Rigorous derivation of the Kuramoto–Sivashinsky equation from a 2D weakly nonlinear Stefan problem

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We are interested in a rigorous derivation of the Kuramoto–Sivashinsky (K-S) equation from a free boundary problem. As a paradigm, we consider a two-dimensional Stefan problem in a strip, a simplified version of a solid-liquid interface model. Near the instability threshold, we introduce a small parameter ε and define rescaled variables accordingly. At fixed ε, our method is based on: definition of a suitable linear 1D operator, projection with respect to the longitudinal coordinate only, and the Lyapunov–Schmidt method. As a solvability condition, we derive a self-consistent parabolic equation for the front. We prove that, starting from the same configuration, the latter remains close to the solution of K-S on a fixed time interval, uniformly in ε sufficiently small.

1. Introduction

A very challenging problem in free boundary problems is the derivation of a single equation for the interface or moving front which captures the dynamics of the system, at least asymptotically, when a suitable parameter ε tends to 0. The Kuramoto–Sivashinsky equation, which we abbreviate hereafter as the K-S equation, or simply K-S,

$$\Phi_\tau + \nu\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0, \tag{1.1}$$

appears in a variety of domains in physics and chemistry, where it models cellular instabilities, pattern formation, turbulence phenomena and transition to chaos. Among many references, we refer to the pioneering papers [12, 10]. The former was historically the first asymptotical derivation of

K-S, in the framework of the Near Equidiffusive Flames model, in short NEF, in combustion theory. The latter provided the first comprehensive computational study. We also refer the reader to the book [14] and its extensive bibliography. Our purpose here is to provide some rigorous mathematical commentary on the derivation of this well-known model from a two-dimensional Stefan problem. We point out that our method might work for other systems allowing scale separation and thereby reduction of the effective dimensionality of the problem. In this respect, we have considered the NEF problem in [5, 6].

As one would surmise at the outset, the K-S model involves a balance between several effects. Roughly speaking, K-S arises when the competing effects of a destabilizing linear part and a stabilizing nonlinearity are the dominant processes in physical reality. The linear instability is itself the result of a competition between two linear operators, $\mathcal{A} = D_{\eta\eta}$ and $\nu\mathcal{A}^2$ (we call $\nu\mathcal{A}^2 + \mathcal{A}$ the *Kuramoto–Sivashinsky linear operator*).

Put another way, the K-S equation is the simplest, and indeed a paradigm system in which these effects compete equally. It is this dominant balance that is explored rigorously in the present essay. It will turn out that, in deriving K-S as an asymptotic limit of more complex systems, only certain type of terms contribute to the lowest order of approximation. Other types of terms will lead to higher order perturbations. In a forthcoming paper, we intend to consider the effects of these higher order perturbations on the basic K-S system.

As a paradigm two-dimensional problem (see [3, 2, 1] for the one-dimensional case and the Q-S equation in flame front dynamics), we consider a solid-liquid interface model introduced by Frankel in [8]. The solidification front is represented by $x = \xi(t, y)$. The liquid phase occurs when $x < \xi(t, y)$, the solid one when $x > \xi(t, y)$. The dynamics of heat is described by the heat conduction equation

$$T_t(t, x, y) = \Delta T(t, x, y), \quad x \neq \xi(t, y), \tag{1.2}$$

where $y \in [-\ell/2, \ell/2]$ with periodic boundary conditions. At $-\infty$, the temperature of the liquid is normalized to 0. At the front $x = \xi(t, y)$ there are two conditions. First, the balance of energy at the interface is given by the jump

$$\left[\frac{\partial T}{\partial n}\right] = V_n, \tag{1.3}$$

where V_n is the normal velocity. Second, according to the Gibbs–Thompson law, the nonequilibrium interface temperature is defined by

$$T = 1 - \gamma\kappa + r(V_n), \tag{1.4}$$

where the melting temperature has been normalized to 1, κ is the interface curvature and the positive constant γ represents the solid-liquid surface tension. The function r is increasing and such that $r(-1) = 0$, $r'(-1) = 1$ (see [8, 9]). Hereafter, we assume that $r - 1$ is linear and we replace the curvature by the second-order derivative. Therefore, (1.4) becomes

$$T = 1 - \gamma\xi_{yy} + V_n + 1. \tag{1.5}$$

It is not difficult to see that system (1.2), (1.3), (1.4) admits a one-phase planar travelling wave (TW) solution $(T(t, x), \xi(t)) = (\Theta(t + x), -t)$, where $\Theta(z) = e^z$ for $z < 0$, and $\Theta(z) = 1$ for $z > 0$.

As usual, we fix the free boundary. We set $\xi(t, y) = -t + \varphi(t, y)$, $x' = x - \xi(t, y)$ and we will omit primes. In this new framework, (1.2) reads

$$T_t + (1 - \varphi_t)T_x = \Delta_\varphi T, \quad x \neq 0,$$

where $\Delta_\varphi = (1 + (\varphi_y)^2)D_{xx} + D_{yy} - \varphi_{yy}D_x - 2\varphi_y D_{xy}$. The front is now fixed at $x = 0$. The first condition (1.3) reads

$$\varphi_t = 1 + (1 + (\varphi_y)^2)[T_x], \tag{1.6}$$

whereas we replace (1.5) by

$$T = 1 - \gamma\varphi_{yy} + \varphi_t + \frac{1}{2}(\varphi_y)^2.$$

Introducing the temperature perturbation $u = T - \Theta$, the problem for the couple (u, φ) reads

$$u_t + (1 - \varphi_t)u_x - \Delta_\varphi u - \varphi_t\Theta_x = (\Delta_\varphi - \Delta)\Theta, \quad x \neq 0, \tag{1.7}$$

where

$$(\Delta_\varphi - \Delta)\Theta = \{(\varphi_y)^2 - \varphi_{yy}\}e^x\chi_{(-\infty,0)} = ((\varphi_y)^2 - \varphi_{yy})\Theta_x.$$

As in [4], we make further simplifications: (i) we consider a *quasi-steady* problem, dropping the time derivative u_t in (1.7); (ii) we take a linearized problem for u; (iii) we limit ourselves to considering only the second-order terms in the jump conditions at $x = 0$. Actually, as observed in similar problems (see [3]), not far from the instability threshold the time derivative in the temperature equation has a relatively small effect on the solution. Our final system reads

$$u_x - \Delta u - \varphi_t\Theta_x = (\Delta_\varphi - \Delta)\Theta, \quad x \neq 0, \tag{1.8}$$

$$\varphi_t = [u_x] - (\varphi_y)^2, \tag{1.9}$$

$$u_{|x=0} = -\gamma\varphi_{yy} + \varphi_t + \frac{1}{2}(\varphi_y)^2. \tag{1.10}$$

For the convenience of the reader, we recall the main results of [4], where we considered problem (1.8)–(1.10) in the strip $\mathbb{R} \times [-\ell/2, \ell/2]$, with periodic boundary conditions prescribed at $y = \pm\ell/2$. More precisely, we studied the stability of the TW solution and proved the following result: there exists $\gamma_c < 1$ such that

 (i) for $\gamma > \gamma_c$, the TW solution to problem (1.8)–(1.10) is orbitally stable (with asymptotic phase);
(ii) for $0 < \gamma < \gamma_c$, the TW is unstable.

We also showed that $\gamma_c = 1 - 3\lambda_1(\ell) + \cdots$, where $-\lambda_1(\ell) = -4\pi^2/\ell^2$ is the largest eigenvalue of the realization of D_{yy} in $C([-\ell/2, \ell/2])$ with periodic boundary conditions and zero average.

The main tool is the derivation of a self-consistent equation for the front φ:

$$\varphi_t + G((\varphi_y)^2) = \Omega\varphi, \quad |y| \leqslant \ell/2, \tag{1.11}$$

where both Ω and G are linear pseudo-differential operators whose symbols ω_k and g_k are explicit and $g_0 = 1/2$, which is reminiscent of K-S. If we think *formally* of (1.11) in the whole space (i.e. $\ell = +\infty$), then ω_k is the growth rate which expands, for small wave number k, as

$$\omega_k = (1 - \gamma)k^2 + (\gamma - 4)k^4 + \cdots,$$

with exchange of stability at $\gamma = 1$. Therefore, when γ is close to unity, but smaller, it is natural to introduce a small parameter $\varepsilon > 0$, setting

$$\gamma = 1 - \varepsilon,$$

and define the rescaled dependent and independent variables accordingly:

$$t = \tau/\varepsilon^2, \quad y = \eta/\sqrt{\varepsilon}, \quad u = \varepsilon^2 v, \quad \varphi = \varepsilon\psi. \tag{1.12}$$

Then we anticipate, in the limit $\varepsilon \to 0$, that $\psi \simeq \Phi$, where Φ solves the following K-S equation (with $\nu = 3$):

$$\Phi_\tau + 3\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0. \tag{1.13}$$

This is what we have to establish in a rigorous mathematical way. Let us fix $\ell_0 > 0$. The main idea is to link the small parameter ε and the width of the strip, which will become larger and larger as $\varepsilon \to 0$, i.e., as $\gamma \to 1$. Take for ℓ the quantity

$$\ell_\varepsilon = \ell_0/\sqrt{\varepsilon},$$

which blows up as $\varepsilon \to 0$ and the strip $\mathbb{R} \times [-\ell_\varepsilon/2, \ell_\varepsilon/2]$ approaches $\mathbb{R}^2$. We easily see that $\lambda_1(\ell_\varepsilon) = 4\pi^2/\ell_\varepsilon^2 = 4\pi^2\varepsilon/\ell_0^2$; hence,

$$\gamma_c = 1 - \frac{12\pi^2}{\ell_0^2}\varepsilon + \dots$$

Thus, ℓ_0 becomes the new bifurcation parameter. We shall assume that $\ell_0 > \sqrt{12}\pi$ in order to have $\gamma_c \in (1 - \varepsilon, 1)$, i.e., $\gamma > \gamma_c$, otherwise the TW is stable and the dynamics is trivial. Clearly, this is related to the stability of the null solution to K-S. The relevant eigenvalue of the Kuramoto–Sivashinsky linear operator $3\mathcal{A}^2 + \mathcal{A}$ is $3(\lambda_1(\ell_0))^2 - \lambda_1(\ell_0)$ which vanishes for $\lambda_1(\ell_0) = 1/3$, i.e., when $\ell_0 = \sqrt{12}\pi$.

An important feature of this paper is that we work in the *fixed strip* $\mathbb{R} \times [-\ell_0/2, \ell_0/2]$, with the rescaled variables (1.12). We will return to the original variable only in the final section.

The main result is the following.

MAIN THEOREM 1.1 Fix $\alpha \in (0, 1/2)$ and let $\Phi_0 \in C^{6+2\alpha}([-\ell_0/2, \ell_0/2])$ satisfy the condition $D_\eta^{(k)}\Phi_0(-\ell_0/2) = D_\eta^{(k)}\Phi_0(\ell_0/2)$ for any $k = 0, \dots, 6$. Let Φ be the periodic solution of (1.13) (with period ℓ_0) on a fixed time interval $[0, T]$, satisfying the initial condition $\Phi(0, \cdot) = \Phi_0$. Then there exists $\varepsilon_0 = \varepsilon_0(T) \in (0, 1/2)$ such that, for $0 < \varepsilon \leqslant \varepsilon_0$, problem (1.8)–(1.10) admits a unique smooth solution (u, φ) on $[0, T/\varepsilon^2]$, which is periodic with period $\ell_0/\sqrt{\varepsilon}$ with respect to y, and satisfies

$$\varphi(0, y) = \varepsilon\Phi_0(y\sqrt{\varepsilon}), \quad |y| \leqslant \frac{\ell_0}{2\sqrt{\varepsilon}}.$$

Moreover, there exists a positive constant C, independent of $\varepsilon \in (0, \varepsilon_0]$, such that

$$|\varphi(t, y) - \varepsilon\Phi(t\varepsilon^2, y\sqrt{\varepsilon})| \leqslant C\varepsilon^2, \quad 0 \leqslant t \leqslant \frac{T}{\varepsilon^2}, |y| \leqslant \frac{\ell_0}{2\sqrt{\varepsilon}}.$$

For a precise definition of what smooth solution means we refer the reader to Subsection 6.3.

Clearly, the initial condition for φ is of special type, compatible with Φ_0 and (1.1) at $\tau = 0$. Initial conditions of this type have already been considered in [1, 3].

The paper is organized as follows. In Section 2 we introduce some notation and the function spaces we extensively use throughout the paper. In Section 3 we proceed to a formal *Ansatz* in the

spirit of [12]. We set $\gamma = 1 - \varepsilon$, expand $v = v^0 + \varepsilon v^1 + \cdots$, $\psi = \psi^0 + \varepsilon \psi^1 + \cdots$, and show that ψ^0 satisfies the K-S equation (1.13), thanks to an elementary solvability condition. The paper consists in giving a rigorous proof of the Ansatz (i.e., of Main Theorem 1.1), thanks to an abstract solvability condition within the framework of adequate function spaces. In this respect, in Section 4 we transform system (1.8)–(1.10) in an equivalent problem (for the new unknowns) using the techniques of [4], which are based on

 (i) definition of a suitable linear one-dimensional operator;
 (ii) projection with respect to the x coordinate only;
(iii) Lyapunov–Schmidt method.

This allows us to decouple the system into a self-consistent fourth-order (in space) parabolic equation for the front ψ and an elliptic equation which can be easily solved whenever a solution to the front equation is determined. Hence, the rest of the paper is devoted to the parabolic equation. In this direction, according to the Ansatz, we split $\psi = \Phi + \varepsilon \rho_\varepsilon$. In Section 5, we solve the fourth-order equation for ρ_ε, locally in time, with time domain possibly depending on ε. Then, in Section 6, we prove that, for any $T > 0$, the function ρ_ε exists, and is smooth, in the whole of $[0, T]$ provided ε is small enough. This result is obtained as a consequence of some a priori estimates independent of ε, which we prove in Subsection 6.1. The a priori estimates are also used to prove Main Theorem 1.1 (see Subsection 6.3). Finally, some technical tools are deferred to the appendix.

2. Notation and function spaces

In this section we introduce some notation and the function spaces which will be used throughout the paper.

2.1 *Notation*

We define

$$I = \mathbb{R} \times [-\ell_0/2, \ell_0/2], \quad I_- = (-\infty, 0] \times [-\ell_0/2, \ell_0/2], \quad I_+ = [0, +\infty) \times [-\ell_0/2, \ell_0/2].$$

We use the bold notation for elements of both the spaces $C((-\infty, 0]) \times C([0, +\infty))$ and $C(I_-) \times C(I_+)$. Given an element **u** of the previous spaces we denote its components by u_1 and u_2. We write $D_x^{(i)}\mathbf{u}$ (resp. $D_y^{(i)}\mathbf{u}$) ($i = 1, 2, \ldots$) to denote the function whose components are $D_x^{(i)}u_1$ and $D_x^{(i)}u_2$ (resp. $D_y^{(i)}u_1$ and $D_y^{(i)}u_2$).

We extensively use the elements $\mathbf{T}$, $\mathbf{T}'$ (related to the TW), $\mathbf{U}$ and $\mathbf{V}$ of the space $C((-\infty, 0]) \times C([0, +\infty))$, which are defined by

$$\begin{cases} T_1(x) = e^x, & x \leqslant 0, \\ T_2(x) = 1, & x \geqslant 0, \end{cases} \qquad \begin{cases} T_1'(x) = e^x, & x \leqslant 0, \\ T_2'(x) = 0, & x \geqslant 0, \end{cases} \tag{2.1}$$

$$\begin{cases} U_1(x) = \dfrac{1-x}{3}e^x, & x \leqslant 0, \\ U_2(x) = \dfrac{1}{3}, & x \geqslant 0, \end{cases} \qquad \begin{cases} V_1(x) = \left(1 - \dfrac{2}{3}x + \dfrac{x^2}{6}\right)e^x, & x \leqslant 0, \\ V_2(x) = 1 + \dfrac{x}{3}, & x \geqslant 0. \end{cases} \tag{2.2}$$

2.2   *Function spaces*

Here, we introduce the function spaces we use in this paper.

