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Mass conserving Allen–Cahn equation and volume preserving mean curvature flow

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We dedicate this article to the memory of Michelle Schatzman

We consider a mass conserving Allen–Cahn equation $u_t = \Delta u + \varepsilon^{-2}(f(u) - \varepsilon\lambda(t))$ in a bounded domain with no flux boundary condition, where $\varepsilon\lambda(t)$ is the average of $f(u(\cdot,t))$ and $-f$ is the derivative of a double equal well potential. Given a smooth hypersurface γ_0 contained in the domain, we show that the solution u^ε with appropriate initial data tends, as $\varepsilon \searrow 0$, to a limit which takes only two values, with the jump occurring at the hypersurface obtained from the volume preserving mean curvature flow starting from γ_0.

1. Introduction

In this paper, we study the limit, as $\varepsilon \to 0$, of the solution u^ε to the mass conserving Allen–Cahn equation (P^ε)

$$(P^\varepsilon)\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2}\Big(f(u^\varepsilon) - \displaystyle\fint_\Omega f(u^\varepsilon)\Big) & \text{in } \Omega \times \mathbb{R}^+,\\ \partial_\nu u^\varepsilon = 0 & \text{on } \partial\Omega \times \mathbb{R}^+,\\ u^\varepsilon(\cdot,0) = g^\varepsilon(\cdot) & \text{on } \Omega \times \{0\}, \end{cases} \tag{1}$$

where

$$\fint_\Omega f(u^\varepsilon) = \frac{1}{|\Omega|}\int_\Omega f(u^\varepsilon(x,t))\,\mathrm{d}x.$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geqslant 1)$, ∂_ν the outward normal derivative to $\partial\Omega$, and $-f(u)$ is the derivative of a smooth double-well potential with wells of equal depths; more precisely,

$$f \in C^\infty(\mathbb{R}), \quad f(\pm 1) = 0, \quad f'(\pm 1) < 0, \quad \int_{-1}^u f = \int_1^u f < 0 \quad \forall u \in (-1,1). \tag{2}$$

A typical example is $f(u) = u - u^3$.

Problem (1) was proposed, along with its well-posedness, by Rubinstein and Sternberg [22] as a model for phase separation in binary mixture. Note that the well-posedness of (1) follows from the general theory of semilinear parabolic equations [18]. The model is mass conserving and energy decreasing since

$$\forall t \geqslant 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} u^{\varepsilon}(x,t)\,\mathrm{d}x = 0$$

and

$$\forall t \geqslant 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(\frac{\varepsilon|\nabla u^{\varepsilon}|^2}{2} + \frac{1}{\varepsilon}F(u^{\varepsilon})\right)\mathrm{d}x = -\varepsilon\int_{\Omega}(u_t^{\varepsilon})^2\,\mathrm{d}x \leqslant 0,$$

where $F(u) := -\int_{-1}^{u} f(s)\,\mathrm{d}s$ is the double equal well potential.

Formally, one can show that, as $\varepsilon \to 0$, assuming (6), the solution u^{ε} to (1) tends to a limit given by

$$\lim_{\varepsilon\to 0} u^{\varepsilon}(x,t) = \begin{cases} -1, & x \in \Omega_t, \\ +1, & x \in \Omega \setminus \overline{\Omega_t}, \end{cases} \tag{3}$$

where $\Omega_t \subset\subset \Omega$, $\gamma_t = \partial\Omega_t$ and $\Gamma := \bigcup_{t\geqslant 0}(\gamma_t \times \{t\})$ is the solution to the volume preserving mean curvature motion equation

$$V = -(n-1)K_{\gamma_t} + \frac{n-1}{|\gamma_t|}\int_{\gamma_t} K_{\gamma_t}\,\mathrm{d}H^{n-1} \qquad \text{on } \gamma_t. \tag{4}$$

Here V is the normal velocity of γ_t (negative when γ_t is shrinking) and K_{γ_t} the mean curvature (positive at points around which Ω_t is locally convex). Note that the integral of the curvature on γ_t is the sum of the integrals of the curvature on each of the (finitely many) connected components of γ_t, with the orientation given by the outer normal vector on γ_t.

The local in time existence of a unique smooth solution to (4) has been first established in a two-dimensional setting in [12]. The general result in arbitrary space dimension is obtained in [14], where the large time behavior of solutions for initial data close to a sphere was also investigated. When the initial data is convex, it is shown in [19] that (4) admits a unique global in time convex solution. Related properties of other volume-preserving curvature driven flows are established in [13]. In particular it is shown that the solution to (4) will develop singularities in finite time. This was previously established in the case of multiple interfaces in the radial setting in [5].

Concerning the connection between (1) and (4), Bronsard and Stoth [5] considered a radially symmetric case with multiple interfaces (rings) and proved (3). The combination of energy and viscosity methods allowed the authors in [5] to study the convergence of the evolution problem (P^{ε}) even after the formation of singularities, defining "ghost" or "phantom" interfaces. Let us also mention [17] where the Rubinstein–Sternberg model is modified in order to ensure that the solution u^{ε} satisfies $|u^{\varepsilon}| \leqslant 1$. This allows the author to use the method introduced in [3] and to prove convergence to Problem (4).

In this paper we prove the convergence result stated in (3) under the following assumptions about the initial data.

There exists a smooth subdomain $\Omega_0 \subset\subset \Omega$ such that $\gamma_0 = \partial\Omega_0$ is a smooth hypersurface without boundary with finitely many connected components. (5)

Once γ^0 satisfies this assumption, we will construct g^ε satisfying

$$\lim_{\varepsilon\to 0} g^\varepsilon(x) = \begin{cases} -1, & x \in \Omega_0, \\ +1, & x \in \Omega \setminus \overline{\Omega}_0. \end{cases} \tag{6}$$

We establish the following result.

THEOREM 1 Assume that γ_0 satisfies (5). Let $\Gamma = \bigcup_{0\leqslant t\leqslant T}(\gamma_t \times \{t\})$ be a smooth solution to (4) such that $\gamma_t \subset\subset \Omega$ for all $t \in [0, T]$. Then there exists a family of continuous functions $\{g^\varepsilon\}_{0<\varepsilon\leqslant 1}$ satisfying (6) such that the solution u^ε to (1) satisfies (3) for all $t \in [0, T]$.

For the Allen–Cahn equation $u_t^\varepsilon = \Delta u^\varepsilon - \varepsilon^{-2} f(u^\varepsilon)$, (3) holds with Γ being the solution to the motion by mean curvature flow $V = -(n-1)K_{\gamma_t}$. A simple method to verify this is to use a comparison principle and construct sub-super solutions [6, 16]. There are different notions of weak solutions such as viscosity [16] and varifold [20] solutions which can be used to establish the global in time limit. Nevertheless, (1) does not have a comparison principle (due to the volume preserving property) and the simple method does not seem to work. Here we shall employ a method first used by de Mottoni and Schatzman [11] for the Allen–Cahn equation, and later on by Alikakos, Bates, and Chen [1] for the Cahn–Hillard equation and Caginalp and Chen [8] for the phase field system. Namely we first rewrite the equation for u^ε in Problem (P^ε) as

$$u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2}(f(u^\varepsilon) - \varepsilon\lambda_\epsilon(t)) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{7}$$

where we define

$$\forall t \geqslant 0, \qquad \lambda_\epsilon(t) = \frac{1}{\varepsilon} \fint_\Omega f(u^\varepsilon(\cdot, t)). \tag{8}$$

The basic strategy of the proof is as follows [1].

1. For a large enough $k \in \mathbb{N}$, construct an approximate solution $(u_k^\varepsilon, \lambda_k^\varepsilon)$ satisfying

$$\begin{cases} (u_k^\varepsilon)_t - \Delta u_k^\varepsilon - \varepsilon^{-2}(f(u_k^\varepsilon) - \varepsilon\lambda_k^\varepsilon) = \delta_k^\varepsilon \quad \text{in } \Omega_T := \Omega \times [0, T], \\ \int_\Omega (u_k^\varepsilon)_t \,\mathrm{d}x = 0 \quad \forall t \in [0, T], \quad \partial_\nu u_k^\varepsilon = 0 \quad \text{on } \partial\Omega \times [0, T], \end{cases} \tag{9}$$

   where $\delta_k^\varepsilon = O(1)\varepsilon^k$. Note that, by integration,

$$\varepsilon\lambda_k^\varepsilon(t) = \fint_\Omega f(u_k^\varepsilon) + O(1)\varepsilon^{k+2}.$$

2. For each $t \in [0, T]$ and small $\varepsilon > 0$, estimate the lower bound of the spectrum of the self-adjoint operator $-\Delta - \varepsilon^{-2} f'(u_k^\varepsilon(\cdot, t))$; namely, show that for some positive constant C^*,

$$\inf_{0<t\leqslant T} \inf_{0<\varepsilon\leqslant 1} \inf_{\int_\Omega \phi=0,\, \int_\Omega \phi^2=1} \int_\Omega (|\nabla\phi|^2 - \varepsilon^{-2} f'(u_k^\varepsilon(\cdot, t))\phi^2) \geqslant -C^*. \tag{10}$$

3. Set $R = u^\varepsilon - u_k^\varepsilon$ and show that R tends to 0 as $\varepsilon \to 0$.

Note that our analysis establishes the convergence as long as a smooth solution to the limit problem (4) exists, before the formation of singularities.

The organization of the paper is as follows. In Section 2, we present an error estimate required in step 3. In Section 3, we recall a known spectrum estimate [10, 7] that can be adapted here to prove step 2 in the strategy described above. After some preliminary geometrical computations in Section 4, we finally construct the approximate solution in Section 5.

2. Error estimate

The error estimate relies on the following result which is proved in the appendix.

LEMMA 1 Let $\Omega \subset \mathbb{R}^n$ (with $n \geqslant 1$) be a bounded domain. Let $p = \min\{4/n, 1\}$. Then there exists $C = C_n(\Omega) > 0$ such that for every $R \in H^1(\Omega)$ with $\int_\Omega R \, \mathrm{d}x = 0$,

$$\|R\|_{L^{2+p}}^{2+p} \leqslant C \|R\|_{L^2}^{p} \|\nabla R\|_{L^2}^{2}, \tag{11}$$

where $L^q = L^q(\Omega)$ for any $q \geqslant 1$.

