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A semidiscrete scheme for a one-dimensional Cahn–Hilliard equation

CARINA GELDHAUSER
Institut für Angewandte Mathematik, Universität Bonn,
Endenicher Allee 60, 53115 Bonn, Germany
E-mail: carina.geldhauser@ins.uni-bonn.de

MATTEO NOVAGA
Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy
E-mail: novaga@math.unipd.it

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We analyze a semidiscrete scheme for the Cahn–Hilliard equation in one space dimension, when the interface length parameter is equal to zero. We prove convergence of the scheme for a suitable class of initial data, and we identify the limit equation. We also characterize the long-time behavior of the limit solutions.

Keywords: Nonconvex functionals; forward-backward parabolic equations; finite element method.

1. Introduction

Motivated by several models in phase transitions and image processing, Cahn–Hilliard type equations have been extensively studied in recent years. In one space dimension, a typical example is

$$u_t = \frac{1}{2}(W'(u_x))_x \quad \text{in } [0,1]\times[0,T], \tag{1.1}$$

where u_x is the derivative of a Lipschitz continuous, one-periodic function $u : [0,1] \to \mathbb{R}$ and W is the nonconvex energy density $W(p) = \frac{1}{4}(p^2-1)^2$ (double well potential). Equation (1.1) is the formal L^2-gradient flow of the functional

$$E[u] := \frac{1}{2}\int_0^1 W(u_x)\,\mathrm{d}x. \tag{1.2}$$

Notice that, by the change of variables $v = u_x$, equation (1.1) reduces to

$$v_t = \frac{1}{2}(W'(v))_{xx} \quad \text{in } [0,1]\times[0,T], \tag{1.3}$$

which corresponds to the H^{-1}-gradient flow of (1.2). We point out that, due to the nonconvexity of W, equations (1.1) and (1.3) are not well-posed.

In this paper, we deal with the semidiscrete problem

$$\begin{aligned} \frac{du^h}{dt} &= D^+W'(D^-u^h) && \text{in } [0,1]\times[0,T],\\ u^h(\cdot,0) &= \overline{u}^h && \text{on } [0,1]\times\{0\}, \end{aligned} \tag{1.4}$$

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where $h > 0$ is the grid size, D^+, D^- are the difference quotients defined in Definition 2.1 below, and $\overline{u}^h$ is the discretization of a piecewise-smooth function with nondifferentiability points $a_1, \ldots, a_m$. We consider (1.4) coupled with the periodic boundary conditions

$$\begin{aligned} u^h(0,t) &= u^h(1,t) && \text{on } \{0,1\} \times [0,T], \\ D^- u^h(0,t) &= D^- u^h(1,t) && \text{on } \{0,1\} \times [0,T]. \end{aligned} \tag{1.5}$$

In Proposition 2.4 we show that, if the initial datum satisfies

$$\frac{1}{\sqrt{3}} \leqslant |D_h^- \overline{u}^h| \leqslant \alpha := \frac{1}{2}\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) \quad \text{in } \bigcup_{j=1}^{m-1} [a_j^h, a_{j+1}^h], \tag{1.6}$$

then this property holds for all times $t \geqslant 0$. This assumption guarantees that the backward-parabolic zone, which is unstable for the evolution, consists of a finite number of points where the derivative u_x jumps from a region of (local) convexity of W to another one. We believe that weakening assumption (1.6) is an important but difficult task, and may lead to new interesting phenomena. We refer to the end of Section 3.1 for a discussion of this issue.

The main result of this paper, proved in Section 3, is the convergence of solutions to (1.4) and (1.5), as $h \to 0$, for initial data satisfying (1.6). We point out that, even under this simplifying assumption on initial data, we could find in the literature three different notions of solution to (1.1), which we briefly review for the reader's convenience. An important consequence of this work is that the regularization of (1.1) by means of a spatial semidiscrete scheme produces in the limit a solution which coincides with the one proposed by Plotnikov in [13], and further analyzed by Evans and Portilheiro in [8].

