, by the pointwise entropy inequalities (2.12) we obtain the following two results, whose role will be crucial below.

PROPOSITION 2.13 Let (u, v, ξ) be a two-phase solution of problem (2.1). For any $g \in C^1(\mathbb{R})$, let G be the function defined by (2.7). Then for any nondecreasing g the following limit exists:

$$L_g := \lim_{t\to\infty} \int_\Omega G(u)(x,t)\,\mathrm{d}x \qquad (L_g \in \mathbb{R}). \tag{2.34}$$

PROPOSITION 2.14 Let (u, v, ξ) be a two-phase solution of problem (2.1). Then there exists $C > 0$ such that

$$\int_0^\infty \int_\Omega v_x^2(x, t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{2.35}$$

STEP 2: *Convergence results.* Fix any two-phase solution (u, v, ξ) of problem (2.1). Here we try to establish existence (or nonexistence), in some suitable topology, of the limit of the families $v(\cdot, t)$, $\xi(t)$, $u(\cdot, t)$ as t diverges to infinity. To this end, let us define *good sequences* to be the diverging sequences $\{t_n\} \subseteq (0, \infty)$ such that

$$\sup_{n\in\mathbb{N}} \int_\Omega v_x^2(x, t_n)\,\mathrm{d}x < \infty, \tag{2.36}$$

and *bad sequences* to be the diverging sequences $\{t_n\} \subseteq (0, \infty)$ such that

$$\sup_{n\in\mathbb{N}} \int_\Omega v_x^2(x, t_n)\,\mathrm{d}x = \infty. \tag{2.37}$$

The following theorem describes the long-time behaviour of the function $v(\cdot, t)$ along any diverging sequence $\{t_n\}$.

THEOREM 2.15 Let (u, v, ξ) be a two-phase solution of problem (2.1) with initial datum u_0 and let M_{u_0} be defined by (2.32). Then there exists a constant $v^* \in \mathbb{R}$ (uniquely determined by the solution (u, v, ξ) itself) such that:

(i) for any diverging *good* sequence $\{t_n\}$ satisfying (2.36) we have

$$v(\cdot, t_n) \to v^* \quad \text{in } C(\overline{\Omega}); \tag{2.38}$$

(ii) for any diverging bad sequence $\{t_n\}$ satisfying (2.37) we have

$$v(\cdot, t_n) \to v^* \quad \text{in measure}; \tag{2.39}$$

(iii) $A \leqslant v^* \leqslant B$ if and only if $a \leqslant M_{u_0} \leqslant d$; finally, if $M_{u_0} < a$ (respectively, $M_{u_0} > d$) then $v^* = \phi(M_{u_0})$ and for any sufficiently small $\varepsilon > 0$ there exists $T > 0$ such that $v(\cdot, t) < A - \varepsilon$ (respectively, $v(\cdot, t) > B + \varepsilon$) in Ω for any $t \geqslant T$.

The above theorem gives a characterization of the asymptotic behaviour in time of the function $v(\cdot, t)$. The next step is the study of the interface $\xi(t)$ as t diverges. This is the content of the following theorem.

THEOREM 2.16 Let (u, v, ξ) be a two-phase solution of problem (2.1) with initial datum u_0, let M_{u_0} be defined by (2.32) and let v^* be the constant given by Theorem 2.15. Then the limit

$$\lim_{t\to\infty} \xi(t) =: \xi^* \tag{2.40}$$

exists. Moreover:

(i) if $A < v^* < B$ there exists $T > 0$ such that $\xi(t) = \xi^*$ for any $t \geqslant T$;
(ii) if $v^* < A$ (respectively, $v^* > B$) then $\xi^* = \omega_2$ (respectively, $\xi^* = \omega_1$) and there exists $T > 0$ such that $\xi(t) = \omega_2$ (respectively, $\xi(t) = \omega_1$) for any $t \geqslant T$.

REMARK 2.6 Some remarks concerning Theorems 2.15–2.16 are in order:

(i) Theorem 2.16 shows that if $v^* \neq A, B$ the interface $\xi(t)$ stabilizes to the value ξ^* in finite time.
(ii) In the light of Theorems 2.15–2.16, when we consider initial data u_0 of problem (2.1) with mass $M_{u_0} < a$ (respectively, $M_{u_0} > d$), there exists $T > 0$ such that for any $t \geqslant T$ we have $u(\cdot, t) = s_1(v(\cdot, t))$ in Ω (respectively, $u(\cdot, t) = s_2(v(\cdot, t))$ in Ω). Here (u, v, ξ) is any two-phase solution of problem (2.1) with initial datum u_0.

Finally, let us proceed to give a characterization of the long-time behaviour of the function $u(\cdot, t)$. Since by Definition 2.1(ii) for any $t \geqslant 0$ we have

$$u(\cdot, t) = \chi_{(\omega_1, \xi(t))} s_1(v(\cdot, t)) + \chi_{(\xi(t), \omega_2)} s_2(v(\cdot, t)) \quad \text{in } \Omega,$$

we have to take into account the asymptotic behaviour of the interface $\xi(t)$ (here χ_E denotes the characteristic function of any set $E \subseteq \Omega$). Combining Theorems 2.15–2.16 we will show that, in the limit as $t \to \infty$, $u(\cdot, t)$ approaches the function $u^* \in L^\infty(\Omega)$, where

$$u^* = \begin{cases} \chi_{(\omega_1, \xi^*)} s_1(v^*) + \chi_{(\xi^*, \omega_2)} s_2(v^*) & \text{if } a \leqslant M_{u_0} \leqslant d, \\ M_{u_0} & \text{if } M_{u_0} < a \text{ or } M_{u_0} > d. \end{cases} \tag{2.41}$$

This is the content of the following theorem.

THEOREM 2.17 Let (u, v, ξ) be a two-phase solution of problem (2.1) with initial datum u_0. Let M_{u_0} be defined by (2.32) and let u^* be the function defined by (2.41). Then:

(i) for any diverging *good* sequence $\{t_n\}$ satisfying (2.36) we have

$$u(x, t_n) \to u^* \quad \text{for any } x \in \overline{\Omega} \setminus \{\xi^*\} \tag{2.42}$$

if $a \leqslant M_{u_0} \leqslant d$; otherwise,

$$u(\cdot, t_n) \to u^* \equiv M_{u_0} \quad \text{in } C(\overline{\Omega}) \tag{2.43}$$

if either $M_{u_0} < a$ or $M_{u_0} > d$;
(ii) for any diverging *bad* sequence $\{t_n\}$ satisfying (2.37) we have

$$u(\cdot, t_n) \to u^* \quad \text{in measure.} \tag{2.44}$$

3. Proofs of the results in Section 2.1

Proof of Proposition 2.1. Consider any $t_1 < t_2$ and for any $n \in \mathbb{N}$ set

$$h_n(t) = \begin{cases} n(t - t_1 + 1/n) & \text{if } t \in [t_1 - 1/n, t_1], \\ 1 & \text{if } t \in (t_1, t_2), \\ -n(t - t_2 - 1/n) & \text{if } t \in [t_2, t_2 + 1/n]. \end{cases}$$

Fix any $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geqslant 0$ and choose

$$\psi_n(x, t) := h_n(t)\varphi(x)$$

as test function in the entropy inequalities (2.8). We obtain

$$n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_\Omega G(u(x,t))\varphi(x)\ \mathrm{d}x - n\int_{t_2}^{t_2+1/n} \mathrm{d}t \int_\Omega G(u(x,t))\varphi(x)\ \mathrm{d}x$$
$$\geqslant \int_{t_1-1/n}^{t_2+1/n} \int_\Omega h_n(t)[g(v)v_x\varphi_x + \varphi g'(v)v_x^2](x,t)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.1)$$

for any $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$. Let us prove that

$$n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_\Omega G(u(x,t))\varphi(x)\,\mathrm{d}x \to \int_\Omega G(u(x,t_1))\varphi(x)\,\mathrm{d}x \qquad (3.2)$$

as $n \to \infty$. Fix any $\varepsilon > 0$ small enough and write the left-hand side of (3.2) in the following way:

$$\begin{aligned} n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_\Omega G(u(x,t))\varphi(x)\,\mathrm{d}x &= n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_{\omega_1}^{\xi(t_1)-\varepsilon} G(u(x,t))\varphi(x)\,\mathrm{d}x \\ &\quad + n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_{\xi(t_1)-\varepsilon}^{\xi(t_1)+\varepsilon} G(u(x,t))\varphi(x)\,\mathrm{d}x \\ &\quad + n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_{\xi(t_1)+\varepsilon}^{\omega_2} G(u(x,t))\varphi(x)\,\mathrm{d}x. \end{aligned} \qquad (3.3)$$

Since $\xi \in \mathrm{Lip}[0,\infty)$, for every ε there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n \geqslant n_\varepsilon$ we have $|\xi(t_1) - \xi(t)| \leqslant \varepsilon$ for all $t \in (t_1 - 1/n, t_1)$. Therefore

$$\{(x,t) \in Q \mid \omega_1 < x < \xi(t_1) - \varepsilon,\ t \in (t_1 - 1/n, t_1)\} \subseteq V_1,$$
$$\{(x,t) \in Q \mid \xi(t_1) + \varepsilon < x < \omega_2,\ t \in (t_1 - 1/n, t_1)\} \subseteq V_2,$$

for any $n \geqslant n_\varepsilon$. Thus, by the continuity of u in V_1 and V_2 (see Remark 2.1) we obtain

$$n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_{\omega_1}^{\xi(t_1)-\varepsilon} G(u(x,t))\varphi(x)\,\mathrm{d}x \to \int_{\omega_1}^{\xi(t_1)-\varepsilon} G(u(x,t_1))\varphi(x)\,\mathrm{d}x, \qquad (3.4)$$
$$n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_{\xi(t_1)+\varepsilon}^{\omega_2} G(u(x,t))\varphi(x)\,\mathrm{d}x \to \int_{\xi(t_1)+\varepsilon}^{\omega_2} G(u(x,t_1))\varphi(x)\,\mathrm{d}x \qquad (3.5)$$

as $n \to \infty$. Moreover, since $u \in L^\infty(Q)$,

$$\left| n\int_{t_1-1/n}^{t_1} \mathrm{d}t \int_{\xi(t_1)-\varepsilon}^{\xi(t_1)+\varepsilon} G(u(x,t))\varphi(x)\,\mathrm{d}x \right| \leqslant 2\varepsilon \|G(u)\|_{L^\infty(Q)} \|\varphi\|_{L^\infty(\Omega)} \leqslant \varepsilon C_{g,\varphi}. \qquad (3.6)$$

In view of the arbitrariness of $\varepsilon > 0$, (3.4)–(3.6) imply (3.2). Similarly, we can prove that

$$n\int_{t_2}^{t_2+1/n} \mathrm{d}t \int_\Omega G(u(x,t))\varphi(x)\,\mathrm{d}x \to \int_\Omega G(u(x,t_2))\varphi(x)\,\mathrm{d}x \qquad (3.7)$$

as $n \to \infty$. In view of (3.2), (3.7), taking the limit as $n \to \infty$ in inequality (3.1) gives (2.12) and concludes the proof. □

4. Proofs of the results in Section 2.2

Proof of Proposition 2.2. Let us begin by proving (2.13). Set

$$g_{AB}(s) := \begin{cases} (s-A)^3 & \text{if } s < A, \\ 0 & \text{if } s \in [A, B], \\ (s-B)^3 & \text{if } s > B. \end{cases} \tag{4.1}$$

Observe that $g_{AB} \in C^1(\mathbb{R})$, $g'_{AB} \geqslant 0$, $g_{AB} < 0$ in $(-\infty, A)$, $g_{AB} \equiv 0$ in $[A, B]$ and $g_{AB} > 0$ in (B, ∞). Let G_{AB} be the function defined by (2.7) for $g \equiv g_{AB}$ and $k = 0$. Choosing $g \equiv g_{AB}$, $t_1 = 0$ and $\varphi \equiv 1$ in the pointwise entropy inequalities (2.12) gives

$$\int_\Omega G_{AB}(u(x,t))\,\mathrm{d}x \leqslant \int_\Omega G_{AB}(u_0(x))\,\mathrm{d}x \tag{4.2}$$

for any $t > 0$. Since in view of assumption (H_3) we have $a \leqslant u_0 \leqslant d$, hence $G_{AB}(u_0) \equiv 0$, inequality (4.2) reads

$$\int_\Omega G_{AB}(u(x,t))\,\mathrm{d}x \leqslant 0. \tag{4.3}$$

On the other hand, we have

$$\begin{cases} G_{AB}(\lambda) > 0 & \text{if either } \lambda < a \text{ or } \lambda > d, \\ G_{AB}(\lambda) = 0 & \text{if } \lambda \in [a, d]. \end{cases} \tag{4.4}$$

Therefore, since $G_{AB} \geqslant 0$ on $\mathbb{R}$, inequality (4.3) implies that $G_{AB}(u(\cdot,t)) = 0$ almost everywhere in Ω; thus $G_{AB}(u) = 0$ almost everywhere in Q by the arbitrariness of $t > 0$. This implies $a \leqslant u \leqslant d$ almost everywhere in Q (see (4.4) again) and concludes the proof of (2.13).

Let us prove (2.14). To do so, we will show that:

(i) $u(x,t) \leqslant b$ for any $x \in (\omega_1, 0)$, $t > 0$;
(ii) $u(x,t) \geqslant c$ for any $x \in (0, \omega_2)$, $t > 0$.