Rubinstein–Sternberg [22] established L^∞ bounds for the solution u^ε to Problem (P^ε) using invariant rectangles. Therefore we can modify f outside a compact interval and assume for simplicity that

$$\lim_{u \to \pm\infty} f(u) = \mp\infty$$

and that there exists $M > 0$ such that

$$\forall |u| \geqslant M, \quad u f''(u) \leqslant 0.$$

Since $p \in (0, 1]$, for any $C_0 > 0$ there exists $C = C(C_0, p)$ such that for all $|u| \leqslant C_0$ and $R \in \mathbb{R}$,

$$(f(u+R) - f(u) - f'(u)R)R \leqslant C|R|^{p+2}.$$

Indeed, note that for R in a compact interval, there is $\theta \in (0, 1)$ such that

$$(f(u+R) - f(u) - f'(u)R)R = \frac{f''(u+\theta R)}{2} R^3 \leqslant C|R|^{p+2},$$

whereas for $|R| \to +\infty$, $f(u+R)R \to -\infty$ uniformly in $|u| \leqslant C_0$ so that

$$(f(u+R) - f(u) - f'(u)R)R \leqslant (-f(u) - f'(u)R)R \leqslant CR^2 \leqslant C|R|^{p+2}.$$

LEMMA 2 Assume that $k > \max\{4, n\}$ and $\{u_k^\varepsilon\}_{0<\varepsilon\leqslant 1}$ satisfies (9) and (10) with

$$\|\delta_k^\varepsilon\|_{L^2(\Omega_T)} \leqslant \varepsilon^k, \quad \|u_k^\varepsilon\|_{L^\infty(\Omega_T)} \leqslant 2.$$

Let $\{u^\varepsilon\}_{0<\varepsilon\leqslant 1}$ be solutions to (1) with initial data $\{g^\varepsilon\}$ satisfying

$$g^\varepsilon(\cdot) = u_k^\varepsilon(\cdot, 0) + \phi^\varepsilon(\cdot), \quad \int_\Omega \phi^\varepsilon = 0, \quad \|\phi^\varepsilon\|_{L^2(\Omega)} \leqslant \varepsilon^k. \tag{12}$$

Then for all sufficiently small positive ε,

$$\sup_{0\leqslant t\leqslant T} \|u^\varepsilon(\cdot, t) - u_k^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leqslant C(T)\varepsilon^k.$$

REMARK 1 By a bootstrap argument using inequality (11), one can show that other norms of $u^\varepsilon - u_k^\varepsilon$ tend to 0 as $\varepsilon \searrow 0$.

Proof. In the following, $C > 0$ denotes a generic strictly positive constant independent of $\varepsilon > 0$.

Set $p = \min\{4/n, 1\}$ and $R = u^\varepsilon - u_k^\varepsilon$. Then $\int_\Omega R(x,t)\,\mathrm{d}x = 0$ for all $t \in [0, T]$. Also,

$$R\{f(u^\varepsilon) - f(u_k^\varepsilon) - f'(u_k^\varepsilon)R\} \leqslant C|R|^{2+p}.$$

Multiplying by R the difference of the equations for u^ε and u_k^ε and integrating the resulting equation over Ω gives, after integration by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|R\|_{L^2}^2 + \int_\Omega \{|\nabla R|^2 - \varepsilon^{-2} f'(u_k^\varepsilon)R^2\} \leqslant \int_\Omega \{C\varepsilon^{-2}|R|^{2+p} + |R\,\delta_k^\varepsilon|\}.$$

By (10),

$$\begin{aligned}\int_\Omega &\{|\nabla R|^2 - \varepsilon^{-2} f'(u_k^\varepsilon)R^2\}\\ &= \varepsilon^2 \int_\Omega \{|\nabla R|^2 - \varepsilon^{-2} f'(u_k^\varepsilon)R^2\} + (1-\varepsilon^2)\int_\Omega \{|\nabla R|^2 - \varepsilon^{-2} f'(u_k^\varepsilon)R^2\}\\ &\geqslant \varepsilon^2\|\nabla R\|_{L^2}^2 - C\|R\|_{L^2}^2.\end{aligned}$$

The interpolation estimate (11) then yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|R\|_{L^2}^2 \leqslant C\|\delta_k^\varepsilon\|_{L^2}\|R\|_{L^2} + C\|R\|_{L^2}^2 - \|\nabla R\|_{L^2}^2\{\varepsilon^2 - C_1\varepsilon^{-2}\|R\|_{L^2}^p\}. \tag{13}$$

We define

$$T_\varepsilon := \sup\{t \in [0, T] : \|R(\cdot,\tau)\|_{L^2} \leqslant \varepsilon^{4/p} C_1^{-1/p} \text{ for all } \tau \in [0, t]\}.$$

Since $k > \max\{n, 4\} = 4/p$, it follows that

$$\|R(\cdot, 0)\|_{L^2} \leqslant \varepsilon^k < \varepsilon^{4/p} C_1^{-1/p}$$

for $\varepsilon > 0$ small enough. Therefore, $T_\varepsilon > 0$. Also, from (13), we have, for all $t \in (0, T_\varepsilon]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|R\|_{L^2} \leqslant C(\|R\|_{L^2} + \|\delta_k^\varepsilon\|_{L^2}).$$

Then Gronwall's inequality shows that

$$\sup_{0\leqslant t\leqslant T_\varepsilon} \|R(\cdot,t)\|_{L^2} \leqslant e^{CT_\varepsilon}\left[\|R(\cdot,0)\|_{L^2} + C\int_0^{T_\varepsilon} \|\delta_k^\varepsilon\|_{L^2}\,\mathrm{d}t\right] \leqslant C(T_\varepsilon)\varepsilon^k.$$

Since for $\varepsilon > 0$ small enough

$$C(T_\varepsilon)\varepsilon^k < \frac{1}{2}\varepsilon^{4/p} C_1^{-1/p},$$

we must have $T_\varepsilon = T$. This completes the proof.

3. The linearized operator

3.1 *A spectrum estimate*

Assume that f satisfies (2). Then there is a unique solution $\theta_0 : \mathbb{R} \to (0, 1)$ to

$$\theta_0'' + f(\theta_0) = 0 \quad \text{on } \mathbb{R}, \quad \theta_0(\pm\infty) = \pm 1, \quad \theta_0(0) = 0. \tag{14}$$

The solution satisfies, for $\alpha = \min\{\sqrt{-f'(1)}, \sqrt{-f'(-1)}\}$,

$$D_\rho^m(\theta_0(\rho) \mp 1) = O(e^{-\alpha|\rho|}) \quad \text{as } \pm\rho \to \infty, \quad \forall m \in \mathbb{N}.$$

Let $\theta_1 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be any function satisfying

$$\int_{\mathbb{R}} \theta_0'^2 f''(\theta_0)\theta_1 \ = 0. \tag{15}$$

Let $\Omega^- \subset\subset \Omega$ be a subset with C^3 boundary $\gamma = \partial\Omega^-$. Denote by $d(x)$ the signed distance (negative in Ω^-) from x to γ, and by $s(x)$, for x close to γ, the projection from x on γ along the normal to γ.

We look for the spectrum of the linearized operator of $-\Delta u - \varepsilon^{-2} f'(u)$ around $u = \psi^\varepsilon$ given by

$$\psi^\varepsilon(x) = \begin{cases} \theta_0(d(x)/\varepsilon) + \varepsilon p^\varepsilon(s(x))\theta_1(d(x)/\varepsilon) + O(1)\varepsilon^2 & \text{if } |d(x)| \leqslant \sqrt{\varepsilon}, \\ \pm 1 + O(1)\varepsilon & \text{if } \pm d(x) \geqslant \sqrt{\varepsilon}. \end{cases} \tag{16}$$

We use the following spectrum estimate.

PROPOSITION 1 Let $\gamma \in C^3$, and p^ε and $O(1)$ in (16) be bounded independently of ε. Then there exists a positive constant C^* depending on $\|\gamma\|_{C^3}$, $\|p^\varepsilon\|_{L^\infty}$ and $\|O(1)\|_{L^\infty}$ such that for every $\varepsilon \in (0, 1]$ and $\phi \in H^1(\Omega)$,

$$\int_\Omega \{|\nabla\phi|^2 - \varepsilon^{-2} f'(\psi^\varepsilon)\phi^2\} \geqslant -C^* \int_\Omega \phi^2.$$

This inequality is established in [7]. Note that such a spectrum estimate was proven by de Mottoni and Schatzman in [10], but around a different approximation ψ_ε. A unified treatment of the spectra was later obtained in [7] in a more general situation and has been used in [1, 8]. Let us also mention previous results in this direction obtained in [4] and also in [2] for the 2D Cahn–Hilliard equation.

We define the linearized operator around $\theta_0(\rho)$ acting on $v = v(\rho)$ by

$$\mathcal{L}v := -v'' - f'(\theta_0)v. \tag{17}$$

In our application, θ_1 is the unique solution to

$$\begin{aligned} \mathcal{L}\theta_1 &= 1 - \sigma\theta_0' \quad \text{in } \mathbb{R}, \\ \theta_1(0) = 0, &\quad \sigma := 2\Big/\int_{\mathbb{R}} \theta_0'^2. \end{aligned} \tag{18}$$

Integrating $\theta_0''\mathcal{L}\theta_1$ by parts over $\mathbb{R}$, one can verify that (15) is satisfied.

We remark that the distance function d in (16) can be replaced by a "quasi-distance" function d^ε given by

$$d^\varepsilon(x) = d(x) - \varepsilon h_1(s(x)) - \varepsilon^2 h_2(s(x)) + O(1)\varepsilon^3$$

where h_1 and h_2 are smooth functions of $s \in \gamma$.

3.2 *Solvability condition*

LEMMA 3 Assume that f satisfies (2). Let θ_0 be the solution to (14), define $\alpha = \min\{\sqrt{-f'(1)}, \sqrt{-f'(-1)}\}$ and let $\mathcal{L}$ be defined in (17). Assume that a function $h(\rho, s, t)$ satisfies, as $\rho \to \pm\infty$,

$$D_\rho^m D_s^n D_t^l [h(\rho, s, t) - h^\pm(t)] = O(|\rho|^i e^{-\alpha|\rho|})$$

for some $i \geqslant 0$ and all $(m, n, l) \in \mathbb{N}^3$ and (s, t) in $U \times [0, T]$, where $U \subset \mathbb{R}^{n-1}$. Then

$$\mathcal{L}Q = h(\cdot, s, t) \quad \text{in } \mathbb{R}, \qquad Q(0, s, t) = 0,$$

has a unique bounded solution $Q(\rho, s, t)$ if and only if

$$\forall (s, t) \in U \times [0, T], \qquad \int_{\mathbb{R}} h(\rho, s, t)\theta_0'(\rho) d\rho = 0. \tag{19}$$

If the solution exists, then it satisfies, for all $(m, n, l) \in \mathbb{N}^3$ and $(s, t) \in U \times [0, T]$,

$$D_\rho^m D_s^n D_t^l \left[Q(\rho, s, t) + \frac{h^\pm(t)}{f'(\pm 1)} \right] = O(|\rho|^i e^{-\alpha|\rho|}) \quad \text{as } \rho \to \pm\infty.$$

Proof. Note that $\mathcal{L}\theta_0' = 0$ due to translation invariance, and that the null-space of $\mathcal{L}$ is spanned by θ_0'. Thus the ode $\mathcal{L}Q = h$ can be solved explicitly assuming that h satisfies the condition (19). We omit the details of the proof; see [10, 7, 1].

