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Interface conditions for limits of the Navier–Stokes–Korteweg model

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We study the behaviour of the pressure across phase boundaries in liquid-vapour flows. As mathematical model, we consider the static version of the Navier–Stokes–Korteweg model, which belongs to the class of diffuse interface models. From this static equation, a formula for the pressure jump across the phase interface can be derived. If we perform the sharp interface limit, we see that the resulting interface condition for the pressure seems to be inconsistent with classical results of hydrodynamics. Therefore we will present two approaches to recover the results of hydrodynamics in the sharp interface limit at least for special situations.

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1. The Navier–Stokes–Korteweg model

In this paper, we consider a mathematical model for liquid-vapour flows including phase transitions which was proposed by Korteweg already in 1901 [Kor01] and which is known as the Navier–Stokes–Korteweg model. It is an extension of the compressible Navier–Stokes equations and given by the following system:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho v) &= 0, \\ \partial_t (\rho v) + \nabla \cdot (\rho v v^t + p(\rho) I) &= \mu \Delta v + \gamma \varepsilon^2 \rho \nabla \Delta \rho. \end{aligned} \tag{1}$$

These equations describe a fluid model for one component, where ρ, v, $p(\rho)$, v^t, I and μ denote the density, the velocity, the pressure, the transpose of v, the identity matrix and the viscosity of liquid/vapour respectively. Compared to the original Navier–Stokes equations, the system (1) contains the term $\gamma \varepsilon^2 \rho \nabla \Delta \rho$, which is supposed to model capillarity effects close to phase transitions, where $\gamma > 0$ is a material constant and $\varepsilon > 0$ is a small value related to the thickness of the interface. The conservation of energy is omitted in (1). The pressure p, as a function

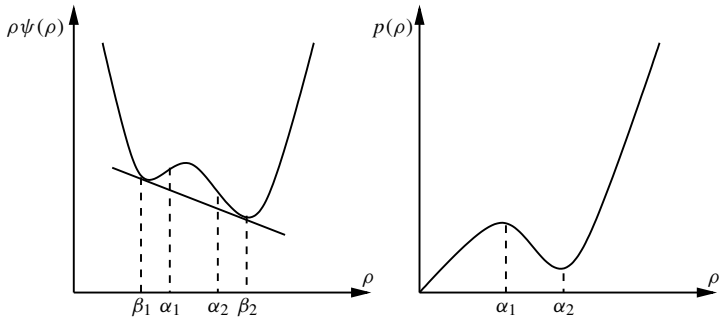

FIG. 1. Graphs of the total free energy density and the pressure.

of the density ρ, is defined by

$$p(\rho) = \rho^2 \psi'(\rho). \tag{2}$$

Here, ψ is a smooth function of ρ and $\rho\psi(\rho)$ is the total free energy density, which has the form of a double well potential (see Figure 1). The values β_1 and β_2 are the Maxwell points, at which the tangent line of $\rho\psi(\rho)$ is equal to the difference quotient and $\frac{\partial^2 \rho\psi(\rho)}{\partial \rho^2}\big|_{\rho=\beta_i} > 0$ for $i \in \{1, 2\}$. The Maxwell line l, $l(\rho) = d_0\rho + d_1$ with $d_0, d_1 \in \mathbb{R}$, satisfies

$$l(\beta_i) = \beta_i \psi(\beta_i) \quad \text{and} \quad l'(\beta_i) = \frac{\partial \rho\psi(\rho)}{\partial \rho}\bigg|_{\rho=\beta_i}, \qquad i = 1, 2. \tag{3}$$

Different phases of the fluid are characterized by the size of ρ. If $\rho \leqslant \alpha_1$ we are in the vapour phase and if $\rho \geqslant \alpha_2$ we are in the liquid phase. The equation (2) is known as the van der Waals equation of state.

For a rigorous derivation of the system (1) one has to consider the equations for conservation of mass, momentum and energy and the equation for the entropy production (second law of thermodynamics). Special conditions for the stress tensor, which appears in the equations for conservation of momentum and energy, ensure that the entropy production is nonnegative (cf. [Di07], [An96] and [DK07]). This leads to the Navier–Stokes–Korteweg model. Then, omitting the equation for the energy, we end up with (1).

The mathematical model (1) for the *static case* can be described alternatively as follows. First we fix some notation:

$$\begin{aligned} \Omega \subset \mathbb{R}^n : &\quad \text{bounded domain with Lipschitz boundary,} \\ \tilde{W}(\rho) : &\quad \text{free energy density (double well)} = \rho\psi(\rho), \\ \tilde{E}_0(\rho) : &\quad \text{total energy.} \end{aligned}$$

For a moment let us consider the minimizers of

$$\tilde{E}_0(\rho) = \int_\Omega \tilde{W}(\rho)\,\mathrm{d}x \tag{4}$$

under the constraint

$$\int_\Omega \rho\,\mathrm{d}x = m \quad \text{(conservation of mass)} \tag{5}$$

for a two-phase fluid at rest.

We define $W(\rho) := \rho\psi(\rho) - l(\rho)$ and

$$E_0(\rho) := \int_\Omega W(\rho(x))\,\mathrm{d}x. \tag{6}$$

Then the functionals $\tilde{E}_0(\rho)$ and $E_0(\rho)$ differ with respect to mass conservation only in the constant $\tilde{m} = d_0 m + d_1|\Omega|$ and have therefore the same minimizers. All functions ρ with

$$\rho(x) = \beta_1 \quad \text{for } x \in \Omega_1, \qquad \rho(x) = \beta_2 \quad \text{for } x \in \Omega_2,$$

such that $\Omega_1 \cup \Omega_2 = \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$, $|\Omega_1|\beta_1 + |\Omega_2|\beta_2 = m$ are minimizers of (4) under mass conservation.

Clearly, since (4) involves no minimal area property for the surface of the interface, there are infinitely many ways to distribute mass of the densities β_1 and β_2 in the domain Ω so that (5) is satisfied. In addition, energy contributions due to surface tension or the curvature of the interface are not included in (4).