2.2.1   *Spaces of one variable only.*

Let us fix $\ell_0 > 0$. Given a real- or complex-valued function $f \in L^2 := L^2(-\ell_0/2, \ell_0/2)$, we denote by $\widehat{f}(k)$ its k-th Fourier coefficient, i.e., we write

$$f(\eta) = \sum_{k=0}^{+\infty} \widehat{f}(k) w_k(\eta), \quad \eta \in (-\ell_0/2, \ell_0/2),$$

where $\{w_k\}$ is a complete set of eigenfunctions of the operator

$$A : D(A) = H^2 \to L^2, \quad Au = D_{\eta\eta} u, \quad u \in D(A),$$

with ℓ_0-periodic boundary conditions, corresponding to the sequence $\{-\lambda_k\}$ of nonpositive eigenvalues, which we label as $0 = -\lambda_0 > -\lambda_1 = -\lambda_2 > -\lambda_3 = -\lambda_4 > \cdots$, for notational convenience.

For integer or arbitrary real s we denote by H^s the usual Sobolev spaces of ℓ_0-periodic (generalized) functions, which we conveniently represent as

$$H^s = \left\{ w = \sum_{k=0}^{+\infty} a_k w_k : \sum_{k=0}^{+\infty} \lambda_k^s a_k^2 < +\infty \right\}, \tag{2.3}$$

with the usual norm. Next, for any $\beta \geqslant 0$, we denote by $C_\sharp^\beta$ the space of all functions $f \in C^\beta := C^\beta([-\ell_0/2, \ell_0/2])$ such that $f^{(j)}(-\ell_0/2) = f^{(j)}(\ell_0/2)$ for any $j = 0, \dots, [\beta]$. The space $C_\sharp^\beta$ is endowed with the Euclidean norm of $C^\beta([-\ell_0/2, \ell_0/2])$.

2.2.2   *Function spaces of two variables.*

Given $h, k \in \mathbb{N} \cup \{0\}$, an interval $J \subset \mathbb{R}$ and a (possibly unbounded) closed set $K \subset \mathbb{R}^d$ (for some $d \in \mathbb{N}$), we denote by $C^{h,k}(J \times K)$ the set of functions $f : J \times K \to \mathbb{R}$ which are h-times continuously differentiable in $J \times K$ with respect to the first variable and k-times continuously differentiable in $J \times K$ with respect to the second set of variables. When $J \times K$ is a compact set, we endow the space $C^{h,k}(J \times K)$ with the norm

$$\|f\|_{C^{h,k}(J \times K)} = \sup_{s \in J} \|f(s, \cdot)\|_{C^k(K)} + \sup_{z \in K} \|f(\cdot, z)\|_{C^h(J)} \tag{2.4}$$

for any $f \in C^{h,k}(J \times K)$. Using (2.4) we can extend the definition of the spaces $C^{h,k}(J \times K)$ to the case when $h, k \notin \mathbb{N}$.

Next, we introduce the space $\mathcal{X}$ defined by

$$\mathcal{X} = \{\mathbf{f} = (f_1, f_2) \in C(I_-) \times C(I_+) : \ \tilde{f}_1 \in C_b(I_-), \ \tilde{f}_2 \in C_b(I_+)\}, \tag{2.5}$$

where "b" stands for bounded and the functions $\tilde{f}_1$ and $\tilde{f}_2$ are defined as follows:

$$\tilde{f}_1(x, \eta) = e^{-x/2} f_1(x, \eta), \quad x \leqslant 0, \ |\eta| \leqslant \ell_0/2,$$

$$\tilde{f}_2(x, \eta) = e^{-x/2} f_2(x, \eta), \quad x \geqslant 0, \ |\eta| \leqslant \ell_0/2.$$

In what follows, we will write $\tilde{\mathbf{f}} := (\tilde{f}_1, \tilde{f}_2)$. The space $\mathcal{X}$ is a Banach space when endowed with the norm

$$\|\mathbf{f}\|_{\mathcal{X}} = \|\tilde{f}_1\|_{C_b(I_-)} + \|\tilde{f}_2\|_{C_b(I_+)} := \sup_{(x,\eta) \in I_-} |\tilde{f}_1(x, \eta)| + \sup_{(x,\eta) \in I_+} |\tilde{f}_2(x, \eta)|$$

for any $\mathbf{f} \in \mathcal{X}$.

2.2.3 *Interpolation spaces.* Let X be a Banach space and $A : D(A) \subset X \to X$ be a sectorial operator. For any $\theta \in (0, 1)$ we denote by $D_A(\theta, \infty)$ the interpolation space between X and $D(A)$, defined as follows:

$$D_A(\theta, \infty) = \left\{ x \in X : [x]_{\theta,\infty} := \sup_{t \in (0,1]} t^{-\theta} \|e^{tA} x - x\| < +\infty \right\}$$
$$= \left\{ x \in X : [[x]]_{\theta,\infty} := \sup_{t \in (0,1]} \|t^{1-\theta} A e^{tA} x\| < +\infty \right\},$$

where $\{e^{tA}\}$ is the analytic semigroup generated by A. $D_A(\theta, \infty)$ is a Banach space when endowed with one of the following equivalent norms:

$$\|x\|_{\theta,\infty,1} = \|x\| + [x]_{\theta,\infty}, \qquad \|x\|_{\theta,\infty,2} = \|x\| + [[x]]_{\theta,\infty}.$$

The interpolation space $D_A(\theta, \infty)$ can also be defined in a more abstract way as follows:

$$D_A(\theta, \infty) := \left\{ x \in X : [[[x]]]_{\theta,\infty} := \sup_{t \in (0,1]} t^{-\theta} K(t, x) < +\infty \right\},$$

where

$$K(t, x) = \inf\{\|a\|_X + t\|b\|_{D(A)} : x = a + b, \ a \in X, \ b \in D(A)\},$$

and $\| \cdot \|_{D(A)}$ denotes the graph-norm. The norm $\|x\|_{\theta,\infty,3} = \|x\| + [[[x]]]_{\theta,\infty}$ is equivalent to the above defined norms. We refer the reader to [11, Chaps. 1 & 2] and [15] for further details.

3. Formal Ansatz

Let us set $\gamma = 1 - \varepsilon$ in (1.10). Applying the change of variables defined by (1.12) to problem (1.8)–(1.10), the problem for the couple (v, ψ) reads (after simplification by ε^2) as follows:

$$v_x - (v_{xx} + \varepsilon v_{\eta\eta}) = (\varepsilon \psi_\tau + \varepsilon(\psi_\eta)^2 - \psi_{\eta\eta})\Theta_x, \quad x \neq 0, \tag{3.1}$$

and at $x = 0$:

$$\varepsilon \psi_\tau = [v_x] - \varepsilon(\psi_\eta)^2, \tag{3.2}$$

$$v_{|x=0} = -\psi_{\eta\eta} + \varepsilon \psi_{\eta\eta} + \varepsilon\left(\psi_\tau + \frac{1}{2}(\psi_\eta)^2\right). \tag{3.3}$$

In the spirit of [12, p. 75], we look for formal expansions:

$$v = v^0 + \varepsilon v^1 + \cdots, \qquad \psi = \psi^0 + \varepsilon \psi^1 + \cdots$$

of the solution to problem (3.1)–(3.3). Considering the zeroth order part of (3.1)–(3.3) (i.e., the terms with no powers of ε in front), it is easy to see that the function v^0 satisfies the system

$$v_x^0 - v_{xx}^0 = -\psi_{\eta\eta}^0 e^x \chi_{(-\infty,0]}, \tag{3.4}$$

$$[v_x^0] = 0, \tag{3.5}$$

$$v_{|x=0}^0 = -\psi_{\eta\eta}^0. \tag{3.6}$$

It is trivial to solve (3.4) together with, e.g., (3.6): it gives

$$v^0 = \begin{cases} -\psi^0_{\eta\eta} e^x (1-x), & x \leqslant 0, \\ -\psi^0_{\eta\eta}, & x > 0. \end{cases}$$

We remark that (3.5) is automatically satisfied. Hence, we are unable to "close" the system for (v^0, ψ^0) at the zeroth order. This situation is quite common in singular perturbation theory when the zeroth order cannot be fully determined (see, e.g., [7]). In such a case, one needs to go to the first order. Most often, the latter demands a solvability condition, for example based on the Fredholm alternative, which provides the missing relation for the zeroth order. Therefore, repeating computations similar to the previous ones, we get the following equation for (v^1, ψ^1):

$$v^1_x - v^1_{xx} - v^0_{\eta\eta} = \{\psi^0_\tau + (\psi^0_\eta)^2 - \psi^1_{\eta\eta}\} e^x \chi_{(-\infty,0]}. \tag{3.7}$$

At $x = 0$ we have the conditions

$$[v^1_x] = \psi^0_\tau + (\psi^0_\eta)^2, \tag{3.8}$$

$$v^1_{|x=0} = -\psi^1_{\eta\eta} + \psi^0_{\eta\eta} + \psi^0_\tau + \frac{1}{2}(\psi^0_\eta)^2. \tag{3.9}$$

Clearly, the solution to (3.7)–(3.9) is given by

$$v^1 = \begin{cases} ae^x + 2\psi^0_{\eta\eta\eta\eta} - \psi^0_\tau - (\psi^0_\eta)^2 + \psi^1_{\eta\eta} xe^x - \frac{1}{2}\psi^0_{\eta\eta\eta\eta} x^2 e^x, & x \leqslant 0, \\ a - \psi^0_{\eta\eta\eta\eta} x, & x \geqslant 0, \end{cases}$$

where a is an arbitrary parameter. There are two remaining unknowns at the first order, namely a and $\psi^1_{\eta\eta}$, and still two relations at $x = 0$. First we use (3.9), which gives

$$a = v^1(0) = -\psi^1_{\eta\eta} + \psi^0_{\eta\eta} + \psi^0_\tau + \frac{1}{2}(\psi^0_\eta)^2. \tag{3.10}$$

Second, we compute

$$v^1_x(0^+) = -\psi^0_{\eta\eta\eta\eta} \qquad v^1_x(0^-) = a + 2\psi^0_{\eta\eta\eta\eta} - \psi^0_\tau - (\psi^0_\eta)^2 + \psi^1_{\eta\eta}.$$

Therefore, from (3.8) we get

$$v^1_x(0^+) - v^1_x(0^-) = -a - \{3\psi^0_{\eta\eta\eta\eta} - \psi^0_\tau - (\psi^0_\eta)^2 + \psi^1_{\eta\eta}\} = \psi^0_\tau + (\psi^0_\eta)^2. \tag{3.11}$$

Obviously (3.10)–(3.11) is a linear system for $(a, \psi^1_{\eta\eta})$ with solvability condition

$$\psi^0_{\eta\eta} + \psi^0_\tau + \frac{1}{2}(\psi^0_\eta)^2 + 3\psi^0_{\eta\eta\eta\eta} = 0,$$

i.e., ψ^0 satisfies a K-S equation.

4. An equivalent problem to (3.1)–(3.3)

The aim of this section is to transform problem (3.1)–(3.3) into an equivalent one. More precisely, we are going to decouple the problem, getting a self-consistent equation for the front ψ and an equation for the other unknown (say $\mathbf{z}$) which can be immediately solved once ψ is known.

In deriving the equivalent problem, we assume that the solution (v, ψ) to problem (3.1)–(3.3) in the time domain $[0, T]$ belongs to the space $\mathcal{V}_T \times \mathcal{Y}_T$, defined as follows:

DEFINITION 4.1 For any $T > 0$, we denote by $\mathcal{V}_T$ the space of all functions $v : [0, T] \times \mathbb{R} \times [-\ell_0/2, \ell_0/2] \to \mathbb{R}$ such that

 (i) v is twice continuously differentiable with respect to the spatial variables in $[0, T] \times I_-$ and in $[0, T] \times I_+$;
(ii) the functions $(\tau, x, \eta) \mapsto e^{-x/2} D_x^{(i)} v(\tau, x, \eta)$ and $(\tau, x, \eta) \mapsto e^{-x/2} D_\eta^{(i)} v(\tau, x, \eta)$ are bounded in $[0, T] \times I_-$ and in $[0, T] \times I_+$ for any $i = 0, 1, 2$.

Further, for any $\alpha \in (0, 1/2)$, we denote by $\mathcal{Y}_T$ the space of all functions $\zeta \in C^{1,4}([0, T] \times [-\ell_0/2, \ell_0/2])$ such that $\zeta_\tau \in C^{0,2+\alpha}([0, T] \times [-\ell_0/2, \ell_0/2])$ and $D_\eta^{(j)} \zeta(\cdot, -\ell_0/2) = D_\eta^{(j)} \zeta(\cdot, \ell_0/2)$ for $j = 0, 1, 2, 3$.

REMARK 4.2 Note that, for any $\zeta \in \mathcal{Y}_T$, the function $\zeta_{\eta\eta}$ is continuously differentiable in $[0, T] \times [-\ell_0/2, \ell_0/2]$ with respect to τ, and $\zeta_{\tau\eta\eta} \equiv \zeta_{\eta\eta\tau}$. Indeed,

$$\zeta(\tau, \eta) = \zeta(0, \eta) + \int_0^\tau \zeta_\tau(s, \eta) \, ds, \quad \tau \in [0, T], \, \eta \in [-\ell/2, \ell/2].$$

Since $\zeta_\tau \in C^{0,2}([0, T] \times [-\ell/2, \ell/2])$ we can differentiate under the integral sign to get

$$\zeta_{\eta\eta}(\tau, \eta) = \zeta_{\eta\eta}(0, \eta) + \int_0^\tau \zeta_{\eta\eta\tau}(s, \eta) \, ds, \quad \tau \in [0, T], \, \eta \in [-\ell/2, \ell/2].$$

This formula clearly shows that $\zeta_{\eta\eta}$ is continuously differentiable with respect to τ in $[0, T] \times [-\ell/2, \ell/2]$ and $\zeta_{\tau\eta\eta} \equiv \zeta_{\eta\eta\tau}$. Hence, in what follows, we always write $\zeta_{\tau\eta\eta}$ instead of $\zeta_{\eta\eta\tau}$.

4.1 *Derivation of a self-consistent equation for the front*

In this subsection we derive a self-consistent equation for the front. Since its derivation is rather long, we split the proof into several steps.