Let us address (i), the proof of (ii) following by similar arguments. Fix any $t > 0$ and observe that for any $\varphi \in C_c^2((\omega_1, 0))$ with $\varphi \geqslant 0$ the pointwise entropy inequalities (2.12) give

$$\int_\Omega G(u(x,t))\varphi(x)\,\mathrm{d}x - \int_\Omega G(u_0(x))\varphi(x)\,\mathrm{d}x \leqslant -\int_0^t \int_\Omega g(v(x,s))v_x(x,s)\varphi_x(x)\,\mathrm{d}x\,\mathrm{d}s \tag{4.5}$$

for any $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$ and G defined by (2.7). By standard regularization arguments the assumption $g \in C^1(\mathbb{R})$ can be dropped so that inequalities (4.5) hold for any *nondecreasing* g. Following [Pl1], for any $\rho \geqslant 0$ we set

$$g_\rho(s) := \begin{cases} \rho^{-1/2} & \text{if } s \geqslant B - \rho, \\ 0 & \text{if } s < B - \rho. \end{cases} \tag{4.6}$$

Let G_ρ be the function defined by (2.7) for $g \equiv g_\rho$ and $k = 0$. Choosing $g \equiv g_\rho$ in inequalities

(4.5) gives

$$\begin{aligned}\int_\Omega G_\rho(u(x,t))\varphi(x)\,\mathrm{d}x - \int_\Omega G_\rho(u_0(x))\varphi(x)\,\mathrm{d}x \\ \leqslant -\int_0^t \int_\Omega g_\rho(v(x,s))v_x(x,s)\varphi_x(x)\,\mathrm{d}x\,\mathrm{d}s \\ = \rho^{-1/2}\int_0^t \int_{\{x\in\Omega \mid v(x,s)>B-\rho\}} (v(x,s)-B+\rho)\varphi_{xx}(x)\,\mathrm{d}x\,\mathrm{d}s \qquad (4.7)\end{aligned}$$

for any φ as above. Let us study the different terms of the previous inequality in the limit $\rho \to 0$. Since in view of (2.13) we have $v \leqslant B$ in Q, taking the limit on the right-hand side of (4.7) gives

$$\begin{aligned}\lim_{\rho\to 0}\left|\rho^{-1/2}\int_0^t \int_{\{x\in\Omega \mid v(x,s)>B-\rho\}} (v(x,s)-B+\rho)\varphi_{xx}(x)\,\mathrm{d}x\,\mathrm{d}s\right| \\ \leqslant \lim_{\rho\to 0}\rho^{1/2}\int_0^t\int_\Omega |\varphi_{xx}(x)|\,\mathrm{d}x\,\mathrm{d}s = 0. \qquad (4.8)\end{aligned}$$

Next, consider the second term on the left-hand side of (4.7). Observe that for almost every $x \in (\omega_1, 0)$,

$$|G_\rho(u_0(x))| \leqslant \left|\int_{s_0(B-\rho)}^{s_1(B-\rho)} |g_\rho(\phi(s))|\,\mathrm{d}s\right| \leqslant \left|\frac{s_1(B-\rho)-s_0(B-\rho)}{\sqrt{\rho}}\right| \qquad (4.9)$$

for any $\rho > 0$. Moreover, for any $x \in (\omega_1, 0)$ such that $u_0(x) < b$ there exists $\rho^* > 0$ (depending on x) such that

$$G_\rho(u_0(x)) = \int_{s_0(B-\rho)}^{s_1(B-\rho)} g_\rho(\phi(s))\,\mathrm{d}s = \frac{s_1(B-\rho)-s_0(B-\rho)}{\sqrt{\rho}}$$

for any $\rho < \rho^*$. On the other hand, for any $x \in (\omega_1, 0)$ such that $u_0(x) = b$, we have

$$G_\rho(u_0(x)) = \int_{s_0(B-\rho)}^{b} g_\rho(\phi(s))\,\mathrm{d}s = \frac{b-s_0(B-\rho)}{\sqrt{\rho}}.$$

In view of assumption (H_1)—in particular $\phi''(b) \neq 0$—we have:

- $\lim_{\rho\to 0} G_\rho(u_0(x)) = \begin{cases} -2\sqrt{2/|\phi''(b)|} & \text{if } u_0(x) < b \\ -\sqrt{2/|\phi''(b)|} & \text{if } u_0(x) = b \end{cases}$ for a.e. $x \in (\omega_1, 0)$;
- in view of (4.9) and (H_1), there exists $\overline{\rho} > 0$ such that $|G_\rho(u_0(x))| \leqslant 4\sqrt{2/|\phi''(b)|}$ for almost every $x \in (\omega_1, 0)$ and for any $\rho < \overline{\rho}$.

Hence, by the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}\lim_{\rho\to 0}\int_\Omega G_\rho(u_0(x))\varphi(x)\,\mathrm{d}x = -2\sqrt{\frac{2}{|\phi''(b)|}}\int_{\{x\in(\omega_1,0) \mid u_0(x)<b\}} \varphi(x)\,\mathrm{d}x \\ -\sqrt{\frac{2}{|\phi''(b)|}}\int_{\{x\in(\omega_1,0) \mid u_0(x)=b\}} \varphi(x)\,\mathrm{d}x. \qquad (4.10)\end{aligned}$$

Finally, let us study the first term on the left-hand side of (4.7). We decompose the function $G_\rho(u(x,t))\varphi(x)$ in the following way:

$$G_\rho(u(\cdot,t))\varphi(\cdot) = G_\rho(u(\cdot,t))\chi_{\{x\in\Omega\,|\,u(x,t)<b\}}\varphi(\cdot) + G_\rho(u(\cdot,t))\chi_{\{x\in\Omega\,|\,u(x,t)=b\}}\varphi(\cdot) + G_\rho(u(\cdot,t))\chi_{\{x\in\Omega\,|\,c\leqslant u(x,t)\leqslant d\}}\varphi(\cdot) \tag{4.11}$$

(recall that in view of (2.13) we have $u(\cdot,t) \leqslant d = s_2(B)$ a.e. in Ω). Arguing as above, passing to the limit as $\rho \to 0$ in the first two terms of the right-hand side of (4.11) gives

$$\lim_{\rho\to 0}\int_\Omega G_\rho(u(x,t))\chi_{\{x\in\Omega\,|\,u(x,t)<b\}}\varphi(x)\,\mathrm{d}x = -2\sqrt{\frac{2}{|\phi''(b)|}}\int_\Omega \chi_{\{x\in\Omega\,|\,u(x,t)<b\}}\varphi(x)\,\mathrm{d}x \tag{4.12}$$

and

$$\lim_{\rho\to 0}\int_\Omega G_\rho(u(x,t))\chi_{\{x\in\Omega\,|\,u(x,t)=b\}}\varphi(x)\,\mathrm{d}x = \lim_{\rho\to 0}\int_\Omega \frac{b - s_0(B-\rho)}{\rho^{1/2}}\chi_{\{x\in\Omega\,|\,u(x,t)=b\}}\varphi(x)\,\mathrm{d}x = -\sqrt{\frac{2}{|\phi''(b)|}}\int_\Omega \chi_{\{x\in\Omega\,|\,u(x,t)=b\}}\varphi(x)\,\mathrm{d}x. \tag{4.13}$$

Concerning the third term on the right-hand side of (4.11) we have

$$\int_\Omega G_\rho(u(x,t))\chi_{\{x\in\Omega\,|\,c\leqslant u(x,t)\leqslant d\}}\varphi(x)\,\mathrm{d}x := I_1^\rho + I_2^\rho, \tag{4.14}$$

where

$$\begin{aligned} I_1^\rho &:= \int_\Omega G_\rho(u(x,t))\chi_{\{x\in\Omega\,|\,c\leqslant u(x,t)\leqslant s_2(B-\rho)\}}\varphi(x)\,\mathrm{d}x \equiv 0,\\ I_2^\rho &:= \int_\Omega G_\rho(u(x,t))\chi_{\{x\in\Omega\,|\,s_2(B-\rho)<u(x,t)\leqslant s_2(B)\}}\varphi(x)\,\mathrm{d}x\\ &= \int_\Omega \frac{s_2(\phi(u(x,t))) - s_2(B-\rho)}{\rho^{1/2}}\chi_{\{x\in\Omega\,|\,s_2(B-\rho)<u(x,t)\leqslant s_2(B)\}}\varphi(x)\,\mathrm{d}x. \end{aligned} \tag{4.15}$$

Taking the limit as $\rho \to 0$ in I_2^ρ gives

$$\lim_{\rho\to 0}|I_2^\rho| \leqslant \lim_{\rho\to 0}\|\varphi\|_{L^1(\omega_1,0)}\frac{s_2(B) - s_2(B-\rho)}{\rho}\rho^{1/2} = 0 \tag{4.16}$$

(here the assumption $s_2'(B) = 1/\phi'(d) < \infty$ has been used). Therefore, in the light of (4.15) and (4.16), taking the limit as $\rho \to 0$ in (4.14) we obtain

$$\lim_{\rho\to 0}\int_\Omega G_\rho(u(x,t))\chi_{\{x\in\Omega\,|\,c\leqslant u(x,t)\leqslant d\}}\varphi(x)\,\mathrm{d}x = 0. \tag{4.17}$$

From (4.12), (4.13) and (4.17) we deduce

$$\lim_{\rho\to 0}\int_\Omega G_\rho(u(x,t))\varphi(x)\,\mathrm{d}x = -2\sqrt{\frac{2}{|\phi''(b)|}}\int_{\{x\in(\omega_1,0)\,|\,u(x,t)<b\}}\varphi(x)\,\mathrm{d}x - \sqrt{\frac{2}{|\phi''(b)|}}\int_{\{x\in(\omega_1,0)\,|\,u(x,t)=b\}}\varphi(x)\,\mathrm{d}x. \tag{4.18}$$

Therefore, in view of (4.8), (4.10) and (4.18), taking the limit as $\rho \to 0$ in inequalities (4.7) gives

$$\int_{\{x\in(\omega_1,0)\,|\,u_0(x)<b\}} \varphi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\{x\in(\omega_1,0)\,|\,u_0(x)=b\}} \varphi(x)\,\mathrm{d}x \leqslant \int_{\{x\in(\omega_1,0)\,|\,u(x,t)<b\}} \varphi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\{x\in(\omega_1,0)\,|\,u(x,t)=b\}} \varphi(x)\,\mathrm{d}x \tag{4.19}$$

for any $\varphi \in C_c^2((\omega_1, 0))$ with $\varphi \geqslant 0$, and for any $t > 0$. Fix any $t > 0$ and, arguing towards a contradiction, assume that the set $\{x \in (\omega_1, 0) \mid u(x,t) > b\}$ has a nonzero Lebesgue measure. Let $K \subset \{x \in (\omega_1, 0) \mid u(x,t) > b\}$ be any compact set with a strictly positive Lebesgue measure, $|K| > 0$, and let $\{\varphi_n\} \subseteq C_c^\infty(\mathbb{R})$ be any sequence of smooth functions such that:

- $0 \leqslant \varphi_n(x) \leqslant 1$ for any $x \in \mathbb{R},\ n \in \mathbb{N}$;
- $\varphi_n(x) = 1$ for any $x \in K,\ n \in \mathbb{N}$;
- $\operatorname{supp}\varphi_n \subset (\omega_1, 0)$ for any $n \in \mathbb{N}$;
- $\varphi_n(x) \to \chi_K(x)$ as $n \to \infty$ for any $x \in \mathbb{R}$.

Write inequality (4.19) for $\varphi = \varphi_n$ and observe that the right-hand side converges to zero as $n \to \infty$. Therefore, passing to the limit as $n \to \infty$ gives

$$\begin{aligned} 0 < \frac{1}{2}|K| &= \lim_{n\to\infty}\frac{1}{2}\int_{\omega_1}^{0}\varphi_n(x)\,\mathrm{d}x = \lim_{n\to\infty}\frac{1}{2}\int_{\{x\in(\omega_1,0)\,|\,u_0(x)\leqslant b\}}\varphi_n(x)\,\mathrm{d}x \\ &\leqslant \lim_{n\to\infty}\left\{\int_{\{x\in(\omega_1,0)\,|\,u_0(x)<b\}}\varphi_n(x)\,\mathrm{d}x + \frac{1}{2}\int_{\{x\in(\omega_1,0)\,|\,u_0(x)=b\}}\varphi_n(x)\,\mathrm{d}x\right\} \\ &\leqslant \lim_{n\to\infty}\left\{\int_{\{x\in(\omega_1,0)\,|\,u(x,t)<b\}}\varphi_n(x)\,\mathrm{d}x + \frac{1}{2}\int_{\{x\in(\omega_1,0)\,|\,u(x,t)=b\}}\varphi_n(x)\,\mathrm{d}x\right\} = 0 \end{aligned}$$

(here the assumption $u_0(x) \leqslant b$ in $(\omega_1, 0)$ has been used). This contradiction proves (i). □

Proof of Proposition 2.3. Let us address only claim (i), (ii) following by similar arguments. Fix any $x_0 \in (\omega_1, 0)$ such that $u_0(x_0) < b$. Then by the continuity of u_0 in $(\omega_1, 0)$—which is ensured by the assumption $\phi(u_0) \in C(\overline{\Omega})$—there exists $r > 0$ such that $u_0(x) \leqslant b_r < b$ in $I_r := (x_0 - r, x_0 + r)$. To prove that $u(x_0, t) < b$ for any $t > 0$, it suffices to observe that in the strip $I_r \times (0, \infty)$ the function u is a weak solution of the porous medium type equation

$$u_t = [\phi(u)]_{xx},$$

with initial datum u_0 subject to the following conditions:

$$u_0 \in C(I_r), \qquad u_0(\cdot) \leqslant b_r < b \quad \text{in } I_r \tag{4.20}$$

(recall that $\phi'(b) = 0$, $\phi''(b) \neq 0$, and $\phi'(s) > 0$ for any $s < b$). Moreover, in view of Proposition 2.2 (see in particular (2.14)), u satisfies the boundary conditions

$$u \leqslant b \quad \text{on } \partial I_r \times (0, \infty). \tag{4.21}$$

In the light of (4.20)–(4.21), by the comparison principle ([DK]) the claim follows (see [dPV]; see also Section 7 in [Va]). □

The proof of Theorem 2.5 needs the following preliminary result.

LEMMA 4.1 Let u_0 satisfy assumption (A_1) and let $(u, v, 0)$ be any two-phase solution of problem (2.1) with initial datum u_0 such that $v_x(\cdot, t) \in BV(\Omega)$ for almost every $t > 0$. Then the limit

$$\lim_{\eta \to 0^\pm} v_x(\eta, t) =: v_x(0^\pm, t) \tag{4.22}$$

exists for almost every $t \in (0, \infty)$. Moreover, for almost every $t \in (0, \infty)$,

$$v_x(0^-, t) = v_x(0^+, t). \tag{4.23}$$

Proof. Let u_0 satisfy assumption (A_1) and let $(u, v, 0)$ be any two-phase solution of problem (2.1) with initial datum u_0. The assumption $v_x(\cdot, t) \in BV(\Omega)$ for almost every $t > 0$ implies the existence of the limits (4.22). Moreover, in view of assumption (A_1), we have $u \in C^{2,1}(V_i)$, where $V_1 = (\omega_1, 0) \times (0, \infty)$ and $V_2 = (0, \omega_2) \times (0, \infty)$ (Corollary 2.4). Therefore, arguing as in [EP] gives the Rankine–Hugoniot conditions (2.9) almost everywhere in t (see Remark 2.2). Thus, equality (4.23) follows, since under assumption (A_1) we have $\xi'(t) \equiv 0$. □

Proof of Theorem 2.5. The proof is almost the same as in [MTT2]. We give it for the convenience of the reader.