There is no classical theory for solutions of forward-backward parabolic equations like (1.1) and (1.3), apart from some results on special solutions (see for instance [11] and references therein). However, several notions of weak solution have been proposed:

1. In [5] the author defines an implicit variational scheme for the functional (1.2) which gives in the limit a solution to

$$u_t = \frac{1}{2}(W^{**\prime}(u_x))_x \quad \text{in } [0,1] \times [0,T],$$

where W^{**} is the convexified potential

$$W^{**} = \max\{f \leqslant W : f \text{ is convex}\}.$$

2. In [4] the following fourth-order regularization of (1.1) is considered:

$$u_t = -\varepsilon u_{xxxx} + W'(u_x)_x \tag{1.7}$$

and the author conjectures the existence of a pointwise limit as $\varepsilon \to 0$. The dynamics of this regularization for small ε, which is quite involved and has at least three relevant scales, was studied in [14, 1], where the asymptotic behavior as $t \to \infty$ is also discussed.

3. In [13] the author considers the regularization

$$u_t = \varepsilon u_{txx} + W'(u_x)_x, \tag{1.8}$$

proving the convergence, as $\varepsilon \to 0$, to a measure-valued solution to (1.1). In [8] further properties of such limit solutions are discussed, with particular emphasis on a hysteresis phenomenon which also appears in our scheme.

Our approach is different from the ones mentioned above: instead of studying continuous regularizations, we perform a spatial semidiscretization using the standard finite element method. In Section 2, we discuss the properties of the Cauchy problem for the semidiscrete scheme (see (2.5)) and provide suitable assumptions on initial data under which solutions are stable. One expects convergence of the scheme to classical solutions of (1.1) at least when the gradient of the initial datum takes values in the forward parabolic region, and we confirm this expectation with the only restriction that the gradient is not too large (see (1.6)). This is an advantage with respect to variational methods like the implicit scheme discussed in [5], which selects a local minimum of (1.2) and automatically forces all 1-Lipschitz functions not to move. Convergence of the scheme as the grid size h goes to zero is proved in Section 3, where we also identify the limit problem. We point out that our limit problem coincides with the limit of the continuous regularization (1.8), but not with the regularization (1.7). Finally, in Section 3.2 we prove the existence of a unique asymptotic state of the solution u, as $t \to +\infty$, whose derivative assumes precisely two values.

In order to keep the focus on the analytical aspects of the problem, we will not discuss the optimal convergence rate of the scheme, or provide numerical simulations. We address the interested reader to [1, 6] for numerical simulations in the one-dimensional case, and to [9] for higher dimensions. A finite element discretization of a simplified granular material model related to (1.1) was performed in [15] (see also [7]), where the authors study the limit profiles as $t \to +\infty$ of the discrete solutions.

2. Spatial semidiscretization

Let $\mathrm{I} := [0, 1]$ and let $\{h, \dots, Nh\}$ be a uniform grid on I with grid size $h = 1/N$, where $N \in \mathbb{N}$. Since we will work with 1-periodic functions, we identify the node 0 with the node N, hence $N + i$ with i. We denote by $PL(\mathrm{I})$ the N-dimensional vector subspace of $W^{1,\infty}(\mathrm{I})$, consisting of all continuous functions $u : \mathrm{I} \to \mathbb{R}$, with $u(0) = u(1)$, which are linear on the intervals $((i-1)h, ih)$ for all $i \in \{1, \dots, N\}$. We also let $PC(\mathrm{I})$ be the N-dimensional vector subspace of $L^2(\mathrm{I})$ of all right-continuous piecewise-constant functions on the grid. Letting $u_i := u(ih)$, we can identify $u \in PL(\mathrm{I})$ (resp. $u \in PC(\mathrm{I})$) with the vector $u^h := (u_1, \dots, u_N)$. Both $PL(\mathrm{I})$ and $PC(\mathrm{I})$ are endowed with the norms

$$\|u^h\|_{L^\infty(\mathrm{I})} := \max\{|u_i| : i = 1, \dots, N\}, \qquad \|u^h\|^2_{L^2_h(\mathrm{I})} := h \sum_{i=1}^{N} u_i^2.$$

Notice that $\|u^h\|_{L^2_h(\mathrm{I})} = \|u^h\|_{L^2(\mathrm{I})}$ for all $u \in PC(\mathrm{I})$, and

$$\|u^h\|_{L^2(\mathrm{I})} \leqslant \|u^h\|_{L^2_h(\mathrm{I})} \leqslant \sqrt{3}\|u^h\|_{L^2(\mathrm{I})} \qquad \forall u \in PL(\mathrm{I}). \tag{2.1}$$

