

## Asymptotic behaviour of a nonlinear parabolic equation with gradient absorption and critical exponent

RAZVAN GABRIEL IAGAR

*Departamento de Matemáticas, Universidad Autónoma de Madrid,  
Campus de Cantoblanco, E-28049 Madrid, Spain*

*E-mail: razvan.iagar@uam.es*

PHILIPPE LAURENÇOT

*Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université de Toulouse,  
F-31062 Toulouse Cedex 9, France*

*E-mail: Philippe.Laurencot@math.univ-toulouse.fr*

JUAN LUIS VÁZQUEZ

*Departamento de Matemáticas, Universidad Autónoma de Madrid,  
Campus de Cantoblanco, E-28049 Madrid, Spain (also affiliated with ICMAT, Madrid)*

*E-mail: juanluis.vazquez@uam.es*

[Received 20 March 2010 and in revised form 8 January 2011]

We study the large-time behaviour of solutions of the evolution equation involving nonlinear diffusion and gradient absorption,

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0.$$

We consider the problem for  $x \in \mathbb{R}^N$  and  $t > 0$  with nonnegative and compactly supported initial data. We take the exponent  $p > 2$  which corresponds to slow  $p$ -Laplacian diffusion. The main feature of the paper is that the exponent  $q$  takes the critical value  $q = p - 1$ , which leads to interesting asymptotics. This is due to the fact that in this case both the Hamilton–Jacobi term  $|\nabla u|^q$  and the diffusive term  $\Delta_p u$  have a similar size for large times. The study performed in this paper shows that a delicate asymptotic equilibrium occurs, so that the large-time behaviour of solutions is described by a rescaled version of a suitable self-similar solution of the Hamilton–Jacobi equation  $|\nabla W|^{p-1} = W$ , with logarithmic time corrections. The asymptotic rescaled profile is a kind of sandpile with a cusp on top, and it is independent of the space dimension.

*2010 Mathematics Subject Classification:* 35B40, 35K65, 35K92, 49L25.

*Keywords:* Nonlinear parabolic equations;  $p$ -Laplacian equation; gradient absorption; asymptotic patterns; Hamilton–Jacobi equation; viscosity solutions.

### 1. Introduction and main results

In this paper we deal with the Cauchy problem associated to the diffusion-absorption equation

$$\partial_t u - \Delta_p u + |\nabla u|^q = 0, \quad (t, x) \in Q, \quad (1.1)$$

posed in  $Q := (0, \infty) \times \mathbb{R}^N$  with initial data

$$u(0, x) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where the  $p$ -Laplacian operator is defined as usual by  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ . To be specific we take  $p > 2$ , which implies finite speed of propagation, and we consider nonnegative weak solutions  $u \geq 0$  with compactly supported initial data  $u_0$  such that

$$u_0 \in W^{1,\infty}(\mathbb{R}^N), \quad u_0 \geq 0, \quad \operatorname{supp}(u_0) \subset B(0, R_0), \quad u_0 \not\equiv 0, \quad (1.3)$$

for some  $R_0 > 0$ . Known properties of the equation ensure that its solutions will be compactly supported with respect to the space variable for every time  $t > 0$ . The goal of the paper is to describe in detail the asymptotic behaviour of the solutions as  $t \rightarrow \infty$ .

The equation (1.1) has been studied by various authors for different values of the parameters  $p \geq 2$  and  $q > 1$  as a model of linear or nonlinear diffusion with gradient-dependent absorption: see [8, 9, 11, 12, 15] for the semilinear case  $p = 2$ , and [1, 7, 16, 20] for the quasilinear case  $p > 2$ . It has been shown that the large-time behaviour of this initial-value problem depends on the relative influence of the diffusion and absorption terms and leads to a classification into the following ranges of  $q$ :

- (i) When  $q > q_2 := p - N/(N + 1)$  the large-time behaviour is purely diffusive and the first order absorption term disappears in the limit  $t \rightarrow \infty$ ; this is a case of asymptotic simplification in the sense of [21].
- (ii) For  $q_1 := p - 1 < q < q_2$  there is a behaviour given by a certain balance of diffusion and absorption in the form of a self-similar solution, its existence being established in [20]; there is no asymptotic simplification.
- (iii) For  $1 < q < p - 1$  the last two authors have recently shown in [16] that the main term is the absorption term, leading to a separate-variables asymptotic behaviour, with diffusion playing a secondary role. We thus have asymptotic simplification, now with absorption as the dominating effect.

The two critical cases  $q = q_2$  and  $q = q_1$  represent limit behaviours, and as is often the case in such situations, they give rise to interesting dynamics due to the curious interaction of two effects of similar strength. Such situations usually lead to phenomena called *resonances* in mechanics, with interesting nontrivial mathematical analysis. Such interesting behaviour has been shown in particular in [11] for  $q = q_2$ , in the linear case  $p = 2$ , with the result that logarithmic factors modify the purely diffusive behaviour found for  $q > q_2$ . A similar situation is expected to be met when  $p > 2$  and  $q = q_2$ .

We devote this paper to the other limit case,  $q = q_1 = p - 1$  when  $p > 2$ , the latter condition guaranteeing that  $q > 1$ . In that case the diffusion and the first order term have similar asymptotic size and logarithmic corrections appear in the asymptotic rates. The mathematical analysis that we perform below is strongly tied to a good knowledge of the expansion of the support of the solution, or in other words, the location of the free boundary, which happens to be approximately a sphere of radius  $|x| \sim C \log t$  for large times  $t$ . From now on, we assume that

$$q = q_1 = p - 1.$$

### 1.1 Bounds in suitable norms

Studying the large time behaviour of solutions and interfaces of our problem relies on suitable precise estimates. The time expansion of the support and the time decay of solutions to the Cauchy

problem (1.1)–(1.2) with nonnegative and compactly supported initial data have recently been investigated in [7]. The following results are proved:

**PROPOSITION 1.1** Consider an initial condition  $u_0$  satisfying (1.3) (and  $q = p - 1$ ). The Cauchy problem (1.1)–(1.2) has a unique nonnegative viscosity solution

$$u \in \mathcal{BC}([0, \infty) \times \mathbb{R}^N) \cap L^\infty(0, \infty; W^{1,\infty}(\mathbb{R}^N))$$

which satisfies

$$0 \leq u(t, x) \leq \|u_0\|_\infty, \quad (t, x) \in Q, \quad (1.4)$$

$$\|\nabla u(t)\|_\infty \leq \|\nabla u_0\|_\infty, \quad t \geq 0, \quad (1.5)$$

$$\text{supp}(u(t)) \subset B(0, C_1 \log t) \quad \text{for all } t \geq 2, \quad (1.6)$$

together with the norm estimates

$$\|u(t)\|_1 \leq C_2 t^{-1/(p-2)} (\log t)^{(p(N+1)-2N-1)/(p-2)} \quad \text{for all } t \geq 2, \quad (1.7)$$

$$\|u(t)\|_\infty \leq C_2 t^{-1/(p-2)} (\log t)^{(p-1)/(p-2)} \quad \text{for all } t \geq 2, \quad (1.8)$$

$$\|\nabla u(t)\|_\infty \leq C_2 t^{-1/(p-2)} (\log t)^{1/(p-2)} \quad \text{for all } t \geq 2, \quad (1.9)$$

for some positive constants  $C_1$  and  $C_2$  depending only on  $p$ ,  $N$ , and  $u_0$ .

Here and below,  $\mathcal{BC}([0, \infty) \times \mathbb{R}^N)$  denotes the space of bounded continuous functions on  $[0, \infty) \times \mathbb{R}^N$  and  $\|\cdot\|_r$  denotes the  $L^r(\mathbb{R}^N)$ -norm for  $r \in [1, \infty]$ . As we shall see, these bounds will be very useful. The well-posedness of (1.1)–(1.2) and the properties (1.4), (1.6), and (1.7) are established in [7, Theorems 1.1 & 1.6, Corollary 1.7], while (1.8) and (1.9) follow from (1.7) and [7, Proposition 1.4]. We will also use the notation  $r_+ = \max\{r, 0\}$  for the positive part of the real number  $r$ .

## 1.2 Main results

We next describe the main contribution of this paper. As already mentioned, our goal is to study the asymptotic behaviour of the solution  $u$  of the resonant problem (1.1) with  $p > 2$  and  $q = p - 1$ , and with compactly supported and nonnegative initial data. Moreover, since the equation has the property of finite speed of propagation, it is natural to raise the question about how the interface and the support of the solution expand in time. We also answer this question in the present paper.

*Asymptotic behaviour.* The main result is the following:

**THEOREM 1.1** Let  $u$  be the solution of the Cauchy problem (1.1)–(1.2) with  $u_0$  as in (1.3). Then  $u$  decays in time like  $O(t^{-1/(p-2)}(\log t)^{(p-1)/(p-2)})$  and the support spreads in space like  $O(\log t)$  as  $t \rightarrow \infty$ . More precisely, we have the limit

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \left| \frac{c_p t^{1/(p-2)}}{(\log t)^{(p-1)/(p-2)}} u(t, x) - \left(1 - \frac{(p-2)|x|}{\log t}\right)_+^{(p-1)/(p-2)} \right| = 0, \quad (1.10)$$

with precise constant

$$c_p = (p-2)^{1/(p-2)} (p-1)^{(p-1)/(p-2)}.$$

In the proof, the expression of the asymptotic profile is obtained after a complicated time scaling of  $u$  and  $x$  in the form of uniform limit

$$\frac{t^{1/(p-2)}}{(\log t)^{(p-1)/(p-2)}}u(t, x) \rightarrow (p - 2)^{-p/(p-2)}W((p - 2)x/\log t), \tag{1.11}$$

where the function

$$W(x) := \left( \frac{p - 2}{p - 1}(1 - |x|)_+ \right)^{(p-1)/(p-2)} \tag{1.12}$$

is the unique viscosity solution to the stationary form of the rescaled problem, which is

$$|\nabla W|^{p-1} - W = 0 \quad \text{in } B(0, 1), \quad W = 0 \quad \text{on } \partial B(0, 1), \quad W > 0 \quad \text{in } B(0, 1). \tag{1.13}$$

Let us notice that, as is usual in resonance cases, the limit profile is not a self-similar solution, but it introduces logarithmic corrections to a self-similar, separate-variables profile (which in our case is  $t^{-1/(p-2)}(p - 2)^{-p/(p-2)}W((p - 2)x)$ ). The uniqueness of  $W$  as viscosity solution of (1.13) is important in the proof and follows from [13].

In accordance with (1.10), we show that the shape of the support of  $u(t)$  gets closer to a ball while expanding as time goes by. This is in sharp contrast with the situation described in [16] for (1.1) in the intermediate range  $q \in (1, p - 1)$ ,  $p > 2$ , where the positivity set stays bounded and can have a very general shape. When  $q = p - 1$ , the diffusion thus acts in three directions: the scaling is different, the support grows unboundedly in time, and the geometry of the positivity set simplifies. Another remarkable consequence of the diffusion-absorption interplay is that the asymptotic profile is radially symmetric and does not depend on the space dimension.

We devote Section 4 to the proof of Theorem 1.1. For the proof, we use a precise estimate for the propagation of the positivity set, described below. Another tool is the existence of a large family of subsolutions having a special, explicit form and allowing for a theoretical argument with viscosity solutions to finish the proof.

*Propagation of the positivity set.* We denote the positivity set and its maximal expansion radius by

$$\mathcal{P}_u(t) := \{x \in \mathbb{R}^N : u(t, x) > 0\}, \quad \gamma(t) = \sup\{|x| : x \in \mathcal{P}_u(t)\} \tag{1.14}$$

respectively. Then:

**THEOREM 1.2** Under the above notations and assumptions, we have

$$\lim_{t \rightarrow \infty} \frac{\gamma(t)}{\log t} = \frac{1}{p - 2}.$$

Moreover, the free boundary of  $u$  has the same speed of expansion in any given direction  $\omega \in \mathbb{R}^N$  with  $|\omega| = 1$ .