Let u_0 satisfy assumption (A_1) and let $(u_1, v_1, 0)$, $(u_2, v_2, 0)$ be two two-phase solutions of problem (2.1) with initial datum u_0 such that $v_{ix}(\cdot, t) \in BV(\Omega)$ for almost every $t > 0$ $(i = 1, 2)$. For any $n \in \mathbb{N}$ set

$$\Omega_1^n := (\omega_1, -1/n), \qquad \Omega_2^n := (1/n, \omega_2), \qquad V_1^n := \Omega_n^1 \times (0, \infty), \qquad V_2^n := \Omega_2^n \times (0, \infty).$$

Since $u_i \in C(V_1)$ and $u_i < b$ in V_1 (Proposition 2.3), we have $u_i \in C^{2,1}(V_1)$ (Corollary 2.4) and for any $n \in \mathbb{N}$ there exist two constants $b_i^n < b$ such that $u_i \leqslant b_i^n$ in V_1^n $(i = 1, 2)$. Analogously, since $u_i \in C(V_2)$ and $u_i > c$ in V_2 (Proposition 2.3), we have $u_i \in C^{2,1}(V_2)$ (Corollary 2.4) and for any $n \in \mathbb{N}$ there exist two constants $c_i^n > c$ such that $u_i \geqslant c_i^n$ in V_2^n $(i = 1, 2)$. This implies that for any $n \in \mathbb{N}$ we can find a constant $C_n > 0$ such that

$$\|u_{ix}(\cdot, t)\|_{L^\infty(\Omega_1^n)} \leqslant C_n \|v_{ix}(\cdot, t)\|_{L^\infty(\Omega_1^n)} \leqslant C_n \|v_{ix}(\cdot, t)\|_{BV(\Omega)}, \tag{4.24}$$

$$\|u_{ix}(\cdot, t)\|_{L^\infty(\Omega_2^n)} \leqslant C_n \|v_{ix}(\cdot, t)\|_{L^\infty(\Omega_2^n)} \leqslant C_n \|v_{ix}(\cdot, t)\|_{BV(\Omega)} \tag{4.25}$$

$(i = 1, 2)$. Next, for any $k \in \mathbb{N}$ let $\{\eta_k\} \subseteq C^2(\mathbb{R})$ be a family of functions such that:

- η_k converges to the absolute value $|\cdot|$ in $C(\mathbb{R})$ as $k \to \infty$;
- $\eta_k'(s) \to \mathrm{sgn}(s)$ as $k \to \infty$ for any $s \neq 0$, and $|\eta_k'(s)| \leqslant 1$ for any $s \in \mathbb{R}$ and $k \in \mathbb{N}$;
- there exists $C > 0$ such that $0 \leqslant \eta''(s) \leqslant Ck$ for any $s \in \mathbb{R}$, and $\eta''(s) = 0$ for any $s \notin (-1/k, 1/k)$.

Since $u_{it} = v_{ixx}$ in V_j $(i, j = 1, 2)$, for any fixed $t > 0$ we obtain

$$\begin{aligned}\int_{\Omega_j^n} [\eta_k(u_1 - u_2)]_t(x, t)\,\mathrm{d}x &= \int_{\Omega_j^n} [\eta_k'(u_1 - u_2)(v_1 - v_2)_{xx}](x, t)\,\mathrm{d}x \\ &= \int_{\Omega_j^n} \{[\eta_k'(u_1 - u_2)(v_1 - v_2)_x]_x\}(x, t)\,\mathrm{d}x \\ &\quad - \int_{\Omega_j^n} \{\eta_k''(u_1 - u_2)(u_{1x} - u_{2x})(v_{1x} - v_{2x})\}(x, t)\,\mathrm{d}x \end{aligned} \tag{4.26}$$

and also

$$
\begin{aligned}
-\int_{\Omega_j^n} & [\eta_k''(u_1-u_2)(u_{1x}-u_{2x})(v_{1x}-v_{2x})](x,t)\,\mathrm{d}x \\
&= -\int_{\Omega_j^n}[\eta_k''(u_1-u_2)\phi'(u_1)(u_{1x}-u_{2x})^2\,](x,t)\,\mathrm{d}x \\
&\quad -\int_{\Omega_j^n}[\eta_k''(u_1-u_2)(\phi'(u_1)-\phi'(u_2))u_{2x}(u_{1x}-u_{2x})](x,t)\,\mathrm{d}x \\
&\leqslant -\int_{\Omega_j^n}[\eta_k''(u_1-u_2)(\phi'(u_1)-\phi'(u_2))u_{2x}(u_{1x}-u_{2x})](x,t)\,\mathrm{d}x \qquad (4.27)
\end{aligned}
$$

($j = 1, 2$). For almost every $t > 0$, taking the limit as $k \to \infty$ on the right-hand side of the above inequalities gives

$$
\begin{aligned}
\limsup_{k\to\infty} & \left\{-\int_{\Omega_j^n}[\eta_k''(u_1-u_2)(\phi'(u_1)-\phi'(u_2))u_{2x}(u_{1x}-u_{2x})](x,t)\,\mathrm{d}x\right\} \\
&\leqslant \lim_{k\to\infty}\|\phi''\|_{L^\infty(a,d)}Ck\|u_{2x}(\cdot,t)\|_{L^\infty(\Omega_j^n)}\int_{I_k^j(t)}[|u_1-u_2|\,|u_{1x}-u_{2x}|](x,t)\,\mathrm{d}x \\
&\leqslant \lim_{k\to\infty}\|\phi''\|_{L^\infty(a,d)}C\|u_{2x}(\cdot,t)\|_{L^\infty(\Omega_j^n)}\int_{I_k^j(t)}[|u_{1x}-u_{2x}|](x,t)\,\mathrm{d}x = 0, \qquad (4.28)
\end{aligned}
$$

where for almost every $t > 0$ we have set

$$
I_k^j(t) = \{x \in \Omega_j^n \mid |u_1(x,t)-u_2(x,t)| \leqslant 1/k\}
$$

($j = 1, 2$). Concerning the left-hand side of (4.26), for any fixed $n \in \mathbb{N}$ and for almost every $t > 0$ we obtain

$$
\begin{aligned}
|[\eta_k(u_1(x,t)-u_2(x,t))]_t| &= |\eta_k'(u_1(x,t)-u_2(x,t))|\,|u_{1t}(x,t)-u_{2t}(x,t)| \\
&\leqslant |u_{1t}(x,t)-u_{2t}(x,t)| \in L^1(\Omega_j^n)
\end{aligned}
$$

($j = 1, 2$), and

$$
[\eta_k(u_1(x,t)-u_2(x,t))]_t \to [\mathrm{sgn}(u_1(x,t)-u_2(x,t))](u_{1t}(x,t)-u_{2t}(x,t))
$$

as $k \to \infty$ for almost every $x \in \Omega_j^n$ ($j = 1, 2$). Therefore, in view of the Lebesgue Theorem and in view of (4.27), (4.28), in the limit as $k \to \infty$ in (4.26) we obtain

$$
\begin{aligned}
\sum_{j=1,2}\int_{\Omega_j^n}[|u_1(\cdot,t)-u_2(\cdot,t)|]_t(x,t)\,\mathrm{d}x &= \lim_{k\to\infty}\sum_{j=1,2}\int_{\Omega_j^n}[\eta_k(u_1-u_2)]_t(x,t) \\
&\leqslant \lim_{k\to\infty}\sum_{j=1,2}\int_{\Omega_j^n}[\eta_k'(u_1-u_2)(v_{1x}-v_{2x})]_x(x,t)\,\mathrm{d}x \\
&= [\mathrm{sgn}(u_1(-1/n,t)-u_2(-1/n,t))](v_{1x}(-1/n,t)-v_{2x}(-1/n,t)) \\
&\quad - [\mathrm{sgn}(u_1(1/n,t)-u_2(1/n,t))](v_{1x}(1/n,t)-v_{2x}(1/n,t)).
\end{aligned}
$$

Integrating the above inequality between 0 and t_0 and letting $n \to \infty$ gives

$$\int_\Omega |u_1(x,t_0) - u_2(x,t_0)|\,dx \leqslant \int_0^{t_0} [\mathrm{sgn}(u_1(0^-,t) - u_2(0^-,t))](v_{1x}(0^-,t) - v_{2x}(0^-,t))\,dt - \int_0^{t_0} [\mathrm{sgn}(u_1(0^+,t) - u_2(0^+,t))](v_{1x}(0^+,t) - v_{2x}(0^+,t))\,dt. \quad (4.29)$$

Observe that since ϕ is nondecreasing in the interval $(-\infty, b)$ and in (c, ∞), and since $v(\cdot, t) \in C(\Omega)$ for almost every $t > 0$, we have: $u_1(0^-,t) > u_2(0^-,t) \Rightarrow v_1(0,t) > v_2(0,t) \Rightarrow u_1(0^+,t) > u_2(0^+,t)$. In other words, for almost every $t > 0$,

$$\mathrm{sgn}(u_1(0^-,t) - u_2(0^-,t)) = \mathrm{sgn}(u_1(0^+,t) - u_2(0^+,t)). \quad (4.30)$$

In view of (4.23) and (4.30), inequality (4.29) reads

$$\int_\Omega |u_1(x,t_0) - u_2(x,t_0)|\,dx \leqslant 0$$

for any $t_0 > 0$, and this concludes the proof. □

5. Proofs of the results in Section 2.3

5.1 *The approximating problems: proofs*

The proofs of Theorem 2.6 and Proposition 2.7 are almost the same as in [NP], so we omit them (see also [Pl3, Sm, ST]).

Proof of Theorem 2.8. It is formally analogous to the proof of Proposition 2.2, so we only sketch it.

Let $g_{AB} \in C^1(\mathbb{R})$ be the nondecreasing function defined by (4.1) and let G_{AB} be the function defined by (2.7) for $g \equiv g_{AB}$ and $k = 0$. Choosing $g \equiv g_{AB}$, $t_1 = 0$ and $\psi \equiv 1$ in the viscous entropy inequalities (2.18) gives

$$\int_\Omega G_{AB}(u^\varepsilon(x,t))\,dx \leqslant \int_\Omega G_{AB}(u_0(x))\,dx = 0$$

for any $t > 0$ (here assumption (H_3) has been used). Since $G_{AB}(s) > 0$ for $s \in \mathbb{R} \setminus [a,d]$ and $G_{AB}(s) = 0$ for $s \in [a,d]$, estimate (2.19) follows.

Next, let us address claim (i), the proof of (ii) following by similar arguments. Firstly, we observe that (2.19) implies $A \leqslant \phi(u^\varepsilon(x,t)) \leqslant B$ for almost every $(x,t) \in Q$. By standard results on elliptic equations, since for any $t \in (0,\infty)$ the function $v(\cdot,t) \in W^{2,\infty}(\Omega)$ solves problem (2.17), it follows that

$$\operatorname*{ess\,inf}_{x\in\Omega} \phi(u^\varepsilon(x,t)) \leqslant v^\varepsilon(x,t) \leqslant \operatorname*{ess\,sup}_{x\in\Omega} \phi(u^\varepsilon(x,t))$$

for any $x \in \Omega$, that is,

$$A \leqslant v^\varepsilon(x,t) \leqslant B \quad (5.1)$$

for any $(x, t) \in Q$. Next, for any $\rho > 0$ let g_ρ be the nondecreasing function defined by (4.6). Let G_ρ be the function defined by (2.7) for $g \equiv g_\rho$ and $k = 0$. Choosing $g \equiv g_\rho$ in the viscous entropy inequalities (2.18) gives

$$\begin{aligned}\int_\Omega G_\rho(u^\varepsilon(x,t))\varphi(x)\,\mathrm{d}x - \int_\Omega G_\rho(u_0(x))\varphi(x)\,\mathrm{d}x \\ \leqslant -\int_0^t \int_\Omega g_\rho(v^\varepsilon(x,s))v_x^\varepsilon(x,s)\varphi_x(x)\,\mathrm{d}x\,\mathrm{d}s \\ = \rho^{-1/2}\int_0^t \int_{\{x\in\Omega \mid v^\varepsilon(x,s)>B-\rho\}} (v^\varepsilon(x,s) - B + \rho)\varphi_{xx}(x)\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{5.2}$$

for any $\varphi \in C_c^2((\omega_1, 0))$ with $\varphi \geqslant 0$ and for any $t > 0$. Then, arguing as in the proof of Proposition 2.2 (in particular, see (4.8), (4.18)), taking the limit as $\rho \to 0$ in inequalities (5.2) gives

$$\begin{aligned}\int_{\{x\in(\omega_1,0)\mid u_0(x)<b\}} \varphi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\{x\in(\omega_1,0)\mid u_0(x)=b\}} \varphi(x)\,\mathrm{d}x \\ \leqslant \int_{\{x\in(\omega_1,0)\mid u^\varepsilon(x,t)<b\}} \varphi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\{x\in(\omega_1,0)\mid u^\varepsilon(x,t)=b\}} \varphi(x)\,\mathrm{d}x\end{aligned} \tag{5.3}$$

for any $\varphi \in C_c^2((\omega_1, 0))$ with $\varphi \geqslant 0$, and for any $t > 0$. Fix any $t > 0$ and, arguing towards a contradiction, assume that the set $\{x \in (\omega_1, 0) \mid u^\varepsilon(x, t) > b\}$ has a nonzero Lebesgue measure. Fix a compact set $K \subset \{x \in (\omega_1, 0) \mid u^\varepsilon(x, t) > b\}$ with $|K| > 0$, and choose a sequence $\{\varphi_n\} \subseteq C_c^\infty(\mathbb{R})$ such that:

- $0 \leqslant \varphi_n(x) \leqslant 1$ for any $x \in \mathbb{R},\ n \in \mathbb{N}$;
- $\varphi_n(x) = 1$ for any $x \in K,\ n \in \mathbb{N}$;
- $\operatorname{supp}\varphi_n \subset (\omega_1, 0)$ for any $n \in \mathbb{N}$;
- $\varphi_n(x) \to \chi_K(x)$ as $n \to \infty$ for any $x \in \mathbb{R}$.