In order to penalize the occurrence of free boundaries between the phases, already van der Waals [Wa94] proposed to add a term of the form

$$\int_\Omega \gamma\varepsilon^2 \frac{|\nabla\rho|^2}{2}\,\mathrm{d}x, \qquad \varepsilon > 0 \text{ small},$$

in (4) and to consider instead of (4), (5) or (6), (5) the following variational problem:

Minimize the functional $\tilde{J}_\varepsilon : H^1(\Omega) \to \mathbb{R}$ *defined by*

$$\tilde{J}_\varepsilon(\rho) = \int_\Omega \left(\tilde{W}(\rho) + \gamma\varepsilon^2 \frac{|\nabla\rho|^2}{2}\right)\mathrm{d}x \quad \textit{(total energy)} \tag{7}$$

under the constraint $\int_\Omega \rho\,\mathrm{d}x = m$ *(conservation of mass).*

It is easy to see that the Euler–Lagrange equation for this variational problem is

$$\tilde{W}'(\rho) = \gamma\varepsilon^2 \Delta\rho + \lambda_\varepsilon, \tag{8}$$

where λ_ε is the Lagrange multiplier corresponding to the mass constraint, which satisfies the relation $\lambda_\varepsilon = d_0 + O(\varepsilon)$ as $\varepsilon \to 0$ (see [LM89]). Taking the gradient of both sides in (8) and multiplying with ρ implies

$$\rho\tilde{W}''(\rho)\nabla\rho = \gamma\varepsilon^2 \rho\nabla\Delta\rho. \tag{9}$$

The definition of $\tilde{W}$ and a simple calculation using (2) shows that $p'(\rho) = \rho\tilde{W}''(\rho)$. Therefore, we get from (9)

$$\nabla p(\rho) = \gamma\varepsilon^2 \rho\nabla\Delta\rho. \tag{10}$$

This together with the mass constraint gives the static form of (1).

The mathematical model for the *dynamical case* $\partial_t v \neq 0$ can be obtained as follows (see [Ro06] and [Se59]): The Lagrangian is given by

$$L(\rho, v) := \frac{1}{2}\rho|v|^2 - \tilde{W}(\rho) - \frac{\gamma\varepsilon^2}{2}|\nabla\rho|^2$$

and the Euler–Lagrange equation for the action functional

$$\int_0^T \int_{\mathbb{R}^3} L(\rho(x,t), v(x,t))\,\mathrm{d}x\,\mathrm{d}t$$

with respect to the constraint $\partial_t \rho + \nabla \cdot (\rho v) = 0$ has the form

$$\partial_t v + v \nabla v = \nabla(-\tilde{W}'(\rho) + \gamma \varepsilon^2 \Delta \rho).$$

Due to $p'(\rho) = \rho \tilde{W}''(\rho)$ and conservation of mass we obtain

$$\partial_t(\rho v) + \nabla \cdot (\rho v v^t + p(\rho) I) = \gamma \varepsilon^2 \rho \nabla \Delta \rho.$$

Adding some scaled viscosity yields the equation of momentum of (1).

In [BDD07], the authors investigate the Cauchy problem for the nondissipative isothermal case of (1) in multiple space dimensions (Euler–Korteweg problem). They prove wellposedness of the Cauchy problem, where the third order term $\gamma \varepsilon^2 \rho \nabla \Delta \rho$ in (1) may even depend nonlinearly on ρ, i.e. $\rho \nabla (K(\rho) \triangle \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2)$ with a capillarity function K. The corresponding one-dimensional isothermal, inviscid initial value problem has been considered in [BDD06]. Danchin and Desjardins establish global existence and uniqueness of solutions close to a stable equilibrium and furthermore local in time existence for (1). The existence of global weak solutions for periodic boundary conditions without any smallness assumptions on the data has been shown in [BDL03]. Global existence results for weak solutions of (1) in 1-D with $\mu = 0$ and $\gamma = 0$ are available in [AC08]. Kotschote [Kot06] considers existence of solutions for the corresponding initial boundary value problem to (1).

The system (1) can also be considered as a diffusive-dispersive regularization. The analytical and numerical background for the diffusive-dispersive regularizations for scalar conservation laws with nonconvex flux functions is the main subject in [HL00] and [HL97].

Different approaches to modelling two phase flows with phase transitions are considered in [Tr94], [SB04] and [LT98]. Truskinovsky [Tr94] studies a system of conservation laws as a simple model for one-dimensional isothermal elastodynamics without body forces and constant reference density. In this system, the usual nonconvex stress is extended by viscous and capillary stresses and hence a model with phase transitions is obtained. For travelling wave solutions, the limiting behaviour can be controlled when the viscosity and the capillarity coefficient tend to zero. The system reduces to an overdetermined boundary value problem of second order on the whole real axis. Kinetic relations are then derived, which give the desired information on the admissible boundary values. In [SB04], a diffuse interface model is derived for the direct simulation of two-phase flows with surface tension, phase change and different viscosities in the two phases. To this end, the authors use a set averaging procedure on an atomic scale. A further approach to the modelling of two phase flows with phase transitions can be found in [LT98], where a system of Navier–Stokes–Cahn–Hilliard type is considered. This system is in some sense a physically motivated regularization of the Euler equations. A quasi-incompressible version of the Navier–Stokes–Cahn–Hilliard equations is also investigated in [LT98].

In this paper, we study the behaviour of the pressure across the interface. Since a rigorous theory of this question is not available and difficult to establish, we concentrate on the static version of (1). In particular, we investigate the behaviour of the pressure in the sharp interface limit, i.e. if the interfacial thickness ε tends to 0. In Section 2, we quote some recent results for the diffuse phase

field model considered in Section 1. These results show that the difference of the pressures on both sides of the interface is continuous if $\varepsilon \to 0$. This seems to contradict the classical result of Young and Laplace (cf. [Yo1805, Fi86, LL91]), which states that the difference of the pressures on both sides of the interface is proportional to the mean curvature of the interface. For this reason, we study jump conditions for critical points of a related sharp interface model with surface tension. We prove in a BV-setting that for critical points the difference of the pressure is proportional to the mean curvature (see Theorem 5). Then we show that we obtain this jump relation for the pressure from the diffuse phase field model in the sharp interface limit if we use either a scaled surface tension or a modified definition of the pressure on the basis of a special scaling of the free energy density (see Section 4). In this context, we will see that the scaling/capillarity quantity $\gamma\varepsilon^2$ can be related to the Mach number under certain conditions. Using this dependence, we achieve an asymptotic expansion of p in the Mach number M, i.e. $p = p_0 + p_1 M + p_2 M^2 + O(M^3)$, and the expected jump condition for p_2 (see Section 3). Jamet et al. try in [JLCD01] to overcome the problem for the pressure conditions by introducing a modified free energy density. However, they do not establish a thermodynamical consistent approach.