4. Differential geometry: local coordinates

4.1 *Parametrization around the limit interface*

Let $\Gamma := \bigcup_{t \in [0,T]} (\gamma_t \times \{t\}) \subset \Omega_T$ be the smooth solution to (4) on $[0, T]$ with $\gamma_t|_{t=0} = \gamma_0$ satisfying (5). Let $\Omega_t \subset\subset \Omega$ be the domain enclosed by γ_t, with $\gamma_t = \partial\Omega_t$. For each fixed $t \in [0, T]$, we use $d(x, t)$ to denote the signed distance from x to γ_t (negative in Ω_t). Then $d(\cdot, \cdot)$ is smooth in a tubular neighborhood of the interface. We choose a parametrization of γ_t by $X_0(s, t)$ with $s \in U \subset \mathbb{R}^{n-1}$ so that

$$\left(\frac{\partial X_0}{\partial s_1}, \ldots, \frac{\partial X_0}{\partial s_{n-1}} \right) \tag{20}$$

is a basis of the tangent space to γ_t at $X_0(s, t)$, for each $s \in U$. We denote by $\mathbf{n}(s, t)$ the unit outer normal vector on $\partial\Omega_t = \gamma_t$ so that

$$\mathbf{n}(s, t) = \nabla d(X_0(s, t), t).$$

Up to a suitable multiplication factor $s_1 \to \lambda s_1$, we may assume that

$$\det\left(\mathbf{n}(s, t), \frac{\partial X_0}{\partial s_1}, \ldots, \frac{\partial X_0}{\partial s_{n-1}} \right) = 1. \tag{21}$$

Next for each fixed $t \in [0, T]$, a local parametrization by coordinates $(s, r) \in U \times (-3\delta, 3\delta)$ is obtained by

$$x = X_0(s, t) + r\mathbf{n}(s, t) = X(r, s, t), \tag{22}$$

which defines a local diffeomorphism from $(-3\delta, 3\delta) \times U$ onto the tubular neighbourhood of γ_t,

$$V^t_{3\delta} = \{x \in \Omega : |d(x, t)| < 3\delta\}. \tag{23}$$

We denote the inverse by

$$r = d(x, t), \qquad s = \mathbf{S}(x, t) = (S^1(x, t), \dots, S^{n-1}(x, t)). \tag{24}$$

In particular, since for all fixed $s \in U$, $t \in [0, T]$ and for all $r \in (-3\delta, 3\delta)$,

$$d(X_0(s, t) + r\mathbf{n}(s, t), t) = r,$$

it follows by differentiation with respect to r that for all $r \in (-3\delta, 3\delta)$,

$$\nabla d(X_0(s, t) + r\mathbf{n}(s, t), t) \cdot \mathbf{n}(s, t) = 1.$$

Since

$$|\nabla d(x, t)| = 1 \qquad \text{for } x \text{ close to } \gamma_t, \tag{25}$$

this equality imposes that for all $(r, s) \in (-3\delta, 3\delta) \times U$,

$$\nabla d(X_0(s, t) + r\mathbf{n}(s, t), t) = \mathbf{n}(s, t), \tag{26}$$

proving that ∇d is constant along the normal lines to γ_t. Thus the projection $\mathbf{S}(x, t)$ from x on γ_t is defined by

$$X_0(\mathbf{S}(x, t), t) = x - d(x, t)\nabla d(x, t). \tag{27}$$

It also follows from (25) that for all $i = 1, \dots, n$ and for $x \in V^t_{3\delta}$,

$$\sum_{j=1}^{n} \frac{\partial^2 d}{\partial x_i \partial x_j}(x, t)\frac{\partial d}{\partial x_j}(x, t) = 0. \tag{28}$$

Thus the symmetric matrix $D^2_x d(x, t)$ has eigenvalues $\{\kappa_1, \dots, \kappa_{n-1}, 0\}$ with unit eigenvectors $\{\tau_1, \dots, \tau_{n-1}, \nabla d\}$ forming an orthonormal basis of $\mathbb{R}^n$ for $x \in V^t_{3\delta}$. In particular, for $x \in \gamma_t$, the τ_i are the principal directions and the κ_i are the principal curvatures of γ_t. Note that $\{\tau_1, \dots, \tau_{n-1}\}$ form a basis of the tangent hyperplane to γ_t at $x = X_0(s, t)$. By definition, K and K_{γ_t} are respectively the sum of the principal curvatures and the mean curvature of γ_t, given by

$$K = (n-1)K_{\gamma_t} = \Delta d(X_0(s, t), t) = \sum_{i=1}^{n-1} \kappa_i(s, t). \tag{29}$$

Note that using (28), for $x \in \gamma_t$, we have

$$\begin{aligned} \nabla d \cdot \nabla \Delta d &= \sum_{ij} \frac{\partial}{\partial x_j}\left(\frac{\partial d}{\partial x_i}\frac{\partial^2 d}{\partial x_i \partial x_j}\right) - \sum_{ij}\left(\frac{\partial^2 d}{\partial x_i \partial x_j}\right)^2 \\ &= -\sum_{ij}\left(\frac{\partial^2 d}{\partial x_i \partial x_j}\right)^2 = -\operatorname{Trace}((D^2_x d)^2) = -\sum_{i=1}^{n-1} \kappa_i^2. \end{aligned}$$

We denote

$$b(s,t) = -\nabla d \cdot \nabla \Delta d|_{X_0(s,t),t} = \sum_{i=1}^{n-1} \kappa_i^2(s,t). \tag{30}$$

Let $V(s,t)$ be the normal velocity of the interface at the point $X_0(s,t)$ defined by

$$V(s,t) = (X_0)_t(s,t) \cdot \mathbf{n}(s,t).$$

Using (26), we have

$$V(s,t) = (X_0)_t(s,t) \cdot \nabla d(X_0(s,t) + r\mathbf{n}(s,t), t) = -d_t(X(r,s,t),t), \tag{31}$$

where the last equality follows from differentiating with respect to t the identity

$$d(X_0(s,t) + r\mathbf{n}(s,t), t) = r.$$

It follows that $d_t(x,t)$ is independent of $r = d(x,t)$ for $|r|$ small enough. Changing coordinates from (x,t) to (r,s,t), we associate to any function $\phi(x,t)$ the function

$$\tilde{\phi}(r,s,t) = \phi(X_0(s,t) + r\mathbf{n}(s,t), t) \tag{32}$$

or equivalently

$$\phi(x,t) = \tilde{\phi}(d(x,t), \mathbf{S}(x,t), t).$$

By differentiation we obtain the formulas

$$\begin{aligned} \partial_t \phi &= (-V\partial_r + \partial_t^{\Gamma})\tilde{\phi}, \\ \nabla \phi &= (\mathbf{n}\partial_r + \nabla^{\Gamma})\tilde{\phi}, \\ \Delta \phi &= (\partial_{rr} + \Delta d\, \partial_r + \Delta^{\Gamma})\tilde{\phi}, \end{aligned} \tag{33}$$

with

$$\begin{aligned} \partial_t^{\Gamma}\tilde{\phi} &= \Big(\partial_t + \sum_{i=1}^{n-1} S_t^i \partial_{s^i}\Big)\tilde{\phi}, \\ \nabla^{\Gamma}\tilde{\phi} &= \Big(\sum_{i=1}^{n-1} \nabla S^i \partial_{s^i}\Big)\tilde{\phi}, \\ \Delta^{\Gamma}\tilde{\phi} &= \Big(\sum_{i=1}^{n-1} \Delta S^i \partial_{s^i} + \sum_{i,j=1}^{n-1} \nabla S^i \cdot \nabla S^j \partial_{s^i s^j}\Big)\tilde{\phi}, \end{aligned} \tag{34}$$

where ∇S^i, S_t^i, Δd, d_t are evaluated at $x = X(r,s,t)$ and are viewed as functions of (r,s,t). Note that the mixed derivatives of the form $\partial^2_{rs^j}\tilde{\phi}$ do not appear eventually in (33) because for all $j = 1, \dots, n-1$ and $x \in V_{3\delta}^t$,

$$\nabla S^j(x,t) \cdot \nabla d(x,t) = 0.$$

(This follows from differentiating with respect to r the identity

$$\forall r \in (-3\delta, 3\delta), \qquad S^j(X_0(s,t) + r\mathbf{n}(s,t), t) = s^j,$$

which holds for all fixed $s \in U$, $t \in [0,T]$ and $j = 1, \dots, n-1$.)

4.2 *The stretched variable*

Following the method used in [9], we now define the stretched variable ρ by considering a graph over γ_t of the form

$$\tilde{\gamma}_t^\varepsilon = \{X(r, s, t) : r = \varepsilon h_\varepsilon(s, t),\ s \in U\}, \tag{35}$$

which is (formally) expected to be a representation of the 0-level set at time t of the solution u^ε of Problem (P^ε).