Since $u_0 \leqslant b$ almost everywhere in $(\omega_1, 0)$, passing to the limit as $n \to \infty$ in inequality (5.3) (for $\varphi = \varphi_n$) gives $0 < \frac{1}{2}|K| \leqslant 0$. This contradiction proves (i). □

5.2 *Vanishing viscosity limit: proofs*

Fix any $u_0 \in L^\infty(\Omega)$ and for any $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) given by Theorem 2.6. In what follows we will study the limiting points of both the families $\{u^\varepsilon\}$, $\{v^\varepsilon\}$ as the regularization parameter ε goes to 0. Let us begin by proving Theorem 2.9.

Proof of Theorem 2.9. (i) In view of estimates (2.21)–(2.23) there exist three subsequences $\{u^{\varepsilon_k}\}$, $\{\phi(u^{\varepsilon_k})\}$, $\{v^{\varepsilon_k}\}$ and three functions $u \in L^\infty(Q)$, $v \in L^\infty(Q)$, $\tilde{v} \in L^\infty(Q)$, with $\tilde{v}_x \in L^2(Q)$, such that

$$u^{\varepsilon_k} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q), \tag{5.4}$$

$$\phi(u^{\varepsilon_k}) \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(Q), \tag{5.5}$$

$$v^{\varepsilon_k} \overset{*}{\rightharpoonup} \tilde{v} \quad \text{in } L^\infty(Q), \tag{5.6}$$

$$v_x^{\varepsilon_k} \rightharpoonup \tilde{v}_x \quad \text{in } L^2(Q). \tag{5.7}$$

Moreover, by (2.16) and estimate (2.23) we have

$$\left|\iint_Q (\tilde{v}-v)\psi \,\mathrm{d}x\,\mathrm{d}t\right| = \lim_{k\to\infty}\left|\iint_Q (\phi(u^{\varepsilon_k})-v^{\varepsilon_k})\psi \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant \lim_{k\to\infty} \varepsilon_k^{1/2}\iint_Q \varepsilon_k^{1/2}|u_t^{\varepsilon_k}|\,|\psi|\,\mathrm{d}x\,\mathrm{d}t = 0$$

for any $\psi \in C_c(Q)$. This implies $v = \tilde{v}$ almost everywhere in Q and concludes the first part of the proof.

(ii) In view of (2.24)–(2.26), passing to the limit as $\varepsilon_k \to 0$ in the weak formulation of problems (2.15)–(2.16) gives equality (2.27). □

Next, let $\{\varepsilon_k\}$ be the vanishing sequence given by Theorem 2.9. Let $\{\eta^{\varepsilon_k}\}$ be the sequence of *Young measures* over $Q \times \mathbb{R}$ associated to the family $\{\phi(u^{\varepsilon_k})\}$. In view of the uniform estimate $\|\phi(u^{\varepsilon_k})\|_{L^\infty(Q)} \leqslant C$ (see (2.21)), for any $T > 0$ the sequence $\{\eta^{\varepsilon_k}\}$ is relatively compact with respect to the *narrow topology* of Young measures over $Q_T = \Omega \times (0, T)$. This is the content of the following proposition (see [GMS, V] for the proof).

PROPOSITION 5.1 For any $\varepsilon > 0$ and $u_0 \in L^\infty(\Omega)$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) with initial datum u_0. Let $\{\eta^\varepsilon\}$ be the family of Young measures associated to the family $\{\phi(u^\varepsilon)\}$ and let $\{\varepsilon_k\}$ be the vanishing sequence given by Theorem 2.9. Then:

(i) there exist a subsequence $\{\varepsilon_h\} \equiv \{\varepsilon_{k_h}\} \subseteq \{\varepsilon_k\}$ and a Young measure η on $Q \times \mathbb{R}$ such that for any $T > 0$,

$$\eta^{\varepsilon_h} \to \eta \quad \text{narrowly in } Q_T \times \mathbb{R},$$

where $Q_T = \Omega \times (0, T)$;

(ii) for any $f \in C(\mathbb{R})$,

$$f(\phi(u^{\varepsilon_h})) \rightharpoonup f^* \quad \text{in } L^\infty(Q), \tag{5.8}$$

where, for almost every $(x, t) \in Q$,

$$f^*(x,t) := \int_{\mathbb{R}} f(\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi). \tag{5.9}$$

Here $\sigma_{(x,t)}$ is the *disintegration* of the Young measure η.

The main result of Plotnikov in [Pl1] is the characterization of the Young measure disintegration $\sigma_{(x,t)}$, which allows one to define weak entropy measure-valued solutions to the original Neumann (or Dirichlet) problem associated to equation (1.1). Analogously, with methods of proof slightly different from those used in [Pl1], we will investigate the structure of the measure $\sigma_{(x,t)}$, proving in this way the existence of a two-phase solution to problem (2.1). In this direction, we begin by the following three lemmata, whose proofs will be postponed until the end of this subsection.

LEMMA 5.2 Let $u_0 \in L^\infty(\Omega)$ and for any $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) with initial datum u_0. For any $g \in C^1(\mathbb{R})$ let G be the function defined by (2.7). Then for any $T > 0$, for any g as above and for any $\varphi \in C^1(\overline{\Omega})$, there exists a constant $C_{g,\varphi}$ (depending on g and φ) such that for any $\varepsilon > 0$,

$$\int_0^T \left|\int_\Omega [G(u^\varepsilon)]_t \varphi \,\mathrm{d}x\right| \mathrm{d}t \leqslant C_{g,\varphi}[\|\sqrt{\varepsilon} u_t^\varepsilon\|_{L^2(Q)}^2 + \sqrt{T}\|v_x^\varepsilon\|_{L^2(Q)} + \|v_x^\varepsilon\|_{L^2(Q)}^2]. \tag{5.10}$$

LEMMA 5.3 Let assumption (H_3) hold, let $u_0 \in L^\infty(\Omega)$ and for any $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) with initial datum u_0. Let $\{\varepsilon_h\}$ and η be respectively the vanishing sequence and the Young measure over $Q \times \mathbb{R}$ given by Proposition 5.1. Finally, let $\sigma_{(x,t)}$ be the disintegration of the Young measure η, defined for almost every $(x,t) \in Q$. Then:

(i) $\operatorname{supp} \sigma_{(x,t)} \subseteq [A, B]$ for almost every $(x,t) \in Q$;
(ii) there exist a subsequence $\{\varepsilon_{h_l}\} \subseteq \{\varepsilon_h\}$ and a set $E^1 \subseteq (0,\infty)$ with $|E^1| = 0$ such that for any $t \in (0,\infty) \setminus E^1$,

$$\lim_{l\to\infty} \int_\Omega G(u^{\varepsilon_{h_l}}(x,t))\varphi(x)\,\mathrm{d}x = \int_{\omega_1}^0 \left(\int_{[A,B]} [G \circ s_1](\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2} \left(\int_{[A,B]} [G \circ s_2](\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x \tag{5.11}$$

for any $\varphi \in C^1(\overline{\Omega})$ and for any G defined by (2.7) with $g \in C^1([A, B])$.

LEMMA 5.4 Let assumption (H_3) hold, let $u_0 \in L^\infty(\Omega)$ and for any $\varepsilon > 0$ let $(u^\varepsilon, v^\varepsilon)$ be the solution of problem (2.15)–(2.16) with initial datum u_0. Let $\{\varepsilon_{h_l}\}$ be the vanishing sequence given by Lemma 5.3. Then there exists a subset $E^2 \subseteq (0,\infty)$ with $|E^2| = 0$ and with the following property: for any $t \in (0,\infty) \setminus E^2$ there exist a subsequence $\{\varepsilon_{h_{l,t}}\} \subseteq \{\varepsilon_{h_l}\}$, possibly depending on t, and a function $v^t \in H^1(\Omega)$ such that

$$\phi(u^{\varepsilon_{h_{l,t}}}(\cdot,t)) \to v^t(\cdot) \quad \text{a.e. in } \Omega. \tag{5.12}$$

We can now prove Theorem 2.10.

Proof of Theorem 2.10. Let $E^1, E^2 \subseteq (0,\infty)$ be the sets of zero Lebesgue measure given respectively by Lemma 5.3 and Lemma 5.4. Let us define the set

$$E := E^1 \cup E^2$$

(clearly of zero Lebesgue measure) and fix any $t \in (0,\infty) \setminus E$. Let $\{\varepsilon_{h_{l,t}}\}$ and $v^t \in H^1(\Omega)$ be respectively the sequence and the function given by Lemma 5.4 so that convergence (5.12) holds. Since we have assumed that the initial datum u_0 satisfies assumption (H_3), from Theorem 2.8 we deduce that $a \leqslant u^{\varepsilon_{h_{l,t}}}(\cdot,t) \leqslant d$ almost everywhere in Ω, and

$$u^{\varepsilon_{h_{l,t}}}(\cdot,t) = \begin{cases} s_1(\phi(u^{\varepsilon_{h_{l,t}}}(\cdot,t))) & \text{a.e. in } (\omega_1, 0), \\ s_2(\phi(u^{\varepsilon_{h_{l,t}}}(\cdot,t))) & \text{a.e. in } (0, \omega_2). \end{cases}$$

Therefore we have

$$u^{\varepsilon_{h_{l,t}}}(\cdot,t) \to u^t(\cdot,t) =: \begin{cases} s_1(v^t(\cdot)) & \text{a.e. in } (\omega_1, 0), \\ s_2(v^t(\cdot)) & \text{a.e. in } (0, \omega_2) \end{cases}$$

(here Lemma 5.4 has been used). This implies that for any $\varphi \in C^1(\overline{\Omega})$ we have

$$\lim_{\varepsilon_{h_{l,t}}\to 0}\int_\Omega G(u^{\varepsilon_{h_{l,t}}}(x,t))\varphi(x)\,\mathrm{d}x = \lim_{\varepsilon_{h_{l,t}}\to 0}\int_{\omega_1}^{0}[G\circ s_1](\phi(u^{\varepsilon_{h_{l,t}}}(x,t)))\varphi(x)\,\mathrm{d}x$$
$$+\lim_{\varepsilon_{h_{l,t}}\to 0}\int_0^{\omega_2}[G\circ s_2](\phi(u^{\varepsilon_{h_{l,t}}}(x,t)))\varphi(x)\,\mathrm{d}x$$
$$=\int_{\omega_1}^{0} G(s_1(v^t(x)))\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2} G(s_2(v^t(x)))\varphi(x)\,\mathrm{d}x \qquad (5.13)$$

for any G defined by (2.7) in terms of $g \in C^1([A,B])$. Combining (5.11) and (5.13) we obtain

$$\int_{\omega_1}^{0}\left(\int_{[A,B]}[G\circ s_1](\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2}\left(\int_{[A,B]}[G\circ s_2](\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x$$
$$=\int_{\omega_1}^{0} G(s_1(v^t(x)))\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2} G(s_2(v^t(x)))\varphi(x)\,\mathrm{d}x \qquad (5.14)$$

for any G and φ as above. Here for almost every $(x,t)\in Q$ the measure $\sigma_{(x,t)}$ is the disintegration of the limiting Young measure η given by Proposition 5.1. We proceed in three steps.

STEP (i). Let us prove the characterization (2.29). Let $u \in L^\infty(Q)$ be the weak* limit of the sequence $\{u^{\varepsilon_k}\}$ given by Theorem 2.9 and let $\{\varepsilon_h\}\subseteq\{\varepsilon_k\}$ be the vanishing sequence given by Proposition 5.1. Observe that in view of Theorem 2.8 and Proposition 5.1 (in particular see (2.20) and (5.8)–(5.9)),

$$\iint_Q u(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \lim_{h\to\infty}\iint_Q u^{\varepsilon_h}(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\lim_{h\to\infty}\int_0^\infty\int_{\omega_1}^0 s_1(\phi(u^{\varepsilon_h}(x,t)))\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \lim_{h\to\infty}\int_0^\infty\int_0^{\omega_2} s_2(\phi(u^{\varepsilon_h}(x,t)))\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^\infty\int_{\omega_1}^0\left(\int_{[A,B]} s_1(\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^\infty\int_0^{\omega_2}\left(\int_{[A,B]} s_2(\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

for any $\psi\in C_c^1(Q)$. The above equalities imply

$$u(x,t)=\left(\int_{[A,B]} s_1(\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\chi_{(\omega_1,0)\times(0,\infty)} + \left(\int_{[A,B]} s_2(\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\chi_{(0,\omega_2)\times(0,\infty)} \qquad (5.15)$$

for almost every $(x,t)\in Q$. Therefore, the characterization (2.29) will follow if we prove that $\sigma_{(x,t)}$ is the Dirac mass:

$$\sigma_{(x,t)} \equiv \delta_{v(x,t)} \qquad (5.16)$$

for almost every $(x,t)\in Q$, where $v\in L^\infty(\Omega)$ is the limit of both the sequences $\{\phi(u^{\varepsilon_h})\}$, $\{v^{\varepsilon_h}\}$ in the weak* topology of $L^\infty(Q)$ (see Theorem 2.9). In fact, in the light of (5.16) equality (5.15) will then read

$$u(x,t)=s_1(v(x,t))\chi_{(\omega_1,0)\times(0,\infty)} + s_2(v(x,t))\chi_{(0,\omega_2)\times(0,\infty)}$$

for almost every $(x,t)\in Q$, proving (2.29).