4.1.1 *Elimination of ψ_τ.* First we eliminate ψ_τ from (3.1) thanks to (3.3), getting the equation

$$v_x - v_{xx} - \varepsilon v_{\eta\eta} - v(\cdot, 0, \cdot)\Theta_x = \left(\frac{1}{2}\varepsilon(\psi_\eta)^2 - \varepsilon\psi_{\eta\eta}\right)\Theta_x, \quad x \neq 0. \tag{4.1}$$

Let us set $\mathbf{v}(\tau, x, \eta) := (v_1(\tau, x, \eta), v_2(\tau, x, \eta))$, where $v_1(\cdot, x, \cdot) = v(\cdot, x, \cdot)$ for any $x \leqslant 0$ and $v_2(\cdot, x, \cdot) = v(\cdot, x, \cdot)$ for any $x \geqslant 0$, and

$$\mathbf{F}_0 = \left(\psi_{\eta\eta} - \frac{1}{2}(\psi_\eta)^2\right)\mathbf{T}', \quad g = \psi_\tau + (\psi_\eta)^2,$$

where $\mathbf{T}'$ is given by (2.1). Taking (3.3) and (4.1) into account, we can easily show that $\mathbf{v}$ solves the problem

$$\begin{cases} \mathcal{L}\mathbf{v} = \varepsilon\mathbf{F}_0 - \varepsilon\mathbf{v}_{\eta\eta}, \\ v_2(\cdot, 0, \cdot) - v_1(\cdot, 0, \cdot) = 0, \\ D_x v_2(\cdot, 0, \cdot) - D_x v_1(\cdot, 0, \cdot) = \varepsilon g, \end{cases} \tag{4.2}$$

where

$$(\mathcal{L}\mathbf{v})(\cdot, x, \eta) = \begin{cases} D_{xx} v_1(\cdot, x, \eta) - D_x v_1(\cdot, x, \eta) + e^x v_1(\cdot, 0, \eta), & x \leqslant 0, \ |\eta| \leqslant \ell_0/2, \\ D_{xx} v_2(\cdot, x, \eta) - D_x v_2(\cdot, x, \eta), & x \geqslant 0, \ |\eta| \leqslant \ell_0/2. \end{cases}$$

4.1.2 *Lifting up the boundary conditions.* We introduce the new unknown $\mathbf{w} = \mathbf{v} - \varepsilon\mathcal{N}(g)$, where $\mathcal{N}(g) = g(\mathbf{V} - \mathbf{T})$, and $\mathbf{T}$ and $\mathbf{V}$ are defined in (2.1) and (2.2). It is easy to see that $\mathbf{w}$ solves the problem

$$\begin{cases} \mathcal{L}\mathbf{w} = \varepsilon\mathbf{F}_0 - \varepsilon\mathbf{w}_{\eta\eta} - \varepsilon^2 g_{\eta\eta}\mathcal{N}(1) - \varepsilon g\mathcal{L}\mathcal{N}(1), \\ w_2(\cdot, 0, \cdot) - w_1(\cdot, 0, \cdot) = 0, \\ D_x w_2(\cdot, 0, \cdot) - D_x w_1(\cdot, 0, \cdot) = 0. \end{cases} \tag{4.3}$$

Since $\mathbf{v} \in \mathcal{V}_T$, $\mathbf{v}(\tau, \cdot)$, $\mathcal{L}\mathbf{v}(\tau, \cdot) \in \mathcal{X}$ (see (2.5) for the definition of the space $\mathcal{X}$) for any $\tau \in [0, T]$, a straightforward computation shows that $\mathbf{w}(\tau, \cdot)$ belongs to $\mathcal{X}$ for any $\tau \in [0, T]$, and hence to the set

$$\{\mathbf{h} \in C^{2,0}(I_-) \times C^{2,0}(I_+) : \mathbf{h}, \mathcal{L}\mathbf{h} \in \mathcal{X}, \ D_x^{(j)} h_1(0, \cdot) = D_x^{(j)} h_2(0, \cdot), \ j = 0, 1\},$$

which is the domain of the realization L of the operator $\mathcal{L}$ in $\mathcal{X}$ (see Section A.1).

4.1.3 *A Lyapunov–Schmidt method.* From the results in the previous subsection, we know that $\mathbf{w}(\tau, \cdot) \in D(L)$ for any $\tau \in [0, T]$, and it solves the equation

$$L\mathbf{w} = \varepsilon\mathbf{F}_0 - \varepsilon\mathbf{w}_{\eta\eta} - \varepsilon^2 g_{\eta\eta}\mathcal{N}(1) - \varepsilon g\mathcal{L}\mathcal{N}(1). \tag{4.4}$$

We are going to project (4.4) along a suitable subspace of $\mathcal{X}$, to derive a self-consistent equation for the front ψ.

According to Theorem A.1, the equation $L\mathbf{z} = \mathbf{f} \in \mathcal{X}$ has a solution if and only if $\mathcal{P}(\mathbf{f}) = 0$, where

$$\mathcal{P}(\mathbf{f}) = \left(\int_{-\infty}^0 f_1(x, \cdot)\, dx + \int_0^{+\infty} e^{-x} f_2(x, \cdot)\, dx \right)\mathbf{U} = Q(\mathbf{f})\mathbf{U}, \quad \mathbf{f} \in \mathcal{X}.$$

Since $\mathbf{w}$ is any of such solutions, it follows that

$$\mathcal{P}(\varepsilon\mathbf{F}_0 - \varepsilon\mathbf{w}_{\eta\eta} - \varepsilon g\mathcal{L}\mathcal{N}(1) - \varepsilon^2 g_{\eta\eta}\mathcal{N}(1)) = 0,$$

or equivalently, after division by $\varepsilon > 0$,

$$0 = Q(\mathbf{F}_0 - \mathbf{w}_{\eta\eta} - g\mathcal{L}\mathcal{N}(1) - \varepsilon g_{\eta\eta}\mathcal{N}(1)). \tag{4.5}$$

Observing that

$$Q(\mathbf{F}_0) = \psi_{\eta\eta} - \frac{1}{2}(\psi_\eta)^2, \tag{4.6a}$$

$$Q(g_{\eta\eta}\mathcal{N}(1)) = \frac{4}{3}g_{\eta\eta} = \frac{4}{3}(\psi_{\tau\eta\eta} + ((\psi_\eta)^2)_{\eta\eta}), \tag{4.6b}$$

$$Q(g\mathcal{L}\mathcal{N}(1)) = -g = -\psi_\tau - (\psi_\eta)^2, \tag{4.6c}$$

we can rewrite (4.5) as follows:

$$\psi_\tau - \frac{4}{3}\varepsilon\psi_{\tau\eta\eta} + \frac{1}{2}(\psi_\eta)^2 + \psi_{\eta\eta} - \frac{4}{3}\varepsilon((\psi_\eta)^2)_{\eta\eta} = Q(\mathbf{w}_{\eta\eta}). \tag{4.7}$$

To get a self-contained equation for the front ψ, we have to give a representation of $Q(\mathbf{w}_{\eta\eta})$ on the right-hand side of (4.7). For this purpose, in the spirit of the Lyapunov–Schmidt method, we split $\mathbf{w}(\tau, \cdot)$ ($\tau \in [0, T]$) along $\mathcal{P}(\mathcal{X})$ and $(I - \mathcal{P})(\mathcal{X})$. Writing

$$\mathbf{w}(\tau, x, \eta) = a(\tau, \eta)\mathbf{U}(x) + \varepsilon\mathbf{z}(\tau, x, \eta),$$

and observing that our assumptions on v guarantee that $\mathbf{z}_{\eta\eta}$ belongs to $(I - \mathcal{P})(\mathcal{X})$ (indeed, $\mathcal{P}\mathbf{z} = 0$ and $\mathcal{P}$ commutes with D_η), we get

$$Q(\mathbf{w}_{\eta\eta}) = Q(a_{\eta\eta}\mathbf{U} + \varepsilon\mathbf{z}_{\eta\eta}) = a_{\eta\eta}. \tag{4.8}$$

Let us compute formally the function a and its second spatial derivative. We use the relation in (3.3) to obtain

$$\frac{1}{3}a + \varepsilon z_1(\cdot, 0, \cdot) = (\varepsilon - 1)\psi_{\eta\eta} + \varepsilon\psi_\tau + \frac{1}{2}\varepsilon(\psi_\eta)^2.$$

Thus,

$$a_{\eta\eta} = -3\varepsilon D_{\eta\eta}z_1(\cdot, 0, \cdot) + 3(\varepsilon - 1)\psi_{\eta\eta\eta\eta} + 3\varepsilon\psi_{\tau\eta\eta} + \frac{3}{2}\varepsilon((\psi_\eta)^2)_{\eta\eta}. \tag{4.9}$$

From (4.8) and (4.9) it follows that

$$Q(\mathbf{w}_{\eta\eta}) = -3\varepsilon D_{\eta\eta}z_1(\cdot, 0, \cdot) + 3(\varepsilon - 1)\psi_{\eta\eta\eta\eta} + 3\varepsilon\psi_{\tau\eta\eta} + \frac{3}{2}\varepsilon((\psi_\eta)^2)_{\eta\eta}.$$

Inserting this into (4.7) we get the following equation for ψ:

$$\psi_\tau - \frac{13}{3}\varepsilon\psi_{\tau\eta\eta} + 3(1 - \varepsilon)\psi_{\eta\eta\eta\eta} + \psi_{\eta\eta} + \frac{1}{2}(\psi_\eta)^2 + 3\varepsilon D_{\eta\eta}z_1(\cdot, 0, \cdot) = \frac{17}{6}\varepsilon((\psi_\eta)^2)_{\eta\eta}. \tag{4.10}$$

We already see that (4.10) reduces to K-S if $\varepsilon = 0$. However, we still have z_1 on the left-hand side of (4.10). In the next subsection, we write it in terms of ψ.

4.1.4 *The equation for* $\mathbf{z}$. To write $D_{\eta\eta}z_1(\cdot, 0, \cdot)$ in terms of ψ, we determine the equation satisfied by $\mathbf{z}$. Projecting equation (4.4) on $(I - \mathcal{P})(\mathcal{X})$, we see that $\mathbf{z}(\tau, \cdot) = (I - \mathcal{P})(\mathbf{z}(\tau, \cdot)) \in D(L)$ ($\tau \in [0, T]$) solves the equation

$$L\mathbf{z} = (I - \mathcal{P})(\mathbf{F}_0) - g(I - \mathcal{P})(\mathcal{L}\mathcal{N}(1)) - \varepsilon g_{\eta\eta}(I - \mathcal{P})(\mathcal{N}(1)) - \varepsilon\mathbf{z}_{\eta\eta}. \tag{4.11}$$

Taking (4.6a)–(4.6c) into account, we can rewrite (4.11) as

$$L\mathbf{z} + \varepsilon\mathbf{z}_{\eta\eta} = \left(\psi_{\eta\eta} - \frac{1}{2}(\psi_\eta)^2\right)(\mathbf{T}' - \mathbf{U}) - \varepsilon(\psi_{\tau\eta\eta} + ((\psi_\eta)^2)_{\eta\eta})\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right). \tag{4.12}$$

We now observe that the operator $L + \varepsilon A := L + \varepsilon D_{\eta\eta}$ with domain

$$D(L + \varepsilon A) = \{\mathbf{u} \in D(L) : \mathbf{u}_{\eta\eta} \in \mathcal{X},\ D_\eta^{(j)}u_i(\cdot, -\ell_0/2) = D_\eta^{(j)}u_i(\cdot, \ell_0/2),\ i = 1, 2,\ j = 0, 1\} \tag{4.13}$$

is closable and its closure, denoted by L_ε, is sectorial and 0 is in the resolvent set of the restriction of L_ε to $(I - \mathcal{P})(\mathcal{X})$ (see Theorem A.2). Hence, we can invert (4.12) using $R(0, L_\varepsilon) = (-L_\varepsilon)^{-1}$, collecting linear and nonlinear terms in ψ:

$$
\begin{aligned}
\mathbf{z} = R(0, L_\varepsilon)&\left(-\psi_{\eta\eta}(\mathbf{T}' - \mathbf{U}) + \varepsilon\psi_{\tau\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right) \\
&+ R(0, L_\varepsilon)\left(\frac{1}{2}(\psi_\eta)^2(\mathbf{T}' - \mathbf{U}) + \varepsilon((\psi_\eta)^2)_{\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right).
\end{aligned} \tag{4.14}
$$

4.1.5   *The fourth-order equation for the front.*   Using (4.14), we can compute $z_1(\cdot, 0, \cdot)$ getting

$$
\begin{aligned}
z_1(\cdot, 0, \cdot) = &-(R(0, L_\varepsilon)[\psi_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot) + \varepsilon\left(R(0, L_\varepsilon)\left[\psi_{\tau\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right]\right)_1(\cdot, 0, \cdot) \\
&+ \left\{R(0, L_\varepsilon)\left(\frac{1}{2}(\psi_\eta)^2(\mathbf{T}' - \mathbf{U}) + \varepsilon((\psi_\eta)^2)_{\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right)\right\}_1(\cdot, 0, \cdot).
\end{aligned}
$$

Since z_1 is as smooth as v_1 is, we can differentiate the previous formula twice with respect to η obtaining

$$
\begin{aligned}
D_{\eta\eta}z_1(\cdot, 0, \cdot) = &-(D_{\eta\eta}R(0, L_\varepsilon)[\psi_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot) \\
&+ \varepsilon\left(D_{\eta\eta}R(0, L_\varepsilon)\left[\psi_{\tau\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right]\right)_1(\cdot, 0, \cdot) \\
&+ \frac{1}{2}\{D_{\eta\eta}R(0, L_\varepsilon)((\psi_\eta)^2(\mathbf{T}' - \mathbf{U}))\}_1(\cdot, 0, \cdot) \\
&+ \varepsilon\left\{D_{\eta\eta}R(0, L_\varepsilon)\left(((\psi_\eta)^2)_{\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right)\right\}_1(\cdot, 0, \cdot).
\end{aligned} \tag{4.15}
$$

Estimate (A2) and our assumptions on ψ (which guarantee that $\psi_{\eta\eta} \in C^1([0, T]; C^\alpha)$, see Remark 4.2) show that $(D_{\eta\eta}R(0, L_\varepsilon)[\psi_{\eta\eta}(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})])_1(\cdot, 0, \cdot)$ belongs to $C^{1,0}([0, T] \times [-\ell_0/2, \ell_0/2])$ and its derivative is $(D_{\eta\eta}R(0, L_\varepsilon)[\psi_{\tau\eta\eta}(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})])_1(\cdot, 0, \cdot)$. Hence, inserting (4.15) into (4.10), we conclude that the function ψ solves the fourth-order equation

$$
\frac{\partial}{\partial\tau}\mathcal{B}_\varepsilon\psi = \mathcal{S}_\varepsilon\psi + \mathcal{F}_\varepsilon((\psi_\eta)^2), \tag{4.16}
$$

where

$$
\mathcal{B}_\varepsilon\psi = \psi - \frac{13\varepsilon}{3}\psi_{\eta\eta} + 3\varepsilon^2\left(D_{\eta\eta}R(0, L_\varepsilon)\left[\psi_{\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right]\right)_1(\cdot, 0, \cdot), \tag{4.17a}
$$

$$
\mathcal{S}_\varepsilon\psi = -3(1 - \varepsilon)\psi_{\eta\eta\eta\eta} - \psi_{\eta\eta} + 3\varepsilon(D_{\eta\eta}R(0, L_\varepsilon)[\psi_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot), \tag{4.17b}
$$

$$
\begin{aligned}
\mathcal{F}_\varepsilon(\psi) = &-3\varepsilon D_{\eta\eta}\left\{R(0, L_\varepsilon)\left(\frac{1}{2}\psi(\mathbf{T}' - \mathbf{U}) + \varepsilon\psi_{\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right)\right\}_1(\cdot, 0, \cdot) \\
&+ \frac{17\varepsilon}{6}\psi_{\eta\eta} - \frac{1}{2}\psi.
\end{aligned} \tag{4.17c}
$$

Clearly, (4.16) reduces to K-S when we set $\varepsilon = 0$.

4.2 *Equivalence between problem* (3.1)–(3.3) *and equation* (4.16)

The following theorem states the equivalence of problem (3.1)–(3.3) and equation (4.16).

THEOREM 4.3 Fix $\varepsilon, T > 0$ and $\alpha \in (0, 1/2)$. Further, let $(v, \psi) \in \mathcal{V}_T \times \mathcal{Y}_T$ be a solution to problem (3.1)–(3.3) (see Definition 4.1). Then the function ψ solves (4.16). Conversely, if $\psi \in \mathcal{Y}_T$ is a solution to (4.16), then there exists a function $v \in \mathcal{V}_T$ such that the pair (v, ψ) solves the Cauchy problem (3.1)–(3.3).