The *stretched variable* ρ is then defined by

$$\rho = \rho^\varepsilon(x, t) = \frac{d(x, t) - \varepsilon h_\varepsilon(\mathbf{S}(x, t), t)}{\varepsilon}, \tag{36}$$

which represents the distance from x to $\tilde{\gamma}_t^\varepsilon$ in the normal direction divided by ε. From now on, we use (ρ, s, t) as independent variables for the inner expansions. The relation between the old and new variables is

$$x = \hat{X}(\rho, s, t) = X(\varepsilon(\rho + h_\varepsilon(s, t)), s, t) = X_0(s, t) + \varepsilon(\rho + h_\varepsilon(s, t))\mathbf{n}(s, t). \tag{37}$$

We associate to any function $w(x, t)$ the function

$$\hat{w}(\rho, s, t) = w(X_0(s, t) + \varepsilon(\rho + h_\varepsilon(s, t))\mathbf{n}(s, t), t) \tag{38}$$

or equivalently

$$w(x, t) = \hat{w}\left(\frac{d(x, t) - \varepsilon h_\varepsilon(\mathbf{S}(x, t), t)}{\varepsilon}, \mathbf{S}(x, t), t\right).$$

Note that

$$\tilde{w}(r, s, t) = \hat{w}\left(\frac{r - \varepsilon h_\varepsilon(s, t)}{\varepsilon}, s, t\right).$$

The relationship between w, $\tilde{w}$, $\hat{w}$ is summarized as follows:

$$w(x, t) = \tilde{w}(\varepsilon\rho + \varepsilon h_\varepsilon, s, t) = \hat{w}(\rho, s, t) = \hat{w}\left(\frac{d - \varepsilon h_\varepsilon(s, t)}{\varepsilon}, s, t\right).$$

In view of (33), we obtain the following formulas for differentiation:

$$\begin{aligned}
\partial_t w &= (-V\varepsilon^{-1} - \partial_t^\Gamma h_\varepsilon)\hat{w}_\rho + \partial_t^\Gamma \hat{w}, \\
\nabla w &= (\mathbf{n}\varepsilon^{-1} - \nabla^\Gamma h_\varepsilon)\hat{w}_\rho + \nabla^\Gamma \hat{w}, \\
\Delta w &= (\varepsilon^{-2} + |\nabla^\Gamma h_\varepsilon|^2)\hat{w}_{\rho\rho} + (\Delta d\, \varepsilon^{-1} - \Delta^\Gamma h_\varepsilon)\hat{w}_\rho - 2\nabla^\Gamma h_\varepsilon \cdot \nabla^\Gamma \hat{w}_\rho + \Delta^\Gamma \hat{w},
\end{aligned} \tag{39}$$

where in the above formula for Δw,

$$\begin{aligned}
\Delta d &= \Delta d|_{x = X_0(s,t) + \varepsilon(\rho + h_\varepsilon(s,t))\mathbf{n}(s,t)} \\
&\approx K(s, t) - \varepsilon(\rho + h_\varepsilon(s, t))b(s, t) + \sum_{i \geqslant 2} \varepsilon^i b_i(s, t)(\rho + h_\varepsilon(s, t))^i,
\end{aligned} \tag{40}$$

with b defined in (30), K defined in (29), and for some given functions $(b_i(s, t))_{i \geqslant 2}$ only depending on γ_t. Therefore

$$\begin{aligned}
\varepsilon^2(\partial_t w - \Delta w) = {} & -\hat{w}_{\rho\rho} - \varepsilon(V + \Delta d)\hat{w}_\rho + \varepsilon^2[(\partial_t^\Gamma \hat{w} - \Delta^\Gamma \hat{w}) - (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon)\hat{w}_\rho] \\
& - \varepsilon^2[|\nabla^\Gamma h_\varepsilon|^2 \hat{w}_{\rho\rho} - 2\nabla^\Gamma h_\varepsilon \cdot \nabla^\Gamma \hat{w}_\rho].
\end{aligned} \tag{41}$$

The Jacobian. For later purposes, we need to compute the Jacobian of the transformation $\hat{X}$. In the (ρ, s) coordinates, $\mathrm{d}x = \varepsilon J^\varepsilon(\rho, s, t)\,\mathrm{d}s\,\mathrm{d}\rho$ where $\varepsilon J^\varepsilon(\rho, s, t) = \partial\hat{X}(\rho, s, t)/\partial(\rho, s)$ is the Jacobian. We now prove

LEMMA 4 For all $\rho \in \mathbb{R}$, $s \in U$ and $t \in [0, T]$,

$$J^\varepsilon(\rho, s, t) = \prod_{i=1}^{n-1} [1 + \varepsilon(\rho + h_\varepsilon(s, t))\kappa_i(s, t)]. \tag{42}$$

Proof. The equality (42) is obtained in two steps. First, we consider the function $X = X(r, s, t)$ defined in (22), denote its Jacobian by $J = J(r, s, t)$ and prove that for all $\rho \in \mathbb{R}$, $s \in U$ and $t \in [0, T]$,

$$J^\varepsilon(\rho, s, t) = J(\varepsilon(\rho + h_\varepsilon(s, t)), s, t). \tag{43}$$

Second, we compute J and show that for all $s \in U$ and all $t \in [0, T]$,

$$J(r, s, t) = \prod_{i=1}^{n-1} [1 + r\kappa_i(s, t)] = 1 + \Delta d(X_0(s, t), t)r + \sum_{i=2}^{n-1} r^i j_i(s, t), \tag{44}$$

for some given functions j_i depending on γ_t. Consequently, (42) follows directly from (43) and (44).

In order to establish (43), note that by definition (37),

$$\hat{X}(\rho, s, t) = X(\varepsilon(\rho + h_\varepsilon(s, t)), s, t),$$

so that

$$\frac{\partial \hat{X}}{\partial \rho} = \varepsilon \frac{\partial X}{\partial r},$$

and for $i = 1, \ldots, n-1$,

$$\frac{\partial \hat{X}}{\partial s_i} = \frac{\partial X}{\partial s_i} + \varepsilon \frac{\partial h_\varepsilon}{\partial s_i} \frac{\partial X}{\partial r}.$$

Thus for all $\rho \in \mathbb{R}$, $s \in U$ and $t \in [0, T]$,

$$\begin{aligned} \varepsilon J^\varepsilon(\rho, s, t) &= \varepsilon \det\left[\frac{\partial X}{\partial r}, \frac{\partial X}{\partial s_1} + \varepsilon \frac{\partial h_\varepsilon}{\partial s_1} \frac{\partial X}{\partial r}, \ldots, \frac{\partial X}{\partial s_{n-1}} + \varepsilon \frac{\partial h_\varepsilon}{\partial s_{n-1}} \frac{\partial X}{\partial r}\right] \\ &= \varepsilon \det\left[\frac{\partial X}{\partial r}, \frac{\partial X}{\partial s_1}, \ldots, \frac{\partial X}{\partial s_{n-1}}\right](\varepsilon(\rho + h_\varepsilon(s, t)), s, t) = \varepsilon J(\varepsilon(\rho + h_\varepsilon(s, t)), s, t), \end{aligned}$$

which is (43).

In order to establish (44), we consider the Hessian matrix of d on γ_t and denote, for $s \in U$ and $t \in [0, T]$,

$$A = A(s, t) = D_x^2 d(X_0(s, t), t),$$

so that (28) reads

$$A \cdot \mathbf{n}(s, t) = 0. \tag{45}$$

Moreover, differentiating the identity (26) at $r = 0$ with respect to s_i for $i = 1, \ldots, n-1$ yields

$$A \cdot \frac{\partial X_0}{\partial s_i} = \frac{\partial \mathbf{n}}{\partial s_i}. \tag{46}$$

From

$$X(r, s, t) = X_0(s, t) + r\mathbf{n}(s, t),$$

it follows, by using (45), that

$$\frac{\partial X}{\partial r} = \mathbf{n}(s, t) = (I_n + rA(s, t))(\mathbf{n}(s, t)),$$

and by (46), for $i = 1, \ldots, n-1$,

$$\frac{\partial X}{\partial s_i} = \frac{\partial X_0}{\partial s_i} + r\frac{\partial \mathbf{n}}{\partial s_i} = (I_n + rA(s,t))\Big(\frac{\partial X_0}{\partial s_i}\Big).$$

Therefore for all $s \in U$ and $t \in [0, T]$,

$$\begin{aligned} J(r,s,t) &= \det\Big[\frac{\partial X}{\partial r}, \frac{\partial X}{\partial s_1}, \ldots, \frac{\partial X}{\partial s_{n-1}}\Big] \\ &= \det\Big[(I_n + rA)(\mathbf{n}), (I_n + rA)\Big(\frac{\partial X_0}{\partial s_1}\Big), \ldots, (I_n + rA)\Big(\frac{\partial X_0}{\partial s_{n-1}}\Big)\Big] \\ &= \det[I_n + rA(s,t)]\det\Big[\mathbf{n}, \frac{\partial X_0}{\partial s_1}, \ldots, \frac{\partial X_0}{\partial s_{n-1}}\Big], \end{aligned}$$

which in view of (21) proves that

$$J(r,s,t) = \det[I_n + rA(s,t)],$$

which yields (44), since the eigenvalues of $A(s,t)$ are $\kappa_1, \ldots, \kappa_{n-1}, 0$.

5. The approximate solution

5.1 *Asymptotic expansions*

Let $k > \max\{2, n/2\}$ be a fixed integer. In what follows, we use $\approx$ to represent asymptotic expansion: $\phi^\varepsilon \approx \sum_{i\geqslant 0} \varepsilon^i \phi_i$ means that for every integer $j \in \mathbb{N}$, we have $\phi^\varepsilon = \sum_{i=0}^{j} \varepsilon^i \phi_i + O(1)\varepsilon^{j+1}$ where $O(1)$ is bounded independently of $\varepsilon \in (0,1)$. For example, since f is smooth, for any bounded sequence $\{b, a_0, a_1, a_2, \ldots\}$, we have the asymptotic expansion

$$\begin{aligned} f\Big(b + \varepsilon\sum_{i\geqslant 0}\varepsilon^i a_i\Big) &\approx \sum_{j\geqslant 0}\varepsilon^j f^{(j)}(b)\Big(\sum_{i\geqslant 0}\varepsilon^i a_i\Big)^j/j\,! \\ &\approx f(b) + \varepsilon f'(b)\sum_{i\geqslant 0}\varepsilon^i a_i + \varepsilon^2\sum_{i\geqslant 0}\varepsilon^i f_i(b, a_0, \ldots, a_i), \end{aligned} \tag{47}$$

where for any fixed b, $f_i(b, a_0, \ldots, a_i)$ is a polynomial in $(a_0, \ldots, a_i)$ of degree $\leqslant i+2$.

Outer expansion. We expand $\lambda^\varepsilon(t)$ and $u^\varepsilon(x,t)$ for $|d(x,t)| \geqslant 3\delta$ as

$$\lambda^\varepsilon(t) \approx \lambda_0(t) + \varepsilon\,\lambda_1(t) + \varepsilon^2\,\lambda_2(t) + \cdots, \tag{48}$$

$$u^\varepsilon(x,t) \approx u_\varepsilon^\pm(t) := \pm 1 + \varepsilon\{u_0^\pm(t) + \varepsilon u_1^\pm(t) + \cdots\}. \tag{49}$$

Substituting (48) and (49) into (7) gives

$$f(u_\varepsilon^\pm(t)) = \varepsilon\lambda^\varepsilon(t) + \varepsilon^2(u_\varepsilon^\pm)'(t),$$

which yields, for all $i \geqslant 0$,

$$u_i^\pm(t) = \{\lambda_i - f_{i-1}(\pm 1, u_0^\pm, \ldots, u_{i-1}^\pm) + (u_{i-2}^\pm)'(t)\}/f'(\pm 1), \tag{50}$$

where $f_{-1} = u_{-2}^\pm = 0$, $u_{-1}^\pm = \pm 1$, and f_i $(i \geqslant 0)$ is defined in (47). Hence, $u_i^\pm$ are determined by $\{\lambda_0, \ldots, \lambda_i\}$.