In fact, we give more precise estimates for the expansion of the positivity region, obtained via comparison with some well-chosen travelling waves. The proof of Theorem 1.2 is given in Section 3. It is worth pointing out here that, while the assumption of compact support for  $u_0$  is of utmost importance for Theorems 1.1 and 1.2 to hold, the size of the support of  $u_0$  is irrelevant for large times and plays no role in the description of the asymptotic state.

*Two scalings.* In order to prove the two theorems, we have to perform two different scaling steps. The first scaling, described in formula (2.2) below, is the natural one corresponding to standard scaling invariance; such a scaling has also been used in [16] in the case  $q \in (1, p - 1)$  to obtain the correct scale of the solutions. But for  $q = p - 1$ , we observe that a grow-up phenomenon appears, which is typical for resonance cases: the effect of the resonance implies that the rescaled solution does not stabilize in time; on the contrary, it grows and becomes unbounded in infinite time. That is why we need a second scaling, given by the new functions  $w$  and  $y$  defined in (4.1) and (4.2), which is less natural but turns out to be adapted to our problem: it takes into account the logarithmic corrections (suggested by the a priori estimates of Proposition 1.1, which turn out to be sharp), and it is adapted to the size of the grow-up phenomenon; thus, in the rescaled variables we can describe the real form and behaviour of the solution.

## 2. Scaling variables I

We recall that  $p > 2$  and  $q = p - 1$ . We introduce a first set of *self-similar* variables; we keep the space variable  $x$  and introduce logarithmic time

$$\tau := \frac{1}{p-2} \ln(1 + (p-2)t), \quad (2.1)$$

as well as the new unknown function  $v = v(\tau, x)$  defined by

$$u(t, x) = (1 + (p-2)t)^{-1/(p-2)} v(\tau, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^N. \quad (2.2)$$

Clearly,  $v$  solves the rescaled equation

$$\partial_\tau v - \Delta_p v + |\nabla v|^q - v = 0, \quad (\tau, x) \in Q, \quad (2.3)$$

with the same initial condition

$$v(0) = u_0, \quad x \in \mathbb{R}^N. \quad (2.4)$$

We next translate the a priori bounds (1.7), (1.8), and (1.9) in terms of the rescaled function  $v$ : there is  $C_3 > 0$  depending only on  $p, N$ , and  $u_0$  such that

$$\frac{\|v(\tau)\|_1}{\tau^{(p(N+1)-2N-1)/(p-2)}} + \frac{\|v(\tau)\|_\infty}{\tau^{(p-1)/(p-2)}} + \frac{\|\nabla v(\tau)\|_\infty}{\tau^{1/(p-2)}} \leq C_3 \quad \text{for } \tau \geq 1. \quad (2.5)$$

### 2.1 The positivity set: time monotonicity

We define the positivity set  $\mathcal{P}_v(\tau)$  of the function  $v$  at time  $\tau \geq 0$  by

$$\mathcal{P}_v(\tau) := \{x \in \mathbb{R}^N : v(\tau, x) > 0\}. \quad (2.6)$$

PROPOSITION 2.1 For  $\tau_1 \in [0, \infty)$  and  $\tau_2 \in (\tau_1, \infty)$  we have

$$\mathcal{P}_v(\tau_1) \subseteq \mathcal{P}_v(\tau_2) \quad \text{and} \quad \bigcup_{\tau \geq 0} \mathcal{P}_v(\tau) = \mathbb{R}^N. \quad (2.7)$$

In addition, for each  $x \in \mathbb{R}^N$  there are  $T_x \geq 0$  and  $\varepsilon_x > 0$  such that

$$v(\tau, x) \geq \varepsilon_x \tau^{(p-1)/(p-2)} \quad \text{for } \tau \geq T_x. \quad (2.8)$$

The proof relies on the availability of suitable subsolutions which we describe next.

LEMMA 2.1 Define two positive real numbers  $R_p$  and  $T_p$  by

$$R_p := \frac{p-2}{2^p(p-1)} \quad \text{and} \quad T_p := \frac{2(p-1)}{p-2}(2+2^{p-1}(N+p-2)).$$

If  $R \in (0, R_p]$  and  $T \geq T_p$ , the function  $s_{R,T}$  given by

$$s_{R,T}(\tau, x) := \frac{p-2}{R(p-1)}(T+\tau)^{(p-1)/(p-2)} \left( R^2 - \frac{|x|^2}{(T+\tau)^2} \right)_+^{(p-1)/(p-2)}, \quad (\tau, x) \in Q,$$

is a (viscosity) subsolution to (2.3).

*Proof.* We have  $s_{R,T}(\tau, x) = (T+\tau)^{(p-1)/(p-2)}\sigma(\xi)$  with  $\xi := x/(T+\tau)$  and  $\sigma(\xi) := (p-2)(R^2 - |\xi|^2)_+^{(p-1)/(p-2)}/(R(p-1))$ . Since  $p-1 > p-2 > 0$ , we observe that  $\sigma$  and  $|\nabla\sigma|^{p-2}\nabla\sigma$  both belong to  $\mathcal{C}^1(\mathbb{R}^N)$ . Therefore,

$$L(\tau, x) := R(T+\tau)^{-(p-1)/(p-2)}\{\partial_\tau s_{R,T} - \Delta_p s_{R,T} + |\nabla s_{R,T}|^{p-1} - s_{R,T}\}$$

is well-defined for  $(\tau, x) \in [0, \infty) \times \mathbb{R}^N$  and

$$\begin{aligned} L(\tau, x) &= \frac{R}{T+\tau} \left\{ \frac{p-1}{p-2} \sigma(\xi) - \xi \cdot \nabla\sigma(\xi) - \Delta_p \sigma(\xi) \right\} + R|\nabla\sigma(\xi)|^{p-1} - R\sigma(\xi) \\ &= (R^2 - |\xi|^2)_+^{(p-1)/(p-2)} \left\{ \frac{1}{T+\tau} (1 + 2^{p-1}(N+p-2)) \frac{|\xi|^{p-2}}{R^{p-2}} \right\} \\ &\quad + (R^2 - |\xi|^2)_+^{(p-1)/(p-2)} \left\{ \frac{2}{T+\tau} \frac{|\xi|^2}{R^2 - |\xi|^2} \left( 1 - \frac{2^{p-1}(p-1)}{p-2} \frac{|\xi|^{p-2}}{R^{p-2}} \right) \right\} \\ &\quad + (R^2 - |\xi|^2)_+^{(p-1)/(p-2)} \left\{ 2^{p-1} \frac{|\xi|^{p-1}}{R^{p-2}} - \frac{p-2}{p-1} \right\} \\ &\leq (R^2 - |\xi|^2)_+^{(p-1)/(p-2)} \left\{ \frac{1 + 2^{p-1}(N+p-2)}{T} + 2^{p-1}R - \frac{p-2}{p-1} \right\} \\ &\quad + (R^2 - |\xi|^2)_+^{(p-1)/(p-2)} \left\{ \frac{2}{T+\tau} \frac{|\xi|^2}{R^2 - |\xi|^2} \left( 1 - \frac{2^{p-1}(p-1)}{p-2} \frac{|\xi|^{p-2}}{R^{p-2}} \right) \right\}_+. \end{aligned}$$

We next note that

$$1 - \frac{2^{p-1}(p-1)}{p-2} \frac{|\xi|^{p-2}}{R^{p-2}} \leq 0 \quad \text{if } |\xi| \geq R/2,$$

so that the last term of the right-hand side of the previous inequality is bounded from above by  $2(R^2 - |\xi|^2)_+^{(p-1)/(p-2)}/(3T)$ . Consequently, owing to the choice of  $R$  and  $T$ ,

$$L(\tau, x) \leq (R^2 - |\xi|^2)_+^{(p-1)/(p-2)} \left\{ \frac{1 + 2^{p-1}(N+p-2)}{T_p} + 2^{p-1}R_p - \frac{p-2}{p-1} + \frac{2}{3T_p} \right\} \leq 0,$$

whence the claim.  $\square$

*Proof of Proposition 2.1.* (i) Fix  $\tau_1 \geq 0$  and  $x_1 \in \mathcal{P}_v(\tau_1)$ . Owing to the continuity of  $x \mapsto v(\tau_1, x)$  there are  $\delta > 0$  and  $r_1 > 0$  such that  $v(\tau_1, x) \geq \delta$  for  $x \in B(x_1, r_1)$ . Take now  $R > 0$  small enough such that  $R < \min\{r_1, R_p\}$  and satisfying

$$R < \frac{r_1}{T_p + \tau_1} \quad \text{and} \quad \frac{p-2}{p-1}(T_p + \tau_1)^{(p-1)/(p-2)} R^{p/(p-2)} \leq \delta,$$

the parameters  $R_p$  and  $T_p$  being defined in Lemma 2.1. Then we have  $s_{R, T_p}(\tau_1, x - x_1) = 0 \leq v(\tau_1, x)$  if  $|x - x_1| \geq R(T_p + \tau_1)$ , while

$$s_{R, T_p}(\tau_1, x - x_1) \leq \frac{p-2}{R(p-1)}(T_p + \tau_1)^{(p-1)/(p-2)} R^{(2p-2)/(p-2)} \leq \delta \leq v(\tau_1, x)$$

if  $|x - x_1| \leq R(T_p + \tau_1)$  as  $R(T_p + \tau_1) \leq r_1$ . Moreover, if  $\tau_2 > \tau_1$ ,  $\tau \in [\tau_1, \tau_2]$  and  $x \in \partial B(x_1, R(T_p + \tau_2))$ , then  $s_{R, T_p}(\tau, x - x_1) = 0 \leq v(\tau, x)$ . Recalling that  $s_{R, T_p}$  is a subsolution to (2.3) by Lemma 2.1, we infer from the comparison principle that  $s_{R, T_p}(\tau, x - x_1) \leq v(\tau, x)$  for  $(\tau, x) \in [\tau_1, \tau_2] \times B(x_1, R(T_p + \tau_2))$ . As  $s_{R, T_p}(\tau, x - x_1) = 0 \leq v(\tau, x)$  for  $\tau \in [\tau_1, \tau_2]$  and  $x \notin B(x_1, R(T_p + \tau_2))$  we actually have  $s_{R, T_p}(\tau, x - x_1) \leq v(\tau, x)$  for  $(\tau, x) \in [\tau_1, \tau_2] \times \mathbb{R}^N$ . Since  $\tau_2 > \tau_1$  is arbitrary and neither  $R$  nor  $T_p$  depends on  $\tau_2$ , we end up with

$$s_{R, T_p}(\tau, x - x_1) \leq v(\tau, x), \quad (\tau, x) \in [\tau_1, \infty) \times \mathbb{R}^N. \quad (2.9)$$

A first consequence of (2.9) is that, if  $\tau_2 > \tau_1$ , then  $v(\tau_2, x_1) \geq s_{R, T_p}(\tau_2, 0) > 0$  so that  $x_1$  also belongs to  $\mathcal{P}_v(\tau_2)$ .

Next, given  $x \in \mathbb{R}^N$ , we have  $x \in B(x_1, R(T_p + \tau))$  for  $\tau$  large enough, and it follows from (2.9) that  $v(\tau, x) \geq s_{R, T_p}(\tau, x - x_1) > 0$  for  $\tau$  large enough. Consequently,  $x$  belongs to  $\mathcal{P}_v(\tau)$  for  $\tau$  large enough, which proves the second assertion of (2.7).

(ii) Consider  $x_0 \in \mathbb{R}^N$ . According to (2.7) there is  $\tau_0$  large enough such that  $x_0 \in \mathcal{P}_v(\tau_0)$ . Arguing as in the proof of (2.7), we may find  $r_0$  small enough (depending on  $x_0$ ) such that  $s_{r_0, T_p}(\tau, x - x_0) \leq v(\tau, x)$  for  $(\tau, x) \in [\tau_0, \infty) \times \mathbb{R}^N$ . Consequently,

$$v(\tau, x_0) \geq \frac{p-2}{r_0(p-1)}(T_p + \tau)^{(p-1)/(p-2)} r_0^{(2p-2)/(p-2)} \geq \frac{p-2}{p-1} r_0^{p/(p-2)} \tau^{(p-1)/(p-2)},$$

which gives the lower bound (2.8). □

**COROLLARY 2.1** Assume that  $u_0(0) > 0$ . Then there is  $r_* > 0$  such that

$$v(\tau, x) \geq \frac{(p-2)}{r_*(p-1)}(1 + \tau)^{(p-1)/(p-2)} \left( r_*^2 - \frac{|x|^2}{(1 + \tau)^2} \right)_+^{(p-1)/(p-2)}, \quad (\tau, x) \in \mathcal{Q}. \quad (2.10)$$

*Proof.* Arguing as in the proof of (2.7) and using the positivity of  $u_0(0)$ , we find  $r_* > 0$  small enough such that  $s_{r_*, T_p}(\tau, x) \leq v(\tau, x)$  for  $(\tau, x) \in \mathcal{Q}$ . Since  $T_p > 1$ , the previous inequality implies (2.10). □

## 2.2 Eventual radial symmetry

We prove the following classical monotonicity lemma (see [3, Proposition 2.1] for instance).