Proof. In view of the arguments in Subsection 4.1, we just need to show that to any solution $\psi \in \mathcal{Y}_T$ to equation (4.16) there corresponds a unique function $v \in \mathcal{V}_T$ such that the pair (v, ψ) solves problem (3.1)–(3.3). For this purpose, let $\mathbf{z}$ be defined by (4.14). By assumptions, $\psi_{\eta\eta}$, $(\psi_\eta)^2$, $((\psi_\eta)^2)_{\eta\eta}$ and $\psi_{\tau\eta\eta}$ are bounded in $[0, T]$ with values in $C_\sharp^\alpha$. Moreover, the functions $\mathbf{T}' - \mathbf{U}$ and $\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}$ are in $(I - \mathcal{P})(\mathcal{X})$. Hence, we can apply Theorem A.2(iii) to conclude that $\mathbf{z}(\tau, \cdot)$ is in $D(L + \varepsilon A)$ (see (4.13)) for any $\tau \in [0, T]$.

Clearly, the components z_1 and z_2 of $\mathbf{z}$ are continuous in $[0, T] \times I_-$ and $[0, T] \times I_+$, respectively. Let us show that also their spatial derivatives (up to the second order) are continuous in $[0, T] \times I_-$ and $[0, T] \times I_+$, respectively. This follows from the estimate (A2) provided one shows that the functions $\psi_{\eta\eta}$, $((\psi_\eta)^2)_{\eta\eta}$ and $\psi_{\tau\eta\eta}$ belong to $C([0, T]; C_\sharp^\theta)$ for some $\theta \in (0, \alpha)$. This property can be proved using an interpolation argument. Indeed, it is well-known that, for any $\theta \in (0, \alpha)$, there exists a positive constant K such that

$$\|\psi\|_{C^\theta} \leqslant K \|\psi\|_{C^0}^{1-\theta/\alpha} \|\psi\|_{C^\alpha}^{\theta/\alpha}$$

for any $\psi \in C^\alpha$ (see, e.g., [15]). Applying this estimate to the function $\psi_{\eta\eta}(\tau_2, \cdot) - \psi_{\eta\eta}(\tau_1, \cdot)$, with $\tau_1, \tau_2 \in [0, T]$, shows that $\psi_{\eta\eta}$ is continuous in $[0, T]$ with values in C^θ (and, hence, in $C_\sharp^\theta$) for any $\theta \in (0, \alpha)$. The same argument shows that the functions $(\psi_\eta)^2$, $((\psi_\eta)^2)_{\eta\eta}$ and $\psi_{\tau\eta\eta}$ are in $C([0, T]; C_\sharp^\theta)$ as well. Finally, since $\mathbf{z}(\tau, \cdot) \in D(L + \varepsilon A)$ for any $\tau \in [0, T]$, the functions $(\tau, x, \eta) \mapsto e^{-x/2} D_x^{(i)} z_1(\tau, x, \eta)$ and $(\tau, x, \eta) \mapsto e^{-x/2} D_\eta^{(i)} z_2(\tau, x, \eta)$ are bounded in $[0, T] \times I_-$ and $[0, T] \times I_+$, respectively, for any $i = 0, 1, 2$.

$\mathbf{z}$ will represent the component along $(I - \mathcal{P})(\mathcal{X})$ of $\mathbf{v} - \varepsilon \mathcal{N}(\psi_\tau + (\psi_\eta)^2)$, where $v_1(\cdot, x, \cdot) = v(\cdot, x, \cdot)$ for any $x \leqslant 0$ and $v_2(\cdot, x, \cdot) = v(\cdot, x, \cdot)$ for any $x \geqslant 0$, and v is the solution to problem (3.1)–(3.3) we are looking for. The computations in Subsection 4.1 suggest setting $\mathbf{v} := \mathbf{w} + \varepsilon \mathcal{N}(\psi_\tau + (\psi_\eta)^2) := a\mathbf{U} + \varepsilon \mathbf{z} + \varepsilon \mathcal{N}(\psi_\tau + (\psi_\eta)^2)$, where

$$a = -3\varepsilon z_1(\cdot, 0, \cdot) + 3(\varepsilon - 1)\psi_{\eta\eta} + 3\varepsilon\psi_\tau + \frac{3}{2}\varepsilon(\psi_\eta)^2. \tag{4.18}$$

Using formulae (4.6a)–(4.6c) and (4.15) we can show that

$$\mathcal{P}(\varepsilon\mathbf{F}_0 - \varepsilon\mathbf{w}_{\eta\eta} - \varepsilon\mathcal{L}\mathcal{N}(\psi_\tau + (\psi_\eta)^2) - \varepsilon^2\mathcal{N}((\psi_\tau + (\psi_\eta)^2)_{\eta\eta})) = 0.$$

Hence, $\mathbf{v}$ solves the equation

$$\begin{aligned}
\mathcal{L}\mathbf{v} &= L(a\mathbf{U}) + \varepsilon L\mathbf{z} + \varepsilon\mathcal{L}\mathcal{N}(\psi_\tau + (\psi_\eta)^2) \\
&= (I - \mathcal{P})\{\varepsilon\mathbf{F}_0 - \varepsilon\mathbf{w}_{\eta\eta} - \varepsilon\mathcal{L}\mathcal{N}(\psi_\tau + (\psi_\eta)^2) - \varepsilon^2\mathcal{N}((\psi_\tau + (\psi_\eta)^2)_{\eta\eta})\} + \varepsilon\mathcal{L}(\mathcal{N}(\psi_\tau + (\psi_\eta)^2)) \\
&= \varepsilon\mathbf{F}_0 - \varepsilon\mathbf{v}_{\eta\eta}.
\end{aligned}$$

Moreover, it is easy to check that **v** also satisfies the boundary conditions of the Cauchy problem (4.2).

Clearly, the function v defined above belongs to $\mathcal{V}_T$ and the pair (v, ψ) solves the differential equation (3.1). Using the second boundary condition in (4.2), it follows immediately that (v, ψ) satisfies condition (3.2). Finally, to check condition (3.3) it suffices to use (4.18), recalling that $\mathcal{N}(\psi_\tau + (\psi_\eta)^3)$ vanishes when $\eta = 0$. This completes the proof. □

4.3 *The equation for the remainder*

In view of Theorem 4.3, in the rest of the paper we deal only with equation (4.16) with periodic boundary conditions. To begin, we recall the following result about K-S:

THEOREM 4.4   Let $\Phi_0 \in C_\sharp^{6+\alpha}$ for some $\alpha \in (0, 1/2)$. Then the Cauchy problem

$$\begin{cases} \Phi_\tau(\tau, \eta) = -3\Phi_{\eta\eta\eta\eta}(\tau, \eta) - \Phi_{\eta\eta}(\tau, \eta) - \frac{1}{2}(\Phi_\eta(\tau, \eta))^2, & \tau \geqslant 0, \ |\eta| \leqslant \ell_0/2, \\ D_\eta^k \Phi(\tau, -\ell_0/2) = D_\eta^k \Phi(\tau, \ell_0/2), & \tau \geqslant 0, \ k = 0, 1, 2, 3, \\ \Phi(0, \eta) = \Phi_0(\eta), & |\eta| \leqslant \ell_0/2, \end{cases} \quad (4.19)$$

admits a unique solution $\Phi \in C^{1,4}([0, +\infty) \times [-\ell_0/2, \ell_0/2])$. In fact, $\Phi \in \mathcal{Y}_T$ for any $T > 0$.

Most of the literature is about the differentiated version of K-S. For this reason and the reader's convenience, we provide a full proof of Theorem 4.4 in the appendix.

According to the Ansatz, we split

$$\psi = \Phi + \varepsilon\rho_\varepsilon,$$

which defines the remainder ρ_ε. To avoid cumbersome notation, we simply write ρ for ρ_ε. From Theorem 4.4 we know that $\rho \in \mathcal{Y}_T$ and it solves the equation

$$\frac{\partial}{\partial\tau}\mathcal{B}_\varepsilon(\rho) = \mathcal{S}_\varepsilon(\rho) - \Phi_\eta\rho_\eta - \frac{\varepsilon}{2}(\rho_\eta)^2 + \mathcal{G}_\varepsilon((\Phi_\eta + \varepsilon\rho_\eta)^2) + \mathcal{H}_\varepsilon(\Phi), \quad (4.20)$$

where

$$\mathcal{G}_\varepsilon(\xi) = \frac{17}{6}\xi_{\eta\eta} - 3\left\{D_{\eta\eta}R(0, L_\varepsilon)\left(\frac{1}{2}\xi(\mathbf{T}' - \mathbf{U}) + \varepsilon\xi_{\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right)\right\}_1(\cdot, 0, \cdot),$$

$$\mathcal{H}_\varepsilon(\Phi) = 3\Phi_{\eta\eta\eta\eta} + 3(D_{\eta\eta}R(0, L_\varepsilon)[\Phi_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot) + \frac{13}{3}\Phi_{\tau\eta\eta}$$
$$- 3\varepsilon\left(D_{\eta\eta}R(0, L_\varepsilon)\left[\Phi_{\tau\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right]\right)_1(\cdot, 0, \cdot).$$

Equation (4.20) on $[-\ell_0/2, \ell_0/2]$ is supplemented by periodic boundary conditions and by an initial condition ρ_0 at $\tau = 0$. For simplicity, to avoid lengthly computations, we take hereafter $\rho_0 = 0$, that is, $\psi(0, \cdot) = \Phi(0, \cdot) = \Phi_0$. In other words, the front ψ and the solution of K-S start from the same configuration, which is physically reasonable. More general *compatible* initial data can be considered as in [3, 1].

5. Local in time solvability of equation (4.20)

As has been remarked in the introduction, except for small ℓ_0, where the TW is stable, global existence of ρ is not guaranteed.

In this section, we prove the following local in time existence and uniqueness result.

THEOREM 5.1   For any $\varepsilon \in (0, 1/2]$ there exist $T_\varepsilon > 0$ and a unique solution ρ to equation (4.20) which belongs to $\mathcal{Y}_{T_\varepsilon}$ and vanishes at $\tau = 0$.

The proof is rather long and needs many preliminary results. For this reason, we split it into several steps. Before entering into the details, we sketch here the strategy of the proof.

As a first step, for any fixed $\varepsilon > 0$, we transform (4.20) into a semilinear equation associated with a sectorial operator. Employing classical tools from the theory of analytic semigroups we prove that such a semilinear equation admits a unique solution $\rho = \rho_\varepsilon$ defined in some time domain $[0, T_\varepsilon]$, which vanishes at $\tau = 0$. Using some bootstrap arguments, we then regularize ρ, showing that it actually belongs to $\mathcal{Y}_{T_\varepsilon}$. These regularity properties of ρ allow us to show that it is in fact a solution to (4.20).

5.1   *The semilinear equation*

In this subsection, we show that we can transform equation (4.20) into a semilinear equation associated with a second order elliptic operator. We obtain it by inverting the operator $\mathcal{B}_\varepsilon$ in (4.17a), i.e., the operator defined by

$$\mathcal{B}_\varepsilon \psi = \psi - \frac{13\varepsilon}{3} \psi_{\eta\eta} + 3\varepsilon^2 \left(D_{\eta\eta} R(0, L_\varepsilon) \left[\psi_{\eta\eta} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right) \right] \right)_1 (\cdot, 0, \cdot).$$

By Theorem A.2 and the results in the proof of Theorem 4.3, we know that the operator $\mathcal{B}_\varepsilon$ is well-defined in $C_\sharp^{2+\theta}$ for any $\theta \in (0, 1)$. We will show that $\mathcal{B}_\varepsilon$ can be extended to the whole of $C_\sharp^2$ as an operator which is invertible. For this purpose, we compute the symbol of the operator $\mathcal{B}_\varepsilon$.

Throughout the section, given a function $f : J \times [-\ell_0/2, \ell_0/2] \to \mathbb{R}$, where $J \subset \mathbb{R}$ is an interval, we denote by $\widehat{f}(x, k)$ the k-th Fourier coefficient of the function $f(x, \cdot)$. Moreover, we set

$$X_{\varepsilon,k} = \sqrt{1 + 4\varepsilon\lambda_k}, \quad k = 0, 1, \ldots. \tag{5.1}$$

LEMMA 5.2   Fix $\varepsilon \in (0, 1/2]$. Then the k-th Fourier multiplier $b_{\varepsilon,k}$ of the operator $\mathcal{B}_\varepsilon$ is given by

$$b_{\varepsilon,k} = \frac{3}{4} \frac{(X_{\varepsilon,k} + 1)(X_{\varepsilon,k}^2 + 2X_{\varepsilon,k} - 1)}{X_{\varepsilon,k} + 2} \sim 3\varepsilon\lambda_k \quad (k \to +\infty). \tag{5.2}$$

Proof.   Even if the proof can be obtained by arguing as in the proof of [4, Prop. 4.2], for the reader's convenience we provide the details.

The main step of the proof is the computation of the symbol of the operator $\varphi \mapsto u_1 := (R(0, L_\varepsilon)[\varphi(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})])_1 (0, \cdot)$ for any $\varepsilon > 0$. To lighten notation, throughout the proof we do not stress explicitly the dependence on ε of the quantities we consider.

We claim that

$$\widehat{u_1}(0, k) = -\frac{4}{9} \frac{4X_k + 7}{(X_k + 1)^2(X_k + 2)} \widehat{\varphi}(k), \quad k = 0, 1, \ldots. \tag{5.3}$$

Let us first assume that φ is smooth enough. Since the function $\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}$ belongs to $(I - \mathcal{P})(\mathcal{X})$, from Theorem A.2(iii) it follows that $\mathbf{u} \in D(L + \varepsilon A)$, so that $L\mathbf{u} + \varepsilon A\mathbf{u} = -(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})\varphi$. Moreover, the function $\widehat{\mathbf{u}}(\cdot, k)$ belongs to $(I - \mathcal{P})(D(L))$ and solves the equation $(\varepsilon\lambda_k - L)\widehat{\mathbf{u}}(\cdot, k) = (\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U})\widehat{\varphi}(k)$ for any $k = 0, 1, \ldots$. Since λ_k is in the resolvent set of the operator L for any $k = 0, 1, \ldots$, by Theorem A.1 it follows that

$$\widehat{\mathbf{u}}(\cdot, k) = R(\varepsilon\lambda_k, L)\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\widehat{\varphi}(k), \quad k = 0, 1, \ldots. \tag{5.4}$$

Formula (5.4) can be extended to all functions $\varphi \in C_\sharp$ by a straightforward approximation argument.

From formula (A1) it is immediate to check that

$$(R(\varepsilon\lambda_k, L)\mathbf{f})_1(0, \cdot) = \frac{1}{X_k}\left(\frac{4\varepsilon\lambda_k}{(1 + (2\varepsilon\lambda_k - 1)X_k)(X_k + 1)} + 1\right)$$
$$\times \left[\int_{-\infty}^0 e^{-\nu_{1,k}t} f_1(t, \cdot)\,dt + \int_0^{+\infty} e^{-\nu_{2,k}t} f_2(t, \cdot)\,dt\right]$$

for any $\mathbf{f} = (f_1, f_2) \in \mathcal{X}$, where

$$\nu_{1,k} = \frac{1}{2} - \frac{1}{2}X_k, \quad \nu_{2,k} = \frac{1}{2} + \frac{1}{2}X_k, \quad k = 0, 1, \ldots.$$

Hence, from the definition of the functions $\mathbf{V}$, $\mathbf{T}$ and $\mathbf{U}$ (see (2.1) and (2.2)), we get

$$\int_{-\infty}^0 e^{-\nu_{1,k}t}\left(V_1(t) - T_1(t) - \frac{4}{3}U_1(t)\right)dt + \int_0^{+\infty} e^{-\nu_{2,k}t}\left(V_2(t) - T_2(t) - \frac{4}{3}U_2(t)\right)dt$$
$$= -\frac{4}{9}\frac{(4X_k + 7)(X_k - 1)}{(X_k + 1)^3}.$$

Since 0 is in the resolvent set of the restriction of L to $(I - \mathcal{P})(\mathcal{X})$, we can extend the previous formula, by continuity, to $\lambda = 0$. Thus,

$$\widehat{u}_1(0, k) = -\frac{4}{9}\left(\frac{4\varepsilon\lambda_k}{(1 + (2\varepsilon\lambda_k - 1)X_k)(X_k + 1)} + 1\right)\frac{(4X_k + 7)(X_k - 1)}{X_k(X_k + 1)^3}\widehat{\varphi}(k)$$
$$= -\frac{4}{9}\frac{4X_k + 7}{(X_k + 2)(X_k + 1)^2}\widehat{\varphi}(k)$$

for any $k = 0, 1, \ldots$, and the assertion follows.