Inner expansion. We shall assume that h_ε has the asymptotic expansion

$$\varepsilon h_\varepsilon(s,t) \approx \varepsilon h_1(s,t) + \varepsilon^2 h_2(s,t) + \cdots, \quad (s,t) \in U \times [0,T]. \tag{51}$$

Near the interface, we assume that the function $\hat{u^\varepsilon}$ associated to u^ε by (38) has the asymptotic expansion

$$\hat{u^\varepsilon}(\rho,s,t) \approx \theta_0(\rho) + \varepsilon\{u_0(\rho,s,t) + \varepsilon u_1(\rho,s,t) + \cdots\}. \tag{52}$$

In the following, the zeroth order expansion refers to

$$\{d(x,t), \lambda_0(t), u_0(\rho,s,t), u_0^\pm(t)\}$$

and the i-th order expansion refers to

$$\{h_i(s,t), \lambda_i(t), u_i(\rho,s,t), u_i^\pm(t)\}.$$

We shall use $(\ldots)_{i-1}$ to denote a generic function of (ρ,s,t) depending only on expansions of order $\leqslant i-1$.

Matching condition. We suppose that for all $i \in \mathbb{N}$,

$$\forall (s,t) \in U \times [0,T], \qquad \lim_{\rho\to\pm\infty} u_i(\rho,s,t) = u_i^\pm(t). \tag{53}$$

Translation. We also impose, for all $i \in \mathbb{N}$,

$$\forall (s,t) \in U \times [0,T], \qquad u_i(0,s,t) = 0, \tag{54}$$

to be consistent with the assumption that $\rho = 0$ is the 0-level set of u^ε.

5.2 *The u-equation in the new variables*

The equation (7) reads

$$-f(u) = -\varepsilon^2(u_t - \Delta u) - \varepsilon\lambda_\varepsilon(t).$$

In the new variables (ρ,s,t), using (41), it becomes the following equation for the function $u = \hat{u^\varepsilon}$ associated to u^ε by (38):

$$\begin{aligned} -f(u) = u_{\rho\rho} &+ \varepsilon[(V(s,t) + \Delta d)u_\rho - \lambda_\varepsilon] + \varepsilon^2[(\Delta^\Gamma u - \partial_t^\Gamma u) + (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon)u_\rho] \\ &+ \varepsilon^2[|\nabla^\Gamma h_\varepsilon|^2 u_{\rho\rho} - 2\nabla^\Gamma h_\varepsilon \cdot \nabla^\Gamma u_\rho], \end{aligned} \tag{55}$$

where $V(s,t)$ is given by (31) and Δd is expanded using (40) and (51) as

$$\Delta d \approx K(s,t) - \sum_{i\geqslant 1} \varepsilon^i [b(s,t)h_i(s,t) + \delta_{i-1}(\rho,s,t)], \tag{56}$$

with δ_{i-1} depending only on expansions of order $\leqslant i-1$ (in particular, $\delta_0(\rho,s,t) = \rho b(s,t)$). Note that $\delta_{i-1}(\rho,s,t)$ is a polynomial in ρ of degree $\leqslant i$, whose coefficients are polynomials in $(h_1,\ldots,h_{i-1})$ with (s,t)-dependent coefficients.

5.3 *The recursive i-th equations*

The zeroth order expansion. Since θ_0 defined in (14) satisfies

$$-f(\theta_0) = (\theta_0)_{\rho\rho}, \quad \theta_0(\pm\infty) = \pm 1, \quad \theta_0(0) = 0,$$

the equation (55) is satisfied at zeroth order as also are the matching and translation conditions (53)–(54).

The first order expansion. At first order (ε^1), the equation (55) imposes

$$\mathcal{L}u_0 = (K(s,t) + V(s,t))\theta_0'(\rho) - \lambda_0(t), \tag{57}$$

with $\mathcal{L}$ defined in (17). The solvability condition stated in Lemma 3 reads

$$(K(s,t) + V(s,t)) \int_{\mathbb{R}} (\theta_0')^2(z)\, \mathrm{d}z = 2\lambda_0(t),$$

or, by definition of σ in (18),

$$V(s,t) = -K(s,t) + \sigma\lambda_0(t) \quad \text{for } s \in U, \tag{58}$$

also equivalent in view of (31) to

$$d_t = \Delta d - \sigma\lambda_0(t) \quad \text{on } \gamma_t. \tag{59}$$

Moreover equation (57) has then a unique solution satisfying (53)–(54) which is given by

$$u_0(\rho, s, t) = -\lambda_0(t)\theta_1(\rho) \tag{60}$$

for all $(s,t) \in U \times [0,T]$. Note that for all non-negative m, n, l,

$$D_\rho^m D_s^n D_t^l [u_0(\rho, s, t) - u_0^\pm(t)] = O(e^{-\alpha|\rho|}) \quad \text{as } \rho \to \pm\infty.$$

Higher order expansion. Plugging the expansions (47), (51), (52) into (55) and using (58) and (56) leads to the identity

$$\begin{aligned}
&-f(\theta_0) - \varepsilon f'(\theta_0)\Big(\sum_{i\geqslant 0}\varepsilon^i u_i\Big) - \varepsilon^2 \sum_{i\geqslant 0}\varepsilon^i f_i(\theta_0, u_0, \dots, u_i)\\
&\quad = \theta_0'' + \varepsilon\Big(\sum_{i\geqslant 0}\varepsilon^i (u_i)_{\rho\rho}\Big) + \varepsilon\Big[\Big(\sigma\lambda_0(t) - \sum_{i\geqslant 1}\varepsilon^i(bh_i + \delta_{i-1})\Big)u_\rho - \sum_{i\geqslant 0}\varepsilon^i\lambda_i\Big] && (61)\\
&\qquad + \varepsilon^3\sum_{i\geqslant 0}\varepsilon^i(\Delta^\Gamma - \partial_t^\Gamma)u_i - \varepsilon\Big(\sum_{i\geqslant 1}\varepsilon^i(\Delta^\Gamma - \partial_t^\Gamma)h_i\Big)\Big(\theta_0' + \varepsilon\sum_{i\geqslant 0}\varepsilon^i(u_i)_\rho\Big) && (62)\\
&\qquad + \Big[\varepsilon^2|\nabla^\Gamma h_\varepsilon|^2 u_{\rho\rho} - 2\varepsilon\Big(\sum_{i\geqslant 1}\varepsilon^i\nabla^\Gamma h_i\Big)\cdot\nabla^\Gamma u_\rho\Big]. && (63)
\end{aligned}$$

Define the operator $\mathcal{N}^\Gamma$ acting on functions $h = h(s,t)$ by

$$\mathcal{N}^\Gamma h := \partial_t^\Gamma h - \Delta^\Gamma h - bh. \tag{64}$$

We derive below the $(i+1)$-th order expansion for $i \geqslant 1$ and obtain the following result.

LEMMA 5 The cancellation of the term of order ε^{i+1}, with $i \geqslant 1$, in (55) is equivalent to

$$\mathcal{L}u_i = \mathcal{N}^\Gamma(h_i)\theta_0' - \lambda_i(t) + b_{12}(\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_i)\theta_0'' + R_{i-1}(\rho, s, t), \tag{65}$$

with R_{i-1} only depending on expansions of order $\leqslant i-1$. Moreover $R_{i-1}(\rho, s, t)$ is a polynomial in ρ of degree $\leqslant i$ (whose coefficients are polynomials in $(h_1, \dots, h_{i-1}, u_1, \dots, u_{i-1})$ and in their derivatives with respect to (ρ, s, t)).

Proof. First note that using (58), the coefficient of order ε^{i+1} in (61) is

$$\begin{aligned}(u_i)_{\rho\rho} + \sigma\lambda_0(t)(u_{i-1})_\rho - b(s,t)h_i(s,t)\theta_0' - \lambda_i(t) - \delta_{i-1}(s,t)\theta_0' \\ = (u_i)_{\rho\rho} - b(s,t)h_i(s,t)\theta_0' - \lambda_i(t) + (\dots)_{i-1},\end{aligned} \tag{66}$$

with $(\dots)_{i-1}$ depending only on expansions of order $\leqslant i-1$. Moreover in view of (56), it is a polynomial in ρ of degree $\leqslant i$ (whose coefficients are polynomials in $(h_1, \dots, h_{i-1}, u_1, \dots, u_{i-1})$ and in their derivatives with respect to (ρ, s, t)).

Next, in view of (51), the coefficient of order ε^{i+1} in (62) is

$$(\Delta^\Gamma - \partial_t^\Gamma)u_{i-2} + (\partial_t^\Gamma - \Delta^\Gamma)h_i\theta_0' + (\dots)_{i-2} = (\partial_t^\Gamma - \Delta^\Gamma)h_i\theta_0' + (\dots)_{i-2}. \tag{67}$$

To obtain the term of order ε^{i+1} in (63), note that

$$\begin{aligned}\varepsilon^2|\nabla^\Gamma h_\varepsilon|^2 &\approx \Big|\sum_{i\geqslant 1}\varepsilon^i\nabla^\Gamma h_i\Big|^2 \approx \sum_{i\geqslant 2}\varepsilon^i\Big(\sum_{j=1}^{i-1}\nabla^\Gamma h_j \cdot \nabla^\Gamma h_{i-j}\Big) \\ &\approx \varepsilon^2|\nabla^\Gamma h_1|^2 + \sum_{i\geqslant 3}\varepsilon^i(2\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_{i-1} + (\dots)_{i-2}),\end{aligned}$$

so that

$$\varepsilon^2|\nabla^\Gamma h_\varepsilon|^2 u_{\rho\rho} \approx \Big[\varepsilon^2|\nabla^\Gamma h_1|^2 + \sum_{i\geqslant 3}\varepsilon^i(2\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_{i-1} + (\dots)_{i-2})\Big]\Big[\theta_0'' + \varepsilon\sum_{i\geqslant 0}\varepsilon^i(u_i)_{\rho\rho}\Big].$$

Hence the coefficient of order ε^{i+1} in $\varepsilon^2|\nabla^\Gamma h_\varepsilon|^2 u_{\rho\rho}$ is

$$b_{1,2}(\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_i)\theta_0'' + (\dots)_{i-2} \tag{68}$$

with $b_{1,2} = 1$ or 2 for $i = 1$ or $i \geqslant 2$ respectively.