2. Phase transitions and sharp interface conditions

We denote the space of functions of bounded variation by $BV(\Omega)$. The symbol $\partial^* S$ stands for the reduced boundary of a Borel set $S \subset \Omega$ with finite perimeter $P_\Omega(S)$, i.e.

$$P_\Omega(S) := \int_\Omega \mathrm{d}|D\chi_S| = \mathcal{H}^{n-1}(\partial^* S), \qquad \chi_S : \text{characteristic function of } S,$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$. For details we refer to [Giu84] and [AFP00]. As before we consider the variational problem (assume that $\gamma = 1$):

Minimize the functional $J_\varepsilon : H^1(\Omega) \to [0, \infty)$ *defined by*

$$J_\varepsilon(\rho) = \int_\Omega \left(W(\rho(x)) + \frac{\varepsilon^2}{2} |\nabla \rho(x)|^2 \right) \mathrm{d}x \tag{11}$$

under the constraint $\int_\Omega \rho(x)\,\mathrm{d}x = m$.

Note that J_ε and $\tilde{J}_\varepsilon$ (see (7)) differ only in the constant $\tilde{m} = d_0 m + d_1|\Omega|$ under mass conservation and have the same minimizers.

The term $(\varepsilon^2/2)|\nabla\rho(x)|^2$ in (11) penalizes the occurrence of large interfaces. In particular, minimizers of (11) try to minimize the interfacial area. This phenomenon and the following asymptotic properties have been shown by Modica:

THEOREM 1 ([Mo87]) Let $\beta_1|\Omega| \leqslant m \leqslant \beta_2|\Omega|$, where β_1, β_2 are the Maxwell points (cf. Figure 1). In addition, let ρ_ε be a global minimizer of (11) with $\int_\Omega \rho_\varepsilon\,\mathrm{d}x = m$. Then the following statements are satisfied:

(a) There exists a sequence $(\varepsilon_k)_k$, $\varepsilon_k > 0$, with $\lim_{k\to\infty} \varepsilon_k = 0$ such that the corresponding sequence $(\rho_{\varepsilon_k})_k$ of global minimizers ρ_{ε_k} converges in $L^1(\Omega)$ as $k \to \infty$.
(b) If $\rho_{\varepsilon_j} \to \rho_0$ in $L^1(\Omega)$ as $j \to \infty$ then $\rho_0(x) = \beta_1$ or $\rho_0(x) = \beta_2$ for a.e. $x \in \Omega$, where $\beta_1|A| + \beta_2|\Omega\backslash A| = m$ and $A := \{x \in \Omega : \rho_0(x) = \beta_1\}$.

(c) The set A is a solution of the following geometric variational problem:

$$P_\Omega(A) = \min\left\{ P_\Omega(F) : F \subset \Omega,\ |F| = \frac{\beta_2|\Omega| - m}{\beta_2 - \beta_1} \right\}.$$

(d) If $\rho_{\varepsilon_j} \to \rho_0$ in $L^1(\Omega)$ as $j \to \infty$ then the energy J_{ε_j} satisfies

$$J_{\varepsilon_j}(\rho_{\varepsilon_j}) = \int_\Omega \left(W(\rho_{\varepsilon_j}) + \frac{\varepsilon_j^2}{2}|\nabla \rho_{\varepsilon_j}|^2 \right) \mathrm{d}x = c_0 P_\Omega(A)\varepsilon_j + o(\varepsilon_j)$$

with $c_0 := \int_{\beta_1}^{\beta_2} \sqrt{2W(t)}\,\mathrm{d}t$.

REMARK 2 Roughly speaking, item (c) of Theorem 1 expresses the fact that the boundary ∂A of A has minimal area since it can be shown by the theory of minimal surfaces that the reduced boundary $\partial^* A$ is smooth and $\mathcal{H}^{n-1}((\partial A \backslash \partial^* A) \cap \Omega) = 0$ (cf. [Giu84]).

From Theorem 1 we conclude that the energy J_ε converges to E_0 (see (6)) as $\varepsilon \to 0$ in the sense of Γ-convergence. This implies that there is no contribution of interfacial energy in the sharp interface limit. In addition, we infer from the following theorem that the pressure p across the interface is continuous in the limit $\varepsilon \to 0$.

In what follows, we denote by k_m the mean curvature (times $n-1$) of the interface I between liquid and vapour. The mean curvature is given by the sum of the principal curvatures with the sign convention that k_m is positive if I is curved in the direction of the outer unit normal ν of the region with the lower density.

THEOREM 3 (see [DK07]) Let items (a) and (b) of Theorem 1 be satisfied. Furthermore, let $U \subset\subset A$ and $V \subset\subset \Omega \backslash \bar{A}$ be open sets and $\psi \in C^3(\mathbb{R})$. Then

$$\rho_{\varepsilon_k}(x) = \rho^0(x) + \rho^1(x)\varepsilon_k + o(\varepsilon_k), \qquad k \to \infty,$$

for $x \in U \cup V$ with $\rho^0(x) = \beta_1$ if $x \in U$, $\rho^0(x) = \beta_2$ if $x \in V$ and $\rho^1(x) = -\frac{c_0(n-1)k_m}{W''(\rho^0(x))(\beta_2-\beta_1)}$, where the mean curvature k_m of $I = \partial^* A$ is constant. Moreover,

$$p(\rho_{\varepsilon_k}(x_2)) - p(\rho_{\varepsilon_k}(x_1)) = -c_0 k_m \varepsilon_k + o(\varepsilon_k) \tag{12}$$

for $x_1 \in U$ and $x_2 \in V$ as $k \to \infty$.