Now, using formula (5.3) it is immediate to complete the proof. $\qquad\square$

PROPOSITION 5.3 For any $\varepsilon \in (0, 1/2]$, the operator $\mathcal{B}_\varepsilon$ is invertible from $C_\sharp^{2+\theta}$ into $C_\sharp^\theta$ for any $\theta \in (0, 1)$.

Proof. From Lemma 5.2, we know that $b_{\varepsilon,k} \neq 0$ for any $k = 0, 1, \ldots$. Hence, the operator $\mathcal{B}_\varepsilon$ admits a realization in L^2 which is invertible from H^2 into L^2. We still denote by $\mathcal{B}_\varepsilon$ such a realization. To prove that $\mathcal{B}_\varepsilon$ is invertible from $C_\sharp^2$ into $C_\sharp$, let us fix $f \in C_\sharp$ and let $u \in H^2$ be the unique solution to the equation $\mathcal{B}_\varepsilon u = f$. Taking (5.2) into account, it is immediate to check

that we can split $\mathcal{B}_\varepsilon = -3\varepsilon D_{\eta\eta} + \mathcal{B}_\varepsilon^{(1)}$, where $\mathcal{B}_\varepsilon^{(1)} \in L(H^1, L^2)$, since its symbol $(b_{\varepsilon,k}^{(1)})$ satisfies

$$b_{\varepsilon,k}^{(1)} \sim \frac{3}{2}\sqrt{\varepsilon\lambda_k} \quad (k \to +\infty).$$

It follows that the function $\mathcal{B}_\varepsilon^{(1)}(u)$ is in $C_\sharp^{1/2}$. Consequently, $u_{\eta\eta}$ is in $C_\sharp$.

Let us now suppose that $f \in C_\sharp^\theta$ with $\theta \leqslant 1/2$ and let $u \in H^2$ satisfy $\mathcal{B}_\varepsilon(u) = f$. Since $\mathcal{B}_\varepsilon^{(1)}(u) \subset C_\sharp^{1/2}$, $u_{\eta\eta}$ belongs to $C_\sharp^\theta$ and hence $u \in C_\sharp^{2+\theta}$. Let now $\theta \in (1/2, 1)$. The above result shows that $u \in C_\sharp^{5/2}$. Since $C_\sharp^{5/2} \subset H^{5/2}$, $\mathcal{B}_\varepsilon^{(1)}u \in H^{3/2} \subset C_\sharp^\theta$. Hence, also in this case $u_{\eta\eta} \in C_\sharp^\theta$ and we are done. $\qquad\square$

In view of Proposition 5.3, we can invert the operator $\mathcal{B}_\varepsilon$ from $C_\sharp^{2+\theta}$ into $C_\sharp^\theta$ for any $\theta \in (0, 1)$, getting the following equation for ρ:

$$\rho_\tau(\tau, \cdot) = \mathcal{R}_\varepsilon(\rho(\tau, \cdot)) + \mathcal{K}_\varepsilon(\tau, \rho_\eta(\tau, \cdot)), \quad \tau \in [0, T], \tag{5.5}$$

where

$$\mathcal{R}_\varepsilon(\rho) = \mathcal{B}_\varepsilon^{-1}(\mathcal{S}_\varepsilon(\rho)),$$

$$\mathcal{K}_\varepsilon(\tau, \rho) = \mathcal{B}_\varepsilon^{-1}(\mathcal{G}_\varepsilon((\Phi_\eta(\tau, \cdot))^2)) - \mathcal{B}_\varepsilon^{-1}(\Phi_\eta(\tau, \cdot)\rho) + 2\varepsilon\mathcal{B}_\varepsilon^{-1}(\mathcal{G}_\varepsilon(\Phi_\eta(\tau, \cdot)\rho))$$

$$- \frac{\varepsilon}{2}\mathcal{B}_\varepsilon^{-1}(\rho^2) + \varepsilon^2\mathcal{B}_\varepsilon^{-1}(\mathcal{G}_\varepsilon(\rho^2)) + 3\mathcal{B}_\varepsilon^{-1}(\Phi_{\eta\eta\eta\eta})$$

$$+ 3\mathcal{B}_\varepsilon^{-1}\big((D_{\eta\eta}R(0, L_\varepsilon)[\Phi_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot)\big) + \frac{13}{3}\mathcal{B}_\varepsilon^{-1}(\Phi_{\tau\eta\eta})$$

$$- 3\varepsilon\mathcal{B}_\varepsilon^{-1}\left(\left(D_{\eta\eta}R(0, L_\varepsilon)\left[\Phi_{\tau\eta\eta}\left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U}\right)\right]\right)_1(\cdot, 0, \cdot)\right). \tag{5.6}$$

5.2   *Solving equation* (5.5)

Here, we prove an existence and uniqueness result for equation (5.5) with initial condition $\rho(0, \cdot) = 0$. For this purpose, we need to thoroughly study the operators $\mathcal{R}_\varepsilon$ and $\mathcal{K}_\varepsilon$. To lighten notation, we do not stress explicitly the dependence on ε of the symbols of the operators we are going to consider. In particular, we simply write X_k for $X_{\varepsilon,k}$ (see (5.1)).

We begin by considering the operator $\mathcal{R}_\varepsilon$. Taking Proposition 5.3 and Theorem A.2(iii) into account, it is immediate to check that $\mathcal{R}_\varepsilon$ is well-defined in $C_\sharp^4$. Actually, we show that it can be extended to $C_\sharp^1 \cap C^2$ as a bounded operator which is sectorial.

PROPOSITION 5.4   For any $\varepsilon \in (0, 1/2]$, the operator $\mathcal{R}_\varepsilon$ can be extended to a sectorial operator R_ε with domain $C_\sharp^1 \cap C^2$. Moreover, $D_{R_\varepsilon}(\theta, \infty) = C_\sharp^{2\theta}$ for any $\theta \in (0, 1) \setminus \{1/2\}$, with equivalence of the corresponding norms.

Proof. To begin, we compute the symbol of the operator $\mathcal{S}_\varepsilon$ (see (4.17b)). For this purpose we set $v_1 := (R(0, L_\varepsilon)[\varphi(\mathbf{T}' - \mathbf{U})])_1(0, \cdot)$ and observe that, arguing as in the proof of (5.3), we can easily show that

$$\widehat{v}_1(0, k) = \frac{2}{3}\frac{1}{(X_k + 1)(X_k + 2)}\widehat{\varphi}(k), \quad k = 0, 1, \dots. \tag{5.7}$$

Hence, we have

$$s_k = \frac{3(X_k - 1)(X_k + 1)^2\{(\varepsilon - 1)X_k^2 + (\varepsilon - 1)X_k + 2\}}{16\varepsilon^2(X_k + 2)} \sim 48(\varepsilon - 1)\varepsilon^2\lambda_k^2 \qquad (5.8)$$

as $k \to +\infty$. From (5.2) and (5.8) it follows that the k-th symbol of the operator $\mathcal{R}_\varepsilon$ is

$$r_k = \frac{(X_k^2 - 1)\{(\varepsilon - 1)X_k^2 + (\varepsilon - 1)X_k + 2\}}{4\varepsilon^2(X_k^2 + 2X_k - 1)}, \quad k = 0, 1, \dots.$$

For any fixed $\varepsilon \in (0, 1/2]$, $r_k \sim (1 - \varepsilon^{-1})\lambda_k$ as $k \to +\infty$. Hence, we can split

$$\mathcal{R}_\varepsilon\varphi = \frac{1 - \varepsilon}{\varepsilon}\varphi_{yy} + \mathcal{R}_\varepsilon^{(1)}\varphi,$$

where the symbol of $\mathcal{R}_\varepsilon^{(1)}$ is

$$r_k^{(1)} = \frac{(X_k^2 - 1)\{(1 - \varepsilon)X_k + \varepsilon + 1\}}{4\varepsilon^2(X_k^2 + 2X_k - 1)} \sim \frac{(1 - \varepsilon)\sqrt{\varepsilon}}{2\varepsilon^2}\sqrt{\lambda_k} \quad (k \to +\infty).$$

We claim that the operator $\mathcal{R}_\varepsilon^{(1)}$ admits a realization in C^0 which is in $L(C_\sharp^{1+\alpha}, C_\sharp)$ for any $\alpha \in (1/2, 1)$. As a first step, we observe that, due to the characterization of the spaces H^s given in (2.3), the operator $\mathcal{R}_\varepsilon^{(1)}$ admits a realization $R_\varepsilon^{(1)} \in L(H^s, H^{s-1})$ for any $s \geqslant 1$. It is well-known that $C_\sharp^m \subset H^m \subset C_\sharp^{m-1/2}$ with continuous embeddings for any $m > 1/2$ such that $m - 1/2 \notin \mathbb{N}$. As a consequence, $R_\varepsilon^{(1)} \in L(C_\sharp^s, C_\sharp^{s-3/2})$ for any $s > 3/2$ such that $s - 3/2 \notin \mathbb{N}$. Therefore, $\mathcal{R}_\varepsilon$ can be extended with a bounded operator R_ε from $D(R_\varepsilon) = C_\sharp^1 \cap C^2$ into C^0.

Let us now prove that R_ε is sectorial. For this purpose, we note that $C_\sharp^\theta$ belongs to the class $J_{\theta/2}$ between C^0 and $C_\sharp^1 \cap C^2$, for any $\theta \in (0, 2)$, i.e., there exists a positive constant K such that

$$\|f\|_{C^\theta} \leqslant K\|f\|_{C^0}^{(2-\theta)/2}\|f\|_{C^2}^{\theta/2} \qquad (5.9)$$

for any $f \in C_\sharp^1 \cap C^2$, and the realization of the second-order derivative in $C([-\ell_0/2, \ell_0/2])$ with domain $C_\sharp^1 \cap C^2$ is sectorial. Hence, we can apply [11, Prop. 2.4.1(i)] to conclude that the operator R_ε is sectorial in C^0. It is now clear that the graph norm of R_ε is equivalent to the Euclidean norm of $C_\sharp^1 \cap C^2$. Hence, [11, Prop. 2.2.2] implies that $D_{R_\varepsilon}(\theta, \infty) = C_\sharp^{2\theta}$ for any $\theta \in (0, 1) \setminus \{1/2\}$. $\quad\square$

We now consider the operator $\mathcal{K}_\varepsilon$ of (5.6). From Proposition 5.3 and Theorems 4.4 and A.2(iii), we know that $\mathcal{K}_\varepsilon$ is continuous from $C_\sharp^{2+\alpha}$ into $[0, +\infty) \times C_\sharp$ for any $\alpha > 0$. Let us show that it can be extended to a larger domain.

PROPOSITION 5.5   Let the assumptions of Theorem 4.4 be satisfied. Then, for any $\varepsilon \in (0, 1/2]$, the operator $\mathcal{K}_\varepsilon$ can be extended to a continuous operator mapping $C_\sharp^s$ into $[0, +\infty) \times C_\sharp^s$ for any $s \geqslant 0$. Moreover, for any $r, T > 0$, there exists a positive constant $K = K(T, r)$ such that

$$\|\mathcal{K}_\varepsilon(\tau_2, \psi) - \mathcal{K}_\varepsilon(\tau_1, \psi)\|_\infty + \|\mathcal{K}_\varepsilon(\tau, \psi) - \mathcal{K}_\varepsilon(\tau, \xi)\|_\infty \leqslant K(|\tau_2 - \tau_1| + \|\psi - \xi\|_\infty)$$

for any $\tau, \tau_1, \tau_2 \in [0, T]$ and any $\psi, \xi \in B(0, r) \subset C_\sharp$.

Proof. As a first step, we observe that, using formulae (5.3) and (5.7), one can easily show that the k-th symbol g_k of the operator $\mathcal{G}_\varepsilon$ is

$$g_k = -\lambda_k \frac{3X_k^2 + 15X_k + 4}{2(X_k + 1)(X_k + 2)}. \tag{5.10}$$

From (5.2) and (5.10), it follows that the symbol of the operator $\mathcal{Z}_\varepsilon := \mathcal{B}_\varepsilon^{-1}\mathcal{G}_\varepsilon$ is

$$z_k = -\frac{2}{3}\lambda_k \frac{3X_k^2 + 15X_k + 4}{(X_k + 1)^2(X_k^2 + 2X_k - 1)} = -\frac{1}{2\varepsilon} + z_k^{(1)},$$

where

$$z_k^{(1)} \sim -\frac{1}{4\sqrt{\varepsilon^3 \lambda_k}}, \quad k \to +\infty. \tag{5.11}$$

We can thus split

$$\mathcal{Z}_\varepsilon = -\frac{1}{2\varepsilon}\mathrm{Id} + \mathcal{Z}_\varepsilon^{(1)},$$

where $\mathcal{Z}_\varepsilon^{(1)}$ is the operator whose symbol is $(z_k^{(1)})$. Formula (5.11) shows that $\mathcal{Z}_\varepsilon^{(1)} \in L(H^s, H^{s+1})$ for any $s \geqslant 0$. Hence, $\mathcal{Z}_\varepsilon^{(1)} \in L(C_\sharp^s, C_\sharp^{s+\theta})$ for any $s \in \mathbb{N} \cup \{0\}$ and any $\theta \in (0, 1/2)$. As a byproduct, $\mathcal{Z}_\varepsilon \in L(C_\sharp^s)$ for any $s \geqslant 0$. The assertion now follows from (5.6) on taking Proposition 5.3 and Theorem 4.4 into account. $\square$

From all the previous results, we get the following:

THEOREM 5.6 For any $\varepsilon \in (0, 1/2]$, equation (5.5) admits a unique solution ρ, defined in a maximal time domain $[0, T_\varepsilon)$, which vanishes at $\tau = 0$, belongs to $C^{1,2}([0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2])$ and satisfies $D_\eta^{(j)}\rho(\cdot, -\ell_0/2) \equiv D_\eta^{(j)}\rho(\cdot, \ell_0/2)$ for $j = 0, 1$.

Proof. Combining [11, Thms. 7.1.2 & 4.3.8], we can easily show that equation (5.5) admits a unique solution ρ, defined in a maximal time domain $[0, T_\varepsilon)$, which belongs to $C^{1,\beta}([0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2])$ for any $\beta < 2$, vanishes at $\tau = 0$, and satisfies $D_\eta^{(j)}\rho(\cdot, -\ell_0/2) \equiv D_\eta^{(j)}\rho(\cdot, \ell_0/2)$ for $j = 0, 1$. Moreover, $R_\varepsilon(\rho)$ is continuous in $[0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2]$. Since $R_\varepsilon(\rho)$ and $\frac{1-\varepsilon}{\varepsilon}D_{\eta\eta}$ differ by the the lower order operator $R_\varepsilon^{(1)}$ (see the proof of Proposition 5.4), $\rho_{\eta\eta} \in C([0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2])$ as well, and this completes the proof. $\square$

5.3 *Proof of Theorem 5.1*

This subsection is the final step of the proof of Theorem 5.1. Using some bootstrap arguments, we show that the solution ρ to equation (5.5) given by Theorem 5.6 is actually a solution to equation (4.20). Of course, we just need to show that both the functions $\rho_{\eta\eta}$ and ρ_τ belong to $C^{0,2}([0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2])$. Throughout the proof, we assume that T' is any fixed number in the interval $[0, T_\varepsilon)$.