LEMMA 2.2 If  $x \in \mathbb{R}^N$  and  $r > 0$  satisfy  $|x| > 2R_0$  and  $r < |x| - 2R_0$ , then

$$v(\tau, x) \leq \inf_{|y|=r} v(\tau, y) \quad \text{for } \tau \geq 0. \tag{2.11}$$

Here,  $R_0$  is the radius of the initial ball defined in (1.3).

*Proof.* The proof relies on Aleksandrov’s reflection principle. Let  $(x, r) \in \mathbb{R}^N \times (0, \infty)$  fulfil the assumptions and consider  $y \in \mathbb{R}^N$  such that  $|y| = r$ . Let  $H$  be the hyperplane of points of  $\mathbb{R}^N$  which are equidistant from  $x$  and  $y$ :

$$H := \left\{ z \in \mathbb{R}^N : \left\langle z - \frac{x+y}{2}, x-y \right\rangle = 0 \right\}.$$

Introducing

$$H_- := \left\{ z \in \mathbb{R}^N : \left\langle z - \frac{x+y}{2}, x-y \right\rangle \leq 0 \right\}$$

and

$$\tilde{v}(\tau, z) := v\left(\tau, z - 2\left\langle z - \frac{x+y}{2}, x-y \right\rangle \frac{x-y}{|x-y|^2}\right), \quad (\tau, z) \in Q,$$

it readily follows from the rotational and translational invariance of (2.3) that  $\tilde{v}$  also solves (2.3). In addition,  $y \in H_-$  and  $\mathcal{P}_v(0) \subseteq B(0, R_0) \subseteq H_-$  by (1.3). Now, on the one hand, if  $z \in H_-$ , then

$$z - 2\left\langle z - \frac{x+y}{2}, x-y \right\rangle \frac{x-y}{|x-y|^2} \notin H_-$$

and  $\tilde{v}(0, z) = 0 \leq v(0, z)$ . On the other hand, if  $z \in H = \partial H_-$  and  $\tau \geq 0$ , we clearly have  $\tilde{v}(\tau, z) = v(\tau, z)$ . We can thus apply the comparison principle to (2.3) on  $(0, \infty) \times H_-$  and conclude that

$$\tilde{v}(\tau, z) \leq v(\tau, z), \quad (\tau, z) \in [0, \infty) \times H_-. \tag{2.12}$$

Recalling that  $y \in H_-$ , we infer from (2.12) that  $v(\tau, y) \geq \tilde{v}(\tau, y) = v(\tau, x)$  for  $\tau \geq 0$ , which is the expected result.  $\square$

REMARK 2.1 Although Lemma 2.2 will not be used in the main proofs, this is an interesting result for the qualitative theory, since it shows that the dynamics symmetrizes the solution.

### 3. Propagation of the positivity set

We next turn to the speed of expansion of the positivity set  $\mathcal{P}_v$  of  $v$  and put

$$\varrho(\tau) := \sup\{|x| : x \in \mathcal{P}_v(\tau)\}, \tag{3.1}$$

so that  $\mathcal{P}_v(\tau) \subseteq B(0, \varrho(\tau))$  for  $\tau \geq 0$ . The purpose of this section is to prove that the expansion speed  $\varrho(\tau)$  of  $\mathcal{P}_v(\tau)$  is asymptotically equal to  $\tau$ , in other words,

$$\lim_{\tau \rightarrow \infty} \frac{\varrho(\tau)}{\tau} = 1,$$

and, more precisely, to prove Theorem 1.2.



The proof relies on the existence of “nice” travelling wave solutions of (2.3), which may be used as subsolutions and supersolutions for the Cauchy problem (2.3)–(2.4). The construction of such travelling waves is inspired by the technique used in the so-called KPP problems [14], which has developed a wide literature; see, e.g., [2], [22] for applications to porous media, and [18] for blow-up problems. We thus begin with a phase-plane analysis, proving the existence of the desired travelling waves.

### 3.1 Travelling wave analysis for $N = 1$

We look for travelling waves of the form

$$v(\tau, x) = f(z), \quad z = x - c\tau, \quad c > 0,$$

solving (2.3) in dimension  $N = 1$ . Then the profile  $f$  solves the ordinary differential equation

$$-cf' - (|f'|^{p-2}f')' + |f'|^{p-1} - f = 0. \quad (3.2)$$

We are actually only interested in travelling waves which present an interface, that is,  $f$  vanishes for  $z$  sufficiently large. As we shall see below, the profile  $f$  is nonmonotone in general, but is nonnegative and decreasing near the interface. We transform (3.2) into a first order system, by introducing the notation  $U = f$  and  $V = -f'$ . We arrive at

$$\begin{cases} (p-1)|V|^{p-2}U' = -(p-1)|V|^{p-2}V, \\ (p-1)|V|^{p-2}V' = -cV - |V|^{p-1} + U, \end{cases} \quad (3.3)$$

where, for the orbits, the term  $(p-1)|V|^{p-2}$  on the right-hand side has no influence (since we work with  $dV/dU$ ) and can be ignored after a change of the time variable. We next perform the phase-plane analysis of the system (3.3).

*Local analysis in the plane.* The system (3.3) has a unique critical point,  $P = (0, 0)$ , and the Jacobian matrix  $J(0, 0)$  at this point is given by

$$J(0, 0) = \begin{pmatrix} 0 & 0 \\ 1 & -c \end{pmatrix}$$

with eigenvalues  $\lambda_1 = 0$  and  $\lambda_2 = -c$ , and corresponding eigenvectors  $e_1 = (c, 1)$  and  $e_2 = (0, 1)$ . By a careful analysis, we notice that the centre manifold at  $P$  is tangent to  $e_1$ , and is asymptotically stable. It follows that  $P$  is a stable node for every  $c > 0$ . There is a unique orbit entering  $P$  and tangent to  $e_2$ , forming the stable manifold; its local behaviour is  $U(z) \sim C(-z)^{(p-1)/(p-2)}$  as  $z \rightarrow 0$ , hence this orbit contains all the travelling waves with velocity  $c$  and having an interface. By standard theory (see, e.g., [17]), all the other orbits approach the centre manifold, tangent to  $e_1$ , and exhibit exponential decay, but no interface:  $U(z) \sim e^{-cz}$  as  $z \rightarrow \infty$ .

*Local analysis at infinity.* We investigate the behaviour of the system when  $U$  is very large. For monotone travelling waves, we make the following inversion of the plane:

$$Z = \frac{1}{U}, \quad W = \frac{|V|^{p-2}V}{U},$$

and we are interested in the local behaviour near  $Z = 0$ . After straightforward calculations, (3.3) becomes

$$\begin{cases} Z' = Z^{(2p-3)/(p-1)} W |W|^{-(p-2)/(p-1)}, \\ W' = Z^{(p-2)/(p-1)} |W|^{p/(p-1)} - c Z^{(p-2)/(p-1)} W |W|^{-(p-2)/(p-1)} + 1 - |W|. \end{cases} \quad (3.4)$$

We find two critical points with  $Z = 0$ , namely  $Q_1 = (0, 1)$  and  $Q_2 = (0, -1)$ . We will analyze only  $Q_1$ , i.e. the decreasing travelling waves. Let us also remark that, in the second equation of (3.4), the terms with  $Z$  are dominated by  $1 - |W|$  near  $Q_1$  and  $Q_2$ , hence we can study the local behaviour by using the approximate equation only with  $1 - |W|$  on the right-hand side. The linearization near  $Q_1$  has eigenvalues  $\lambda_1 = 0$  and  $\lambda_2 = -1$ , and the centre manifold, which is tangent to the line  $W = 1$ , is unstable. Hence, the point  $Q_1$  behaves like a saddle, and the orbits which are interesting for our study are those going out of  $Q_1$ . These orbits are tangent to  $W = 1$ , and in the original system they satisfy  $U \sim V^{p-1}$ , hence, by integration,

$$U(z) \sim |z|^{(p-1)/(p-2)} \quad \text{as } z \rightarrow -\infty,$$

and are decreasing. The local analysis around  $Q_2$  is similar, but not interesting for our goals.

Let us notice that not all solutions passing through a point in the first quadrant come from  $Q_1$ . Indeed, the orbits touching the curve  $U = cV + V^{p-1}$  change monotonicity as functions  $V = V(U)$ , hence they have previously reached the axis  $V = 0$ , meaning a change of monotonicity as  $f = f(z)$ , and they enter through this change in the first quadrant. Analyzing the curve  $U = cV + V^{p-1}$ , we observe that it connects in the phase-plane the points  $P = (0, 0)$  and  $Q_1$ , being tangent at  $Q_1$  to the axis  $W = 1$ . In particular, there exist nonmonotone solutions, and this is the object we are interested in.

*Global behaviour.* This is now not difficult to establish, by merging the previous local analysis with the following important remarks:

- (a) The evolution of the system (3.3) with respect to the parameter  $c$  is monotone. Indeed, we calculate

$$\frac{d}{dc} \left( \frac{dV}{dU} \right) = \frac{1}{(p-1)|V|^{p-2}} > 0.$$

- (b) There exists an explicit family of travelling wave solutions with speed  $c = 1$ :

$$f_{1,K}(z) = \left( \frac{p-2}{p-1} \right)^{(p-1)/(p-2)} (K-z)_+^{(p-1)/(p-2)}, \quad K \geq 0. \quad (3.5)$$

This function is obviously decreasing and exhibits an interface at  $z = K$ . It is immediate to check that this orbit satisfies  $U = V^{p-1}$ , hence it comes from the point  $Q_1$  along the centre manifold of it, and it enters  $P$ , being the unique orbit entering  $P$  and tangent to the eigenvector  $e_2 = (0, 1)$  (unique for  $c = 1$ ), as discussed above.

- (c) Moreover, the vectors of the direction field of (3.3) over the curve  $U = V^{p-1}$  (which gives the explicit orbit (3.5)) have the same direction. Indeed, the normal vector to this curve is  $(1, -(p-1)V^{p-2})$  and we calculate

$$(1, -(p-1)V^{p-2}) \cdot (-(p-1)V^{p-1}, -cV - V^{p-1} + U) = (p-1)(c-1)V^{p-1}.$$

For  $c = 1$  we obtain the explicit trajectory, and for  $c < 1$ , the above scalar product is negative, hence all these vectors have the same direction, contrary to  $(1, -(p-1)V^{p-2})$ . For  $c > 1$ , all these vectors have the same direction as  $V$ .

Since we are interested only in travelling waves with an interface, we analyze only the unique (for  $c$  fixed) orbit entering  $P = (0, 0)$  tangent to  $e_2 = (0, 1)$ . For  $c = 1$ , it is explicit and connects  $P$  and  $Q_1$  in the first quadrant. We draw the phase-plane for  $c = 1$  in Figure 1 below; it is clear that the explicit connection will not change sign and monotonicity.

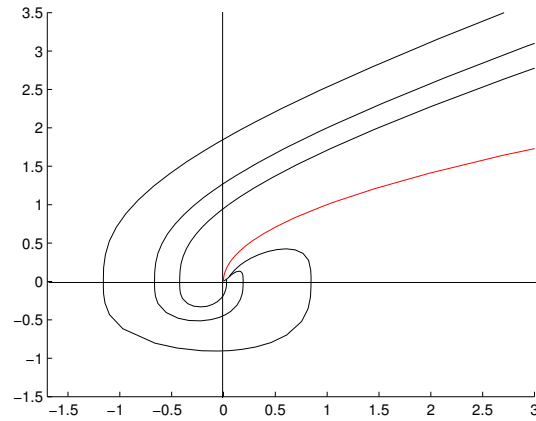


FIG. 1. Phase portrait around the origin for  $c = 1$ . Experiment for  $p = 3$ ,  $N = 1$ .