REMARK 4 Equation (12) implies that for a two-phase system the pressure p_- of the enclosed phase is always higher than the pressure p_+ of the surrounding phase (see Figure 2).

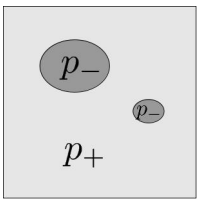

FIG. 2. Pressure condition $p_- > p_+$.

Theorem 3 seems to be inconsistent with classical results of hydrodynamics (cf. [Yo1805, Fi86, LL91]), i.e. the pressure has to satisfy the Young–Laplace equation

$$[p]_-^+ = p_+ - p_- = -\hat{\sigma} k_m \tag{13}$$

at the interface I, where $[\cdot]_-^+$ denotes the jump of the quantity in brackets across the interface and $\hat{\sigma}$ is a constant surface tension. The corresponding interface condition for the dynamical case with phase transitions is

$$p_+ - p_- = -\hat{\sigma} k_m - [\rho(v_\nu - v_I)^2]_-^+ + \left[\mu \frac{\partial v_\nu}{\partial \nu}\right]_-^+, \tag{14}$$

where $v_\nu := v \cdot \nu$ and v_I is the velocity of the interface I (see [GCNB07, formula (13)]).

In order to illuminate this apparent controversy and to get more insight into jump conditions across the interface, we study necessary conditions for minimizers of a related sharp interface functional including surface tension.

THEOREM 5 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and let $\rho \in BV(\Omega)$ such that the reduced boundary of $A = \{x \in \Omega : \rho(x) \leqslant a\} \subsetneq \Omega$ is a C^2 boundary for $a \in (\alpha_1, \alpha_2)$. Furthermore, let $\rho \in W^{1,1}(A) \cup W^{1,1}(\Omega \backslash \overline{A})$ with $\rho(x) \in (0, \alpha_1)$ for $x \in A$ (vapour phase) and $\rho(x) \in (\alpha_2, \infty)$ for $x \in \Omega \setminus \overline{A}$ (liquid phase). Then, any critical point $\hat{\rho}$ of the energy functional $\hat{J} : BV(\Omega) \to [0, \infty)$,

$$\hat{J}(\rho) = \hat{\sigma} \int_I \mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \rho \psi(\rho)\, \mathrm{d}x, \qquad \int_\Omega \rho \, \mathrm{d}x = m, \tag{15}$$

with $I = \partial^* A$, $\hat{\sigma} > 0$ and $\psi \in C^3(\mathbb{R})$, fulfills the condition

$$p(\hat{\rho}^+) - p(\hat{\rho}^-) = -\hat{\sigma} k_m \quad \text{on } I,$$

where $\hat{\rho}^-$ and $\hat{\rho}^+$ denote the inner and outer traces of $\hat{\rho}$ on I.

Proof. To obtain the pressure condition we choose variations by means of a one-parameter family of diffeomorphisms $\Phi : [-\tau_0, \tau_0] \times \Omega \to \Omega$, given by the initial value problem

$$\Phi(0, x) = x \quad \text{and} \quad \Phi_{,\tau}(\tau, x) = \xi(\Phi(\tau, x))$$

for $x \in \Omega$ and $\tau \in [-\tau_0, \tau_0]$, where $\xi \in C_c^\infty(\Omega, \mathbb{R}^n)$ is arbitrary. Then Φ satisfies the following properties:

(i) $\Phi(\tau, \cdot)$ is the inverse of $\Phi(-\tau, \cdot)$, i. e. $\Phi(\tau, \Phi(-\tau, x)) = x$. Consequently,

$$\mathrm{Id} = \Phi_{,x}(\tau, \Phi(-\tau, x)) \Phi_{,x}(-\tau, x).$$

(ii)
$$\frac{\mathrm{d}}{\mathrm{d}\tau}(|\det \Phi_{,x}(\tau, x)|)\Big|_{\tau=0} = (\nabla \cdot \xi)(x).$$

(iii)
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big((\Phi_{,x}(\tau, x))^{-1}\big)\Big|_{\tau=0} = -\nabla \xi(x).$$

We consider for any fixed $h \in BV(\Omega)$ with $\int_\Omega h\,\mathrm{d}x \neq 0$, $h(x) \in (0, \alpha_1)$ for $x \in A$ and $h(x) \in (\alpha_2, \infty)$ for $x \in \Omega \setminus \overline{A}$,

$$\begin{aligned} j(\tau, \omega) &:= \int_\Omega \big(\hat\rho(\Phi(-\tau, y)) + \omega h(\Phi(-\tau, y))\big)\,\mathrm{d}y - m \\ &= \int_\Omega (\hat\rho(x) + \omega h(x))|\det \Phi_{,x}(\tau, x)|\,\mathrm{d}x - m, \qquad (\tau, \omega) \in [-\tau_0, \tau_0] \times \mathbb{R}. \end{aligned}$$

Clearly, $j(0, 0) = 0$. Moreover, $j \in C^1([-\tau_0, \tau_0] \times \mathbb{R})$ and

$$\begin{aligned} \frac{\partial j}{\partial \tau}(\tau, \omega)\bigg|_{\tau=0} &= \int_\Omega (\hat\rho(x) + \omega h(x))\nabla \cdot \xi(x)\,\mathrm{d}x, \\ \frac{\partial j}{\partial \omega}(\tau, \omega) &= \int_\Omega h(x)|\det\Phi_{,x}(\tau, x)|\,\mathrm{d}x \quad \text{with} \quad \frac{\partial j}{\partial \omega}(0, 0) \neq 0. \end{aligned}$$

By the Implicit Function Theorem, there exists a C^1 function $\eta : \mathbb{R} \to \mathbb{R}$ with $\eta(0) = 0$ such that

$$j(\tau, \eta(\tau)) = 0 \tag{16}$$

for τ sufficiently small. Without loss of generality we may assume that (16) holds for $\tau \in [-\tau_0, \tau_0]$. Differentiating (16) gives

$$\frac{\partial j}{\partial \tau}(\tau, \eta(\tau)) + \frac{\partial j}{\partial \omega}(\tau, \eta(\tau))\eta'(\tau) = 0$$

for $\tau \in [-\tau_0, \tau_0]$. Consequently,

$$\eta'(0) = -\frac{\frac{\partial j}{\partial \tau}(0, 0)}{\frac{\partial j}{\partial \omega}(0, 0)}.$$