To begin, we observe that Propositions 5.4 and 5.5 show that $\mathcal{K}_\varepsilon(\cdot, \rho) \in C([0, T']; D_{R_\varepsilon}(\theta, \infty))$ for any $\theta \in (0, 1)$. Therefore, we can apply [11, Thm. 4.3.8] and conclude that $\rho_\tau, R_\varepsilon(\rho)$ are bounded in $[0, T']$ with values in $C_\sharp^{2\theta}$ for any $\theta \in (0, 1)$. Since $R_\varepsilon(\rho)$ and $\frac{1-\varepsilon}{\varepsilon}D_{\eta\eta}$ differ by the lower order operator $R_\varepsilon^{(1)}$, $\rho_{\eta\eta} \in C^{0,2\theta}([0, T'] \times [-\ell_0/2, \ell_0/2])$ for any θ as above. In particular, ρ_τ and $\rho_{\eta\eta}$ belong to $C^{0,1}([0, T'] \times [-\ell_0/2, \ell_0/2])$.

Let us now set $\zeta := \rho_\eta$. Clearly, $\zeta \in C([0, T_\varepsilon); C_\sharp^2) \cap C^1([0, T_\varepsilon); C_\sharp)$ and $\zeta_\tau = \rho_{\eta\tau}$. Moreover,

$$\zeta_\tau = (1 - \varepsilon^{-1})\zeta_{\eta\eta} + D_\eta R_\varepsilon^{(1)}\rho + D_\eta \mathcal{K}_\varepsilon(\cdot, \rho_\eta)$$

in $[0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2]$, and $\zeta(0, \cdot) \equiv 0$. Since $R_\varepsilon^{(1)} \in L(C_\sharp^{3+\theta}, C_\sharp^{3/2+\theta})$ for any $\theta \in (0, 1)\backslash\{1/2\}$, the function $D_\eta R_\varepsilon^{(1)}(\rho)$ is bounded in $[0, T']$ with values in $C_\sharp^\alpha$ for any $\alpha < 3/2$. Using Proposition 5.5 we now deduce that $D_\eta R_\varepsilon^{(1)}(\rho) + D_\eta \mathcal{K}_\varepsilon(\cdot, \rho_\eta(\tau, \cdot))$ is bounded in $[0, T']$ with values in $C_\sharp^\alpha$ for any α as above. Hence, Theorem 4.3.9(iii) of [11] implies that the functions ζ_τ and $\zeta_{\eta\eta}$ are bounded (in fact, continuous) in $[0, T']$ with values in $C_\sharp^\alpha$. This completes the proof.

6. Uniform existence of ρ and proof of the main result

So far we have only proved a local existence-uniqueness result for equation (4.20) on an ε-dependent time interval. In this section, we want to prove that, for any fixed $T > 0$, the local solution ρ exists in the whole of $[0, T]$, at least for sufficiently small values of ε. The main tool in this direction is represented by the a priori estimates in the next subsection.

6.1   *A priori estimates*

The main result of this subsection is contained in the following theorem.

THEOREM 6.1   For any $T > 0$, there exist $\varepsilon_0 = \varepsilon_0(T) \in (0, 1/2)$ and $K = K(T) > 0$ such that, for any $T' < T$ and any $\rho \in \mathcal{Y}_{T'}$ (see Definition 4.1) which solves equation (4.20), we have

$$\sup_{\substack{\tau \in (0,T'] \\ \eta \in [-\ell_0/2, \ell_0/2]}} |\rho_\eta(\tau, \eta)| + \sup_{\tau \in (0,T']} \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\eta\eta}(\tau, \eta))^2 \, d\eta$$

$$+ \int_0^{T'} \int_{-\ell_0/2}^{\ell_0/2} (\rho_\tau(\tau, \eta))^2 \, d\eta \, d\tau + \int_0^{T'} \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\tau\eta}(\tau, \eta))^2 \, d\eta \, d\tau \leqslant K$$

for all $\tau \in (0, T']$, whenever $\varepsilon \leqslant \varepsilon_0$.

The proof of Theorem 6.1 employs an energy method. Let $\rho \in \mathcal{Y}_{T'}$ solve (4.20), i.e.,

$$\frac{\partial}{\partial \tau} \mathcal{B}_\varepsilon(\rho) = \mathcal{S}_\varepsilon(\rho) - \Phi_\eta \rho_\eta - \frac{\varepsilon}{2}(\rho_\eta)^2 + \mathcal{G}_\varepsilon((\Phi_\eta + \varepsilon\rho_\eta)^2) + \mathcal{H}_\varepsilon(\Phi). \tag{6.1}$$

Multiplying both sides of (6.1) by ρ_τ and integrating over $[-\ell_0/2, \ell_0/2]$, we get

$$\int_{-\ell_0/2}^{\ell_0/2} \mathcal{B}_\varepsilon(\rho_\tau)\rho_\tau \, d\eta = \int_{-\ell_0/2}^{\ell_0/2} \mathcal{S}_\varepsilon(\rho)\rho_\tau \, d\eta - \int_{-\ell_0/2}^{\ell_0/2} \Phi_\eta \rho_\eta \rho_\tau \, d\eta - \frac{\varepsilon}{2} \int_{-\ell_0/2}^{\ell_0/2} (\rho_\eta)^2 \rho_\tau \, d\eta$$

$$+ \int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon((\Phi_\eta + \varepsilon\rho_\eta)^2)\rho_\tau \, d\eta + \int_{-\ell_0/2}^{\ell_0/2} \mathcal{H}_\varepsilon(\Phi)\rho_\tau \, d\eta. \tag{6.2}$$

Using the definition of the operator $\mathcal{S}_\varepsilon$ (see (4.17b)) and then integrating by parts yields

$$\int_{-\ell_0/2}^{\ell_0/2} \mathcal{S}_\varepsilon(\rho)\rho_\tau \, \mathrm{d}\eta$$

$$= \int_{-\ell_0/2}^{\ell_0/2} \{-3(1-\varepsilon)\rho_{\eta\eta\eta\eta} - \rho_{\eta\eta} + 3\varepsilon(D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot)\}\rho_\tau \, \mathrm{d}\eta$$

$$= -\frac{3}{2}(1-\varepsilon)\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\eta\eta})^2 \, \mathrm{d}\eta - \int_{-\ell_0/2}^{\ell_0/2} \rho_{\eta\eta}\rho_\tau \, \mathrm{d}\eta$$

$$+ 3\varepsilon \int_{-\ell_0/2}^{\ell_0/2} (D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot)\rho_\tau \, \mathrm{d}\eta.$$

Therefore, we can write equation (6.2) in the following equivalent form:

$$\frac{3}{2}(1-\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\eta\eta})^2 \, \mathrm{d}\eta + \int_{-\ell_0/2}^{\ell_0/2} \mathcal{B}_\varepsilon(\rho_\tau)\rho_\tau \, \mathrm{d}\eta$$

$$= -\frac{\varepsilon}{2} \int_{-\ell_0/2}^{\ell_0/2} (\rho_\eta)^2 \rho_\tau \, \mathrm{d}\eta - \int_{-\ell_0/2}^{\ell_0/2} \rho_{\eta\eta}\rho_\tau \, \mathrm{d}\eta$$

$$+ 3\varepsilon \int_{-\ell_0/2}^{\ell_0/2} (D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot)\rho_\tau \, \mathrm{d}\eta$$

$$+ \int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon((\Phi_\eta + \varepsilon\rho_\eta)^2)\rho_\tau \, \mathrm{d}\eta - \int_{-\ell_0/2}^{\ell_0/2} \Phi_\eta\rho_\eta\rho_\tau \, \mathrm{d}\eta + \int_{-\ell_0/2}^{\ell_0/2} \mathcal{H}_\varepsilon(\Phi)\rho_\tau \, \mathrm{d}\eta. \quad (6.3)$$

In the following lemmata we estimate the terms

$$\mathcal{I}_1 := \int_{-\ell_0/2}^{\ell_0/2} \mathcal{B}_\varepsilon(\rho_\tau)\rho_\tau \, \mathrm{d}\eta,$$

$$\mathcal{I}_2 := \int_{-\ell_0/2}^{\ell_0/2} (D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\cdot, 0, \cdot)\rho_\tau \, \mathrm{d}\eta,$$

$$\mathcal{I}_3 := \int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon((\Phi_\eta + \varepsilon\rho_\eta)^2)\rho_\tau \, \mathrm{d}\eta,$$

$$\mathcal{I}_4 := \int_{-\ell_0/2}^{\ell_0/2} \mathcal{H}_\varepsilon(\Phi)\rho_\tau \, \mathrm{d}\eta.$$

The main issue is to control $\mathcal{I}_1$:

LEMMA 6.2 We have

$$\mathcal{I}_1(\tau) \geqslant \int_{-\ell_0/2}^{\ell_0/2} (\rho_\tau(\tau, \cdot))^2 \, \mathrm{d}\eta + 3\varepsilon \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\tau\eta}(\tau, \cdot))^2 \, \mathrm{d}\eta$$

for any $\tau \in [0, T']$ and any $\varepsilon \in (0, 1/2]$.

Proof. Of course, we can limit ourselves to proving the estimate with $\rho_\tau(\tau, \cdot)$ replaced by $\varphi \in H^2$.
    It is immediate to check that

$$\int_{-\ell_0/2}^{\ell_0/2} \mathcal{B}_\varepsilon(\varphi)\varphi \, d\eta = \sum_{k=0}^{+\infty} b_{\varepsilon,k}|\widehat{\varphi}(k)|^2,$$

where the symbol $(b_{\varepsilon,k})$ of the operator $\mathcal{B}_\varepsilon$ is defined by (5.2). Note that $b_{\varepsilon,k} = h(X_{\varepsilon,k})$ for any
$k = 0, 1, \ldots$, where the function $h : [1, +\infty) \to \mathbb{R}$ is defined by

$$h(s) = \frac{3}{4}\frac{(s+1)(s^2 + 2s - 1)}{s+2}, \quad s \geqslant 1.$$

Since $h(s) \geqslant (3s^2 + 1)/4$ for any $s \geqslant 1$, we can estimate

$$\int_{-\ell_0/2}^{\ell_0/2} \mathcal{B}_\varepsilon(\varphi)\varphi \, d\eta \geqslant \sum_{k=0}^{+\infty}(1 + 3\varepsilon\lambda_k)|\widehat{\varphi}(k)|^2 = \int_{-\ell_0/2}^{\ell_0/2} |\varphi|^2 \, d\eta + 3\varepsilon \int_{-\ell_0/2}^{\ell_0/2} |\varphi_\eta|^2 \, d\eta,$$

and we are done.   □

We now consider the terms $\mathfrak{I}_2$, $\mathfrak{I}_3$ and $\mathfrak{I}_4$.

LEMMA 6.3   For any $\tau \in [0, T']$ and any $\varepsilon \in (0, 1/2]$,

$$|\mathfrak{I}_2(\tau)| \leqslant \frac{1}{12\varepsilon}\|\rho_\tau(\tau, \cdot)\|_2\|\rho_{\eta\eta}(\tau, \cdot)\|_2.$$

Proof. It is immediately seen that for any $\tau \in [0, T]$ we can estimate

$$\left|\int_{-\ell_0/2}^{\ell_0/2}(D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\tau, 0, \cdot)\rho_\tau(\tau, \cdot) \, d\eta\right|$$

$$\leqslant \frac{1}{3\varepsilon}\|\varrho_\tau(\tau, \cdot)\|_2\|(D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\tau, 0, \cdot)\|_2.$$

To compute the L^2-norm of the function $(D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\tau, 0, \cdot)$, we take advantage
of formula (5.7), which allows us to estimate

$$\|(D_{\eta\eta}R(0, L_\varepsilon)[\rho_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1(\tau, 0, \cdot)\|_2^2 = \frac{1}{4}\sum_{k=0}^{+\infty}\left|\frac{X_k + 1}{X_k + 2}\lambda_k^2\widehat{\rho}(\tau, k)\right|^2 \leqslant \frac{1}{4}\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2,$$

where, as usual, $X_k = \sqrt{1 + 4\varepsilon\lambda_k}$ and $\widehat{\rho}(\tau, k)$ is the k-th Fourier coefficient of the function $\rho(\tau, \cdot)$.
This accomplishes the proof.   □

LEMMA 6.4   There exists a positive constant C, independent of $\varepsilon \in (0, 1/2]$, τ and T', such that

$$|\mathfrak{I}_3(\tau)| \leqslant C\big(\|\rho_\tau(\tau, \cdot)\|_2 + \|\rho_\tau(\tau, \cdot)\|_2\|\rho_{\eta\eta}(\tau, \cdot)\|_2 + \varepsilon\|\rho_\tau(\tau, \cdot)\|_2\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2$$
$$+ \varepsilon\|\rho_{\tau\eta}(\tau, \cdot)\|_2\|\rho_{\eta\eta}(\tau, \cdot)\|_2 + \varepsilon^2\|\rho_{\tau\eta}(\tau, \cdot)\|_2\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2\big)$$

for any $\tau \in [0, T']$.

Proof. As in the proof of the previous lemma, it is enough to estimate the L^2-norm of the function $\mathcal{G}_\varepsilon((\Phi_\eta(\tau, \cdot) + \varepsilon\rho_\eta(\tau, \cdot))^2)$. For this purpose, we observe that (see (5.10))

$$\|\mathcal{G}_\varepsilon(\psi)\|_2^2 = \sum_{k=0}^{+\infty} \lambda_k^2 \left(\frac{3X_k^2 + 15X_k + 4}{2(X_k + 1)(X_k + 2)} \right)^2 |\widehat{\psi}(k)|^2 \leqslant \frac{121}{36} \sum_{k=0}^{+\infty} \lambda_k^2 |\widehat{\psi}(k)|^2$$

for any $\psi \in H^2$, where $X_k = \sqrt{1 + 4\varepsilon\lambda_k}$ for any $k = 0, 1, \ldots$. It follows that

$$\|\mathcal{G}_\varepsilon(\psi)\|_2 \leqslant 2\|\psi_{\eta\eta}\|_2. \tag{6.4}$$

Moreover, the symbol g_k can be split as follows:

$$g_k = -\frac{3}{2}\lambda_k + \frac{1}{4\varepsilon}h(X_k), \quad k = 0, 1, \ldots,$$

where the function $h : [1, +\infty) \to \mathbb{R}$ is defined by

$$h(s) = \frac{(1 - 3s)(s - 1)}{s + 2}, \quad s \geqslant 1.$$

Clearly, $-3s \leqslant h(s) \leqslant 0$ for any $s \geqslant 1$. Hence, we can split

$$\mathcal{G}_\varepsilon(\psi) = \frac{3}{2}\psi_{\eta\eta} + \frac{1}{4\varepsilon}\mathcal{G}_\varepsilon^{(1)}(\psi), \tag{6.5}$$

where $\mathcal{G}_\varepsilon^{(1)} \in L(H^1, L^2)$ and

$$\|\mathcal{G}_\varepsilon^{(1)}(\psi)\|_2 \leqslant 3\left(\sum_{k=0}^{+\infty}(1 + 4\varepsilon\lambda_k)|\widehat{\psi}(k)|^2 \right)^{1/2} = 3(\|\psi\|_2^2 + 4\varepsilon\|\psi_\eta\|_2^2)^{1/2} \leqslant 6\|\psi\|_{H^1}. \tag{6.6}$$