Similarly, the coefficient of order ε^{i+1} in the term $-2\varepsilon^2\nabla^\Gamma h_\varepsilon \cdot \nabla^\Gamma u_\rho$ is

$$\nabla^\Gamma h_{i-1} \cdot \nabla^\Gamma(u_0)\rho + \nabla^\Gamma h_{i-2} \cdot \nabla^\Gamma(u_1)\rho + \dots + \nabla^\Gamma h_1 \cdot \nabla^\Gamma(u_{i-2})\rho$$

where the first term cancels out since $\nabla^\Gamma(u_0)' = 0$ in view of (60); thus it only depends on expansions of order $\leqslant i-2$, so that the term of order ε^{i+1} in (63) is given by (68).

Finally at order ε^{i+1}, with $i \geqslant 1$, using (66)–(68), the equation (55) reads

$$\begin{aligned}-f'(\theta_0)u_i - f_{i-1}(\theta_0, u_0, \dots, u_{i-1}) = (u_i)_{\rho\rho} - \lambda_i(t) \\ + (\partial_t^\Gamma h_i - \Delta^\Gamma h_i - bh_i)\theta_0' + b_{12}(\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_i)\theta_0'' + (\dots)_{i-1}(\rho, s, t),\end{aligned}$$

which is exactly (65), with R_{i-1} only depending on expansions of order $\leqslant i-1$. Moreover $R_{i-1}(\rho, s, t)$ is a polynomial in ρ of degree $\leqslant i$ as described in Lemma 5.

The solvability condition. According to Lemma 3, the equation (65) has a solution if and only if the following solvability condition is satisfied:

$$\forall (s,t) \in U \times [0,T], \qquad \int_{\mathbb{R}} \mathcal{L}u_i(\rho, s, t)\theta_0'(\rho)\,\mathrm{d}\rho = 0. \tag{69}$$

Note that

$$\int_{\mathbb{R}} b_{12}(\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_i)\theta_0''(\rho)\theta_0'(\rho)\,\mathrm{d}\rho = b_{12}(\nabla^\Gamma h_1 \cdot \nabla^\Gamma h_i)(s,t)\int_{\mathbb{R}} \frac{1}{2}[(\theta_0')^2]'(\rho)\,\mathrm{d}\rho = 0,$$

so that (69) reads

$$\mathcal{N}^\Gamma(h_i) = \sigma\lambda_i(t) + r_{i-1}(s,t), \tag{70}$$

with

$$r_{i-1}(s,t) = -\frac{\sigma}{2}\int_{\mathbb{R}} R_{i-1}(\rho, s, t)\theta_0'(\rho)\,\mathrm{d}\rho$$

only depending on expansions of order $\leqslant i-1$. We summarize the construction by induction in the next lemma.

LEMMA 6 Let $k \geqslant 1$ be given. Assume that for all $i \leqslant k-1$, (65) has a solution u_i satisfying

$$D_\rho^m D_s^n D_t^l[u_i(\rho, s, t) - u_i^\pm(t)] = O(\rho^i e^{-\alpha|\rho|}) \quad \text{as } \rho \to \pm\infty. \tag{71}$$

Also assume that for $i = k$, $\{h_i(s,t), \lambda_i(t)\}$ satisfies (70). Then for $i = k$, (65) admits a unique solution satisfying $u_i(0, s, t) = 0$ and (71).

The proof follows from Lemma 3 and an induction argument and is omitted. Just note that in the limit $\rho \to \pm\infty$, the equation $0 = \varepsilon^2(u_t^\varepsilon - \Delta u^\varepsilon) + f(u^\varepsilon) - \varepsilon\lambda^\varepsilon|_{x=\hat{X}(\rho,s,t)}$ becomes the outer expansion equation, so that $u_i(\pm\infty, s, t) = u_i^\pm(t)$. Furthermore since R_{i-1} is a polynomial in ρ of degree $\leqslant i$, (71) is satisfied for each $i \geqslant 0$ and $(s,t) \in U \times [0,T]$.

5.4 *Equation for λ^ε*

To find $\lambda^\varepsilon(t)$, we use an asymptotic expansion for $0 = \int_\Omega u_t^\varepsilon(x,t)\,\mathrm{d}x$. We denote by $\Omega_\varepsilon^\pm(t)$ the two domains separated by $\tilde{\gamma}_t^\varepsilon$ defined in (35), with $\tilde{\gamma}_t^\varepsilon = \partial\Omega_\varepsilon^-(t)$. Hence in view of (36),

$$\begin{aligned}\Omega_\varepsilon^+(t) &= \{x \in \Omega : d(x,t) > 3\delta\} \cup \{x \in V_{3\delta}^t : d(x,t) - \varepsilon h_\varepsilon(\mathbf{S}(x,t),t) > 0\}\\ &= \{x \in \Omega : d(x,t) > 3\delta\} \cup \{x \in V_{3\delta}^t : \rho^\varepsilon(x,t) > 0\}\end{aligned} \tag{72}$$

and

$$\Omega_\varepsilon^-(t) = \Omega \setminus \overline{\Omega_\varepsilon^+(t)} = \{x \in \Omega : d(x,t) < -3\delta\} \cup \{x \in V_{3\delta}^t : \rho^\varepsilon(x,t) < 0\}. \tag{73}$$

We write

$$\int_\Omega u_t^\varepsilon(x,t)\,\mathrm{d}x = \int_{|d(x,t)|\geqslant 3\delta} u_t^\varepsilon(x,t)\,\mathrm{d}x + \int_{|d(x,t)|<3\delta} u_t^\varepsilon(x,t)\,\mathrm{d}x \tag{74}$$

where

$$\int_{|d(x,t)|<3\delta} u_t^\varepsilon(x,t)\,\mathrm{d}x$$
$$= \int_{\{x\in V_{3\delta}^t\,:\,|\rho^\varepsilon(x,t)|\geqslant\delta/\varepsilon\}} u_t^\varepsilon(x,t)\,\mathrm{d}x + \int_{\{x\in V_{3\delta}^t\,:\,|\rho^\varepsilon(x,t)|<\delta/\varepsilon\}} u_t^\varepsilon(x,t)\,\mathrm{d}x. \tag{75}$$

In the following we choose $0<\varepsilon\leqslant\varepsilon_0$ small enough so that for all $\varepsilon\in(0,\varepsilon_0]$,

$$\max_{s\in U,\,t\in[0,T)} |\varepsilon h_\varepsilon(s,t)| \leqslant \delta/2, \tag{76}$$

and consequently

$$|\rho^\varepsilon(x,t)|\geqslant\delta/\varepsilon \;\Rightarrow\; |d(x,t)|\geqslant\delta/2.$$

Thus at points (x,t) where either $|d(x,t)|\geqslant 3\delta$ or $|\rho^\varepsilon(x,t)|\geqslant\delta/\varepsilon$, it follows that $|d(x,t)|\geqslant\delta/2$, so that

$$u_t^\varepsilon(x,t)\approx (u_\varepsilon^+)'(t)\chi_{\{d(x,t)>0\}} + (u_\varepsilon^-)'(t)\chi_{\{d(x,t)<0\}}$$

since exponentially small terms of order $O(e^{-\alpha\delta/(2\varepsilon)})$ do not affect the asymptotic expansion in the ε power series. Note moreover that if $|\rho^\varepsilon(x,t)|\geqslant\delta/\varepsilon$, then $d(x,t)$ and $\rho=\rho^\varepsilon(x,t)$ have the same sign. To simplify the notations, we denote

$$\{|\rho|\geqslant\delta/\varepsilon\} = \{x\in V_{3\delta}^t : |\rho^\varepsilon(x,t)|\geqslant\delta/\varepsilon\},$$
$$\{|\rho|<\delta/\varepsilon\} = \{x\in V_{3\delta}^t : |\rho^\varepsilon(x,t)|<\delta/\varepsilon\}.$$

Therefore in view of (74)–(75),

$$\int_\Omega u_t^\varepsilon(x,t)\,\mathrm{d}x \approx \int_{|d(x,t)|\geqslant3\delta} [(u_\varepsilon^+)'(t)\chi_{\{d>0\}} + (u_\varepsilon^-)'(t)\chi_{\{d<0\}}]\,\mathrm{d}x \tag{77}$$
$$+ \int_{|\rho|\geqslant\delta/\varepsilon} [(u_\varepsilon^+)'(t)\chi_{\{\rho>0\}} + (u_\varepsilon^-)'(t)\chi_{\{\rho<0\}}]\,\mathrm{d}x + \int_{|\rho|<\delta/\varepsilon} u_t^\varepsilon(x,t)\,\mathrm{d}x \tag{78}$$
$$\approx I_1 + \int_{|\rho|<\delta/\varepsilon} [u_t^\varepsilon - (u_\varepsilon^+)'(t)\chi_{\{d(x,t)>0\}} - (u_\varepsilon^-)'(t)\chi_{\{d(x,t)<0\}}]\,\mathrm{d}x, \tag{79}$$

where

$$I_1 = (u_\varepsilon^+)'(t)|\Omega_\varepsilon^+(t)| + (u_\varepsilon^-)'(t)|\Omega_\varepsilon^-(t)|. \tag{80}$$

In the second integral, we make the change of variables given in (37) and substitute the expression for u_t^ε in formula (39) to obtain

$$\int_{|\rho|<\delta/\varepsilon} [u_t^\varepsilon - (u_\varepsilon^+)'(t)\chi_{\{\rho>0\}} - (u_\varepsilon^-)'(t)\chi_{\{\rho<0\}}]\,\mathrm{d}x$$
$$= \int_{0<\rho<\delta/\varepsilon} \partial_t^\Gamma[\hat{u^\varepsilon}(\rho,s,t) - u_\varepsilon^+(t)]\varepsilon J^\varepsilon(\rho,s,t)\,\mathrm{d}\rho\,\mathrm{d}s$$
$$+ \int_{-\delta/\varepsilon<\rho<0} \partial_t^\Gamma[\hat{u^\varepsilon}(\rho,s,t) - u_\varepsilon^-(t)]\varepsilon J^\varepsilon(\rho,s,t)\,\mathrm{d}\rho\,\mathrm{d}s$$
$$+ \int_{|\rho|<\delta/\varepsilon} (-V\varepsilon^{-1} - \partial_t^\Gamma h_\varepsilon)\frac{\partial\hat{u^\varepsilon}}{\partial\rho}\varepsilon J^\varepsilon(\rho,s,t)\,\mathrm{d}\rho\,\mathrm{d}s. \tag{81}$$