By remarks (a) and (c) above, it follows that for  $c < 1$ , this unique orbit escapes from  $Q_1$ , hence it should cross at some point the curve  $U = cV + V^{p-1}$  (which still connects  $P = (0, 0)$  and  $Q_1$ ); as explained before, this orbit previously had a change of sign (crossing the axis  $U = 0$ ) and then a change of monotonicity (crossing the axis  $V = 0$ ). In particular, we can say that the explicit orbit (3.5) is a separatrix between the monotone and the nonmonotone orbits. We draw the local phase portrait for  $c < 1$ , around the origin, in Figure 2 below. We gather the discussion above in the following result.

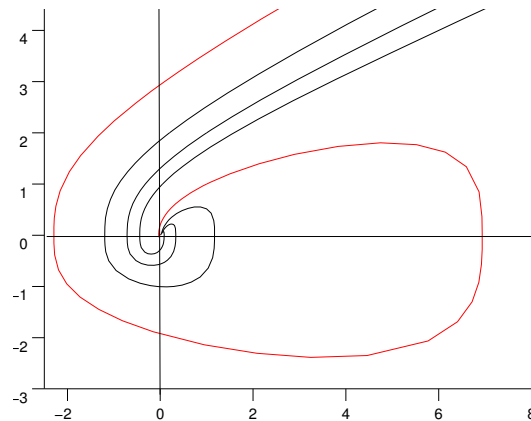


FIG. 2. Phase portrait around the origin for  $c < 1$ . Experiment for  $p = 3$ ,  $N = 1$ ,  $c = 0.9$ .

- LEMMA 3.1 (i) For any  $c \in (0, 1)$  and  $K \geq 0$ , there exists a unique travelling wave solution  $\tilde{f}_{c,K}(z) = \tilde{f}_{c,K}(x - c\tau)$  of (2.3) in dimension  $N = 1$ , having an interface at  $z = K$  (that is,  $\tilde{f}_{c,K}(z) = 0$  for  $z \geq K$ ) and moving with speed  $c$ . In addition,  $\tilde{f}_{c,K}(z) = \tilde{f}_{c,0}(z - K)$  for  $z \in \mathbb{R}$ .
- (ii) For  $c = 1$  and for any  $K \geq 0$ , there exists a unique nonnegative travelling wave  $f_{1,K}(z) = f_{1,K}(x - \tau)$  of (2.3) in dimension  $N = 1$  with interface at  $z = K$ , having the explicit formula

$$f_{1,K}(x - \tau) = \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (K + \tau - x)_+^{(p-1)/(p-2)}. \tag{3.6}$$

Here again,  $f_{1,K}(z) = f_{1,0}(z - K)$  for  $z \in \mathbb{R}$ .

- (iii) For any  $c > 1$  and  $K \geq 0$ , there exists a unique travelling wave solution  $f_{c,K} = f_{c,K}(x - c\tau)$  of (2.3) in dimension  $N = 1$  with interface at  $z = K$  and moving with speed  $c$ . Moreover,  $f_{c,K}$  is nonnegative and nonincreasing, and  $f_{c,K}(z) = f_{c,0}(z - K)$  for  $z \in \mathbb{R}$ .

*Compactly supported subsolutions for  $0 < c < 1$ .* We are looking for nonnegative and compactly supported subsolutions travelling with any speed  $0 < c < 1$ . These subsolutions are constructed in the following way: from the analysis above, we know that, given  $c \in (0, 1)$  and  $K \geq 0$ , there are two points  $z_{c,K} \in (-\infty, K)$  and  $\tilde{z}_{c,K} \in (z_{c,K}, K)$  such that

$$z_{c,K} := \inf \{z \in (-\infty, K) : \tilde{f}_{c,K} > 0 \text{ in } (z, K)\} > -\infty,$$

and

$$\tilde{f}'_{c,K} > 0 \text{ in } (z_{c,K}, \tilde{z}_{c,K}) \text{ and } \tilde{f}'_{c,K} < 0 \text{ in } (\tilde{z}_{c,K}, K).$$

We then define

$$f_{c,K}(z) = \begin{cases} \tilde{f}_{c,K}(z) & \text{for } z_{c,K} \leq z \leq K, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.7}$$

In other words, we consider the positive hump of the graph of  $\tilde{f}_{c,K}$  located between its last change of sign and the interface. It is immediate to check that  $f_{c,K}$  is a compactly supported subsolution to (2.3) in dimension  $N = 1$ , and that it has an increasing part until reaching its maximum at  $\tilde{z}_{c,K}$ , and then decreases to the interface point  $K$ . The notation  $f_{c,K}$  will designate these subsolutions if  $0 < c < 1$  and the solutions to (2.3) in dimension  $N = 1$  given by Lemma 3.1 if  $c \geq 1$ .

### 3.2 Construction of subsolutions in dimension $N \geq 1$

We turn to equation (2.3) posed in dimension  $N \geq 1$  for which we aim at constructing some special subsolutions having an interface that moves out in all directions with a given velocity  $c < 1$ . The construction is based on the travelling waves  $f_{c,K}$  identified in the previous subsection. The first attempt is to try the form  $V(\tau, x) = f_{c,K}(|x| - c\tau)$ ,  $c \in (0, 1)$ , which satisfies

$$\begin{aligned} & \partial_\tau V - \Delta_p V + |\nabla V|^{p-1} - V \\ &= -cf'_{c,K} - (|f'_{c,K}|^{p-2} f'_{c,K})' + |f'_{c,K}|^{p-1} - f_{c,K} - \frac{N-1}{|x|} |f'_{c,K}|^{p-2} f'_{c,K} \\ &\leq -\frac{N-1}{|x|} |f'_{c,K}|^{p-2} f'_{c,K}. \end{aligned}$$

Thus,  $V$  is a subsolution of (2.3) in the region of  $Q$  where  $f'_{c,K} \geq 0$ . We therefore have to modify the profile in the decreasing part of  $f_{c,K}$  and we proceed as follows.

*Travelling wave solutions to a modified equation in dimension  $N = 1$ .* For  $\alpha \in (0, 1/2)$ , we consider the following perturbation of (2.3):

$$\partial_\tau \zeta - \partial_x (|\partial_x \zeta|^{p-2} \partial_x \zeta) + |\partial_x \zeta|^{p-1} - \alpha |\partial_x \zeta|^{p-2} \partial_x \zeta - \zeta = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (3.8)$$

and look for travelling wave solutions  $\zeta(\tau, x) = g(x - c\tau)$ . Then  $g$  solves

$$-cg' - (|g'|^{p-2} g')' + |g'|^{p-1} - \alpha |g'|^{p-2} g' - g = 0. \quad (3.9)$$

The phase-plane analysis for (3.9) is similar to that of (3.2), with the difference that an extra term  $-\alpha |V|^{p-2} V$  appears on the right-hand side of the second equation in (3.3). This is only reflected in the analysis at infinity, where the point  $Q_1$  changes into  $(0, 1/(1 + \alpha))$  and the explicit separatrix is obtained for  $c = 1/(1 + \alpha) < 1$ . In particular, we have the following analogue of Lemma 3.1(i).

**LEMMA 3.2** For any  $\alpha > 0$  sufficiently small,  $c \in (0, 1/(1 + \alpha))$  and  $K \geq 0$ , there exists a unique travelling wave solution  $g_{c,K,\alpha}(z) = g_{c,K,\alpha}(x - c\tau)$  of (3.8) having an interface at  $z = K$  and moving with speed  $c$ . In addition,  $g_{c,K,\alpha}(z) = g_{c,0,\alpha}(z - K)$  for  $z \in \mathbb{R}$  and there are two points  $z_{c,K,\alpha} \in (-\infty, K)$  and  $\tilde{z}_{c,K,\alpha} \in (z_{c,K,\alpha}, K)$  such that

$$z_{c,K,\alpha} := \inf \{z \in (-\infty, K) : g_{c,K,\alpha} > 0 \text{ in } (z, K)\} > -\infty,$$

and

$$g'_{c,K,\alpha} > 0 \text{ in } (z_{c,K,\alpha}, \tilde{z}_{c,K,\alpha}) \quad \text{and} \quad g'_{c,K,\alpha} < 0 \text{ in } (\tilde{z}_{c,K,\alpha}, K).$$

Setting

$$M_{c,\alpha} := \sup_{z \in [z_{c,0,\alpha}, 0]} g_{c,0,\alpha}(z),$$

we notice that

$$z_{c,K,\alpha} = z_{c,0,\alpha} + K, \quad \tilde{z}_{c,K,\alpha} = \tilde{z}_{c,0,\alpha} + K, \quad \sup_{z \in [z_{c,K,\alpha}, K]} g_{c,K,\alpha}(z) = M_{c,\alpha}. \quad (3.10)$$

If we now put  $V(\tau, x) = g_{c,K,\alpha}(|x| - c\tau)$ , we calculate that

$$\partial_\tau V - \Delta_p V + |\nabla V|^{p-1} - V = \left( \alpha - \frac{N-1}{|x|} \right) (|g'_{c,K,\alpha}|^{p-2} g'_{c,K,\alpha})(|x| - c\tau),$$

and it is a subsolution where  $g'_{c,K,\alpha} \leq 0$  and  $\alpha \geq (N-1)/|x|$ . Matching these two conditions turns out to be possible as we show now.

Fix  $c \in (1/2, 1)$  and  $\alpha_c := (1 - c)/(1 + c)$  and define

$$\tau_0(c) := \max\{2(N-1)/\alpha_c - 2\tilde{z}_{c,0,\alpha_c}, -\tilde{z}_{c,0,\alpha_c}/c\} > 2(N-1)/\alpha_c, \quad (3.11)$$

the point  $\tilde{z}_{c,0,\alpha_c} \in (-\infty, 0)$  being defined in Lemma 3.2. Then  $c < 1/(1 + \alpha_c)$  and, for  $K \geq 0$ ,  $\tau \geq \tau_0(c)$ , and  $|x| \geq \tilde{z}_{c,K,\alpha_c} + c\tau = \tilde{z}_{c,0,\alpha_c} + K + c\tau$ , we have

$$\frac{N-1}{|x|} \leq \frac{N-1}{\tilde{z}_{c,0,\alpha_c} + c\tau_0(c)} \leq \frac{2(N-1)}{2\tilde{z}_{c,0,\alpha_c} + \tau_0(c)} \leq \alpha_c,$$

and

$$\begin{aligned} g'_{c,K,\alpha_c}(|x| - c\tau) &< 0 && \text{if } \tilde{z}_{c,K,\alpha_c} + c\tau \leq |x| < K + c\tau, \\ g'_{c,K,\alpha_c}(|x| - c\tau) &= 0 && \text{if } K + c\tau \leq |x|. \end{aligned}$$

Consequently, for  $c \in (1/2, 1)$ ,  $\alpha_c = (1 - c)/(1 + c)$ , and  $K > 0$ , the function  $V$  defined by  $V(\tau, x) = g_{c,K,\alpha_c}(|x| - c\tau)$  is a subsolution to (2.3) for  $\tau \geq \tau_0(c)$  and  $|x| \geq \tilde{z}_{c,K,\alpha_c} + c\tau$ . Observing that any positive constant is a subsolution to (2.3), we construct a compactly supported subsolution  $v_{c,K}$  to (2.3) by setting

$$v_{c,K}(\tau, x) := \begin{cases} M_{c,\alpha_c} & \text{if } 0 \leq |x| < \tilde{z}_{c,K,\alpha_c} + c\tau, \\ g_{c,K,\alpha_c}(|x| - c\tau) & \text{if } |x| > \tilde{z}_{c,K,\alpha_c} + c\tau, \end{cases} \tag{3.12}$$

for  $\tau \geq \tau_0(c)$ . It is easy to check that the function  $v_{c,K}$  is a subsolution to (2.3) in  $[\tau_0(c), \infty) \times \mathbb{R}^N$ . It will be used for comparison from below, as indicated in the next subsection.