So, let us prove (5.16). Fix any $\bar{t} \in (0, \infty) \setminus E$ and observe that for any $\varphi \in C_c^1((\omega_1, 0))$ equality (5.14) reads

$$\int_{\omega_1}^0 \left(\int_{[A,B]} [G \circ s_1](\xi) \, \mathrm{d}\sigma_{(x,\bar{t})}(\xi) \right) \varphi(x) \, \mathrm{d}x = \int_{\omega_1}^0 [G \circ s_1](v^{\bar{t}}(x)) \varphi(x) \, \mathrm{d}x \tag{5.17}$$

for any G defined by (2.7) in terms of $g \in C^1([A, B])$. The above equality implies that there exists a set $F \subseteq (\omega_1, 0)$ of zero Lebesgue measure such that for any G as above,

$$\int_{[A,B]} [G \circ s_1](\xi) \, \mathrm{d}\sigma_{(x,\bar{t})}(\xi) = [G \circ s_1](v^{\bar{t}}(x)) \tag{5.18}$$

for any $x \in (\omega_1, 0) \setminus F$. Fix any $x \in (\omega_1, 0) \setminus F$ such that (5.18) holds. Choosing

$$\underline{g}(s) \begin{cases} \equiv 0 & \text{if } s \in [A, v^{\bar{t}}(x)], \\ > 0 & \text{if } s \in (v^{\bar{t}}(x), B], \end{cases} \qquad \underline{G}(\lambda) := \int_a^\lambda \underline{g}(\phi(s)) \, \mathrm{d}s$$

in (5.18) gives

$$\int_{(v^{\bar{t}}(x), B]} [\underline{G} \circ s_1](\xi) \, \mathrm{d}\sigma_{(x,\bar{t})}(\xi) = 0.$$

Since $\underline{G}(\lambda) > 0$ for $\lambda \in (s_1(v^{\bar{t}}(x)), b]$, the above equality implies that $\operatorname{supp} \sigma_{(x,\bar{t})} \subseteq [A, v^{\bar{t}}(x)]$. On the other hand, if we choose

$$\overline{g}(s) \begin{cases} \equiv 0 & \text{if } s \in [v^{\bar{t}}(x), B], \\ > 0 & \text{if } s \in [A, v^{\bar{t}}(x)), \end{cases} \qquad \overline{G}(\lambda) := \int_b^\lambda \overline{g}(\phi(s)) \, \mathrm{d}s,$$

equality (5.18) reads

$$\int_{[A, v^{\bar{t}}(x))} [\overline{G} \circ s_1](\xi) \, \mathrm{d}\sigma_{(x,\bar{t})}(\xi) = 0.$$

Since $\overline{G}(\lambda) < 0$ for $\lambda \in [a, s_1(v^{\bar{t}}(x)))$, the above equality implies that $\operatorname{supp} \sigma_{(x,\bar{t})} \subseteq [v^{\bar{t}}(x), B]$. In other words, we have obtained

$$\operatorname{supp} \sigma_{(x,\bar{t})} = \{v^{\bar{t}}(x)\}$$

for almost every $x \in (\omega_1, 0)$. Moreover, by similar arguments we also obtain

$$\operatorname{supp} \sigma_{(x,\bar{t})} = \{v^{\bar{t}}(x)\}$$

for almost every $x \in (0, \omega_2)$. Therefore for almost every $x \in \Omega$ the probability measure $\sigma_{(x,\bar{t})}$ is the Dirac mass concentrated at the point $v^{\bar{t}}(x)$:

$$\sigma_{(x,\bar{t})} = \delta_{v^{\bar{t}}(x)}. \tag{5.19}$$

Combining (5.19) with the basic properties of the narrow convergence of Young measures (in particular see (5.8)–(5.9) in Proposition 5.1), we obtain

$$v(x, \bar{t}) = \int_{[A,B]} \xi \, \mathrm{d}\sigma_{(x,\bar{t})}(\xi) = v^{\bar{t}}(x) \tag{5.20}$$

for almost every $x \in \Omega$. Here the function $v \in L^\infty(Q)$ is the limit of both the sequences $\{\phi(u^{\varepsilon_k})\}$, $\{v^{\varepsilon_k}\}$ in the weak* topology of $L^\infty(Q)$ (see Theorem 2.9). Since $v \in L^2((0,T); H^1(\Omega))$ for any $T > 0$, and $\bar{t} \in (0,\infty) \setminus E$ where $|E| = 0$, we can suppose that $v(\cdot,\bar{t}) \in H^1(\Omega) \subseteq C(\overline{\Omega})$. Since $v^{\bar{t}}(\cdot) \in H^1(\Omega) \subseteq C(\overline{\Omega})$ too, the above equality holds *for all* $x \in \Omega$, so $\sigma_{(x,\bar{t})} = \delta_{v(x,\bar{t})}$ *for all* $x \in \Omega$. As $\bar{t}$ is arbitrary, the characterization (5.16) follows.

STEP (ii). Let us prove the convergences in (2.30)–(2.31). To this end, let $\{\varepsilon_h\}$ be the vanishing sequence given by Proposition 5.1. In view of standard results on the narrow convergence of Young measures, the characterization (5.16) implies that

$$\phi(u^{\varepsilon_h}) \to v \quad \text{in measure}$$

in every cylinder $Q_T := \Omega \times (0,T)$, for any $T > 0$ ([V]). Since Q_T has finite Lebesgue measure for any $T > 0$ and the sequence $\{u^{\varepsilon_h}\}$ is uniformly bounded in $L^\infty(Q)$, we deduce that

$$\phi(u^{\varepsilon_h}) \to v \quad \text{in } L^p(Q_T) \tag{5.21}$$

for any $1 \leqslant p < \infty$ ([B]). By (2.23) we also obtain

$$\begin{aligned} \|v^{\varepsilon_h} - v\|^2_{L^2(Q_T)} &\leqslant \|v^{\varepsilon_h} - \phi(u^{\varepsilon_h})\|^2_{L^2(Q)} + \|\phi(u^{\varepsilon_h}) - v\|^2_{L^2(Q_T)} \\ &\leqslant \varepsilon_h \|\sqrt{\varepsilon_h} u_t^{\varepsilon_h}\|^2_{L^2(Q)} + \|\phi(u^{\varepsilon_h}) - v\|^2_{L^2(Q_T)} \to 0 \end{aligned}$$

as $h \to \infty$, that is,

$$v^{\varepsilon_h} \to v \quad \text{in } L^2(Q_T) \tag{5.22}$$

for any $T > 0$. In view of (5.21), (5.22) there exists a subsequence $\{\varepsilon_j\} \equiv \{\varepsilon_{h_j}\} \subseteq \{\varepsilon_h\}$ such that

$$\phi(u^{\varepsilon_j}), v^{\varepsilon_j} \to v \quad \text{almost everywhere in } Q_T, \tag{5.23}$$

therefore by the arbitrariness of $T > 0$ the convergence in (2.30) follows. Let us address (2.31). For this purpose, observe that by the convergence in (2.30) we obtain

$$u^{\varepsilon_j}(x,t) = s_1(\phi(u^{\varepsilon_j}(x,t))) \to s_1(v(x,t)) \tag{5.24}$$

for almost every $(x,t) \in (\omega_1, 0) \times (0,\infty)$ and

$$u^{\varepsilon_j}(x,t) = s_2(\phi(u^{\varepsilon_j}(x,t))) \to s_2(v(x,t)) \tag{5.25}$$

for almost every $(x,t) \in (0,\omega_2) \times (0,\infty)$ (here Theorem 2.8 has been used). The above convergences imply (2.31).

STEP (iii). Let us prove the limiting entropy inequalities (2.8). To do so, fix any $T > 0$ and pass to the limit as $j \to \infty$ in the viscous entropy inequalities (2.18) for $\varepsilon = \varepsilon_j$, $t_1 = 0$, $t_2 = T$ and for $\psi \in C^1(\overline{Q}_T)$, $\psi \geqslant 0$, $\psi(\cdot,T) = 0$ in $\overline{\Omega}$ (here $Q_T = \Omega \times (0,T)$). In view of (2.21) and the convergence in (2.31), we have

$$\iint_{Q_T} G(u^{\varepsilon_j})\psi_t \,\mathrm{d}x\,\mathrm{d}t \to \iint_Q G(u)\psi_t \,\mathrm{d}x\,\mathrm{d}t \tag{5.26}$$

for any $g \in C^1(\mathbb{R})$ and G defined by (2.7). Moreover, since $v^{\varepsilon_j} \to v$ almost everywhere in Q and $v_x^{\varepsilon_j} \rightharpoonup v_x$ in $L^2(Q)$ (see (2.26)), it follows that

$$g(v^{\varepsilon_j})v_x^{\varepsilon_j} \rightharpoonup g(v)v_x \quad \text{in } L^2(Q_T)$$

(here (2.22) has been used), therefore

$$\iint_{Q_T} g(v^{\varepsilon_j})v_x^{\varepsilon_j}\psi_x \,\mathrm{d}x\,\mathrm{d}t \to \iint_{Q_T} g(v)v_x\psi_x \,\mathrm{d}x\,\mathrm{d}t \tag{5.27}$$

for any g and ψ as above. Finally, observe that since

$$\sqrt{g'(v^{\varepsilon_j})}v_x^{\varepsilon_j} \rightharpoonup \sqrt{g'(v)}v_x \quad \text{in } L^2(Q_T)$$

(recall that $g' \geqslant 0$), we have

$$\iint_{Q_T} g'(v)v_x^2\psi \,\mathrm{d}x\,\mathrm{d}t \leqslant \liminf_{j\to\infty} \iint_{Q_T} g'(v^{\varepsilon_j})(v_x^{\varepsilon_j})^2\psi \,\mathrm{d}x\,\mathrm{d}t \tag{5.28}$$

for any $\psi \in C^1(\overline{Q}_T)$ with $\psi \geqslant 0$. Combining (5.26)–(5.28) gives the entropy inequalities (2.8) and concludes the proof. □

Proof of Lemma 5.2. Fix any $T > 0$, $\varphi \in C^1(\overline{\Omega})$, and $g \in C^1(\mathbb{R})$, and let G be the function defined by (2.7). In view of (2.16) and (2.17),

$$\begin{aligned}\int_0^T \left| \int_\Omega [G(u^\varepsilon(x,t))]_t \varphi(x)\,\mathrm{d}x \right| \mathrm{d}t &= \int_0^T \left| \int_\Omega g(\phi(u^\varepsilon(x,t)))u_t^\varepsilon(x,t)\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t \\ &\leqslant \int_0^T \left| \int_\Omega [g(\phi(u^\varepsilon(x,t))) - g(v^\varepsilon(x,t))]u_t^\varepsilon(x,t)\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t \\ &\quad + \int_0^T \left| \int_\Omega g(v^\varepsilon(x,t))v_{xx}^\varepsilon(x,t)\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t. \end{aligned} \tag{5.29}$$

Since the family $\{u^\varepsilon\}$ is uniformly bounded in $L^\infty(Q)$ (see (2.21)), we can find a constant $C^* > 0$ such that

$$C^* > \sup_{\varepsilon>0} \|\phi(u^\varepsilon)\|_{L^\infty(Q)}.$$

Let us study the different terms on the right-hand side of (5.29). We have

$$\begin{aligned}&\int_0^T \left| \int_\Omega [g(\phi(u^\varepsilon(x,t))) - g(v^\varepsilon(x,t))]u_t^\varepsilon(x,t)\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t \\ &\qquad \leqslant \|g'\|_{C([-C^*,C^*])} \int_0^T \left| \int_\Omega \varepsilon[u_t^\varepsilon(x,t)]^2\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t \\ &\qquad \leqslant \|g'\|_{C([-C^*,C^*])} \|\sqrt{\varepsilon}u_t^\varepsilon\|_{L^2(Q_T)}^2 \|\varphi\|_{C(\overline{\Omega})} \end{aligned} \tag{5.30}$$

(here (2.23) has been used). Moreover,

$$\int_0^T \left| \int_\Omega g(v^\varepsilon(x,t)) v^\varepsilon_{xx}(x,t)\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t \leqslant \int_0^T \left| \int_\Omega g(v^\varepsilon(x,t)) v^\varepsilon_{x}(x,t)\varphi_x(x)\,\mathrm{d}x \right| \mathrm{d}t + \int_0^T \left| \int_\Omega g'(v^\varepsilon(x,t)) [v^\varepsilon_{x}(x,t)]^2\varphi(x)\,\mathrm{d}x \right| \mathrm{d}t \leqslant \|g\|_{C^1([-C^*,C^*])} \|\varphi\|_{C^1(\overline{\Omega})} [\sqrt{T}\|v^\varepsilon_x\|_{L^2(Q)} + \|v^\varepsilon_x\|^2_{L^2(Q)}] \tag{5.31}$$

(here (2.23) has been used). Set

$$C_{g,\varphi} := \|g\|_{C^1([-C^*,C^*])} \|\varphi\|_{C^1(\overline{\Omega})}.$$

Then combining (5.29)–(5.31) gives estimate (5.10) and concludes the proof. □

Proof of Lemma 5.3. (i) Firstly we observe that in view of assumption (H_3),

$$A \leqslant \phi(u^\varepsilon) \leqslant B \tag{5.32}$$

almost everywhere in Q (see Theorem 2.8). Let $\{\varepsilon_h\}$ be the vanishing sequence given by Proposition 5.1 and choose any $f \in C(\mathbb{R})$ such that $f \equiv 0$ in $[A, B]$ and $f > 0$ in $\mathbb{R} \setminus [A, B]$. In view of (5.8)–(5.9) and (5.32),

$$0 \equiv f(\phi(u^{\varepsilon_h})) \overset{*}{\rightharpoonup} \int_{\mathbb{R}} f(\xi)\,\mathrm{d}\sigma_{(\cdot,\cdot)}(\xi) \quad \text{in } L^\infty(Q),$$

so

$$\int_{\mathbb{R}} f(\xi)\,\mathrm{d}\sigma_{(x,t)}(\xi) \equiv 0 \tag{5.33}$$

for almost every $(x, t) \in Q$. Fix any $(x, t) \in Q$ such that the above equality holds. Since $f \geqslant 0$, equation (5.33) implies $f = 0$ $\sigma_{(x,t)}$-a.e. in $\mathbb{R}$. On the other hand, $f > 0$ in $\mathbb{R} \setminus [A, B]$, therefore $\operatorname{supp} \sigma_{(x,t)} \subseteq [A, B]$. By the arbitrariness of (x, t), claim (i) follows.