Finally,

$$\int_\Omega u_t^\varepsilon(x,t)\,\mathrm{d}x \approx I_1 + I_2 + I_3,$$

where

$$I_2 = \int_{|\rho|<\delta/\varepsilon} \partial_t^\Gamma [\hat{u^\varepsilon}(\rho,s,t) - u_\varepsilon^+(t)\chi_{\{\rho>0\}} - u_\varepsilon^-(t)\chi_{\{\rho<0\}}]\varepsilon J^\varepsilon(\rho,s,t)\,\mathrm{d}\rho\,\mathrm{d}s \tag{82}$$

and

$$I_3 = \int_{|\rho|<\delta/\varepsilon} (-V\varepsilon^{-1} - \partial_t^\Gamma h_\varepsilon)\frac{\partial \hat{u^\varepsilon}}{\partial\rho}\varepsilon J^\varepsilon(\rho,s,t)\,\mathrm{d}\rho\,\mathrm{d}s. \tag{83}$$

The calculation for I_1. The boundary of $\Omega_\varepsilon^-(t)$ is $\tilde{\gamma}_t^\varepsilon$, which according to (35) is given in local coordinates (r,s) by $r = \varepsilon h_\varepsilon(s,t)$. Therefore in view of (44),

$$\begin{aligned}|\Omega_\varepsilon^-(t)| &= |\Omega_t| + \int_U \int_0^{\varepsilon h_\varepsilon(s,t)} J(r,s,t)\,\mathrm{d}r\,\mathrm{d}s \\ &\approx |\Omega_t| + \sum_{i\geqslant 1}\varepsilon^i\left\{\int_U h_i(s,t)\,\mathrm{d}s + (\ldots)_{i-1}\right\},\end{aligned}$$

where $(\ldots)_{i-1}$ only depends on expansions of order $\leqslant i-1$. Hence

$$|\Omega_\varepsilon^+(t)| = |\Omega| - |\Omega_\varepsilon^-(t)| \approx |\Omega| - |\Omega_t| - \sum_{i\geqslant 1}\varepsilon^i\left\{\int_U h_i(s,t)\,\mathrm{d}s + (\ldots)_{i-1}\right\}.$$

From the outer expansion (49), it follows that

$$(u_\varepsilon^\pm)'(t) \approx \varepsilon\sum_{i\geqslant 0}\varepsilon^i(u_i^\pm)'(t) \approx \sum_{i\geqslant 1}\varepsilon^i(u_{i-1}^\pm)'(t),$$

with $(u_{i-1}^\pm)'(t)$ given by (50) and depending only on expansions of order $\leqslant i-1$. Therefore

$$I_1 = (u_\varepsilon^+)'(t)|\Omega_\varepsilon^+(t)| + (u_\varepsilon^-)'(t)|\Omega_\varepsilon^-(t)| \approx \sum_{i\geqslant 1}\varepsilon^i(\ldots)_{i-1}$$

where $(\ldots)_{i-1}$ depends only on expansions of order $\leqslant i-1$.

The calculation for I_2. Using the expression for $\partial_t^\Gamma\hat{u^\varepsilon}$ in formula (34) and (71), we compute

$$\begin{aligned}&\partial_t^\Gamma[\hat{u^\varepsilon}(\rho,s,t) - u_\varepsilon^+(t)\chi_{\{\rho>0\}} - u_\varepsilon^-(t)\chi_{\{\rho<0\}}] \\ &\qquad\approx \varepsilon\sum_{i\geqslant 1}\varepsilon^i\partial_t^\Gamma[u_i(\rho,s,t) - u_i^+(t)\chi_{\{\rho>0\}} - u_i^-(t)\chi_{\{\rho<0\}}] \\ &\qquad\approx \varepsilon\sum_{i\geqslant 1}\varepsilon^i\Big(\partial_t + \sum_{j=1}^{n-1}S_t^j\partial_{s^j}\Big)[u_i(\rho,s,t) - u_i^+(t)\chi_{\{\rho>0\}} - u_i^-(t)\chi_{\{\rho<0\}}] \\ &\qquad\approx \sum_{i\geqslant 2}\varepsilon^i O(\rho^{i-1}e^{-\alpha|\rho|})\end{aligned}$$

with $O(\rho^{i-1}e^{-\alpha|\rho|})$ depending only on expansions of order $\leqslant i-1$. Therefore by definition of I_2 in (82),

$$I_2 \approx \sum_{i\geqslant 3} \varepsilon^i (\dots)_{i-2},$$

where $(\dots)_{i-2}$ depends only on expansions of order $\leqslant i-2$.

The calculation for I_3. Using the expansions

$$\frac{\partial \hat{u^\varepsilon}}{\partial\rho} \approx \theta_0' + \varepsilon \sum_{i\geqslant 0} \varepsilon^i \frac{\partial u_i}{\partial\rho},$$

$$-V - \varepsilon\partial_t^\Gamma h_\varepsilon = d_t(X_0(s,t),t) - \sum_{i\geqslant 1} \varepsilon^i \partial_t^\Gamma h_i$$

and rewriting the expression for J^ε in (42) as

$$\begin{aligned} J^\varepsilon(\rho,s,t) &= \prod_{i=1}^{n-1} [1+\varepsilon(\rho+h_\varepsilon(s,t))\kappa_i(s,t)] \\ &\approx 1 + \Delta d(X_0(s,t),t)\varepsilon(\rho+h_\varepsilon(s,t)) + \sum_{i\geqslant 2}\varepsilon^i(\dots)_{i-1}, \end{aligned}$$

with $(\dots)_{i-1}$ depending only on expansions of order $\leqslant i-1$, we obtain

$$\begin{aligned} &(-V - \varepsilon\partial_t^\Gamma h_\varepsilon)\frac{\partial \hat{u^\varepsilon}}{\partial\rho} J^\varepsilon(\rho,s,t) \\ &\quad\approx d_t(X_0(s,t),t)\theta_0'(\rho) + \sum_{i\geqslant 1}\varepsilon^i\theta_0'(\rho)(-\partial_t^\Gamma h_i + d_t(X_0(s,t),t)h_i\Delta d) + \sum_{i\geqslant 1}\varepsilon^i(\dots)_{i-1} \end{aligned}$$

so that

$$\begin{aligned} I_3 &\approx \int_U \int_{\mathbb{R}} \Big\{\theta_0' d_t(s,t) + \sum_{i\geqslant 1}\varepsilon^i[\theta_0'(-\partial_t^\Gamma h_i + d_t(s,t)\Delta d(s,t)h_i) + (\dots)_{i-1}]\Big\}\,\mathrm{d}\rho\,\mathrm{d}s \\ &\approx 2\int_U d_t(s,t)\,\mathrm{d}s + \sum_{i\geqslant 1}\varepsilon^i\Big\{2\int_U\{-\partial_t^\Gamma h_i + (d_t\Delta d)h_i\}\,\mathrm{d}s + (\dots)_{i-1}\Big\}. \end{aligned}$$

Finally, substituting d_t and $\partial_t^\Gamma h_i$ by (59) and (70), and using $\int_U \Delta^\Gamma h_i\,\mathrm{d}s = 0$, we obtain

$$\frac{1}{2}\int_\Omega u_t^\varepsilon \approx \int_U (\Delta d - \sigma\lambda_0)\,\mathrm{d}s + \sum_{i\geqslant 1}\varepsilon^i\Big\{\int_U [(-b + d_t\Delta d)h_i - \sigma\lambda_i]\,\mathrm{d}s + (\dots)_{i-1}\Big\}.$$

Thus the condition $\int_\Omega u_t^\varepsilon\,\mathrm{d}x \approx 0$ is equivalent to

$$\sigma\lambda_0(t) = \overline{\Delta d(\cdot,t)}, \tag{84}$$

$$\sigma\lambda_i(t) = -\overline{[b(\cdot,t) - d_t(\cdot,t)\Delta d(\cdot,t)]h_i(\cdot,t)} + \Lambda_{i-1}(t), \quad i\geqslant 1, \tag{85}$$

where $\Lambda_{i-1}(t)$ depends only on expansions of order $\leqslant i-1$, and

$$\overline{\phi(\cdot)} := \frac{1}{|U|}\int_U \phi = \frac{1}{|\gamma_t|}\int_{\gamma_t} \phi,$$

using that $J(0, s, t) = 1$. Hence we obtain closed systems for $d, h_1, \ldots, h_i$, namely

$$d_t(s,t) = \Delta d(s,t) - \overline{\Delta d(s,t)}, \tag{86}$$

$$\partial_t^\Gamma h_i = \Delta^\Gamma h_i + bh_i - \overline{[b(\cdot,t) - d_t(\cdot,t)\Delta(\cdot,t)]h_i(\cdot,t)} + \Lambda_{i-1}(t) \tag{87}$$

on $U \times [0, T]$.

5.5 *Construction of expansions of arbitrary order*

We can now use induction to construct each order of expansion as follows:

1) *Zeroth order*. Given a smooth initial interface γ_0, (86) is equivalent to the volume preserving mean curvature flow (4). By the result established in [14], there is a time $T > 0$ such that there is a unique smooth solution on a time interval $[0, T]$. Consequently, $\Gamma = \bigcup_{0\leqslant t\leqslant T}(\gamma_t \times \{t\})$ and the modified distance function d are well defined. Set $\lambda_0(t)$ as in (84), $u_0(\rho, s, t)$ as in (60) and $u_0^\pm(t) = \lambda_0(t)/f'(\pm 1)$ as in (50). We obtain the zeroth order expansion $\{d(x,t), \lambda_0(t), u_0(\rho, s, t), u_0^\pm(t)\}$.

2) *Higher order expansion*. Fix $i \geqslant 1$. Assume that all expansions of order $\leqslant i-1$ are constructed. Then $\Lambda_{i-1}(t)$ in (87) is known. Since γ_t is a smooth hypersurface without boundary, it follows from standard parabolic PDE theory [21] that (87) admits a unique smooth solution (assuming an initial condition such as $h_i(\cdot, 0) = 0$ on U is given). Consequently, we can define $\lambda_i(t)$ as in (85), $u_i^\pm(t)$ as in (50) and u_i as the solution of (65) given by Lemma 6. This gives the i-th order expansion $\{h_i(s,t), \lambda_i(t), u_i(\rho, s, t), u_i^\pm(t)\}$ and completes the induction.

5.6 *Construction of the approximate solution*

We now fix an arbitrary positive integer $k > \max(n, 4)$. We construct an approximate solution u_k^ε such that Lemma 2 can be applied.