We set

$$\hat\rho^\tau(x) = \hat\rho(\Phi(-\tau, x)) + \eta(\tau)h(\Phi(-\tau, x)) \quad \text{for } \tau \in [-\tau_0, \tau_0]$$

and may assume

$$\hat\rho^\tau(x) \in (0, a) \quad \text{for } x \in A \quad \text{and} \quad \hat\rho^\tau(x) \in (a, \infty) \quad \text{for } x \in \Omega \setminus \overline{A},$$

According to (16), $\hat\rho^\tau$, $\tau \in [-\tau_0, \tau_0]$, are admissible comparison functions. This implies

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\hat J(\hat\rho^\tau)\Big|_{\tau=0}$$

since $\hat\rho = \hat\rho^0$.

Next, we determine the above τ-derivative. The first variation of the area integral, i.e. $\hat\sigma \int_I \mathrm{d}\mathcal{H}^{n-1}$, is computed in the BV-setting (see for instance [Giu84] and [Gar08]). For completeness we sketch the arguments. In the following, χ denotes the characteristic function of A. We define

$$\chi^\tau(y) = \chi(\Phi(-\tau, y)), \qquad y \in \Omega. \tag{17}$$

Then,

$$\hat\sigma \int_\Omega \mathrm{d}|D\chi^\tau(y)| = \hat\sigma \int_\Omega \mathrm{d}|D\chi(\Phi(-\tau, y))| = \hat\sigma \int_\Omega |(\Phi_{,x}(\tau, x))^{-T}\nu(x)|\,|\det \Phi_{,x}(\tau, x)|\,\mathrm{d}|D\chi(x)|,$$

where $\nu = -D\chi/|D\chi|$ is the generalized unit outer normal of A, which is a $|D\chi|$-measurable function. From properties (i)–(iii) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_\Omega \mathrm{d}|D\chi^\tau|\right)\Big|_{\tau=0} = \int_\Omega (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)\,\mathrm{d}|D\chi|.$$

Applying the generalized divergence theorem on submanifolds ([AFP00, p. 359]) gives

$$\begin{aligned}\int_\Omega (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)\,\mathrm{d}|D\chi| &= \int_I (\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu)\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_I (\mathrm{div}_I\,\xi)\,\mathrm{d}\mathcal{H}^{n-1} = \int_I (\mathrm{div}_I\,\nu)(\xi\cdot\nu)\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

where div_I denotes the tangential divergence with respect to the interface I.

Now we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega \hat\rho^\tau\psi(\hat\rho^\tau)\,\mathrm{d}y\Big|_{\tau=0} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega (\hat\rho(x)+\eta(\tau)h(x))\psi(\hat\rho(x)+\eta(\tau)h(x))|\det\Phi_{,x}(\tau,x)|\,\mathrm{d}x\Big|_{\tau=0}\\ &= \int_\Omega \hat\rho(x)\psi(\hat\rho(x))\,\nabla\cdot\xi(x)\,\mathrm{d}x + \int_\Omega \frac{\partial\big(\hat\rho(x)\psi(\hat\rho(x))\big)}{\partial\hat\rho}\eta'(0)h(x)\,\mathrm{d}x\\ &= \int_\Omega \hat\rho\psi(\hat\rho)\nabla\cdot\xi\,\mathrm{d}x - \int_\Omega \frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho}\,\frac{\int_\Omega\hat\rho\,\nabla\cdot\xi\,\mathrm{d}x}{\int_\Omega h\,\mathrm{d}x}h\,\mathrm{d}x\\ &= \int_\Omega \hat\rho\psi(\hat\rho)\nabla\cdot\xi\,\mathrm{d}x + \lambda\int_\Omega\hat\rho\nabla\cdot\xi\,\mathrm{d}x\end{aligned}$$

with $\lambda = -\int_\Omega \frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho}h\,\mathrm{d}x/\int_\Omega h\,\mathrm{d}x$. By Green's formula for BV-functions we derive

$$\begin{aligned}\int_\Omega \big(\hat\rho(x)\psi(\hat\rho(x))+\lambda\hat\rho\big)\nabla\cdot\xi\,\mathrm{d}x = &-\int_A\left(\frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho}+\lambda\right)\nabla\hat\rho\cdot\xi\,\mathrm{d}x\\ &-\int_{\Omega\setminus\overline{A}}\left(\frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho}+\lambda\right)\nabla\hat\rho\cdot\xi\,\mathrm{d}x\\ &+\int_I\big(\hat\rho^-\psi(\hat\rho^-)-\hat\rho^+\psi(\hat\rho^+)+\lambda(\hat\rho^--\hat\rho^+)\big)\xi\cdot\nu\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\hat J(\hat\rho^\tau)\Big|_{\tau=0} = \hat\sigma&\int_I(\mathrm{div}_I\,\nu)(\xi\cdot\nu)\,\mathrm{d}\mathcal{H}^{n-1}\\ &-\int_A\left(\frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho}+\lambda\right)\nabla\hat\rho\cdot\xi\,\mathrm{d}x - \int_{\Omega\setminus\overline{A}}\left(\frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho}+\lambda\right)\nabla\hat\rho\cdot\xi\,\mathrm{d}x\\ &+\int_I\big(\hat\rho^-\psi(\hat\rho^-)-(\hat\rho^+\psi(\hat\rho^+)+\lambda(\hat\rho^--\hat\rho^+)\big)\xi\cdot\nu\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Since ξ may be arbitrarily chosen we get

$$\frac{\partial\hat\rho\psi(\hat\rho)}{\partial\hat\rho} = -\lambda \quad \text{for a.e. } x\in\Omega.$$

Now, we take arbitrary variations ξ with compact support in the neighbourhood of a point of I to obtain

$$\hat{\sigma}\,\mathrm{div}_I\,\nu = \hat{\rho}^+\psi(\hat{\rho}^+) - \hat{\rho}^-\psi(\hat{\rho}^-) + \lambda(\hat{\rho}^+ - \hat{\rho}^-) \quad \text{on } I.$$

Consequently,

$$[p(\hat{\rho})]_-^+ = p(\hat{\rho}^+) - p(\hat{\rho}^-) = -\hat{\sigma}k_m \quad \text{on } I$$

since $k_m = \mathrm{div}_I\,\nu$ and $p(\rho) = \rho\frac{\partial\rho\psi(\rho)}{\partial\rho} - \rho\psi(\rho)$. This completes the proof. □

From Theorem 5, we conclude that minimizers of the energy functional $\hat{J}$ satisfy the classical jump conditions for pressures of hydrodynamics.