### 3.3 Proof of Theorem 1.2

We conclude the proof of Theorem 1.2 by a comparison argument, using the subsolutions and supersolutions constructed in the previous subsections. First, we identify a class of solutions of (2.3) that is representative for the general solutions.

We say that a function  $V = V(\tau, x)$  is *radially nonincreasing* if  $V(\tau, \cdot)$  is radially symmetric for all  $\tau$ , and it is nonincreasing in the radial variable  $r := |x|$ . For example, the subsolutions  $v_{c,K}$  are radially nonincreasing. The next results show that the class of radially nonincreasing solutions of (2.3) is sufficient for our aims.

**LEMMA 3.3** Let  $u_0 = u_0(r)$  be a radially nonincreasing function satisfying (1.3). Then the solution  $v$  of (2.3) with initial condition  $u_0$  is also radially nonincreasing.

*Proof.* The radial symmetry of the solution  $v$  follows from the invariance of the equation (2.3) with respect to rotations. We now write the equation satisfied by  $\xi = \partial_r v$ , obtained by differentiating (2.3) with respect to  $r$ :

$$\partial_t \xi - \partial_r^2(|\xi|^{p-2}\xi) - \frac{N-1}{r} \partial_r(|\xi|^{p-2}\xi) + \frac{N-1}{r^2} |\xi|^{p-2}\xi + (p-1)|\xi|^{p-3}\xi \partial_r \xi - \xi = 0,$$

which is a parabolic equation (of porous medium type) and satisfies a maximum principle. Since 0 is a solution to the above equation, the derivative  $\xi = \partial_r v$  remains nonpositive if it is initially nonpositive, and it follows that  $v$  is radially nonincreasing.  $\square$

We are now in a position to end the proof of Theorem 1.2 for radially nonincreasing initial data. More precisely, we have the following upper and lower bounds for  $\varrho(\tau)$  defined in (3.1), the support of  $v(\tau)$  being included in the ball  $B(0, \varrho(\tau))$ .

**LEMMA 3.4** Let  $u_0 = u_0(r)$  be a radially nonincreasing function satisfying (1.3) and denote by  $v$  the solution of (2.3) with initial condition  $u_0$ . For any  $c \in (1/2, 1)$ , there exists  $\tau_1(c) > 0$  such that, for any  $\tau \geq \tau_1(c)$ ,

$$1 + c(\tau - \tau_1(c)) \leq \varrho(\tau) \leq R_0 + \frac{p-1}{p-2} \|u_0\|_\infty^{(p-2)/(p-1)} + \tau. \tag{3.13}$$

In particular,  $\varrho(\tau)/\tau \rightarrow 1$  as  $\tau \rightarrow \infty$ .

*Proof.* The upper bound follows by comparison with the explicit travelling wave solutions (3.6). More precisely, we define

$$R_1 := R_0 + \frac{p-1}{p-2} \|u_0\|_\infty^{(p-2)/(p-1)} \tag{3.14}$$

and consider the function  $\bar{v}(\tau, x) = f_{1,R_1}(x_1 - \tau)$ , which is a solution of (2.3) by Lemma 3.1. If  $x = (x_i)_{1 \leq i \leq N} \in \mathbb{R}^N$  is such that  $x_1 \geq R_0$ , then  $|x| \geq R_0$  and  $u_0(x) = 0 \leq \bar{v}(0, x)$ , while if  $x_1 \leq R_0$ ,

$$\begin{aligned} u_0(x) &\leq \|u_0\|_\infty \leq \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (R_1 - R_0)^{(p-1)/(p-2)} \\ &\leq \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (R_1 - x_1)^{(p-1)/(p-2)} = \bar{v}(0, x). \end{aligned}$$

The comparison principle then entails that  $v(\tau, x) \leq \bar{v}(\tau, x)$  for  $(\tau, x) \in [0, \infty) \times \mathbb{R}^N$ , from which we conclude that  $\mathcal{P}_v(\tau) \subseteq \{x \in \mathbb{R}^N : x_1 \leq R_1 + \tau\}$ . Owing to the rotational invariance of (2.3), we actually have  $\mathcal{P}_v(\tau) \subseteq \{x \in \mathbb{R}^N : \langle x, \omega \rangle \leq R_1 + \tau\}$  for every  $\omega \in \mathbb{S}^{N-1}$  and  $\tau \geq 0$ , and thus

$$\mathcal{P}_v(\tau) \subseteq B(0, R_1 + \tau). \tag{3.15}$$

The lower bound follows from comparison with the subsolutions constructed in (3.12). Fix  $c \in (1/2, 1)$  and put  $r_1 := 1 + c\tau_0(c)$ ,  $\tau_0(c)$  being defined by (3.11). Since  $v(\tau)$  is radially nonincreasing for all  $\tau \geq 0$  by Lemma 3.3, we infer from Proposition 2.1 that, for  $x \in B(0, r_1)$  and  $\tau \geq T_{r_1}$ ,

$$v(\tau, x) \geq v(\tau, r_1 x/|x|) \geq \varepsilon_{r_1} \tau^{(p-1)/(p-2)}.$$

Define

$$\tau_1(c) := \max\{\tau_0(c), T_{r_1}, (M_{c,(1-c)/(1+c)}/\varepsilon_{r_1})^{(p-2)/(p-1)}\}$$

so that the previous inequality and the properties of  $v_{c,1}$  defined in (3.12) guarantee that

$$v(\tau_1(c), x) \geq M_{c,(1-c)/(1+c)} \geq v_{c,1}(\tau_0(c), x), \quad x \in B(0, r_1).$$

Since  $v_{c,1}(\tau_0(c), x) = 0$  for  $x \notin B(0, r_1)$ , we also have  $v(\tau_1(c), x) \geq v_{c,1}(\tau_0(c), x)$  for  $x \notin B(0, r_1)$ . Recalling that  $v_{c,1}$  is a subsolution to (2.3) in  $(\tau_0(c), \infty) \times \mathbb{R}^N$ , we infer from the comparison principle that

$$v(\tau + \tau_1(c), x) \geq v_{c,1}(\tau + \tau_0(c), x), \quad (\tau, x) \in Q. \tag{3.16}$$

Consequently,  $v(\tau + \tau_1(c), x) > 0$  if  $x \in B(0, r_1 + c\tau)$ , whence

$$B(0, 1 + c(\tau + \tau_0(c) - \tau_1(c))) \subset \mathcal{P}_v(t), \quad \tau \geq \tau_1(c). \tag{3.17}$$

This readily implies that

$$\varrho(\tau) \geq 1 + c(\tau + \tau_0(c) - \tau_1(c)) \geq 1 + c(\tau - \tau_1(c)), \quad \tau \geq \tau_1(c).$$

In particular, we deduce from (3.15) and (3.17) that

$$\liminf_{\tau \rightarrow \infty} \frac{\varrho(\tau)}{\tau} \geq c \quad \text{for any } c \in (1/2, 1) \quad \text{and} \quad \limsup_{\tau \rightarrow \infty} \frac{\varrho(\tau)}{\tau} \leq 1,$$

which implies that  $\varrho(\tau)/\tau \rightarrow 1$  as  $\tau \rightarrow \infty$ . □

Reverting the rescaling and coming back to the notation with  $t = (e^{(p-2)\tau} - 1)/(p - 2)$  and  $\gamma(t) = \varrho(\tau)$ , we find the result of Theorem 1.2 for radially nonincreasing initial data. The extension to arbitrary initial data satisfying (1.3) is performed in Section 5. Moreover, we notice that the speed is the same in any direction  $\omega \in \mathbb{S}^{N-1}$ , as stated.

**4. Proof of Theorem 1.1**

4.1 *Scaling variables II*

According to Proposition 2.1, as  $\tau \rightarrow \infty$  the solution  $v$  to (2.3), (2.4) expands in space and grows unboundedly in time. In order to take into account such phenomena, we next introduce a further scaling of the space variable

$$y := \frac{x}{1 + \tau}, \tag{4.1}$$

together with the new unknown function  $w = w(\tau, y)$  defined by

$$v(\tau, x) = (1 + \tau)^{(p-1)/(p-2)} w\left(\tau, \frac{x}{1 + \tau}\right), \quad (\tau, x) \in [0, \infty) \times \mathbb{R}^N. \tag{4.2}$$

It follows from (2.3) and (2.4) that  $w$  solves

$$\partial_\tau w - \frac{1}{1 + \tau} \left( \Delta_p w + y \cdot \nabla w - \frac{p-1}{p-2} w \right) + |\nabla w|^{p-1} - w = 0, \quad (\tau, y) \in Q, \tag{4.3}$$

with the same initial condition

$$w(0) = u_0, \quad y \in \mathbb{R}^N. \tag{4.4}$$

Throughout this section, besides (1.3), we assume that  $u_0$  is radially nonincreasing. In particular,  $u_0(0) > 0$ . We gather several properties of  $w$  in the next lemma.

LEMMA 4.1 There is a positive constant  $C_4$  depending only on  $p, N$ , and  $u_0$  such that

$$\|w(\tau)\|_1 + \|w(\tau)\|_\infty + \|\nabla w(\tau)\|_\infty \leq C_4, \quad \tau \geq 0, \tag{4.5}$$

$$w(\tau, y) \geq \frac{1}{C_4} (r_*^2 - |y|^2)_+^{(p-1)/(p-2)}, \quad (\tau, y) \in Q, \tag{4.6}$$

the radius  $r_*$  being defined in Corollary 2.1. Moreover,

$$\mathcal{P}_w(\tau) := \{y \in \mathbb{R}^N : w(\tau, y) > 0\} \subseteq B\left(0, 1 + \frac{R_1}{1 + \tau}\right) \tag{4.7}$$

for  $\tau \geq 0$  where  $R_1$  is defined by (3.14). In addition, for any  $c \in (1/2, 1)$ ,

$$B\left(0, c - \frac{\tau_1(c)}{1 + \tau}\right) \subset \mathcal{P}_w(\tau) \quad \text{for } \tau \geq \tau_1(c), \tag{4.8}$$

the time  $\tau_1(c) > 0$  being defined in Lemma 3.4.

*Proof.* The estimates (4.5) and (4.6) readily follow from (2.5) and (2.10), while (4.7) is a consequence of (3.15). The assertion about the ball  $B(0, c - \tau_1(c)/(1 + \tau))$  follows from (3.17).  $\square$



At this point, (4.3) indicates that  $w(\tau)$  behaves as  $\tau \rightarrow \infty$  as the solution  $\tilde{w}$  to the Hamilton–Jacobi equation  $\partial_\tau \tilde{w} + |\nabla \tilde{w}|^{p-1} - \tilde{w} = 0$  in  $Q$  which is known to converge to a stationary solution uniquely determined by the limit of the support of  $\tilde{w}(\tau)$  as  $\tau \rightarrow \infty$  (see, e.g., [15, Theorem A.2]). As an intermediate step, we thus have to identify the limit of the support of  $w(\tau)$  as  $\tau \rightarrow \infty$ . Thanks to (4.7), we already know that it is included in  $B(0, 1)$  but the information in (4.8) is yet too weak to exclude the vanishing of  $w(\tau)$  outside a ball of radius smaller than one. To complete the proof of Theorem 1.1 for radially nonincreasing initial data, we show first that the asymptotic limit is supported exactly in the ball  $B(0, 1)$ . Then we use a viscosity technique, the same that has been used in the previous paper [16] to establish the convergence to the expected stationary solution.

4.2 Proof of Theorem 1.1:  $N = 1$

We first consider the one-dimensional case  $N = 1$  and divide the proof into several technical steps.

*Step 1. A special family of subsolutions.* Given  $c \in (1/2, 1)$ , we have

$$v(\tau, x) \geq v_{c,1}(\tau + \tau_0(c) - \tau_1(c), x), \quad (\tau, x) \in [\tau_1(c), \infty) \times \mathbb{R},$$

by (3.16), the times  $\tau_0(c)$  and  $\tau_1(c)$  being defined in (3.11) and Lemma 3.4, respectively. Then

$$w(\tau, y) \geq w_c(\tau, y) := \frac{1}{(1 + \tau)^{(p-1)/(p-2)}} v_{c,1}(\tau + \tau_0(c) - \tau_1(c), y(1 + \tau)) \quad (4.9)$$

for  $(\tau, y) \in [\tau_1(c), \infty) \times \mathbb{R}$ .