(ii) Fix any $T > 0$ and set $Q_T := \Omega \times (0, T)$. Fix any $g \in C^1([A, B])$ and let G be defined by (2.7). For any $\varepsilon > 0$ and $\varphi \in C^1(\overline{\Omega})$ let $\mathcal{G}^{\varepsilon,\varphi} \in W^{1,1}(0, T)$ be the function defined by setting

$$\mathcal{G}^{\varepsilon,\varphi}(t) := \int_\Omega G(u^\varepsilon(x,t))\varphi(x)\,\mathrm{d}x. \tag{5.34}$$

In view of estimate (2.23) and Lemma 5.2 (in particular see (5.10)) the family $\{\mathcal{G}^{\varepsilon,\varphi}\}$ is uniformly bounded in $W^{1,1}(0, T)$, therefore there exist a subsequence $\{\varepsilon_h^{g,\varphi}\} \subseteq \{\varepsilon_h\}$, in general depending on g and φ, and a function $\mathcal{G}^\varphi \in L^1(0, T)$ such that

$$\mathcal{G}^{\varepsilon_h^{g,\varphi},\varphi} \to \mathcal{G}^\varphi \quad \text{in } L^1(0, T) \tag{5.35}$$

as $\varepsilon_h^{g,\varphi} \to 0$. On the other hand, in view of Theorem 2.8 we have

$$u^{\varepsilon_h^{g,\varphi}}(x,t) = \begin{cases} s_1(\phi(u^{\varepsilon_h^{g,\varphi}}(x,t))) & \text{for a.e. } (x,t) \in (\omega_1, 0) \times (0, \infty), \\ s_2(\phi(u^{\varepsilon_h^{g,\varphi}}(x,t))) & \text{for a.e. } (x,t) \in (0, \omega_2) \times (0, \infty). \end{cases}$$

Therefore, by Proposition 5.1 (see (5.8)–(5.9)) we obtain

$$\lim_{\varepsilon_h^{g,\varphi}\to 0}\int_0^T h(t)\,\mathrm{d}t\int_\Omega G(u^{\varepsilon_h^{g,\varphi}}(x,t))\varphi(x)\,\mathrm{d}x$$

$$= \int_0^T h(t)\,\mathrm{d}t\int_{\omega_1}^0\Big(\int_{[A,B]}G(s_1(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x + \int_0^T h(t)\,\mathrm{d}t\int_0^{\omega_2}\Big(\int_{[A,B]}G(s_2(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x \qquad (5.36)$$

for any $h \in C^1([0,T])$. Combining (5.35) and (5.36) we have

$$\int_0^T \mathcal{G}^\varphi(t)h(t)\,\mathrm{d}t = \int_0^T h(t)\,\mathrm{d}t\int_{\omega_1}^0\Big(\int_{[A,B]}G(s_1(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x + \int_0^T h(t)\,\mathrm{d}t\int_0^{\omega_2}\Big(\int_{[A,B]}G(s_2(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x$$

for any h as above, hence

$$\mathcal{G}^\varphi(t) = \int_{\omega_1}^0\Big(\int_{[A,B]}G(s_1(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2}\Big(\int_{[A,B]}G(s_2(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x$$

for almost every $t \in (0,T)$. This implies that the convergence in (5.35) holds *along the whole sequence* $\{\varepsilon_h\}$, so

$$\mathcal{G}^{\varepsilon_h,\varphi} \to \int_{\omega_1}^0\Big(\int_{[A,B]}G(s_1(\xi))\,\mathrm{d}\sigma_{(x,\cdot)}(\xi)\Big)\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2}\Big(\int_{[A,B]}G(s_2(\xi))\,\mathrm{d}\sigma_{(x,\cdot)}(\xi)\Big)\varphi(x)\,\mathrm{d}x \qquad (5.37)$$

in $L^1(0,T)$. In view of (5.37), there exist a subsequence $\{\varepsilon_{h_l}^{g,\varphi}\} \subseteq \{\varepsilon_h\}$ and a set $E^{g,\varphi} \subseteq (0,T)$ of zero Lebesgue measure such that

$$\mathcal{G}^{\varepsilon_{h_l}^{g,\varphi},\varphi}(t) \to \int_{\omega_1}^0\Big(\int_{[A,B]}G(s_1(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2}\Big(\int_{[A,B]}G(s_2(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\Big)\varphi(x)\,\mathrm{d}x \qquad (5.38)$$

for any $t \in (0,T)\setminus E^{g,\varphi}$. Here in general the sequence $\{\varepsilon_{h_l}^{g,\varphi}\}$ and the set $E^{g,\varphi}$ depend on the choice of $g \in C^1([A,B])$ and $\varphi \in C^1(\overline{\Omega})$. Let us show that the convergence in (5.38) holds along a subsequence $\{\varepsilon_{h_l}\}$ and off a set $E^T \subseteq (0,T)$ of zero Lebesgue measure, both independent of g and φ. To see this, observe that the spaces $C^1([A,B])$ and $C^1(\overline{\Omega})$ are separable, hence there exist countable sets $\mathcal{D}_1 \subseteq C^1([A,B])$ and $\mathcal{D}_2 \subseteq C^1(\overline{\Omega})$ such that

$$\overline{\mathcal{D}}_1 = C^1([A,B]), \quad \overline{\mathcal{D}}_2 = C^1(\overline{\Omega}).$$

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are countable, by standard diagonal arguments, there exist a subsequence $\{\varepsilon_{h_l}\} \subseteq \{\varepsilon_h\}$ and a set $E^T \subseteq (0, T)$ of zero Lebesgue measure such that, *for any* $g \in \mathcal{D}_1$ *and for any* $\varphi \in \mathcal{D}_2$ *convergence* (5.38) *holds along the sequence* $\{\varepsilon_{h_l}\}$ *and for all* $t \in (0, T) \setminus E^T$. In other words,

$$\lim_{l\to\infty} \int_\Omega G(u^{\varepsilon_{h_l}}(x,t))\varphi(x)\,\mathrm{d}x = \int_{\omega_1}^0 \left(\int_{[A,B]} G(s_1(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x + \int_0^{\omega_2} \left(\int_{[A,B]} G(s_2(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x \tag{5.39}$$

for all $g \in \mathcal{D}_1$, $\varphi \in \mathcal{D}_2$, and $t \in (0, T) \setminus E^T$. To prove that (5.39) holds for all $g \in C^1([A, B])$ and $\varphi \in C^1(\overline{\Omega})$, fix any $t \in (0, T) \setminus E^T$, $g \in C^1([A, B])$ and $\varphi \in C^1(\overline{\Omega})$. Choose $\{g_n\} \subseteq \mathcal{D}_1$ and $\{\varphi_n\} \subseteq \mathcal{D}_2$ so that

$$\begin{cases} g_n \to g & \text{in } C^1([A,B]), \\ \varphi_n \to \varphi & \text{in } C^1(\overline{\Omega}). \end{cases} \tag{5.40}$$

Let G and G_n be respectively the functions defined by (2.7) for g and g_n ($n \in \mathbb{N}$) for $k = 0$. Then

$$|G(\lambda) - G_n(\lambda)| = \left|\int_0^\lambda [g(\phi(s)) - g_n(\phi(s))]\,\mathrm{d}s\right| \leqslant \|g - g_n\|_{C^1([A,B])}|\lambda| \tag{5.41}$$

for any $\lambda \in [a, d]$ (see Figure 1). For the sake of simplicity set

$$\Omega_1 := (\omega_1, 0), \qquad \Omega_2 := (0, \omega_2),$$

and recall that in view of assumption (H_3) we have $u^{\varepsilon_{h_l}}(x,t) = s_1(\phi(u^{\varepsilon_{h_l}}(x,t)))$ in Ω_1 and $u^{\varepsilon_{h_l}}(x,t) = s_2(\phi(u^{\varepsilon_{h_l}}(x,t)))$ in Ω_2 (see Theorem 2.8). Then

$$\begin{aligned} &\left|\int_{\Omega_i} G(u^{\varepsilon_{h_l}}(x,t))\varphi(x)\,\mathrm{d}x - \int_{\Omega_i}\left(\int_{[A,B]} G(s_i(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x\right| \\ &\quad\leqslant \int_{\Omega_i} |G(u^{\varepsilon_{h_l}}(x,t)) - G_n(u^{\varepsilon_{h_l}}(x,t))|\,|\varphi(x)|\,\mathrm{d}x + \int_{\Omega_i} |G_n(u^{\varepsilon_{h_l}}(x,t))|\,|\varphi(x) - \varphi_n(x)|\,\mathrm{d}x \\ &\qquad + \left|\int_{\Omega_i}\left[G_n(u^{\varepsilon_{h_l}}(x,t))\varphi_n(x) - \left(\int_{[A,B]} G_n(s_i(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi_n(x)\right]\mathrm{d}x\right| \\ &\qquad + \int_{\Omega_i}\left(\int_{[A,B]} |G_n(s_i(\xi))|\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)|\varphi_n(x) - \varphi(x)|\,\mathrm{d}x \\ &\qquad + \int_{\Omega_i}\left(\int_{[A,B]} |G_n(s_i(\xi)) - G(s_i(\xi))|\,\mathrm{d}\sigma_{(x,t)}(\xi)\right)\varphi(x)\,\mathrm{d}x \end{aligned} \tag{5.42}$$

($i = 1, 2$). Let us study the five terms on the right-hand side of (5.42) first as $l \to \infty$ and then as $n \to \infty$. In view of (5.41),

$$\begin{aligned} &\limsup_{l\to\infty} \int_{\Omega_i} |G(u^{\varepsilon_{h_l}}(x,t)) - G_n(u^{\varepsilon_{h_l}}(x,t))|\,|\varphi(x)|\,\mathrm{d}x \\ &\quad \leqslant \limsup_{l\to\infty} \|g_n - g\|_{C^1([A,B])} \int_{\Omega_i} |u^{\varepsilon_{h_l}}(x,t)|\,\mathrm{d}x \leqslant C_1\|g - g_n\|_{C^1([A,B])} \end{aligned} \tag{5.43}$$

($i = 1, 2$); concerning the second and fourth terms on the right-hand side of (5.42) we obtain respectively

$$\limsup_{l\to\infty} \int_{\Omega_i} |G_n(u^{\varepsilon_{h_l}}(x,t))|\,|\varphi(x) - \varphi_n(x)|\,\mathrm{d}x \leqslant \limsup_{l\to\infty} \|\varphi_n - \varphi\|_{C^1(\overline{\Omega})} \int_{\Omega_i} |G_n(u^{\varepsilon_{h_l}}(x,t))|\,\mathrm{d}x \leqslant C_2 \|\varphi_n - \varphi\|_{C^1(\overline{\Omega})} \tag{5.44}$$

and

$$\int_{\Omega_i} \left(\int_{[A,B]} |G_n(s_i(\xi))|\,\mathrm{d}\sigma_{(x,t)}(\xi) \right) |\varphi_n(x) - \varphi(x)|\,\mathrm{d}x \leqslant C_3 \|\varphi_n - \varphi\|_{C^1(\overline{\Omega})}. \tag{5.45}$$

Finally we address the fifth term on the right-hand side of (5.42). By (5.41) we obtain

$$\int_{\Omega_i} \left(\int_{[A,B]} |G_n(s_i(\xi)) - G(s_i(\xi))|\,\mathrm{d}\sigma_{(x,t)}(\xi) \right) \varphi(x)\,\mathrm{d}x \leqslant C_4 \|g_n - g\|_{C^1([A,B])} \tag{5.46}$$

($i = 1, 2$). Here the constants C_p ($p = 1, 2, 3, 4$) are independent of n. Concerning the third term on the right-hand side of (5.42), observe that in view of (5.39),

$$\lim_{l\to\infty} \left| \int_{\Omega_i} \left[G_n(u^{\varepsilon_{h_l}}(x,t)) - \int_{[A,B]} G_n(s_i(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi) \right] \varphi_n(x)\,\mathrm{d}x \right| = 0$$

for $i = 1, 2$ and any $n \in \mathbb{N}$ (recall that $\{g_n\} \subseteq \mathcal{D}_1$ and $\{\varphi_n\} \subseteq \mathcal{D}_2$). The above equality and (5.43)–(5.46) imply there exists a constant $C > 0$ such that

$$\limsup_{l\to\infty} \left| \int_{\Omega_i} G(u^{\varepsilon_{h_l}}(x,t))\varphi(x)\,\mathrm{d}x - \int_{\Omega_i} \left(\int_{[A,B]} G(s_i(\xi))\,\mathrm{d}\sigma_{(x,t)}(\xi) \right) \varphi(x)\,\mathrm{d}x \right| \leqslant C[\|g_n - g\|_{C^1([A,B])} + \|\varphi_n - \varphi\|_{C^1(\overline{\Omega})}] \tag{5.47}$$

($i = 1, 2$). In view of (5.40) and in view of the arbitrariness of g, t, φ, letting $n \to \infty$ in (5.47) gives equality (5.11) for $t \in (0, T) \setminus E^T$, $g \in C^1([A, B])$, and $\varphi \in C^1(\overline{\Omega})$.

Finally, fix any diverging and strictly increasing sequence $\{T_i\} \subseteq (0, \infty)$. Let $\{\varepsilon^1_{h_l}\}$ and $E^{T_1} \subseteq (0, T_1)$ be respectively the subsequence and the set of zero Lebesgue measure such that the convergence in (5.11) holds for any $g \in C^1([A, B])$, $\varphi \in C^1(\overline{\Omega})$ and $t \in (0, T_1) \setminus E^{T_1}$. Then we can find a subsequence $\{\varepsilon^2_{h_l}\} \subseteq \{\varepsilon^1_{h_l}\}$ and a set $F^1 \subseteq (T_1, T_2)$ of zero Lebesgue measure such that the convergence in (5.11) holds for any $g \in C^1([A, B])$, $\varphi \in C^1(\overline{\Omega})$ and $t \in (0, T_2) \setminus E^{T_2}$, where

$$E^{T_2} := E^{T_1} \cup F^1.$$

Therefore, using an inductive procedure, for any $i \in \mathbb{N}$ we can construct a sequence $\{\varepsilon^i_{h_l}\}$ and a set $E^{T_i} \subseteq (0, T_i)$ with $|E^{T_i}| = 0$ such that (5.11) holds along the sequence $\{\varepsilon^i_{h_l}\}$ and for any $t \in (0, T_i) \setminus E^{T_i}$. By standard diagonal arguments, setting

$$E^1 := \bigcup_{i=1}^{\infty} E^{T_i}$$

gives the claim and concludes the proof. □

Proof of Lemma 5.4. In view of estimate (2.23) and the Fatou Lemma, we have

$$\int_0^\infty \liminf_{l\to\infty}\left\{\int_\Omega [(v_x^{\varepsilon_{h_l}}(x,t))^2+\varepsilon_{h_l}(u_t^{\varepsilon_{h_l}}(x,t))^2]\,\mathrm{d}x\right\}\mathrm{d}t \leqslant C.$$

The above estimate implies that

$$\liminf_{l\to\infty}\left\{\int_\Omega [(v_x^{\varepsilon_{h_l}}(x,t))^2+\varepsilon_{h_l}(u_t^{\varepsilon_{h_l}}(x,t))^2]\,\mathrm{d}x\right\}$$

belongs to the space $L^1(0,\infty)$. Therefore we can find a set $E^2 \subset (0,\infty)$ with $|E^2| = 0$ such that for any $t \in (0,\infty)\setminus E^2$ there exist a subsequence $\{\varepsilon_{h_l,t}\} \subseteq \{\varepsilon_{h_l}\}$ and a constant $C(t) > 0$, both depending on t, such that

$$\sup_{\varepsilon_{h_l,t}>0}\left\{\int_\Omega [(v_x^{\varepsilon_{h_l,t}}(x,t))^2+\varepsilon_{h_l,t}(u_t^{\varepsilon_{h_l,t}}(x,t))^2]\,\mathrm{d}x\right\} \leqslant C(t). \tag{5.48}$$