3. Zero Mach number limit for the Navier–Stokes–Korteweg equations

In this section, we consider the zero Mach number limit for the compressible Navier–Stokes–Korteweg system. It turns out that we get the incompressible Navier–Stokes equations in the limit. For the nondimensionalization of

$$\partial_t\rho + \nabla\cdot(\rho v) = 0,$$
$$\partial_t(\rho v) + \nabla\cdot(\rho v v^t) + \nabla p(\rho) = \mu\Delta v + \gamma^2\varepsilon\rho\nabla\Delta\rho$$

we choose the following reference quantities: x_{ref}, t_{ref}, p_{ref}, ρ_{ref}, $v_{\mathrm{ref}} := x_{\mathrm{ref}}/t_{\mathrm{ref}}$, $c_{\mathrm{ref}} := \sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}$, the Mach number $M := v_{\mathrm{ref}}/c_{\mathrm{ref}}$, the Reynolds number $\mathrm{Re} := \rho_{\mathrm{ref}}v_{\mathrm{ref}}x_{\mathrm{ref}}/\mu$ and the capillarity number $\lambda := \rho_{\mathrm{ref}}^2\gamma\varepsilon^2/(x_{\mathrm{ref}}^2\,p_{\mathrm{ref}})$. Then the nondimensionalized form of the compressible Navier–Stokes–Korteweg equations is given by

$$\partial_t\rho + \nabla\cdot(\rho v) = 0, \tag{18}$$
$$\partial_t(\rho v) + \nabla\cdot(\rho v v^t) + \frac{1}{M^2}\nabla p(\rho) = \frac{1}{\mathrm{Re}}\Delta v + \frac{\lambda}{M^2}\,\rho\nabla\Delta\rho.$$

The corresponding dimensionless total energy has the form

$$\mathcal{E}(v,\rho) = \int_\Omega\left(M^2\frac{\rho}{2}v^2 + \rho\psi(\rho) + \frac{\lambda}{2}|\nabla\rho|^2\right)\mathrm{d}x. \tag{19}$$

Next, we want to concentrate on solutions of (1) for which the scaled energy

$$\frac{1}{\sqrt{\lambda}}(\mathcal{E}(v,\rho) - \tilde{m}) = \int_\Omega\left(\frac{M^2}{\sqrt{\lambda}}\,\frac{\rho}{2}v^2 + \frac{1}{\sqrt{\lambda}}W(\rho) + \frac{\sqrt{\lambda}}{2}|\nabla\rho|^2\right)\mathrm{d}x$$

is uniformly bounded as $\lambda\to 0$. This leads to the condition $M^4 \leqslant C_1\lambda$ for some constant $C_1 > 0$ as $\lambda\to 0$. Therefore we may take for λ the following ansatz:

$$\lambda = C_2M^\delta$$

with $0 < \delta \leqslant 4$ for $M \ll 1$, where $C_2 > 0$ is some constant. In the following, we consider for δ the upper bound. We choose

$$\lambda = M^4 \tag{20}$$

and obtain

$$\partial_t \rho + \nabla \cdot (\rho v) = 0,$$
$$\partial_t(\rho v) + \nabla \cdot (\rho v v^t) + \frac{1}{M^2}\nabla p(\rho) = \frac{1}{\mathrm{Re}}\Delta v + M^2 \rho \nabla \Delta \rho. \tag{21}$$

Now, we formally consider the limit $M \to 0$ and we assume that $\rho > 0$ and the following asymptotic expansions hold:

$$\begin{aligned} \rho(x,t) &= \rho_0(x,t) + \rho_1(x,t)M + \rho_2(x,t)M^2 + O(M^3) \quad \text{for } (x,t) \in \Omega \times (0,T), \\ v(x,t) &= v_0(x,t) + v_1(x,t)M + v_2(x,t)M^2 + O(M^3) \quad \text{for } (x,t) \in \Omega \times (0,T), \end{aligned} \tag{22}$$

where $\rho_i, v_i \in C^1(\Omega \times (0,T))$, $i = 0, 1, 2$. Consequently, p may also be expanded in the Mach number:

$$p(\rho(x,t)) = p_0(\rho_0(x,t)) + p_1(\rho_0(x,t), \rho_1(x,t))M + p_2(\rho_0(x,t), \rho_1(x,t), \rho_2(x,t))M^2 + O(M^3). \tag{23}$$

The existence of the above asymptotic expansions have been proved rigorously in [KM82] for the corresponding inviscid systems without capillarity term. For the viscous system, the arguments are only formal. Inserting the asymptotic expansions of (22) in (21) and comparing terms of the orders M^{-2} and M^{-1}, we obtain

$$\nabla p_0(\rho_0(x,t)) = 0 \quad \text{and} \quad \nabla p_1(\rho_0(x,t), \rho_1(x,t)) = 0.$$

This shows

$$p_0 = p_0(t),\ \rho_0 = \rho_0(t) \quad \text{and} \quad p_1 = p_1(t),\ \rho_1 = \rho_1(t).$$

Conservation of mass in the whole set Ω implies $\int_{\partial\Omega} \nu v_0 = \int_\Omega \nabla \cdot v_0 = 0$. Since

$$\int_\Omega \partial_t \rho_0 + \rho_0 \int_\Omega \nabla \cdot v_0 = 0$$

we have $\partial_t \rho_0(t) = 0$ and therefore,

$$\rho_0(t) = \text{const} \quad \text{and} \quad p_0(t) = \text{const}.$$

Again, we use $\partial_t \rho_0 + \rho_0(t) \nabla \cdot v_0 = 0$ in order to obtain

$$\nabla \cdot v_0 = 0. \tag{24}$$

The momentum equation (21) implies for terms of order M^0,

$$\rho_0 \partial_t v_0 + \rho_0 \nabla \cdot (v_0 v_0^t) + \nabla p_2 = \frac{1}{\mathrm{Re}} \Delta v_0. \tag{25}$$

The equations (24) and (25) are just the incompressible Navier–Stokes equations.