*Step 2. An explicit family of supersolutions.* Let us introduce the following family of functions:

$$F_R(\tau, y) = \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{\tau+R}{\tau+1} - |y|\right)_+^{(p-1)/(p-2)}, \quad (\tau, y) \in Q. \quad (4.10)$$

We easily see by direct calculation that  $F_R$  is a classical solution of (4.3) for  $y \neq 0$ , and for all parameter values  $R \geq 0$ . However, near  $y = 0$ , it is only a supersolution both in the weak and the viscosity sense. The latter is straightforward to verify using the definition of viscosity subsolutions and supersolutions with jets, as in the classical survey [10]. Let us mention at this point that these functions can be used in a comparison argument to give an alternative proof of (4.7).

**REMARK 4.1** This family of functions arises naturally if we think about asymptotics. Indeed, as already mentioned, we formally expect that the asymptotic profiles of (4.3) should be given by solutions of the stationary Hamilton–Jacobi equation

$$|\nabla \tilde{w}|^{p-1} - \tilde{w} = 0, \quad (4.11)$$

supported in some ball  $B(0, R)$ , that is,

$$H_R(y) := \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (R - |y|)_+^{(p-1)/(p-2)}, \quad y \in \mathbb{R}.$$

Making then the “ansatz” that, for large times, the solution of (4.3) should behave in a similar way to its limit, we write

$$w(\tau, y) \sim \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} (C(\tau) - |y|)_+^{(p-1)/(p-2)}.$$

Integrating the resulting ordinary differential equation for  $C(\tau)$ , we arrive at the family of explicit exact profiles  $F_R$  given by (4.10).

*Step 3. Constructing suitable subsolutions.* We now face the problem of finding suitable subsolutions with similar behaviour. Since the  $F_R$ 's are classical solutions to (4.3) except at  $y = 0$ , we expect to be able to construct also a family of subsolutions based on them. To this end, we consider the ‘‘damped’’ family  $F_{R,\vartheta,\beta}$  defined by

$$F_{R,\vartheta,\beta}(\tau, y) := \vartheta \left( \frac{p-2}{p-1} \right)^{(p-1)/(p-2)} \left( \frac{\beta(\tau+R)}{\tau+1} - |y| \right)_+^{(p-1)/(p-2)}, \quad (\tau, y) \in Q, \quad (4.12)$$

for parameters  $R \in (0, 1)$ ,  $\vartheta \in (0, 1]$ , and  $\beta \in (1/2, 1]$ . Observe that, since  $(p-1)/(p-2) > 1$ ,  $F_{R,\vartheta,\beta}$  and  $|\nabla F_{R,\vartheta,\beta}|^{p-2} \nabla F_{R,\vartheta,\beta}$  both belong to  $C^1([0, \infty) \times (\mathbb{R} \setminus \{0\}))$ . For  $\vartheta \in (0, 1)$ ,  $\beta \in (1/2, 1]$ ,  $\tau > 0$  and  $y \neq 0$ , we calculate

$$\begin{aligned} \partial_\tau F_{R,\vartheta,\beta} &- \frac{1}{1+\tau} \left( \Delta_p F_{R,\vartheta,\beta} + y \cdot \nabla F_{R,\vartheta,\beta} - \frac{p-1}{p-2} F_{R,\vartheta,\beta} \right) + |\nabla F_{R,\vartheta,\beta}|^{p-1} - F_{R,\vartheta,\beta} \\ &= \vartheta \beta \frac{1-R}{(1+\tau)^2} F_{R,1,\beta}^{1/(p-1)} - \frac{\vartheta}{1+\tau} \left( \vartheta^{p-2} - \frac{\beta(\tau+R)}{\tau+1} \right) F_{R,1,\beta}^{1/(p-1)} - \vartheta(1-\vartheta^{p-2}) F_{R,1,\beta} \\ &= \vartheta \left( \frac{\beta - \vartheta^{p-2}}{1+\tau} - (1-\vartheta^{p-2}) F_{R,1,\beta}^{(p-2)/(p-1)} \right) F_{R,1,\beta}^{1/(p-1)} \\ &\leq \vartheta(1-\vartheta^{p-2}) F_{R,1,\beta}^{1/(p-1)} \left[ \frac{1}{1+\tau} - \frac{p-2}{p-1} \left( \frac{\beta(\tau+R)}{\tau+1} - |y| \right) \right]. \end{aligned}$$

Analyzing the sign of the last expression and taking into account that  $\vartheta \in (0, 1)$ , we find that  $F_{R,\vartheta,\beta}$  has the following properties:

$$\begin{aligned} &F_{R,\vartheta,\beta} \text{ is a classical subsolution to (4.3) in} \\ &\{(\tau, y) \in Q : \tau \geq \tau_2(R, \beta), 0 < |y| \leq K_{R,\beta}(\tau)\} \end{aligned} \quad (4.13)$$

with

$$\tau_2(R, \beta) := \frac{p-1}{\beta(p-2)} - R \quad \text{and} \quad K_{R,\beta}(\tau) := \frac{\beta(\tau+R)}{\tau+1} - \frac{p-1}{p-2} \frac{1}{\tau+1}, \quad (4.14)$$

and

$$F_{R,\vartheta,\beta} \text{ vanishes for } |y| \geq \frac{\beta(\tau+R)}{\tau+1} \text{ and } \tau \geq 0. \quad (4.15)$$

Let us notice here that both the edge of the support of  $F_{R,\vartheta,\beta}$  and the constant  $K_{R,\beta}(\tau)$ , where the behaviour changes, do not depend on  $\vartheta$ . While the two properties (4.13) and (4.15) are suitable for our purpose, the function  $F_{R,\vartheta,\beta}$  does not behave in a suitable way near  $y = 0$  (where it is a viscosity supersolution) and in an asymptotically small region near the edge of its support (where it is a classical supersolution). However, we already have a positive bound from below for  $w$  in a small neighbourhood of  $y = 0$  by (4.6), which allows us to remedy the first bad property of  $F_{R,\vartheta,\beta}$ . More precisely, we infer from (4.6) that

$$w(\tau, y) \geq C_5 := \frac{1}{C_4} \left( \frac{3r_*^2}{4} \right)^{(p-1)/(p-2)} > 0, \quad (\tau, y) \in [0, \infty) \times B(0, r_*/2),$$

whence

$$w(\tau, y) \geq \vartheta \geq F_{R,\vartheta,\beta}(\tau, y), \quad (\tau, y) \in [0, \infty) \times B(0, r_*/2), \quad (4.16)$$

provided that

$$0 < \vartheta < \min\{1, C_5\}. \quad (4.17)$$

Consider next

$$\tau \geq \tau_2(R, \beta) \quad \text{and} \quad K_{R, \beta}(\tau) \leq |y| \leq \frac{\beta(\tau + R)}{\tau + 1}.$$

Then

$$\begin{aligned} F_{R, \vartheta, \beta}(\tau, y) &\leq \vartheta \left( \frac{p-2}{p-1} \right)^{(p-1)/(p-2)} \left( \frac{p-1}{p-2} \frac{1}{1+\tau} \right)^{(p-1)/(p-2)} \\ &= \frac{\vartheta}{(1+\tau)^{(p-1)/(p-2)}}. \end{aligned} \quad (4.18)$$

Now, if  $c \in (\beta, 1)$ , we have

$$|y|(1+\tau) \leq \beta(\tau + R) \leq \tilde{z}_{c, 1, (1-c)/(1+c)} + c(\tau + \tau_0(c) - \tau_1(c))$$

as soon as

$$\tau \geq \tau_3(c, R, \beta) := \frac{\beta R + c(\tau_1(c) - \tau_0(c)) - \tilde{z}_{c, 1, (1-c)/(1+c)}}{c - \beta}. \quad (4.19)$$

In that case,

$$w_c(\tau, y) = \frac{1}{(1+\tau)^{(p-1)/(p-2)}} v_{c, 1}(\tau + \tau_0(c) - \tau_1(c), y(1+\tau)) = \frac{M_{c, (1-c)/(1+c)}}{(1+\tau)^{(p-1)/(p-2)}}$$

according to the properties (3.12) of  $v_{c, 1}$ . Recalling (4.9) and (4.18) we realize that

$$F_{R, \vartheta, \beta}(\tau, y) \leq w_c(\tau, y) \leq w(\tau, y), \quad K_{R, \beta}(\tau) \leq |y| \leq \frac{\beta(\tau + R)}{\tau + 1}, \quad (4.20)$$

provided

$$c \in (\beta, 1), \quad \vartheta < \min\{1, M_{c, (1-c)/(1+c)}\}, \quad \tau \geq \max\{\tau_1(c), \tau_2(R, \beta), \tau_3(c, R, \beta)\}. \quad (4.21)$$

After this preparation, we are in a position to establish a positive lower bound for  $w$  on the ball  $B(0, 1 - \varepsilon)$  for any  $\varepsilon \in (0, 1/4)$ . Indeed, we fix  $\varepsilon \in (0, 1/4)$ , choose  $c = 1 - \varepsilon$ ,  $R = \beta = 1 - 2\varepsilon$ , and define

$$\tau_4(\varepsilon) := \max\{\tau_1(1 - \varepsilon)/\varepsilon, \tau_2(1 - 2\varepsilon, 1 - 2\varepsilon), \tau_3(1 - \varepsilon, 1 - 2\varepsilon, 1 - 2\varepsilon)\}.$$

As  $\tau_4(\varepsilon) > \tau_1(1 - \varepsilon)/\varepsilon$ , (4.8) guarantees that  $B(0, 1 - 2\varepsilon) \subset \mathcal{P}_w(\tau_4(\varepsilon))$  and there is thus  $m_\varepsilon \in (0, 1)$  such that

$$w(\tau_4(\varepsilon), y) \geq m_\varepsilon, \quad y \in B(0, 1 - 2\varepsilon). \quad (4.22)$$

Now, for  $\vartheta \in (0, 1)$  satisfying

$$0 < \vartheta < \min\{m_\varepsilon, C_5, M_{1-\varepsilon, \varepsilon/(2-\varepsilon)}\} \quad (4.23)$$

we infer from (4.14), (4.16), (4.17), and (4.20)–(4.22) that

$$F_{1-2\varepsilon, \vartheta, 1-2\varepsilon}(\tau, y) \leq w(\tau, y), \quad |y| \in \{r_*/2, K_{1-2\varepsilon, 1-2\varepsilon}(\tau)\}, \quad \tau \geq \tau_4(\varepsilon),$$

and

$$F_{1-2\varepsilon, \vartheta, 1-2\varepsilon}(\tau_4(\varepsilon), y) \leq \vartheta \leq m_\varepsilon \leq w(\tau_4(\varepsilon), y), \quad r_*/2 \leq |y| \leq K_{1-2\varepsilon, 1-2\varepsilon}(\tau_4(\varepsilon)) \leq 1 - 2\varepsilon.$$

It then follows from (4.3), (4.13), and the comparison principle that

$$F_{1-2\varepsilon, \vartheta, 1-2\varepsilon}(\tau, y) \leq w(\tau, y), \quad r_*/2 \leq |y| \leq K_{1-2\varepsilon, 1-2\varepsilon}(\tau), \quad \tau \geq \tau_4(\varepsilon).$$

Recalling (4.15), (4.16), and (4.20), we have thus established that

$$F_{1-2\varepsilon, \vartheta, 1-2\varepsilon}(\tau, y) \leq w(\tau, y), \quad \tau \in [\tau_4(\varepsilon), \infty) \times \mathbb{R}, \tag{4.24}$$

for all  $\vartheta \in (0, 1)$  satisfying (4.23).