Fix any $t \in (0,\infty)\setminus E^2$ and observe that estimate (5.48) implies that the sequence $\{v^{\varepsilon_{h_l,t}}(\cdot,t)\}$ is uniformly bounded in $C(\overline{\Omega})$ and equicontinuous. Therefore, passing if necessary to a further subsequence that we continue to denote by $\{\varepsilon_{h_l,t}\}$, there exists a function $v^t \in C(\overline{\Omega})$ such that

$$v^{\varepsilon_{h_l,t}}(\cdot,t) \to v^t(\cdot) \quad \text{in } C(\overline{\Omega}) \tag{5.49}$$

as $l \to \infty$. Moreover, in view of (5.48) again (possibly extracting another subsequence) we obtain

$$\varepsilon_{h_l,t}\, u_t^{\varepsilon_{h_l,t}}(x,t) \to 0$$

for almost every $x \in \Omega$. The above convergences imply that

$$\varphi(u^{\varepsilon_{h_l,t}})(x,t) \to v^t(x)$$

for almost every $x \in \Omega$ (here (2.16) has been used). By the arbitrariness of $t \in (0,\infty)\setminus E^2$, (5.12) follows and this concludes the proof. □

6. Proofs of the results in Section 2.4

6.1 *Basic estimates: proofs*

Proof of Proposition 2.12. Fix any $t > 0$ and for any $n \in \mathbb{N}$ set

$$h_n^t(s) = \begin{cases} 1 & \text{if } t \in [0,t), \\ -n(s-t-1/n) & \text{if } s \in [t, t+1/n]. \end{cases} \tag{6.1}$$

The choice of

$$\psi_n(x,s) := h_n^t(s)$$

as test function in the weak formulation (2.6) of problem (2.1) gives

$$n\int_t^{t+1/n}\int_\Omega u(x,t)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x. \tag{6.2}$$

Arguing as in the proof of Proposition 2.1, letting $n \to \infty$ in (6.2) gives (2.33) and concludes the proof. □

Proof of Proposition 2.13. Choosing $\varphi \equiv 1$ in the pointwise entropy inequalities (2.12) gives

$$\int_{\Omega} G(u(x,t_1))\,\mathrm{d}x \geqslant \int_{\Omega} G(u(x,t_2))\,\mathrm{d}x \tag{6.3}$$

for all $t_1 \leqslant t_2$ and $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$ (recall that G is defined in terms of g by (2.7)). By standard arguments of approximation with smooth functions, the assumption $g \in C^1(\mathbb{R})$ can be dropped. Inequalities (6.3) imply that the map

$$t \mapsto \int_{\Omega} G(u(x,t))\,\mathrm{d}x$$

is nonincreasing in $(0, \infty)$ for any nondecreasing g, hence the claim follows. □

Proof of Proposition 2.14. Let us choose in inequalities (2.12) $g(\lambda) = \lambda$, $t_1 = 0$, $t_2 = T$ and $\varphi(\cdot) \equiv 1$ in $\overline{\Omega}$. We obtain

$$\int_0^T \int_{\Omega} v_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{\Omega} I(u_0)\,\mathrm{d}x - \int_{\Omega} I(u(x,T))\,\mathrm{d}x, \tag{6.4}$$

where

$$I(\lambda) := \int_0^{\lambda} \phi(s)\,\mathrm{d}s.$$

Since $u \in L^\infty(Q)$ (see Definition 2.1(i)) and $T > 0$ is arbitrary, inequalities (6.4) imply estimate (2.35). □

6.2 *Convergence results: proofs*

Most proofs of the results in Subsection 2.4 need the following technical proposition.

PROPOSITION 6.1 Let $v^1, v^2 \in [A, B]$ and $\xi^1,\ \xi^2 \in \overline{\Omega} = [\omega_1, \omega_2]$ be such that

$$(\xi^1 - \omega_1) \int_0^{s_1(v^1)} g(\phi(s))\,\mathrm{d}s + (\omega_2 - \xi^1) \int_0^{s_2(v^1)} g(\phi(s))\,\mathrm{d}s$$
$$= (\xi^2 - \omega_1) \int_0^{s_1(v^2)} g(\phi(s))\,\mathrm{d}s + (\omega_2 - \xi^2) \int_0^{s_2(v^2)} g(\phi(s))\,\mathrm{d}s$$

for any $g \in BV(\mathbb{R})$. Then $v^1 = v^2$ and $\xi^1 = \xi^2$.

Proof. The proof is almost the same as in [ST], so we omit it. □

Proof of Theorem 2.15. (i) Let $\{t_n\}$ be any diverging *good* sequence satisfying (2.36). Then the claim is a consequence of Theorem 3.6 in [ST]. Precisely, there exists a constant $v^* \in \mathbb{R}$, uniquely determined by the solution (u, v, ξ) itself, such that for any diverging good sequence $\{t_n\}$ we have $v(\cdot, t_n) \to v^*$ uniformly in $\overline{\Omega}$.

(ii) Let $\{t_n\}$ be any diverging *bad* sequence satisfying (2.37). Also in this case the claim is a consequence of the results obtained in Theorem 3.7 in [ST] for general weak entropy measure-valued solutions to problem (2.1). However, in the case of two-phase solutions the techniques and the methods of proof are much easier than those used in [ST]. Therefore we give below the details of the proof.

To start with, observe that when $\{t_n\}$ is a bad sequence, the main complication in comparison to the proof of claim (i) is the weakening of the a priori estimates for the family $\{v(\cdot, t_n)\}$. In particular, comparing (2.37) to (2.36), it is easily seen that the sequence $\{v(\cdot, t_n)\}$ need not be relatively compact in the strong topology of $C(\overline{\Omega})$. Therefore, in the investigation of the asymptotic behaviour in time along bad sequences it is natural to look for weaker convergence results, in particular *convergence in measure*. In this connection, let us define

$$v_{t_n}(x, t) := v(x, t + t_n)$$

where $x \in \overline{\Omega},\ t \in (-1, 1)$. Since

$$\int_{-1}^{1} \int_{\Omega} (v_{t_n})_x^2(x, t)\,\mathrm{d}x\,\mathrm{d}t = \int_{t_n-1}^{t_n+1} \int_{\Omega} v_x^2(x, s)\,\mathrm{d}x\,\mathrm{d}s \to 0$$

as $n \to \infty$ (see estimate (2.35) in Proposition 2.14), there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and a set $E \subseteq (-1, 1)$ with $|E| = 0$ such that

$$\int_{\Omega} (v_{t_{n_k}})_x^2(x, t)\,\mathrm{d}x \equiv \int_{\Omega} v_x^2(x, t + t_{n_k})\,\mathrm{d}x \to 0$$

for any $t \in (-1, 1) \setminus E$. This implies that there exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ and $t_1 \in (-1, 0) \setminus E$, $t_2 \in (0, 1) \setminus E$ such that if we set

$$s_k^1 := t_{n_k} + t_1, \quad s_k^2 := t_{n_k} + t_2,$$

then $\{s_k^j\}$ are diverging *good* sequences and

$$s_k^1 < t_{n_k} < s_k^2, \quad |t_{n_k} - s_k^j| \leqslant 1, \tag{6.5}$$

$$v(\cdot, s_k^j) \to v^* \quad \text{in } C(\overline{\Omega}) \tag{6.6}$$

$(j = 1, 2)$, where $v^* \in \mathbb{R}$ is the constant given by (i). Moreover, there exist subsequences $\{s_{k_h}^j\} \subseteq \{s_k^j\}$ (denoted again $\{s_k^j\}$ for simplicity) and $\xi^j \in \overline{\Omega}$ such that

$$\xi(s_k^j) \to \xi^j \tag{6.7}$$

as $k \to \infty$ $(j = 1, 2)$. Observe that in view of Proposition 2.13, for any $g \in BV(\mathbb{R})$,

$$\begin{aligned}(\xi^1 - \omega_1) \int_0^{s_1(v^*)} g(\phi(s))\,\mathrm{d}s + (\omega_2 - \xi^1) \int_0^{s_2(v^*)} g(\phi(s))\,\mathrm{d}s \\ = \lim_{k\to\infty} \int_{\Omega} G(u(x, s_k^1))\,\mathrm{d}x = \lim_{k\to\infty} \int_{\Omega} G(u(x, s_k^2))\,\mathrm{d}x \\ = (\xi^2 - \omega_1) \int_0^{s_1(v^*)} g(\phi(s))\,\mathrm{d}s + (\omega_2 - \xi^2) \int_0^{s_2(v^*)} g(\phi(s))\,\mathrm{d}s.\end{aligned}$$

The above equality easily implies that

$$\xi^1 = \xi^2 =: \xi^*. \tag{6.8}$$

Finally, there exist a subsequence $\{t_{n_{k_h}}\} \subseteq \{t_{n_k}\}$, which we will continue to denote by $\{t_{n_k}\}$, and a constant $\xi^{\{t_{n_k}\}} \in \overline{\Omega}$ such that

$$\xi(t_{n_k}) \to \xi^{\{t_{n_k}\}} \tag{6.9}$$

as $k \to \infty$. In view of the pointwise entropy inequalities (2.12), for any $\varphi \in C^1(\overline{\Omega})$ with $\varphi \geqslant 0$, we obtain

$$\int_\Omega G(u(x, s_k^1))\varphi(x)\,\mathrm{d}x - \int_\Omega G(u(x, t_{n_k}))\varphi(x)\,\mathrm{d}x \geqslant \int_{s_k^1}^{t_{n_k}} \int_\Omega g(v(x,t))v_x(x,t)\varphi_x(x)\,\mathrm{d}x\,\mathrm{d}t \tag{6.10}$$

(recall that $s_k^1 \leqslant t_{n_k}$ for any k) and

$$\int_\Omega G(u(x, t_{n_k}))\varphi(x)\,\mathrm{d}x - \int_\Omega G(u(x, s_k^2))\varphi(x)\,\mathrm{d}x \geqslant \int_{t_{n_k}}^{s_k^2} \int_\Omega g(v(x,t))v_x(x,t)\varphi_x(x)\,\mathrm{d}x\,\mathrm{d}t, \tag{6.11}$$

for any G defined by (2.7) in terms of $g \in C^1(\mathbb{R})$ with $g' \geqslant 0$ (recall also that $t_{n_k} \leqslant s_k^2$ for any k). Since $|t_{n_k} - s_k^j| \leqslant 1$ for $j = 1, 2$, by estimate (2.35) we have

$$\left| \int_{t_{n_k}}^{s_k^j} \int_\Omega g(v(x,t))v_x(x,t)\varphi_x(x)\,\mathrm{d}x\,\mathrm{d}t \right|$$
$$\leqslant \|g(v)\|_{L^\infty(Q)}\|\varphi\|_{C^1(\overline{\Omega})} \left| \int_{t_{n_k}}^{s_k^j} \int_\Omega v_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t \right|^{1/2} |t_{n_k} - s_k^j|^{1/2} |\Omega|^{1/2} \to 0$$

as $k \to \infty$ ($j = 1, 2$). Therefore passing to the limit as $k \to \infty$ in (6.10)–(6.11) gives

$$\lim_{k\to\infty} \int_\Omega G(u(x, t_{n_k}))\varphi(x)\,\mathrm{d}x = \int_{\omega_1}^{\xi^*} G(s_1(v^*))\varphi(x)\,\mathrm{d}x + \int_{\xi^*}^{\omega_2} G(s_2(v^*))\varphi(x)\,\mathrm{d}x \tag{6.12}$$

for any G and φ as above. Let us proceed in two steps.

(α) First we will prove that

$$\xi^{\{t_{n_k}\}} = \xi^*. \tag{6.13}$$

(β) Then we will show that along the subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ we have

$$v(\cdot, t_{n_k}) \to v^* \quad \text{in } L^p(\Omega) \tag{6.14}$$

for any $1 \leqslant p < \infty$.

Observe that the constant v^* is uniquely determined for any fixed two-phase solution of problem (1.1), therefore the convergence in (6.14) holds along the whole sequence $\{v(\cdot, t_n)\}$. Finally, since the set Ω has finite Lebesgue measure, (6.14) *is equivalent* to (2.39), namely $v(\cdot, t_n)$ converges to v^* in measure ([B, GMS, V]). This concludes the proof of claim (ii).

Proof of (α). Towards a contradiction, assume that

$$\xi^{\{t_{n_k}\}} < \xi^*$$

(the case $\xi^{\{t_{n_k}\}} > \xi^*$ can be treated in a similar way). Since $\xi(t_{n_k}) \to \xi^{\{t_{n_k}\}}$ and $\xi(s_k^j) \to \xi^*$ as $k \to \infty$ ($j = 1, 2$), for any $\delta > 0$ such that $\xi^{\{t_{n_k}\}} + \delta < \xi^* - \delta$ we can find $k_\delta \in \mathbb{N}$ such that for any $k \geqslant k_\delta$,

$$\xi^{\{t_{n_k}\}} - \delta < \xi(t_{n_k}) < \xi^{\{t_{n_k}\}} + \delta < \xi^* - \delta < \xi(s_k^j) < \xi^* + \delta \tag{6.15}$$

($j = 1, 2$). Therefore choosing $\varphi \in C_c^1((\xi^{\{t_{n_k}\}} + \delta, \xi^* - \delta))$ with $\varphi \geqslant 0$ and $g(s) \equiv 1$ in equality (6.12) gives

$$\lim_{k\to\infty} \int_\Omega u(x, t_{n_k})\varphi(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\xi^{\{t_{n_k}\}}+\delta}^{\xi^*-\delta} s_2(v(x, t_{n_k}))\varphi(x)\,\mathrm{d}x = \int_{\xi^{\{t_{n_k}\}}+\delta}^{\xi^*-\delta} s_1(v^*)\varphi(x)\,\mathrm{d}x.$$

Since the family $\{v(\cdot, t_{n_k})\}$ is bounded in $L^\infty(\Omega)$ (see Definition 2.1), the above equality implies

$$s_2(v(\cdot, t_{n_k})) \overset{*}{\rightharpoonup} s_1(v^*) \quad \text{in } L^\infty(\xi^{\{t_{n_k}\}} + \delta, \xi^* - \delta). \tag{6.16}$$

Observe that $s_2(v(x, t_{n_k})) \geqslant c$ for any $x \in (\xi^{\{t_{n_k}\}} + \delta, \xi^* - \delta)$, while $s_1(v^*) < b < c$. Therefore (6.16) gives a contradiction and equality (6.13) holds.