Let $\delta > 0$ be a small fixed constant such that $d(x,t)$ is smooth in the 3δ-neighborhood of Γ, and for each $t \in [0, T]$, γ_t is a distance at least 3δ away from $\partial\Omega$. We define

$$\rho_k^\varepsilon(x,t) = \varepsilon^{-1}\Big\{d(x,t) - \sum_{i=1}^{k+1} \varepsilon^i h_i(\mathbf{S}(x,t), t)\Big\},$$

$$u_{\varepsilon,k}^{\text{in}}(x,t) = \theta_0(\rho_k^\varepsilon) + \varepsilon \sum_{i=0}^{k+1} \varepsilon^i u_i(\rho_k^\varepsilon(x,t), \mathbf{S}(x,t), t),$$

$$u_{\varepsilon,k,\pm}^{\text{out}}(t) = \pm 1 + \varepsilon \sum_{i=0}^{k+1} \varepsilon^i u_i^\pm(t),$$

$$\lambda_k^\varepsilon(t) = \sum_{i=0}^{k+1} \varepsilon^i \lambda_i(t).$$

We note that ρ_k^ε and $u_{\varepsilon,k}^{\text{in}}$ are smooth in the 3δ-neighborhood of Γ.

Now let $\zeta \in C^\infty(\mathbb{R})$ be a cut-off function (depending only on δ) satisfying

$$\zeta(s) = 1 \quad \text{if } |s| \leqslant \delta, \quad \zeta(s) = 0 \quad \text{if } |s| > 2\delta,$$
$$0 \leqslant s\zeta'(s) \leqslant 4 \quad \text{if } \delta \leqslant |s| \leqslant 2\delta.$$

We define the needed approximate solution u_k^ε by

$$\tilde{u}_k^\varepsilon(x,t) := \zeta(d(x,t))u_k^{\text{in}} + [1 - \zeta(d(x,t))]\{u_{\varepsilon,k,+}^{\text{out}}\chi_{\{d(x,t)>0\}} + u_{\varepsilon,k,-}^{\text{out}}\chi_{\{d(x,t)<0\}}\},$$
$$u_k^\varepsilon(x,t) := \tilde{u}_k^\varepsilon(x,t) + \fint_\Omega \{\tilde{u}_k^\varepsilon(\cdot,0) - \tilde{u}_k^\varepsilon(\cdot,t)\}$$

for all $(x,t) \in \bar{\Omega} \times [0,T]$.

The admissible initial data g^ε are then defined as in (12). Then by construction $(u_k^\varepsilon, \lambda_k^\varepsilon)$ is an approximation of order k satisfying the assumptions in Lemma 2. Here we just remark that (i) in the set $\{(x,t) : \delta \leqslant \pm d(x,t) \leqslant 2\delta\}$, the limiting behavior (71) guarantees that $u_k^\varepsilon(x,t) = u_{\varepsilon,k,\pm}^{\text{out}}(t) + O(e^{-\alpha\delta/(4\varepsilon)})$, valid also after differentiation, (ii) $\partial_n u_k^\varepsilon = 0$ on $\partial\Omega_T$ since u_k^ε is a function of t near $\partial\Omega_T$, and (iii) the correction

$$\fint_\Omega \{\tilde{u}_k^\varepsilon(\cdot,0) - \tilde{u}_k^\varepsilon(\cdot,t)\} = -\fint_\Omega \int_{[0,t]} (\tilde{u}_k^\varepsilon)_t(y,\tau)\,\mathrm{d}\tau\,\mathrm{d}y$$

is of order $O(\varepsilon^{k+1})$, valid also after differentiation. The remaining part of the proof follows the same lines as in [1].

This completes the construction of the approximate solution. The proof of Theorem 1 then follows from the conclusion of Lemma 2 by letting $\varepsilon \to 0$.

Appendix

Proof of Lemma 1. We first consider the case $n \geqslant 4$ so that $p = 4/n$. The Gagliardo–Nirenberg–Sobolev inequality (see [15, Theorem 2, p. 265]) states that there exists $C > 0$ such that for every $R \in H^1(\Omega)$,

$$\|R\|_{L^{2^*}} \leqslant C\|R\|_{H^1},$$

with $2^* = 2n/(n-2)$. Using the Poincaré–Wirtinger inequality (see [15, Theorem 1, p. 275]), it follows that there exists $C > 0$ such that for every $R \in H^1(\Omega)$ with $\int_\Omega R\,\mathrm{d}x = 0$,

$$\|R\|_{L^{2^*}} \leqslant C\|\nabla R\|_{L^2}. \tag{88}$$

Using Hölder's inequality, we have

$$\|R\|_{L^{2+p}}^{2+p} = \int_\Omega |R|^2|R|^p \leqslant \left(\int_\Omega |R|^{2\beta}\right)^{1/\beta}\left(\int_\Omega |R|^{p\beta'}\right)^{1/\beta'}$$

and we choose

$$\beta = \frac{n}{n-2} = \frac{2^*}{2}, \qquad \beta' = \frac{n}{2}$$

to obtain

$$\|R\|_{L^{2+p}}^{2+p} \leqslant \|R\|_{L^{2^*}}^2 \|R\|_{L^2}^p.$$

Combined with (88), this yields the inequality

$$\|R\|_{L^{2+p}}^{2+p} \leqslant C\|R\|_{L^2}^{p}\|\nabla R\|_{L^2}^{2},$$

which is the conclusion of Lemma 1.

Next we consider the case that $1 \leqslant n \leqslant 3$ so that $p = 1$. Schwarz's inequality then gives

$$\|R\|_{L^3}^3 = \int_\Omega |R|^2|R| \leqslant \|R\|_{L^4}^2\|R\|_{L^2}.$$

For $n = 1, 2, 3$, by the Sobolev imbedding, $H^1 \subset L^4$, so that there exists $C > 0$ such that for every $R \in H^1(\Omega)$,

$$\|R\|_{L^4} \leqslant C\|R\|_{H^1}.$$

Using again the Poincaré–Wirtinger inequality, we finally deduce that there exists $C > 0$ such that for every $R \in H^1(\Omega)$ with $\int_\Omega R\,\mathrm{d}x = 0$,

$$\|R\|_{L^3}^3 \leqslant C\|\nabla R\|_{L^2}^2\|R\|_{L^2},$$

which concludes the proof of Lemma 1.

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REFERENCES

1. ALIKAKOS, N. D., BATES, P. W., & CHEN, X. F. Convergence of the Cahn–Hilliard equation to the Hele-Shaw model. *Arch. Ration. Mech. Anal.* **128** (1994), 165–205. Zbl 0828.35105 MR 1308851
2. ALIKAKOS, N. D., & FUSCO, G. The spectrum of the Cahn–Hilliard operator for generic interface in higher space dimensions. *Indiana Univ. Math. J.* **42** (1993), 637–674. Zbl 0798.35123 MR 1237062
3. BARLES, G., SONER, H. M., & SOUGANIDIS, P. E. Front propagation and phase field theory. *SIAM J. Control Optim.* **31** (1993), 439–469. Zbl 0785.35049 MR 1205984
4. BATES, P. W., & FIFE, P. C. Spectral comparison principles for the Cahn–Hilliard and phase-field equations and time scales for coarsening. *Phys. D* **43** (1990), 335–348. Zbl 0706.58074 MR 1067916
5. BRONSARD, L., & STOTH, B. Volume preserving mean curvature flow as a limit of nonlocal Ginzburg–Landau equation. *SIAM J. Math. Anal.* **28** (1997), 769–807. Zbl 0874.35009 MR 1453306
6. CHEN, X. F. Generation and propagation of interfaces for reaction-diffusion equations. *J. Differential Equations* **96** (1992), 116–141. Zbl 0765.35024 MR 1153311
7. CHEN, X. F. Spectrums for the Allen–Cahn, Cahn–Hilliard, and phase field equations for generic interface. *Comm. Partial Differential Equations* **19** (1994), 1371–1395. Zbl 0811.35098 MR 1284813
8. CHEN, X. F., & CAGINALP, G. Convergence of the phase field model to its sharp interface limits. *Eur. J. Appl. Math.* **9** (1998), 417–445. Zbl 0930.35024 MR 1643668
9. CHEN, X. F., CAGINALP, G., & ECK, C. A rapidly converging phase field model. *Discrete Contin. Dynam. Systems* **15** (2006), 1017–1034. Zbl 1119.80009

10. DE MOTTONI, P., & SCHATZMAN, M. Development of interfaces in $\mathbb{R}^n$. *Proc. Roy. Soc. Edinburgh Sect. A* **116** (1990), 207–220. Zbl 0725.35009 MR 1084732
11. DE MOTTONI, P., & SCHATZMAN, M. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.* **347** (1995), 1533–1589. Zbl 0840.35010 MR 1672406
12. ELLIOTT, C. M., & GARCKE, H. Existence results for diffusive surface motion laws. *Adv. Math. Sci. Appl.* **7** (1997), 467–490. Zbl 0876.35050 MR 1454678
13. ESCHER, J., & ITO, K. Some dynamic properties of volume preserving curvature driven flows. *Math. Ann.* **333** (2005), 213–230. Zbl 1083.53059 MR 2169834
14. ESCHER, J., & SIMONETT, G. The volume preserving mean curvature flow near spheres. *Proc. Amer. Math. Soc.* **126** (1998), 2789–2796. Zbl 0909.53043 MR 1485470
15. EVANS, L. C. *Partial Differential Equations*. Grad. Stud. in Math. 19, Amer. Math. Soc. (2002).
16. EVANS, L. C., SONER, H. M., & SOUGANIDIS, P. E. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* **45** (1992), 1097–1123. Zbl 0801.35045 MR 1177477
17. GOLOVATY, D. The volume preserving motion by mean curvature as an asymptotic limit of reaction-diffusion equations. *Quart. Appl. Math.* **55** (1997), 243–298. Zbl 0878.35059 MR 1447577
18. HENRY, D. *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Math. 840, Springer (1981). Zbl 0456.35001 MR 0610244
19. HUISKEN, G. The volume preserving mean curvature flow. *J. Reine Angew. Math.* **382** (1987), 35–48. Zbl 0621.53007 MR 0921165
20. ILMANEN, T. Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* **38** (1993), 417–461. Zbl 0784.53035 MR 1237490
21. LADYZHENSKAYA, O. A., SOLONNIKOV, V. A., & URALTSEVA, N. N. *Linear and Quasilinear Equations of Parabolic Type*. Transl. Math. Monogr. 23, Amer. Math. Soc. (1967). Zbl 0164.12302 MR 0241822
22. RUBINSTEIN, J., & STERNBERG, P. Nonlocal reaction-diffusion equations and nucleation. *IMA J. Appl. Math.* **48** (1992), 249–264. Zbl 0763.35051 MR 1167735