*Step 4. Positive lower bound.* For  $\varepsilon \in (0, 1/4)$ , fix  $\vartheta_\varepsilon \in (0, 1)$  satisfying (4.23). According to (4.24), we have, for  $\tau \geq \tau_4(\varepsilon) + 1$  and  $y \in B(0, 1 - 3\varepsilon)$ ,

$$\begin{aligned} w(\tau, y) &\geq \vartheta_\varepsilon \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{(1-2\varepsilon)(\tau+1-2\varepsilon)}{\tau+1} - |y|\right)_+^{(p-1)/(p-2)} \\ &\geq \vartheta_\varepsilon \left(\frac{p-2}{p-1}\right)^{(p-1)/(p-2)} \left(\frac{\varepsilon(\tau-1+4\varepsilon)}{\tau+1}\right)_+^{(p-1)/(p-2)} \\ &\geq \mu_\varepsilon := \vartheta_\varepsilon \left(\frac{2(p-2)\varepsilon^2}{p-1}\right)^{(p-1)/(p-2)} > 0. \end{aligned}$$

We have thus proved that, for all  $\varepsilon \in (0, 1/4)$ , there are  $\mu_\varepsilon > 0$  and  $\tau_5(\varepsilon) := \tau_4(\varepsilon) + 1$  such that

$$0 < \mu_\varepsilon \leq w(\tau, y), \quad (\tau, y) \in [\tau_5(\varepsilon), \infty) \times B(0, 1 - 3\varepsilon). \tag{4.25}$$

*Step 5. Convergence. Viscosity argument.* To complete the proof, we use an argument relying on the theory of viscosity solutions in a similar way to [16] for the subcritical case of (1.1) with  $q \in (1, p - 1)$ . We thus employ the technique of half-relaxed limits [6] in the same fashion as in [19, Section 3] and [16]. To this end, we pass to the logarithmic time and introduce the new variable  $s := \log(1 + \tau)$  along with the new unknown function

$$w(\tau, y) = \omega(\log(1 + \tau), y), \quad (\tau, y) \in [0, \infty) \times \mathbb{R}.$$

Then  $\partial_\tau w(\tau, y) = e^{-s} \partial_s \omega(s, y)$  and it follows from (4.3) and (4.4) that  $\omega$  solves

$$e^{-s} \left( \partial_s \omega - \Delta_p \omega - y \cdot \nabla \omega + \frac{p-1}{p-2} \omega \right) + |\nabla \omega|^{p-1} - \omega = 0, \quad (s, y) \in Q, \tag{4.26}$$

with initial condition  $\omega(0) = u_0$ . We readily infer from Lemma 4.1 that

$$\|\omega(s)\|_1 + \|\omega(s)\|_\infty + \|\nabla \omega(s)\|_\infty \leq C_4, \quad s \geq 0, \tag{4.27}$$

$$\omega(s, y) = 0 \quad \text{for } s \geq 0 \quad \text{and} \quad |y| \geq 1 + R_1 e^{-s}. \tag{4.28}$$

We next introduce the half-relaxed limits

$$\omega_*(y) := \liminf_{(\sigma, z, \lambda) \rightarrow (s, y, \infty)} \omega(\lambda + \sigma, z) \quad \text{and} \quad \omega^*(y) := \limsup_{(\sigma, z, \lambda) \rightarrow (s, y, \infty)} \omega(\lambda + \sigma, z),$$

for  $(s, y) \in Q$ , which are well-defined according to the uniform bounds in (4.27) and indeed do not depend on  $s > 0$ . Then the definition of  $\omega_*$  and  $\omega^*$  clearly ensures that

$$0 \leq \omega_*(y) \leq \omega^*(y) \quad \text{for } y \in \mathbb{R}, \quad (4.29)$$

while the uniform bounds (4.27) and the Rademacher theorem warrant that  $\omega_*$  and  $\omega^*$  both belong to  $W^{1,\infty}(\mathbb{R})$ . Finally, by Proposition 6.1 applied to (4.26),  $\omega_*$  and  $\omega^*$  are a viscosity supersolution and a viscosity subsolution, respectively, to the Hamilton–Jacobi equation

$$H(\zeta, \nabla \zeta) := |\nabla \zeta|^{p-1} - \zeta = 0 \quad \text{in } \mathbb{R}. \quad (4.30)$$

Our aim now is to show that  $\omega_* \geq \omega^*$  in  $\mathbb{R}$  (which implies that  $\omega_* = \omega^*$  by (4.29)). Since  $\omega^*$  and  $\omega_*$  are a subsolution and a supersolution to (4.30), respectively, such an inequality would follow from a comparison principle which cannot be applied yet without further information on  $\omega^*$  and  $\omega_*$ . We actually need to prove the following two facts:

- (a)  $\omega_*(y) = \omega^*(y) = 0$  if  $|y| \geq 1$ ,
- (b)  $\omega^*(y) \geq \omega_*(y) > 0$  if  $y \in B(0, 1)$ ,

and then to follow the technique used in [16] to conclude that  $\omega_* = \omega^*$  and identify the limit.

To prove (a), take  $y \in \mathbb{R}$  with  $|y| > 1$ . We then deduce from (4.28) that there exists  $s_1(y) > 0$  such that  $\omega(s, y) = 0$  for  $s \geq s_1(y)$ . Pick sequences  $(\sigma_n)_{n \geq 1}$ ,  $(\lambda_n)_{n \geq 1}$ , and  $(z_n)_{n \geq 1}$  such that  $\sigma_n \rightarrow 0$ ,  $\lambda_n \rightarrow \infty$ ,  $z_n \rightarrow y$ , and  $\omega(\sigma_n + \lambda_n, z_n) \rightarrow \omega^*(y)$ . On the one hand, there exists  $n_1(y) > 0$  such that  $\sigma_n + \lambda_n > s_1(y)$  for any  $n \geq n_1(y)$ ; hence  $\omega(\sigma_n + \lambda_n, y) = 0$  for any  $n \geq n_1(y)$ . On the other hand, we can write

$$|\omega(\sigma_n + \lambda_n, z_n) - \omega(\sigma_n + \lambda_n, y)| \leq |y - z_n| \|\nabla \omega(\sigma_n + \lambda_n)\|_\infty \leq C_4 |y - z_n| \rightarrow 0,$$

hence  $\omega^*(y) = 0 = \omega_*(y)$  for any  $y \in \mathbb{R}$  with  $|y| > 1$ . In addition, since  $\omega^*$  and  $\omega_*$  are continuous, it follows that  $\omega^* = \omega_* = 0$  also for  $|y| = 1$ , proving (a).

To prove (b), take  $y \in B(0, 1)$ . Then there exists  $\varepsilon \in (0, 1/4)$  such that  $y \in B(0, 1 - 4\varepsilon)$ . Since  $1 - 3\varepsilon > 1 - 4\varepsilon$ , there is  $r_2(y) > 0$  such that  $B(y, r_2(y)) \subset B(0, 1 - 3\varepsilon)$  and we deduce from (4.25) that there exists  $s_2(\varepsilon) := \log(\tau_5(\varepsilon) + 1) > 0$  such that  $\omega(s, z) \geq \mu_\varepsilon$  for any  $s \geq s_2(\varepsilon)$  and  $z \in B(y, r_2(y))$ . We now pick sequences  $(\sigma_n)_{n \geq 1}$ ,  $(\lambda_n)_{n \geq 1}$  and  $(z_n)_{n \geq 1}$  such that  $\sigma_n \rightarrow 0$ ,  $\lambda_n \rightarrow \infty$ ,  $z_n \rightarrow y$ , and  $\omega(\sigma_n + \lambda_n, z_n) \rightarrow \omega_*(y)$ . Then there exists again  $n_2(y) > 0$  such that  $\sigma_n + \lambda_n > s_2(y)$  and  $z_n \in B(y, r_2(y))$  for any  $n \geq n_2(y)$ . Consequently,  $\omega(\sigma_n + \lambda_n, z_n) \geq \mu_\varepsilon$  for any  $n \geq n_2(y)$ . This readily implies that  $\omega^*(y) \geq \omega_*(y) \geq \mu_\varepsilon > 0$ , hence (b) is proved.

Following [16] we introduce

$$W_*(y) = \frac{p-1}{p-2} \omega_*(y)^{(p-2)/(p-1)}, \quad W^*(y) = \frac{p-1}{p-2} \omega^*(y)^{(p-2)/(p-1)}, \quad (4.31)$$

for any  $y \in B(0, 1)$ . From Proposition 6.2, it follows that  $W_*$  and  $W^*$  are respectively a viscosity supersolution and a viscosity subsolution of the eikonal equation

$$|\nabla \zeta| = 1 \quad \text{in } B(0, 1),$$

with boundary conditions  $W^*(y) = W_*(y) = 0$  for  $|y| = 1$ , and are both positive in  $B(0, 1)$ . Using the comparison principle of Ishii [13], we find that  $W^*(y) \leq W_*(y)$ , hence they should be equal by (4.29). It follows that  $\omega_* = \omega^* = W$  in  $B(0, 1)$ , where  $W$  is the unique viscosity solution to (1.13),

$$|\nabla W|^{p-1} - W = 0 \quad \text{in } B(0, 1), \quad W = 0 \quad \text{on } \partial B(0, 1),$$

which is positive in  $B(0, 1)$  and actually explicit and given by

$$W(x) := \left( \frac{p-2}{p-1} (1 - |x|)_+ \right)^{(p-1)/(p-2)},$$

as stated in Theorem 1.1. In addition, the equality  $\omega_* = \omega^*$  and (4.28) entail the convergence of  $\omega(s)$  as  $s \rightarrow \infty$  towards  $W$  in  $L^\infty(\mathbb{R})$  by Lemma 4.1 in [5] or Lemma V.1.9 in [4]. We end the proof by reverting the two scaling steps and arriving in this way at (1.10).  $\square$

4.3 Proof of Theorem 1.1:  $N \geq 2$

We now prove Theorem 1.1 for radially nonincreasing initial data in dimension  $N \geq 2$ . We follow the same steps as in dimension  $N = 1$ , and we only indicate the main differences. These are mainly due to the appearance of the new term

$$\frac{N-1}{r} |\partial_r w|^{p-2} \partial_r w, \quad r = |y|, \tag{4.32}$$

in the radial form of the  $p$ -Laplacian term. As we shall see, performing carefully the same steps as for dimension  $N = 1$ , we find that this term does not change anything in an essential way. We follow the same division into steps as in the case  $N = 1$ .

*Step 1.* Thanks to the construction in Section 3.2, this step is the same as in dimension  $N = 1$ .

*Step 2.* Due to the appearance of the extra term (4.32) in the radial form of (4.3), we check by direct calculation that, in dimension  $N \geq 2$ , the function  $F_R$  given by (4.10) is now a strict supersolution to (4.3) in  $Q$ . Indeed, for  $y \neq 0$ ,

$$\partial_\tau F_R - \frac{1}{1+\tau} \left( \Delta_p F_R + y \cdot \nabla F_R - \frac{p-1}{p-2} F_R \right) + |\nabla F_R|^{p-1} - F_R = \frac{N-1}{(1+\tau)|y|} F_R.$$

Moreover, its singularity at  $y = 0$  is now stronger. This seems to introduce a new difficulty, but we will see that it can be handled by the same perturbation techniques. Let us notice at this moment that  $F_R$  can be used for upper bounds in the same way as in the case  $N = 1$ , and that  $F_R$  still solves the limit Hamilton–Jacobi equation (4.11).