REMARK 6 The pressure which appears in the incompressible Navier–Stokes equations is p_2, while we have p for the compressible equations. The relation between p_2 and p is given by (23). However, since $p_0 = p_0(t)$ and $p_1 = p_1(t)$, we could also choose p in (25).

In order to derive the pressure condition on the interface between the two phases, we only consider the static case of (21), which is given by

$$\nabla p(\rho) = M^4 \rho \nabla \Delta \rho. \tag{26}$$

This yields, as $p'(\rho) = \rho W''(\rho)$,

$$\nabla W'(\rho) = M^4 \nabla \Delta \rho \tag{27}$$

for $\rho > 0$.

We remark that equations (19), (20), (26) and (27) are related to (7)–(10). Therefore, we can apply the same arguments to (27) as in [DK07] to study the pressure behaviour in the limit $M \to 0$. We obtain

$$W'(\rho) = M^4 \Delta \rho + \lambda_M \quad \text{with} \quad \lambda_M = -\frac{c_0 k_m M^2}{\beta_2 - \beta_1}. \tag{28}$$

Consequently, we derive the pressure condition

$$p(\rho(x_2)) - p(\rho(x_1)) = -c_0 k_m M^2 + o(M^2), \qquad x_1 \in U \subset\subset A,\ x_2 \in V \subset\subset \Omega \setminus \overline{A},$$

under the assumptions of Theorem 3. Using the asymptotic expansion

$$p(\rho(x)) = p_0(\rho_0(x)) + p_1(\rho_0(x), \rho_1(x))M + p_2(\rho_0(x), \rho_1(x), \rho_2(x))M^2 + O(M^3),$$

we get

$$\begin{aligned} p(\rho(x_2)) - p(\rho(x_1)) &= \big(p_2(\rho_0(x_2), \rho_1(x_2), \rho_2(x_2)) - p_2(\rho_0(x_1), \rho_1(x_1), \rho_2(x_1))\big)M^2 + o(M^2) \\ &= -c_0 k_m M^2 + o(M^2). \end{aligned}$$

This shows that we have recovered the relation (13) for the pressure p_2 at least for the special scaling of λ as in (20). Notice that we obtain this relation for the pressure p_2 (sometimes called hydrodynamic pressure), which enters the incompressible equation

$$\rho_0 \partial_t v_0 + \rho_0 \nabla \cdot (v_0 v_0^t) + \nabla p_2(\rho) = \frac{1}{\mathrm{Re}} \Delta v_0. \tag{29}$$

4. Phase field like scaling

In the following, we propose two different ways of scaling to get by means of J_ε (see (11)) a sharp interface model, which includes surface energy. This, in turn, leads to a nonvanishing jump condition for the pressure across the interface.

(i) The scaled surface tension $\hat{\sigma}$

Phase transitions from the liquid to the vapour phase usually take place in a region of small thickness and the layer is so small that it can be approximated by a hypersurface (cf. [Yo1805, Fi86, LL91]). However, this means that, if we replace the small layer of thickness ε between the two media by a hypersurface, we also have to rescale the surface tension σ. The scaled surface tension, which occurs in the sharp interface model, is given by $\hat{\sigma} = \sigma/\varepsilon$, i.e. free energy per unit area. Therefore, if we pass from the phase field model to the sharp interface model, we also have to rescale the

surface tension and hence the pressure condition. To be more precise, from Theorem 1, we deduce the following asymptotic behaviour:

$$\begin{aligned}\tilde{J}_\varepsilon(\rho_\varepsilon) &:= J(\rho_\varepsilon) + \tilde{m} = c_0\varepsilon \int_I \mathrm{d}\mathcal{H}^{n-1} + \tilde{m} + o(\varepsilon) \\ &= c_0\varepsilon \int_I \mathrm{d}\mathcal{H}^{n-1} + \int_A \beta_1\psi(\beta_1) + \int_{\Omega\setminus A} \beta_2\psi(\beta_2) + o(\varepsilon), \qquad \varepsilon \to 0. \end{aligned} \tag{30}$$

This means that in (30) the surface tension is related to the width ε of the interface, i.e. $\sigma = c_0\varepsilon$. Hence, the scaled surface tension $\hat{\sigma}$ in the corresponding sharp interface functional $\hat{J}$ (see (15)) is given by $\hat{\sigma} = \sigma/\varepsilon = c_0$. Analogously, we have to consider the scaled pressure which satisfies the condition

$$\frac{p(\rho_{\varepsilon_k}(x_2)) - p(\rho_{\varepsilon_k}(x_1))}{\varepsilon} = -c_0 k_m + o(1), \qquad \varepsilon \to 0, \tag{31}$$

under the assumptions of Theorem 3. Condition (31) agrees with the pressure condition for the sharp interface model:

$$p(\hat{\rho}^+) - p(\hat{\rho}^-) = -\hat{\sigma} k_m$$

(cf. Theorem 5).