We now split (for any arbitrarily fixed $\tau \in [0, T']$)

$$\int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon((\Phi_\eta(\tau, \cdot) + \varepsilon\rho_\eta(\tau, \cdot))^2)\rho_\tau(\tau, \cdot)\,\mathrm{d}\eta$$

$$= \int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon((\Phi_\eta(\tau, \cdot))^2)\rho_\tau(\tau, \cdot)\,\mathrm{d}\eta + 2\varepsilon\int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon(\Phi_\eta(\tau, \cdot)\rho_\eta(\tau, \cdot))\rho_\tau(\tau, \cdot)\,\mathrm{d}\eta$$

$$+ \varepsilon^2 \int_{-\ell_0/2}^{\ell_0/2} \mathcal{G}_\varepsilon((\rho_\eta(\tau, \cdot))^2)\rho_\tau\,\mathrm{d}\eta$$

$$:= J_1(\tau) + J_2(\tau) + J_3(\tau).$$

To estimate J_1, we use (6.4) and the Hölder inequality to get

$$|J_1(\tau)| \leqslant 2\|\rho_\tau(\tau, \cdot)\|_2\|((\Phi_\eta(\tau, \cdot))^2)_{\eta\eta}\|_2. \tag{6.7}$$

Estimating the terms J_2 and J_3 is a bit more tricky. Using (6.5) and (6.6), we get

$$
\begin{aligned}
|J_2(\tau)| &\leqslant 3\varepsilon \left| \int_{-\ell_0/2}^{\ell_0/2} \rho_\tau(\tau,\cdot)(\Phi_\eta(\tau,\cdot)\rho_\eta(\tau,\cdot))_{\eta\eta}\, \mathrm{d}\eta \right| + \frac{3}{2}\|\Phi_\eta(\tau,\cdot)\rho_\eta(\tau,\cdot)\|_{H^1}\|\rho_\tau(\tau,\cdot)\|_2 \\
&= 3\varepsilon \left| \int_{-\ell_0/2}^{\ell_0/2} \rho_{\tau\eta}(\tau,\cdot)(\Phi_{\eta\eta}(\tau,\cdot)\rho_\eta(\tau,\cdot) + \Phi_\eta(\tau,\cdot)\rho_{\eta\eta}(\tau,\cdot))\, \mathrm{d}\eta \right| \\
&\quad + \frac{3}{2}\|\Phi_\eta(\tau,\cdot)\rho_\eta(\tau,\cdot)\|_{H^1}\|\rho_\tau(\tau,\cdot)\|_2 \\
&\leqslant 3\varepsilon\|\Phi_{\eta\eta}\|_\infty\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_\eta(\tau,\cdot)\|_2 + 3\varepsilon\|\Phi_\eta\|_\infty\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2 \\
&\quad + \frac{3}{2}\{(\|\Phi_\eta\|_\infty + \|\Phi_{\eta\eta}\|_\infty)\|\rho_\eta(\tau,\cdot)\|_2 + \|\Phi_\eta\|_\infty\|\rho_{\eta\eta}(\tau,\cdot)\|_2\}\|\rho_\tau(\tau,\cdot)\|_2. \quad (6.8)
\end{aligned}
$$

Using the Poincaré–Wirtinger inequality, we can continue the estimate to obtain

$$
|J_2(\tau)| \leqslant C_1(\varepsilon\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2 + \|\rho_{\eta\eta}(\tau,\cdot)\|_2\|\rho_\tau(\tau,\cdot)\|_2) \quad (6.9)
$$

for some positive constant C_1, independent of τ, T' and ε.

To estimate J_3 we argue similarly to get

$$
\begin{aligned}
|J_3(\tau)| &\leqslant \frac{3}{2}\varepsilon^2 \left| \int_{-\ell_0/2}^{\ell_0/2} \rho_\tau(\tau,\cdot)((\rho_\eta(\tau,\cdot))^2)_{\eta\eta}\, \mathrm{d}\eta \right| + \frac{3}{2}\varepsilon\|(\rho_\eta(\tau,\cdot))^2\|_{H^1}\|\rho_\tau(\tau,\cdot)\|_2 \\
&= \frac{3}{2}\varepsilon^2 \left| \int_{-\ell_0/2}^{\ell_0/2} \rho_{\tau\eta}(\tau,\cdot)((\rho_\eta(\tau,\cdot))^2)_\eta\, \mathrm{d}\eta \right| + \frac{3}{2}\varepsilon\|(\rho_\eta(\tau,\cdot))^2\|_{H^1}\|\rho_\tau(\tau,\cdot)\|_2 \\
&\leqslant 3\varepsilon^2\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_\eta(\tau,\cdot)\|_\infty\|\rho_{\eta\eta}(\tau,\cdot)\|_2 \\
&\quad + \frac{3}{2}\varepsilon\|\rho_\eta(\tau,\cdot)\|_\infty(\|\rho_\eta(\tau,\cdot)\|_2 + 2\|\rho_{\eta\eta}(\tau,\cdot)\|_2)\|\rho_\tau(\tau,\cdot)\|_2 \\
&\leqslant C_2(\varepsilon^2\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2^2 + \varepsilon\|\rho_{\eta\eta}(\tau,\cdot)\|_2^2\|\rho_\tau(\tau,\cdot)\|_2) \quad (6.10)
\end{aligned}
$$

for some positive constant C_2, independent of τ, T' and ε.

By combining (6.7), (6.8) and (6.10), the assertion follows at once. $\qquad\square$

LEMMA 6.5   There exists a positive constant C, independent of $\varepsilon \in (0, 1/2]$, τ and T', such that

$$
|\mathcal{J}_4(\tau)| \leqslant C(\|\Phi_{\tau\eta\eta}(\tau,\cdot)\|_2 + \|\Phi_{\eta\eta\eta}(\tau,\cdot)\|_2)\|\rho_\tau(\tau,\cdot)\|_2, \quad \tau \in [0, T']. \quad (6.11)
$$

Proof. Of course, we just need to estimate the terms

$$
J_4^{(1)}(\tau) = \varepsilon \int_{-\ell_0/2}^{\ell_0/2} \rho_\tau \left(D_{\eta\eta} R(0, L_\varepsilon) \left[\Phi_{\tau\eta\eta} \left(\mathbf{V} - \mathbf{T} - \frac{4}{3}\mathbf{U} \right) \right] \right)_1 (\cdot, 0, \cdot)\, \mathrm{d}\eta,
$$

$$
J_4^{(2)}(\tau) = \int_{-\ell_0/2}^{\ell_0/2} \rho_\tau (D_{\eta\eta} R(0, L_\varepsilon)[\Phi_{\eta\eta}(\mathbf{T}' - \mathbf{U})])_1 (\cdot, 0, \cdot)\, \mathrm{d}\eta;
$$

the remaining terms are easily handled.

Concerning $J_4^{(1)}$, taking (5.3) into account we can estimate

$$|J_4^{(1)}(\tau)| \leqslant \frac{4}{9}\varepsilon \sum_{k=0}^{+\infty} \lambda_k^2 \frac{4X_k + 7}{(X_k + 1)^2(X_k + 2)}|\widehat{\rho}_\tau(\tau,k)|\,|\widehat{\Phi}_\tau(\tau,k)|$$

$$\leqslant \frac{1}{9} \sum_{k=0}^{+\infty} \lambda_k \frac{X_k^2(4X_k + 7)}{(X_k + 1)^2(X_k + 2)}|\widehat{\rho}_\tau(\tau,k)|\,|\widehat{\Phi}_\tau(\tau,k)|$$

$$\leqslant \frac{4}{9} \sum_{k=0}^{+\infty} \lambda_k|\widehat{\rho}_\tau(\tau,k)|\,|\widehat{\Phi}_\tau(\tau,k)| = \frac{4}{9}\|\rho_\tau(\tau,\cdot)\|_2\|\Phi_{\tau\eta\eta}(\tau,\cdot)\|_2$$

for any $\tau \in [0, T']$. Similarly, using (5.7) we can estimate

$$|J_4^{(2)}(\tau)| \leqslant \frac{2}{3} \sum_{k=0}^{+\infty} \lambda_k^2 \frac{1}{(X_k + 1)(X_k + 2)}|\widehat{\rho}_\tau(\tau,k)|\,|\widehat{\Phi}(\tau,k)| \leqslant \frac{1}{3} \sum_{k=0}^{+\infty} \lambda_k^2|\widehat{\rho}_\tau(\tau,k)|\,|\widehat{\Phi}(\tau,k)|$$

$$= \frac{1}{3}\|\rho_\tau(\tau,\cdot)\|_2\|\Phi_{\eta\eta\eta}(\tau,\cdot)\|_2$$

for any $\tau \in [0, T']$. Now, estimate (6.11) follows immediately. $\square$

Finally, we recall the following technical result proved in [1], which plays a crucial role in the proof of Theorem 6.1.

LEMMA 6.6 (slight extension of [1, Lemma 3.1]) Let A_0, C_0, C_1, C_2 be positive constants. For any $T > 0$, there exist $\varepsilon_0 \in (0, 1/2)$ and a constant K_0 such that, if $A_\varepsilon \in C^1([0, T'])$ $(T' \in (0, T))$ satisfies

$$\begin{cases} A_\varepsilon'(\tau) \leqslant C_0 + C_1 A_\varepsilon(\tau) + C_2\varepsilon(A_\varepsilon(\tau))^2, & \tau \in [0, T'], \\ A_\varepsilon(0) \leqslant A_0, \end{cases}$$

for some $\varepsilon \in (0, \varepsilon_0]$, then $A_\varepsilon(\tau) \leqslant K_0$ for any $\tau \in [0, T']$.

Proof of Theorem 6.1.   To begin, we observe that, invoking the Poincaré–Wirtinger inequality, we can estimate

$$\left|\int_{-\ell_0/2}^{\ell_0/2} (\rho_\eta(\tau,\cdot))^2\rho_\tau(\tau,\cdot)\,\mathrm{d}\eta\right| \leqslant \|\rho_\tau(\tau,\cdot)\|_2\|\rho_\eta(\tau,\cdot)\|_2\|\rho_\eta(\tau,\cdot)\|_\infty \leqslant \sqrt{\ell_0}\|\rho_\tau(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2^2$$

for any $\tau \in [0, T']$. Hence, Lemmata 6.2–6.5, estimate (6.3) and the Hölder inequality give

$$\frac{3}{2}(1 - \varepsilon)\frac{\mathrm{d}}{\mathrm{d}\tau}\|\rho_{\eta\eta}(\tau,\cdot)\|_2^2 + \|\rho_\tau(\tau,\cdot)\|_2^2 + 3\varepsilon\|\rho_{\tau\eta}(\tau,\cdot)\|_2^2$$

$$\leqslant K\big(\|\rho_\tau(\tau,\cdot)\|_2 + \|\rho_\tau(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2 + \varepsilon\|\rho_\tau(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2^2$$

$$+ \varepsilon\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2 + \varepsilon^2\|\rho_{\tau\eta}(\tau,\cdot)\|_2\|\rho_{\eta\eta}(\tau,\cdot)\|_2^2\big) \tag{6.12}$$

for some positive constant K, depending on Φ but independent of $\tau \in [0, T']$. Using the Young inequality $ab \leqslant \frac{1}{4}a^2 + b^2$ we get

$$\frac{3}{2}(1 - \varepsilon)\frac{\mathrm{d}}{\mathrm{d}\tau}\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2 + \|\rho_\tau(\tau, \cdot)\|_2^2 + 3\varepsilon\|\rho_{\tau\eta}(\tau, \cdot)\|_2^2$$

$$\leqslant K^2 + \frac{3}{4}\|\rho_\tau(\tau, \cdot)\|_2^2 + \frac{1}{4}(\varepsilon + \varepsilon^2)\|\rho_{\tau\eta}(\tau, \cdot)\|_2^2$$
$$+ K^2(1 + \varepsilon + \varepsilon^2)\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2 + K^2\varepsilon^2\|\rho_{\eta\eta}(\tau, \cdot)\|_2^4, \quad (6.13)$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2 \leqslant \frac{4}{3}K^2 + 4K^2\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2 + \frac{4}{3}K^2\varepsilon^2\|\rho_{\eta\eta}(\tau, \cdot)\|_2^4$$

$$\leqslant \frac{4}{3}K^2 + 4K^2\|\rho_{\eta\eta}(\tau, \cdot)\|_2^2 + \frac{2}{3}K^2\varepsilon\|\rho_{\eta\eta}(\tau, \cdot)\|_2^4, \quad (6.14)$$

provided that $\varepsilon \leqslant 1/2$. Applying Lemma 6.6 to (6.14) with $(C_0, C_1, C_2) = (4K^2/3, 4K^2, 2K^2/3)$ and $A_\varepsilon(\tau) = \|\rho_{\eta\eta}(\tau, \cdot)\|_2^2$, we deduce that there exist $\varepsilon_0 \in (0, 1/2)$ and $K_0 > 0$ such that

$$\sup_{\tau \in (0,T']} \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\eta\eta}(\tau, \eta))^2 \, \mathrm{d}\eta \leqslant K_0$$

for any $\varepsilon \in (0, \varepsilon_0)$. Now, using again the Poincaré–Wirtinger inequality, we get

$$\sup_{\substack{\tau \in (0,T'] \\ \eta \in [-\ell_0/2, \ell_0/2]}} |\rho_\eta(\tau, \eta)| \leqslant K_1$$

for any $\varepsilon \in (0, \varepsilon_0]$, with K_1 independent of ε and $T' < T$.

Finally, integrating (6.13) and using the estimates obtained so far, we deduce that

$$\int_0^{T'} \int_{-\ell_0/2}^{\ell_0/2} (\rho_\tau(\tau, \eta))^2 \, \mathrm{d}\tau \, \mathrm{d}\eta + \int_0^{T'} \int_{-\ell_0/2}^{\ell_0/2} (\rho_{\tau\eta}(\tau, \eta))^2 \, \mathrm{d}\tau \, \mathrm{d}\eta \leqslant K_2 \quad (6.15)$$

for some constant K_2, independent of ε and $T' < T$. The assertion now follows. $\quad\square$

COROLLARY 6.7   Under the assumptions of Theorem 6.1, there exists a constant $M > 0$, independent of $T' < T$, such that

$$\|\rho\|_{C^{0,1}([0,T'] \times [-\ell_0/2, \ell_0/2])} \leqslant M$$

for any $\varepsilon \in (0, \varepsilon_0]$.

Proof. In view of Theorem 6.1 we only have to estimate the sup-norm of the function ρ, but this is immediate if we observe that (6.15) implies that ρ is bounded in $H^1([0, T']; H^1)$ by a constant independent of T', and $\rho(0, \cdot) = 0$. $\quad\square$

6.2 *Solving equation* (4.20) *in* $[0, T]$

We now consider a fixed time interval $[0, T]$ and $0 < \varepsilon \leqslant \varepsilon_0$, with $\varepsilon_0 = \varepsilon_0(T)$ given by Theorem 6.1. Thanks to the a priori estimates of Subsection 6.1 and a classical result for semilinear problems, we can show that, for any $\varepsilon \in (0, \varepsilon_0]$, the solution $\rho = \rho_\varepsilon$ to problem (4.20), given by Theorem 5.1, can be extended to a function $\rho \in \mathcal{Y}_T$ (see Definition 4.1) which solves the equation in the whole of $[0, T]$.

THEOREM 6.8 Fix $T > 0$ and let $\varepsilon_0 = \varepsilon_0(T)$ be as in Theorem 6.1. Then, for any $\varepsilon \in (0, \varepsilon_0]$, equation (4.20) admits a unique solution $\rho \in \mathcal{Y}_T$.

Proof. Let us fix T as in the statement of the theorem and let $\varepsilon \in (0, \varepsilon_0)$. Suppose for contradiction that $T_\varepsilon < T$. Then, by Theorem 6.1 applied with any $T' < T_\varepsilon$, we get

$$\sup_{\tau \in [0, T_\varepsilon)} \|\rho(\tau, \cdot)\|_{C^1} < K$$

for some positive constant K, independent of ε. Hence, the function $\mathcal{K}_\varepsilon(\cdot, \rho_\eta)$ is bounded in $[0, T_\varepsilon) \times [-\ell_0/2, \ell_0/2]$. In view of [11, Prop. 7.1.8], applied to equation (5.5), and the bootstrap argument in the proof of Theorem 5.1, this leads us to a contradiction. $\qquad\square$

6.3 *Proof of Main Theorem 1.1*

We are now in a position to prove the main result of this paper. Let us fix a function $\Phi_0 \in C_\sharp^{4+4\beta}$ for some $\beta \in (1/2, 1)$.