*Step 3.* In order to construct subsolutions starting from the family of functions  $F_R$ , we follow again the ideas of the case  $N = 1$ . The calculations will be different in some points. We again consider the damped family  $F_{R,\vartheta,\beta}$  defined in (4.12) for  $R \in (0, 1)$ ,  $\vartheta \in (0, 1)$ , and  $\beta \in (1/2, 1]$ . For  $y \neq 0$  we have

$$\begin{aligned} Y &:= \partial_\tau F_{R,\vartheta,\beta} - \frac{1}{1+\tau} \left( \Delta_p F_{R,\vartheta,\beta} + y \cdot \nabla F_{R,\vartheta,\beta} - \frac{p-1}{p-2} F_{R,\vartheta,\beta} \right) + |\nabla F_{R,\vartheta,\beta}|^{p-1} - F_{R,\vartheta,\beta} \\ &= \vartheta F_{R,1,\beta}^{1/(p-1)} \left[ \frac{\beta - \vartheta^{p-2}}{1+\tau} + \frac{(N-1)\vartheta^{p-2}}{(1+\tau)|y|} F_{R,1,\beta}^{(p-2)/(p-1)} - (1 - \vartheta^{p-2}) F_{R,1,\beta}^{(p-2)/(p-1)} \right]. \end{aligned}$$

At this point, we further assume that  $|y| > r_*/2$ , the radius  $r_*$  being defined in Corollary 2.1, and that

$$\vartheta^{p-2} \leq \frac{(1-\beta)r_*}{2(N-1)}. \tag{4.33}$$

Since  $F_{R,1,\beta} \leq 1$ , we obtain

$$\begin{aligned}
 Y &\leq \vartheta F_{R,1,\beta}^{1/(p-1)} \left[ \frac{\beta - \vartheta^{p-2}}{1 + \tau} + \frac{2(N-1)\vartheta^{p-2}}{(1 + \tau)r_*} - (1 - \vartheta^{p-2})F_{R,1,\beta}^{(p-2)/(p-1)} \right] \\
 &\leq \vartheta(1 - \vartheta^{p-2})F_{R,1,\beta}^{1/(p-1)} \left[ \frac{1}{1 + \tau} - \frac{p-2}{p-1} \left( \frac{\beta(\tau + R)}{\tau + 1} - |y| \right) \right],
 \end{aligned}$$

from which we conclude that

$$\begin{aligned}
 &F_{R,\vartheta,\beta} \text{ is a classical subsolution to (4.3) in} \\
 &\{(\tau, y) \in Q : \tau \geq \tau_2(R, \beta), r_*/2 < |y| \leq K_{R,\beta}(\tau)\},
 \end{aligned} \tag{4.34}$$

where  $\tau_2(R, \beta)$  and  $K_{R,\beta}(\tau)$  are still given by (4.14). We now proceed as in the one-dimensional case to establish (4.24) for all  $\vartheta \in (0, 1)$  satisfying (4.23) along with

$$\vartheta^{p-2} \leq \frac{\varepsilon r_*}{N-1},$$

for (4.33) to be satisfied.

*Steps 4 & 5.* The final steps of the proof are similar to the one-dimensional case. □

**5. Arbitrary initial data**

So far, we have proved Theorems 1.1 and 1.2 for radially nonincreasing initial data satisfying (1.3). We now extend these two results to general initial data satisfying (1.3).

*Proof of Theorems 1.1 and 1.2.* Since  $u_0 \not\equiv 0$ , there are  $x_0 \in \mathbb{R}^N$ ,  $r_0 > 0$ , and  $\eta_0 > 0$  such that  $u_0(x) \geq 2\eta_0$  for  $x \in B(x_0, r_0)$ . Then there exists a radially nonincreasing initial condition  $\tilde{u}_0$  satisfying (1.3) but with support in  $B(0, r_0)$  and such that  $\|\tilde{u}_0\|_\infty \leq \eta_0$  and  $\tilde{u}_0(x) \leq u_0(x - x_0)$  for  $x \in \mathbb{R}^N$ . Similarly, there is a radially nonincreasing initial condition  $\tilde{U}_0$  satisfying (1.3) but with support in  $B(0, \tilde{R}_0)$  for some  $\tilde{R}_0 > R_0$  and such that  $\tilde{U}_0(x) \geq \|u_0\|_\infty$  for  $x \in B(0, R_0)$ . Denoting by  $\tilde{u}$  and  $\tilde{U}$  the solutions to (1.1) with initial conditions  $\tilde{u}_0$  and  $\tilde{U}_0$ , respectively, the comparison principle and the translation invariance of (1.1) ensure that

$$\tilde{u}(t, x + x_0) \leq u(t, x) \leq \tilde{U}(t, x), \quad (t, x) \in Q. \tag{5.1}$$

Moreover, since

$$\left| \left( 1 - \frac{(p-2)|x + x_0|}{\log t} \right)_+^{(p-1)/(p-2)} - \left( 1 - \frac{(p-2)|x|}{\log t} \right)_+^{(p-1)/(p-2)} \right| \leq \frac{(p-1)|x_0|}{\log t},$$

and since Theorems 1.1 and 1.2 apply to both  $\tilde{u}$  and  $\tilde{U}$ , the expected results follow from (5.1). □

**Appendix. Some results about viscosity solutions**

We state, for the sake of completeness, some standard results in the theory of viscosity solutions, that we use in the proof of Theorem 1.1. The first one concerns the ‘‘viscosity’’ limit of a family of small perturbations and can be found in [5, Theorem 4.1].

PROPOSITION 6.1 Let  $u_\varepsilon$  be a viscosity subsolution (resp. a viscosity supersolution) of the equation

$$H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon, D^2 u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N,$$

where  $H_\varepsilon$  is uniformly bounded in all variables and degenerate elliptic. Suppose that  $\{u_\varepsilon\}$  is a uniformly bounded family of functions. Then

$$u^*(x) := \limsup_{(y,\varepsilon) \rightarrow (x,0)} u_\varepsilon(y) \quad (6.1)$$

is a subsolution of the equation

$$H_*(x, u, \nabla u, D^2 u) = 0. \quad (6.2)$$

In the same way,

$$u_*(x) := \liminf_{(y,\varepsilon) \rightarrow (x,0)} u_\varepsilon(y)$$

is a supersolution of  $H^*(x, u, \nabla u, D^2 u) = 0$ . Here,  $H_*$  and  $H^*$  are constructed in the same way as  $u_*$  and  $u^*$ .

In other words, this result can be applied to asymptotically small perturbations of a known equation, as in Section 4.

We also use the following result:

PROPOSITION 6.2 Let  $u \in \mathcal{C}(\Omega)$  be a viscosity solution of

$$H(x, u, \nabla u) = 0 \quad \text{in } \Omega, \quad (6.3)$$

where  $\Omega \subset \mathbb{R}^N$  and  $H$  is a continuous function. If  $\Phi \in \mathcal{C}^1(\mathbb{R})$  is an increasing function, then  $v = \Phi(u)$  is a viscosity solution of

$$H(x, \Phi^{-1}(v(x)), (\Phi^{-1})'(v(x))\nabla v(x)) = 0. \quad (6.4)$$

The same result holds true for subsolutions and supersolutions and can be found in [5]. In particular, we use this result in order to pass from the Hamilton–Jacobi equation  $|\nabla u|^{p-1} - u = 0$  to the standard eikonal equation  $|\nabla v| = 1$ . Finally, we also use the (now standard) comparison principle for viscosity subsolutions and supersolutions of the eikonal equation, that can be found in [13].

### Acknowledgements

The first and the third authors are supported by the Spanish Projects MTM2005-08760-C02-01 and MTM2008-06326-C02-01. JLV was partially supported by the ESF Programme “Global and geometric aspects of nonlinear partial differential equations”. This work was initiated while the second author enjoyed the support and hospitality of the Departamento de Matemáticas of the Universidad Autónoma de Madrid.

### REFERENCES

1. ANDREUCCI, D., TEDEEV, A. F., & UGHI, M. The Cauchy problem for degenerate parabolic equations with source and damping. *Ukrainian Math. Bull.* **1** (2004), 1–23. Zbl pre05373808 MR 2180703
2. ARONSON, D. G. Density-dependent interaction-diffusion systems. In: *Dynamics and Modelling of Reactive Systems* (Madison, WI, 1979), Publ. Math. Res. Center Univ. Wisconsin 44, Academic Press, New York (1980), 161–176. MR 0588018



3. ARONSON, D. G., & CAFFARELLI, L. The initial trace of a solution of the porous medium equation. *Trans. Amer. Math. Soc.* **280** (1983), 351–366. Zbl 0556.76084 MR 0712265
4. BARDI, M., & CAPUZZO-DOLCETTA, I. *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*. Systems Control Found. Appl., Birkhäuser, Boston (1997). Zbl 0890.49011 MR 1484411
5. BARLES, G. *Solutions de Viscosité des Équations d’Hamilton–Jacobi*. Math. Appl. 17, Springer, Berlin (1994). Zbl 0819.35002 MR 1613876
6. BARLES, G., & PERTHAME, B. Exit time problems in optimal control and vanishing viscosity method. *SIAM J. Control Optim.* **26** (1988), 1133–1148. Zbl 0674.49027 MR 0957658
7. BARTIER, J.-PH., & LAURENÇOT, PH. Gradient estimates for a degenerate parabolic equation with gradient absorption and applications. *J. Funct. Anal.* **254** (2008), 851–878. Zbl 1145.35049 MR 2381164
8. BENACHOUR, S., KARCH, G., & LAURENÇOT, PH. Asymptotic profiles of solutions to viscous Hamilton–Jacobi equations. *J. Math. Pures Appl.* (9) **83** (2004), 1275–1308. Zbl 1064.35075 MR 2092308
9. BILER, P., GUEDDA, M., & KARCH, G. Asymptotic properties of solutions of the viscous Hamilton–Jacobi equation. *J. Evolution Equations* **4** (2004), 75–97. Zbl 1056.35024 MR 2047307
10. CRANDALL, M. G., ISHII, H., & LIONS, P.-L. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), 1–67. Zbl 0755.35015 MR 1118699
11. GALLAY, TH., & LAURENÇOT, PH. Asymptotic behavior for a viscous Hamilton–Jacobi equation with critical exponent. *Indiana Univ. Math. J.* **56** (2007), 459–479. Zbl 1124.35008 MR 2305943
12. GILDING, B. The Cauchy problem for  $u_t = \Delta u + |\nabla u|^q$ , large-time behaviour. *J. Math. Pures Appl.* (9) **84** (2005), 753–785. Zbl 1100.35046 MR 2138640
13. ISHII, H. A simple, direct proof of uniqueness for solutions of the Hamilton–Jacobi equations of eikonal type. *Proc. Amer. Math. Soc.* **100** (1987), 247–251. Zbl 0644.35017 MR 0884461
14. KOLMOGOROFF, A., PETROVSKY, I., & PISCOUNOFF, N. Étude de l’équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. État Moscou Sér. Int. Sect. A Math. Mécan.* **1** (1937), no. 6, 1–25. Zbl 0018.32106
15. LAURENÇOT, PH. Large time behaviour for diffusive Hamilton–Jacobi equations. In: *Topics in Mathematical Modeling*, M. Beneš and E. Feireisl (eds.), Lect. Notes Nečas Center Math. Modeling 4, Matfyzpress, Praha, 2008, 95–168.
16. LAURENÇOT, PH., & VÁZQUEZ, J. L. Localized non-diffusive asymptotic patterns for nonlinear parabolic equations with gradient absorption. *J. Dynam. Differential Equations* **19** (2007), 985–1005. Zbl 1134.35070 MR 2357535
17. PERKO, L. *Differential Equations and Dynamical Systems*. Texts Appl. Math. 7, Springer, New York (1991). Zbl 0717.34001 MR 1083151
18. QUIRÓS, F., ROSSI, J. D., & VÁZQUEZ, J. L. Thermal avalanche for blow-up solutions of semilinear heat equations. *Comm. Pure Appl. Math.* **57** (2004), 59–98. Zbl 1036.35084 MR 2007356
19. ROQUEJOFFRE, J.-M. Convergence to steady states or periodic solutions in a class of Hamilton–Jacobi equations. *J. Math. Pures Appl.* (9) **80** (2001), 85–104. Zbl 0979.35033 MR 1810510
20. SHI, P. H. Self-similar singular solution of a  $p$ -Laplacian evolution equation with gradient absorption term. *J. Partial Differential Equations* **17** (2004), 369–383. Zbl 1076.35062 MR 2105844
21. VÁZQUEZ, J. L. Singular solutions and asymptotic behaviour of nonlinear parabolic equations. In: *International Conference on Differential Equations, Vol. 1, 2* (Barcelona, 1991), World Sci., River Edge, NJ (1993), 234–249. Zbl 0938.35558 MR 1242245
22. VÁZQUEZ, J. L. Porous medium flow in a tube. Travelling waves and KPP behaviour. *Comm. Contemp. Math.* **9** (2007), 731–751. Zbl 1157.35061 MR 2361739