Proof of (β). Arguing as in the proof of Step (α), for any $\delta > 0$ there exists $k_\delta \in \mathbb{N}$ such that

$$\xi^* - \delta < \xi(t_{n_k}) < \xi^* + \delta$$

for any $k \geqslant k_\delta$ (here (6.13) has been used). For any $1 \leqslant p < \infty$, set

$$g_p(s) := -(p+1)^{-1}[-s_1(s)]^p.$$

Let G_p be the function defined by

$$G_p(\lambda) := \int_b^\lambda g_p(\phi(s))\,\mathrm{d}s.$$

Observe that for any $\lambda < b$,

$$G_p(\lambda) = |\lambda|^{p+1} - |b|^{p+1}.$$

Choosing $\varphi \in C_c^1((\omega_1, \xi^* - \delta))$ and $g = g_p$ in (6.12) gives

$$\lim_{k\to\infty} \int_{\omega_1}^{\xi^*-\delta} |s_1(v(x, t_{n_k}))|^p \varphi(x)\,\mathrm{d}x = \int_{\omega_1}^{\xi^*-\delta} |s_1(v^*)|^p \varphi(x)\,\mathrm{d}x$$

(for any $1 \leqslant p < \infty$). Similarly, we obtain

$$\lim_{k\to\infty} \int_{\xi^*+\delta}^{\omega_2} |s_2(v(x, t_{n_k}))|^p \varphi(x)\,\mathrm{d}x = \int_{\xi^*+\delta}^{\omega_2} |s_2(v^*)|^p \varphi(x)\,\mathrm{d}x$$

for any $\varphi \in C_c^1((\xi^* + \delta, \omega_2))$ and $1 \leqslant p < \infty$. By the arbitrariness of δ, the above convergences imply

$$s_1(v(\cdot, t_{n_k})) \to s_1(v^*) \quad \text{in } L^p(\omega_1, \xi^*), \tag{6.17}$$

$$s_2(v(\cdot, t_{n_k})) \to s_2(v^*) \quad \text{in } L^p(\xi^*, \omega_2). \tag{6.18}$$

In view of the continuity of the two branches s_1 and s_2, (6.17)–(6.18) imply (6.14).

(iii) The proof is almost the same as that of Theorem 3.3 in [ST], so we omit it. □

The following lemma gives properties of monotonicity in time of the interface $\xi(t)$ and can be regarded as the counterpart of Proposition 6.1 in [ST].

LEMMA 6.2 Let (u, v, ξ) be any two-phase solution of problem (2.1) with initial datum u_0 and let v^* be the constant given by Theorem 2.15, corresponding to the solution (u, v, ξ). Then there exists $T > 0$ such that for $t \geqslant T$ the map $t \mapsto \xi(t)$ is:

(i) nondecreasing if $v^* < B$;
(ii) nonincreasing if $v^* > A$.

Proof. (i) Assume $v^* < B$ (the case $v^* > A$ being analogous). Fix any diverging good sequence $\{t_n\}$, so that $v(\cdot, t_n) \to v^*$ in $C(\overline{\Omega})$ by Theorem 2.15(i). Since we have assumed $v^* < B$, there exists $\overline{n} \in \mathbb{N}$ such that $v(\cdot, t_n) \leqslant B$ for any $n \geqslant \overline{n}$. Set

$$T := t_{\overline{n}}.$$

Write the pointwise entropy inequalities (2.12) for $\varphi \equiv 1$ in $\overline{\Omega}$, $t_1 = T$, $t_2 = t$, $g = g_B$ and $G = G_{AB}$, where the nondecreasing function g_B and G_{AB} are defined by

$$g_B(s) := \begin{cases} 0 & \text{if } s \leqslant B, \\ (s - B)^2 & \text{if } s > B, \end{cases}$$

and

$$G_{AB}(\lambda) := \int_0^\lambda g_{AB}(\phi(s))\, \mathrm{d}s.$$

For any $t \geqslant T$ we obtain

$$\int_{\omega_1}^{\xi(t)} G_{AB}(s_1(v(x,t)))\, \mathrm{d}x + \int_{\xi(t)}^{\omega_2} G_{AB}(s_2(v(x,t)))\, \mathrm{d}x$$
$$\leqslant \int_{\omega_1}^{\xi(T)} G_{AB}(s_1(v(x,T)))\, \mathrm{d}x + \int_{\xi(T)}^{\omega_2} G_{AB}(s_2(v(x,T)))\, \mathrm{d}x = 0, \quad (6.19)$$

the last equality in (6.19) following by our choice of T and by the uniform convergence of $v(\cdot, t_n)$ to v^* in $\overline{\Omega}$. On the other hand, observe that the nonnegative function $G_{AB}(\lambda)$ is strictly positive for any $\lambda > s_2(B) = d$, therefore inequality (6.19) implies

$$v(\cdot, t) \leqslant B \quad \text{for any } t \geqslant T. \quad (6.20)$$

Next, for any $\rho > 0$ let g_ρ be the function defined by (4.6) and let G_ρ be the function defined by (2.7) for $g = g_\rho$ and $k = 0$. Writing the pointwise entropy inequalities (2.12) for $g = g_\rho$ and $t_2 \geqslant t_1 \geqslant T$ gives

$$\int_{\omega_1}^{\xi(t_1)} G_\rho(s_1(v(x,t_1)))\varphi(x)\, \mathrm{d}x + \int_{\xi(t_1)}^{\omega_2} G_\rho(s_2(v(x,t_1)))\varphi(x)\, \mathrm{d}x$$
$$- \left(\int_{\omega_1}^{\xi(t_2)} G_\rho(s_1(v(x,t_2)))\varphi(x)\, \mathrm{d}x + \int_{\xi(t_2)}^{\omega_2} G_\rho(s_2(v(x,t_2)))\varphi(x)\, \mathrm{d}x \right)$$
$$\geqslant \int_{t_1}^{t_2} \int_\Omega g_\rho(v(x,t)) v_x(x,t) \varphi_x(x)\, \mathrm{d}x\, \mathrm{d}t = -\int_{t_1}^{t_2} \int_\Omega \varphi_{xx}(x) \left(\int_0^{v(x,t)} g_\rho(s)\, \mathrm{d}s \right) \mathrm{d}x\, \mathrm{d}t \quad (6.21)$$

for any $\varphi \in C_c^2(\Omega)$ with $\varphi \geqslant 0$. Concerning the right-hand side of (6.21), we have

$$\left|\int_{t_1}^{t_2}\int_{\Omega}\varphi_{xx}(x)\left(\int_0^{v(x,t)} g_\rho(s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t\right| = \left|\int_{t_1}^{t_2}\int_{\{v(x,t)\geqslant B-\rho\}}\rho^{-1/2}(v(x,t)-B+\rho)\varphi_{xx}(x)\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\leqslant \rho^{1/2}\int_{t_1}^{t_2}\int_{\Omega}|\varphi_{xx}(x)|\,\mathrm{d}x \to 0 \tag{6.22}$$

as $\rho \to 0$ (here (6.20) has been used). Moreover, for any $t \geqslant T$,

$$\begin{aligned}
&\int_{\omega_1}^{\xi(t)} G_\rho(s_1(v(x,t)))\varphi(x)\,\mathrm{d}x + \int_{\xi(t)}^{\omega_2} G_\rho(s_2(v(x,t)))\varphi(x)\,\mathrm{d}x \\
&\quad = \int_{\omega_1}^{\xi(t)} \chi_{\{v(x,t)<B-\rho\}}(x,t)\left(\int_{s_0(B-\rho)}^{s_1(B-\rho)}\rho^{-1/2}\,\mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \int_{\omega_1}^{\xi(t)} \chi_{\{v(x,t)\geqslant B-\rho\}}(x,t)\left(\int_{s_0(B-\rho)}^{s_1(v(x,t))}\rho^{-1/2}\,\mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \int_{\xi(t)}^{\omega_2} \chi_{\{v(x,t)\geqslant B-\rho\}}(x,t)\left(\int_{s_2(B-\rho)}^{s_2(v(x,t))}\rho^{-1/2}\,\mathrm{d}s\right)\mathrm{d}x.
\end{aligned}\tag{6.23}$$

Since $\phi''(b) \neq 0$, arguing as in the proof of Proposition 2.2 gives

$$\begin{aligned}
&\lim_{\rho\to 0}\int_{\omega_1}^{\xi(t)} G_\rho(s_1(v(x,t)))\varphi(x)\,\mathrm{d}x + \int_{\xi(t)}^{\omega_2} G_\rho(s_2(v(x,t)))\varphi(x)\,\mathrm{d}x \\
&\quad = -\sqrt{\frac{2}{|\phi''(b)|}}\int_{\omega_1}^{\xi(t)}[2\chi_{\{v(x,t)<B\}}(x,t) + \chi_{\{v(x,t)=B\}}(x,t)]\varphi(x)\,\mathrm{d}x.
\end{aligned}\tag{6.24}$$

In view of (6.22)–(6.24), taking the limit as $\rho \to 0$ in (6.21) gives

$$\int_{\omega_1}^{\xi(t_1)}[2\chi_{\{v(x,t_1)<B\}} + \chi_{\{v(x,t_1)=B\}}]\varphi(x)\,\mathrm{d}x \leqslant \int_{\omega_1}^{\xi(t_2)}[2\chi_{\{v(x,t_2)<B\}} + \chi_{\{v(x,t_2)=B\}}]\varphi(x)\,\mathrm{d}x \tag{6.25}$$

for any φ as above. Aiming for a contradiction, suppose that $\xi(t_2) < \xi(t_1)$. Fix any $\overline{\varphi} \in C_c^2((\xi(t_2), \xi(t_1)))$ with $\overline{\varphi} \geqslant 0$, and observe that (6.25) gives the absurd:

$$0 < \int_{\xi(t_2)}^{\xi(t_1)}\varphi(x)\,\mathrm{d}x \leqslant \int_{\xi(t_2)}^{\xi(t_1)}[2\chi_{\{v(x,t_1)<B\}}(x,t_1) + \chi_{\{v(x,t_1)=B\}}(x,t_1)]\overline{\varphi}(x)\,\mathrm{d}x \leqslant 0.$$

This concludes the proof. □

Proof of Theorem 2.16. Let us distinguish the cases $A < v^* < B$, $v^* = A$, $v^* = B$ and $v^* < A$, $v^* > B$.

If $A < v^* < B$, in view of Lemma 6.2 there exists $T > 0$ such that the function $\xi(t)$ is both nonincreasing and nondecreasing for $t \geqslant T$. This implies that $\xi(t)$ is constant for $t \geqslant T$ and the claim follows.

In the case $v^* = A$ (respectively, $v^* = B$), by Lemma 6.2 there exists $T > 0$ such that the map $t \mapsto \xi(t)$ is nondecreasing (respectively, nonincreasing) on (T, ∞). Therefore the claim follows.

If $v^* < A$, by Theorem 2.15(iii) there exists $T > 0$ such that $v(\cdot, t) < A$ in $\overline{\Omega}$ for any $t \geqslant T$. Hence, in view of Definition 2.1(ii) we have $u(\cdot, t) = s_1(v(\cdot, t))$, so $\xi(t) = \omega_2$ for any $t \geqslant T$.

Finally, if $v^* > B$, by Theorem 2.15(iii) there exists $T > 0$ such that $v(\cdot, t) > B$ in $\overline{\Omega}$ for any $t \geqslant T$. In other words, $u(\cdot, t) = s_2(v(\cdot, t))$ and this implies $\xi(t) = \omega_1$ for any $t \geqslant T$. □

Proof of Theorem 2.17. Let (u, v, ξ) be any two-phase solution of problem (2.1) with initial datum u_0 and let $v^* \in \mathbb{R}$, $\xi^* \in \overline{\Omega}$ be the constants given by Theorems 2.15–2.16 corresponding to (u, v, ξ). Fix any diverging sequence $\{t_n\}$ and recall that the sequences $\{u(\cdot, t_n)\}$, $\{v(\cdot, t_n)\}$ and $\{\xi(t_n)\}$ are related as follows:

$$u(x, t_n) = \chi_{(\omega_1, \xi(t_n))} s_1(v(x, t_n)) + \chi_{(\xi(t_n), \omega_2)} s_2(v(x, t_n)) \tag{6.26}$$

for any $x \in \Omega$.

(i) Let $\{t_n\}$ be any diverging *good* sequence satisfying (2.36). Since $v(\cdot, t_n) \to v^*$ in $C(\overline{\Omega})$ (Theorem 2.15(i)) and $\xi(t_n) \to \xi^*$ (Theorem 2.16), taking the limit as $n \to \infty$ in (6.26) gives

$$u(x, t_n) \to u^*$$

for any $x \in \Omega \setminus \xi^*$, the function u^* being defined by (2.41).

Moreover, if $M_{u_0} < a$ (the case $M_{u_0} > d$ being analogous), by Theorem 2.15(iii) we have $v^* = \phi(M_{u_0}) < A$ and equation (6.26) reads

$$u(x, t_n) = s_1(v(x, t_n))$$

for any $x \in \Omega$ and for $n \in \mathbb{N}$ sufficiently large (see Remark 2.6). Thus, for any n large enough,

$$\begin{aligned} \|s_1(v(x, t_n)) - M_{u_0}\|_{C(\overline{\Omega})} &= \|s_1(v(x, t_n)) - s_1(\phi(M_{u_0}))\|_{C(\overline{\Omega})} \\ &\leqslant B_{M_{u_0}} \|v(\cdot, t_n) - \phi(M_{u_0})\|_{C(\overline{\Omega})}, \end{aligned} \tag{6.27}$$

where

$$B_{M_{u_0}} := \|s_1'\|_{L^\infty(\phi(M_{u_0})-\varepsilon, \phi(M_{u_0})+\varepsilon)} < \infty$$

(here we have chosen $\varepsilon > 0$ so that $\phi(M_{u_0}) + \varepsilon < A$). Since the right-hand side of (6.27) vanishes as $n \to \infty$ by Theorem 2.15, the sequence $\{u(\cdot, t_n)\}$ converges to M_{u_0} uniformly in $\overline{\Omega}$.

(ii) Now let $\{t_n\}$ be any diverging *bad* sequence satisfying (2.37). In this case, by Theorem 2.15(ii) and Theorem 2.16 we have $v(\cdot, t_n) \to v^*$ in measure (hence strongly in $L^1(\Omega)$) and $\xi(t_n) \to \xi^*$. Passing to the limit as $n \to \infty$ in (6.26) gives

$$u(\cdot, t_n) \to u^* \text{ in } L^1(\Omega),$$

proving the convergence in measure (2.44) ([B]). □

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