From the results in Subsection 6.2, we know that, for any $T > 0$, there exists $\varepsilon_0 = \varepsilon_0(T)$ such that equation (4.16) admits a unique solution $\psi_\varepsilon \in \mathcal{Y}_T$ such that $\rho(0, \cdot) = \Phi_0$. Moreover, by Corollary 6.7,
$$\|\psi_\varepsilon(\tau, \cdot) - \Phi_0(\tau, \cdot)\|_{C^0} \leqslant \varepsilon M, \quad \tau \in [0, T]$$
for some positive constant M and any $\varepsilon \in (0, \varepsilon_0]$.

In view of Theorem 4.3, there exists a function $v \in \mathcal{V}_T$ such that the pair (v, ψ) is the unique solution to problem (3.1)–(3.3).

Coming back to problem (1.8)–(1.10) and setting $\ell_\varepsilon = \ell_0/\sqrt{\varepsilon}$ and $T_\varepsilon = T/\varepsilon^2$, it is now immediate to conclude that, for any $\varepsilon \in (0, \varepsilon_0]$, it admits a unique solution $(u, \varphi) \in \mathcal{V}_{T_\varepsilon} \times \mathcal{Y}_{T_\varepsilon}$. Moreover,
$$\|\varphi_\varepsilon(t, \cdot) - \varepsilon\Phi(t\varepsilon^2, \sqrt{\varepsilon}\,\cdot)\|_{C([-\ell_\varepsilon/2, \ell_\varepsilon/2])} \leqslant \varepsilon^2 M, \quad t \in [0, T_\varepsilon].$$

This accomplishes the proof of the Main Theorem.

A. Some results from [4]

In this appendix we recall some results from [4] that are used throughout this paper.

A.1 *The operator L*

Let $\mathcal{L}$ be the differential operator defined on smooth functions $\mathbf{u}$ by

$$(\mathcal{L}\mathbf{u})(x, \eta) = \begin{cases} D_{xx}u_1(x, \eta) - D_x u_1(x, \eta) + e^x u_1(0, \eta), & x \leqslant 0, \ |\eta| \leqslant \ell_0/2, \\ D_{xx}u_2(x, \eta) - D_x u_2(x, \eta), & x \geqslant 0, \ |\eta| \leqslant \ell_0/2, \end{cases}$$

and let L be its realization in $\mathcal{X}$, defined by

$$
\begin{cases}
D(L) = \{\mathbf{u} \in C^{2,0}(I_-) \times C^{2,0}(I_+) : \mathbf{u}, \mathcal{L}\mathbf{u} \in \mathcal{X},\ D_x^{(j)} u_1(0, \cdot) = D_x^{(j)} u_2(0, \cdot),\ j = 0, 1\}, \\[2mm]
L\mathbf{u} = \begin{cases} D_{xx}u_1 - D_x u_1 + e^x u_1(0, \cdot), & (x, \eta) \in I_-, \\ D_{xx}u_2 - D_x u_2, & (x, \eta) \in I_+. \end{cases}
\end{cases}
$$

THEOREM A.1   The following properties are satisfied:

(i)   the operator L is sectorial, and hence it generates an analytic semigroup in $\mathcal{X}$;
(ii) the spectrum of the operator L consists of 0 and the halfline $(-\infty, -1/4]$;
(iii) the spectral projection on the kernel of L is the operator $\mathcal{P}$ defined by

$$
\mathcal{P}(\mathbf{f}) = \left(\int_{-\infty}^0 f_1(x, \cdot)\, dx + \int_0^{+\infty} e^{-x} f_2(x, \cdot)\, dx \right) \mathbf{U} = Q(\mathbf{f})\mathbf{U}, \quad \mathbf{f} \in \mathcal{X};
$$

(iv) let $\mathbf{f} \in \mathcal{X}$; then the equation $L\mathbf{u} = \mathbf{f}$ has a solution $\mathbf{u} \in D(L)$ if and only if $\mathcal{P}(\mathbf{f}) = 0$;
(v) for any $\lambda \notin (-\infty, -1/4] \cup \{0\}$ and any $\mathbf{f} = (f_1, f_2) \in \mathcal{X}$, setting $\mathbf{u} := R(\lambda, L)\mathbf{f}$ we have

$$
u_1(0, \eta) = u_2(0, \eta) = g(\lambda) \left(\int_{-\infty}^0 e^{-\nu_1 t} f_1(t, \eta)\, dt + \int_0^{+\infty} e^{-\nu_2 t} f_2(t, \eta)\, dt \right) \tag{A1}
$$

for any $\eta \in [-\ell_0/2, \ell_0/2]$, where

$$
g(\lambda) = \left(\frac{2\lambda}{1 + (2\lambda - 1)X(\lambda)} \frac{1}{\nu_2} + 1 \right) \frac{1}{X(\lambda)}, \quad X(\lambda) = \sqrt{1 + 4\lambda}.
$$

A.2   *The operator L_ε*

For any $\varepsilon > 0$, we consider the operator $L + \varepsilon A$ defined by

$$
\begin{cases}
D(L + \varepsilon A) = \{\mathbf{u} \in C^{2,0}(I_-) \cap C^{0,2}(I_-) \times C^{2,0}(I_+) \cap C^{0,2}(I_+) : \\
\qquad\qquad \mathbf{u}, \mathbf{u}_{\eta\eta}, \mathcal{L}\mathbf{u} \in \mathcal{X},\ D_x^{(j)} u_1(0, \cdot) = D_x^{(j)} u_2(0, \cdot),\ j = 0, 1, \\
\qquad\qquad D_\eta^{(j)} u_i(\cdot, -\ell_0/2) = D_\eta^{(j)} u_i(\cdot, \ell_0/2),\ i = 1, 2,\ j = 0, 1\}, \\[2mm]
(L + \varepsilon A)\mathbf{u} = \begin{cases} D_{xx}u_1 + \varepsilon D_{\eta\eta}u_1 - D_x u_1 + e^x u_1(0, \cdot), & (x, \eta) \in I_-, \\ D_{xx}u_2 + \varepsilon D_{\eta\eta}u_2 - D_x u_2, & (x, \eta) \in I_+. \end{cases}
\end{cases}
$$

THEOREM A.2   The following properties are satisfied:

(i)   the operator $L + \varepsilon A$ is closable and its closure L_ε is sectorial;
(ii) the restriction of L_ε to $(I - \mathcal{P})(\mathcal{X})$ is sectorial and 0 is in its resolvent set;
(iii) let $\mathbf{f} = \mathbf{h}\,\varphi$ for some $\mathbf{h} \in (I - \mathcal{P})(\mathcal{X})$, independent of η, and some $\varphi \in C_\sharp^{2\alpha}$ ($\alpha \in (0, 1) \setminus \{1/2\}$); then the function $R(0, L_\varepsilon)\mathbf{f}$ belongs to $D(L + \varepsilon A)$, and there exists a positive constant C, depending on ε and α but independent of $\mathbf{h}$ and φ, such that

$$
\|D_x^{(i)} R(0, L_\varepsilon)\mathbf{f}\|_{\mathcal{X}} + \|D_\eta^{(i)} R(0, L_\varepsilon)\mathbf{f}\|_{\mathcal{X}} \leqslant C \|\mathbf{h}\|_{\mathcal{X}} \|\varphi\|_{C^{2\alpha}} \tag{A2}
$$

for $i = 0, 1, 2$.

A.3 *Proof of Theorem 4.4*

We split the proof into three steps. In the first one, we show that problem (4.19) admits a unique solution Φ in some time domain $[0, T)$. Since this result can be proved using the same arguments as in Subsection 5.3, we just sketch the proof. Then, in Steps 2 and 3, we show that Φ exists and is smooth in the whole of $[0, +\infty)$.

Step 1. By the proof of Proposition 5.4, the realization A of the second-order derivative in C^0, with domain $C_\sharp^1 \cap C^2$, is a sectorial operator with spectrum contained in $(-\infty, 0]$. Hence, [11, Prop. 2.4.1 & 2.4.4] applies and shows that the operator $B := -3A^2 - A$ is sectorial in C^0 with domain $D(A^2)$. Moreover, $D_B(\alpha, \infty) = C_\sharp^{4\alpha}$, with equivalence of norms, for any $\alpha \in (0, 2)$ such that $4\alpha \notin \mathbb{N}$.

The variation of constants formula shows that any solution $\Phi \in C^{1,4}([0, +\infty) \times [-\ell_0/2, \ell_0/2])$ to the Cauchy problem (4.19) is a fixed point of the operator Γ, formally defined by

$$(\Gamma(\Phi))(\tau, \cdot) = e^{\tau B}\Phi_0 + \int_0^\tau e^{(\tau-s)B}(\Phi_\eta(s, \cdot))^2 \, ds, \quad \tau > 0,$$

where $\{e^{tB}\}$ denotes the semigroup generated by B.

Let us fix $\alpha \in (1/4, 1/2)$. Theorem 7.1.2 in [11] implies that Γ has a unique fixed point Φ in $C([0, T_0]; D_B(\alpha, \infty))$. A bootstrap argument allows us to prove that Φ belongs to $\mathcal{Y}_{T_0}$. Using [11, Prop. 4.2.1] and our assumptions on Φ_0, it can be shown first that $\Phi \in C^{\beta,4\gamma}([0, T_0] \times [-\ell_0/2, \ell_0/2])$ for any $\beta, \gamma \in (0, 1)$, and then that $\Phi_\eta \in C^\beta([0, T_0] \times [-\ell_0/2, \ell_0/2])$ for any $\beta \in (0, 3/4)$. Moreover, $D_\eta^{(j)}\Phi(\cdot, -\ell_0/2) \equiv D_\eta^{(j)}\Phi(\cdot, \ell_0/2)$ for $j = 0, 1, 2, 3$. Next, applying [11, Thm. 4.3.1(i)], we deduce that $\Phi \in C^{1,4}([0, T] \times [-\ell_0/2, \ell_0/2])$ and is a solution to problem (4.19). Moreover, since $\Phi_0 \in D_B(3/2 + \alpha/4, \infty)$, Φ_τ is bounded in $[0, T_0]$ with values in $D_B(1/2 + \alpha/4, \infty)$. Hence, $\Phi_\tau \in C^{0,2+\alpha}([0, T_0] \times [-\ell_0/2, \ell_0/2])$. As a byproduct, $\Phi_{\eta\eta\eta\eta} \in C^{0,2+\alpha}([0, T_0] \times [-\ell_0/2, \ell_0/2])$ and $D_\eta^{(j)}\Phi(\cdot, -\ell_0/2) = D_\eta^{(j)}\Phi(\cdot, \ell_0/2)$ for $j = 4, 5, 6$.

Using a continuation argument, we can extend Φ to a maximal domain $[0, T)$ with a function (still denoted by Φ) which belongs to $\mathcal{Y}_{T'}$ for any $T' < T$.

The rest of the proof is devoted to showing that $T = +\infty$. The main step is an a priori estimate suggested by the proof [13, Thm. 2.4], which deals with L^2-regularity for the K-S equation.

Step 2. Here we show that

$$\|\Phi_\eta(\tau, \cdot)\|_2 \leqslant e^{13\tau/6}\|D_\eta\Phi_0\|_2, \quad \tau \in [0, T). \tag{A3}$$

For this purpose, we introduce the function v defined by $v(\tau, \eta) = e^{-2\tau}\Phi_\eta(\tau, \eta)$ for any $(\tau, \eta) \in [0, T) \times [-\ell_0/2, \ell_0/2]$. The smoothness of Φ implies that $v \in C^{1,4}([0, T) \times [-\ell_0/2, \ell_0/2])$ solves the parabolic equation

$$v_\tau = -3v_{\eta\eta\eta\eta} - v_{\eta\eta} - e^{2\tau}vv_\eta - 2v, \tag{A4}$$

and satisfies the boundary conditions $D_\eta^{(k)}v(\tau, -\ell_0/2) = D_\eta^{(k)}v(\tau, \ell_0/2)$ for any $\tau \in [0, T)$ and $k = 0, 1, 2, 3$. Multiplying both sides of (A4) by $v(\tau, \cdot)$, integrating on $(-\ell_0/2, \ell_0/2)$ and observing that the integral over $(-\ell_0/2, \ell_0/2)$ of $(v(\tau, \cdot))^2 v_\eta(\tau, \cdot)$ vanishes for any $\tau \in (0, T]$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|v(\tau, \cdot)\|_2^2 + 3\|v_{\eta\eta}(\tau, \cdot)\|_2^2 - \|v_\eta(\tau, \cdot)\|_2^2 + 2\|v(\tau, \cdot)\|_2^2 = 0, \quad \tau \in [0, T). \tag{A5}$$

In view of the estimate

$$\|v_\eta(\tau, \cdot)\|_2^2 \leqslant \|v(\tau, \cdot)\|_2 \|v_{\eta\eta}(\tau, \cdot)\|_2 \leqslant 3\|v_{\eta\eta}(\tau, \cdot)\|_2^2 + \frac{5}{3}\|v(\tau, \cdot)\|_2^2, \quad \tau \in [0, T),$$

formula (A5) leads us to the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\|v(\tau, \cdot)\|_2^2 + \frac{1}{3}\|v(\tau, \cdot)\|_2^2 \leqslant 0, \quad \tau \in [0, T),$$

from which estimate (A3) follows at once.

Step 3. Let us consider the function Ψ defined by $\Psi(\tau, \eta) = \Phi(\tau, \eta) - \Pi(\Phi(\tau, \cdot))$ for any $\tau \in [0, T)$ and any $\eta \in [-\ell_0/2, \ell_0/2]$, where $\Pi(\Phi(\tau, \cdot))$ denotes the average of $\Phi(\tau, \cdot)$ over the interval $(-\ell_0/2, \ell_0/2)$. Applying the Poincaré–Wirtinger inequality, we get

$$\|\Phi(\tau, \cdot) - \Pi(\Phi(\tau, \cdot))\|_\infty \leqslant \sqrt{\ell_0} e^{13\tau/6}\|D_\eta\Phi_0\|_2, \quad \tau \in [0, T). \tag{A6}$$

Let us now show that the function $\tau \mapsto \Pi(\Phi(\tau, \cdot))$ satisfies a similar estimate. For this purpose, we fix $T' \in (0, T)$, $\tau \in [0, T')$, and apply the operator Π to both sides of (4.19). Since Φ and its derivatives satisfy periodic boundary conditions,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi(\Phi(\tau, \cdot)) = \Pi(\Phi_\tau(\tau, \cdot)) = -\frac{1}{2\ell_0}\Pi((\Phi_\eta(\tau, \cdot))^2)$$

for any $\tau \in [0, T)$. Taking (A3) into account, we can then estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi(\Phi(\tau, \cdot))\right| \leqslant \frac{1}{2\ell_0}e^{13\tau/3}\|D_\eta\Phi_0\|_2^2, \quad \tau \in [0, T).$$

Hence,

$$|\Pi(\Phi(\tau))| \leqslant |\Pi(\Phi_0)| + \int_0^\tau \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\Pi(\Phi(\tau, \cdot))\right| \mathrm{d}\tau \leqslant |\Pi(\Phi_0)| + \frac{3}{26\ell_0}\|D_\eta\Phi_0\|_2^2 e^{13\tau/3} \tag{A7}$$

for any $\tau \in [0, T)$. Estimates (A6) and (A7) show that Φ is bounded in $[0, T) \times [-\ell_0/2, \ell_0/2]$. Therefore, we can apply [11, Prop. 7.2.2] with $X_\alpha = D_B(\alpha, \infty)$, which implies that $T = +\infty$.

Acknowledgments

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