(ii) The scaled free energy $\rho\psi_\varepsilon(\rho)$

Another possibility is to scale already the free energy density in the phase field model to obtain a nonvanishing contribution of the surface energy in the limit $\varepsilon \to 0$. We modify the phase field energy functional $\tilde{J}_\varepsilon$ in such a way that the structure of the minimizers is kept but the corresponding limit for $\varepsilon \to 0$ is different from $\tilde{m}$. This can be obtained by a suitable scaling of

$$W(\rho) + \frac{\varepsilon^2}{2}|\nabla\rho|^2$$

with some power of ε. In order to get some contribution which is different from 0 and ∞ we have to scale with $1/\varepsilon$. Therefore we consider

$$I_\varepsilon(\rho) := \frac{1}{\varepsilon}\int_\Omega \left(W(\rho) + \frac{\varepsilon^2}{2}|\nabla\rho|^2 \right) \mathrm{d}x + \tilde{m} \to \text{Minimum}. \tag{32}$$

The limit of the energy I_ε for $\varepsilon \to 0$ is now $c_0 P_\Omega(A) + \tilde{m}$. It turns out that the functional in (32) satisfies the identities

$$\begin{aligned} I_\varepsilon(\rho) &= \int_\Omega \left(\frac{1}{\varepsilon} W(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 \right) \mathrm{d}x + \tilde{m} \\ &= \int_\Omega \left(\frac{1}{\varepsilon}(\rho\psi(\rho) - l(\rho)) + \frac{\varepsilon}{2}|\nabla\rho|^2 \right) \mathrm{d}x + d_0 m + d_1|\Omega| \\ &= \int_\Omega \left(\frac{1}{\varepsilon}(\rho\psi(\rho) - l(\rho)) + \frac{\varepsilon}{2}|\nabla\rho|^2 \right) \mathrm{d}x + \int_\Omega l(\rho)\,\mathrm{d}x \\ &= \int_\Omega \left(\frac{1}{\varepsilon}(\rho\psi(\rho) - l(\rho)) + l(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 \right) \mathrm{d}x \\ &= \int_\Omega \left(\rho\psi_\varepsilon(\rho) + \frac{\varepsilon}{2}|\nabla\rho|^2 \right) \mathrm{d}x \end{aligned} \tag{33}$$

with $\rho\psi_\varepsilon(\rho)$ defined by

$$\rho\psi_\varepsilon(\rho) = \frac{1}{\varepsilon}(\rho\psi(\rho) - l(\rho)) + l(\rho). \tag{34}$$

For the following arguments, it is important to notice that the minimizers of I_ε and $\tilde{J}_\varepsilon$ are the same since $I_\varepsilon - \tilde{m} = (1/\varepsilon)(\tilde{J}_\varepsilon - \tilde{m})$, but the values of the minima of I_ε are different. The Euler–Lagrange equation of (33) is given by

$$\frac{\partial}{\partial\rho}(\rho\psi_\varepsilon(\rho)) - \varepsilon\Delta\rho = \tilde{\lambda}_\varepsilon, \tag{35}$$

where $\tilde{\lambda}_\varepsilon$ is the Lagrange multiplier with respect to the constraint $\int_\Omega \rho\,\mathrm{d}x = m$.

To the scaled energy corresponds the scaled pressure

$$p_\varepsilon(\rho) := \rho^2\psi'_\varepsilon(\rho). \tag{36}$$

We have

$$p_\varepsilon(\rho) = \frac{1}{\varepsilon}(p(\rho) + (1-\varepsilon)d_1), \tag{37}$$

where $p(\rho) = \rho^2\psi'(\rho)$ as in (2).

Now let the assumptions of Theorem 3 be satisfied. If $x \in U \subset\subset A$ or $x \in V \subset\subset \Omega \setminus \overline{A}$ then $p_\varepsilon(\rho_\varepsilon)$ converges pointwise in $U \cup V$ for a subsequence as $\varepsilon \to 0$. This can be seen as follows. The pressure may be rewritten as

$$\begin{aligned} p_\varepsilon(\rho) &= -\rho\psi_\varepsilon(\rho) + \frac{\rho}{\varepsilon}W'(\rho) + \rho d_0 \\ &= -\frac{1}{\varepsilon}(\rho\psi(\rho) - l(\rho)) - l(\rho) + \frac{\rho}{\varepsilon}W'(\rho) + \rho d_0 \\ &= -\frac{1}{\varepsilon}W(\rho) + \frac{\rho}{\varepsilon}W'(\rho) - d_1. \end{aligned}$$

Since W is twice differentiable and $|\rho_\varepsilon(x) - \beta_i + c_0 k_m \varepsilon/(W''(\beta_i)(\beta_2 - \beta_1))| = o(\varepsilon)$ in $U \cup V$ for $i \in \{1, 2\}$, we find that $(1/\varepsilon)W(\rho_\varepsilon)$ and $(\rho/\varepsilon)W'(\rho_\varepsilon)$ converge a.e. in $U \cup V$ as $\varepsilon \to 0$.

In order to see the relation to the static version of the Navier–Stokes–Korteweg equation, we use equation (35) and the pressure relation

$$\nabla p_\varepsilon(\rho) = 2\rho\nabla\rho\psi'_\varepsilon(\rho) + \rho^2\psi''_\varepsilon(\rho)\nabla\rho. \tag{38}$$

We infer

$$\nabla p_\varepsilon(\rho) = \rho\nabla(\psi_\varepsilon(\rho) + \rho\psi'_\varepsilon(\rho)) = \rho\nabla\frac{\partial}{\partial\rho}(\rho\psi_\varepsilon(\rho)) = \varepsilon\rho\nabla\Delta\rho. \tag{39}$$

Equation (39) is the static form of the Navier–Stokes–Korteweg equation for the pressure p_ε as defined in (36) with the corresponding energy functional I_ε in (32). From Theorem 3 we obtain

$$p(\rho_{\varepsilon_k}(x_2)) - p(\rho_{\varepsilon_k}(x_1)) = -c_0\,k_m\varepsilon_k + o(\varepsilon_k)$$

for $x_1 \in U$ and $x_2 \in V$ as $k \to \infty$. Then (37) implies for p_ε the jump condition

$$\begin{aligned} p_\varepsilon(\rho(x_2)) - p_\varepsilon(\rho(x_1)) &= \frac{1}{\varepsilon}\big(p(\rho(x_2)) - p(\rho(x_1))\big) \\ &= -c_0\, k_m + o(1), \quad \varepsilon \to 0. \end{aligned}$$

This means that for the pressure p_ε defined in (36), we obtain the same jump condition as in [LL91, Fi86] (see equation (13)). While p defined as in (2) is the thermodynamic pressure and appears in (1), the pressure p_ε in (36) behaves more "incompressibly", since small perturbations for ρ imply large perturbations for the pressure $p_\varepsilon(\rho)$ if $\varepsilon > 0$ is small.

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