Definition 2.1 We define the map $D^- : PL(\mathrm{I}) \to PC(\mathrm{I})$ and its adjoint $D^+ : PC(\mathrm{I}) \to PL(\mathrm{I})$ as

$$(D^- u^h)_i = \frac{u_i - u_{i-1}}{h}, \qquad (D^+ w)_i = \frac{w_{i+1} - w_1}{h}, \qquad i \in \{1, \dots, N\}.$$

With this notation, the space discretization of (1.1) can be expressed by the following system of ODEs on $PL(\mathrm{I})$:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = -\frac{1}{h}\frac{\partial \mathcal{F}}{\partial u_i} = \frac{1}{h}\left(W'\left(\frac{u_{i+1} - u_i}{h}\right) - W'\left(\frac{u_i - u_{i-1}}{h}\right)\right) = (D^+ W'(D^- u))_i \tag{2.2}$$

for all $i \in \{1, \dots, N\}$, with periodic boundary conditions.

We now introduce the class of initial data for (1.1) which we will consider in this paper.

ASSUMPTION 2.1 Let $\{a_j\}_{j=1}^m \in (0,1)$, with $a_1 < \cdots < a_m$. We shall consider initial $\overline{u} \in W^{1,\infty}(\mathrm{I}) \cap C^1(\mathrm{I} \setminus \{a_1, \ldots, a_m\})$ such that $\overline{u}(0) = \overline{u}(1)$ and $\overline{u}_x(0) = \overline{u}_x(1)$.

REMARK 2.1 Notice that, if u solves

$$\begin{aligned} u_t &= W'(u_x)_x && \text{in } \mathrm{I} \times [0,+\infty), \\ u(0,t) &= u(1,t) && \text{on } \partial\mathrm{I} \times [0,+\infty), \\ u_x(0,t) &= u_x(1,t) && \text{on } \partial\mathrm{I} \times [0,+\infty), \end{aligned} \tag{2.3}$$

then $v = u_x$ solves

$$\begin{aligned} v_t &= W'(v)_{xx} && \text{in } \mathrm{I} \times [0,+\infty), \\ v(0) &= v(1) && \text{on } \partial\mathrm{I} \times [0,+\infty), \\ v_x(0,t) &= v_x(1,t) && \text{on } \partial\mathrm{I} \times [0,+\infty). \end{aligned} \tag{2.4}$$

Conversely, if $v = u_x$ solves (2.4) and $\int_{\mathrm{I}} v \, \mathrm{d}x = 0$, then u solves (2.3). To get the full equivalence, i.e. for $\int_{\mathrm{I}} v \, \mathrm{d}x = c$, it is enough to replace the second line in (2.3) with $u(0,t) = u(1,t) + c$. For simplicity of presentation, we restrict to the case $c = 0$.

ASSUMPTION 2.2 Let $\overline{u}$ be as in Assumption 2.1. We denote by $a_1^h, \ldots, a_m^h$ the grid points corresponding to the nondifferentiability points of $\overline{u}$, that is, $a_i \in [a_i^h, a_i^h + h)$ for all $i \in \{1, \ldots, N\}$. For the discrete initial data $\overline{u}^h \in PL(\mathrm{I})$ we require

$$\|\overline{u}^h - \overline{u}\|_{L^\infty(\mathrm{I})} \xrightarrow[h \to 0]{} 0, \qquad \|D^- \overline{u}^h - \overline{u}_x\|_{L^1(\mathrm{I})} \xrightarrow[h \to 0]{} 0, \qquad \|D^- \overline{u}^h\|_{L^\infty(\mathrm{I})} \leqslant C,$$

for some $C > 0$ independent of h.

The Cauchy problem corresponding to (2.2) is

$$\begin{aligned} \frac{\mathrm{d}u^h}{\mathrm{d}t} &= D^+ W'(D^- u^h) && \text{in } \mathrm{I} \times [0,T], \\ u^h(0,t) &= u^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T], \\ D^- u^h(0,t) &= D^- u^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T], \\ u^h(\cdot,0) &= \overline{u}^h && \text{on } \mathrm{I} \times \{0\}, \end{aligned} \tag{2.5}$$

where $\overline{u}^h \in PL(\mathrm{I})$ denotes the discrete initial datum with the properties listed in Assumption 2.2. Note that, due to the smoothness of W, the scheme (2.5) admits a unique solution $u^h \in C^\infty([0,t_0], PL(\mathrm{I}))$ for a suitable $t_0 > 0$. Moreover, by direct integration we get

$$\int_{\mathrm{I}} u^h(x,t) \, \mathrm{d}x = \int_{\mathrm{I}} \overline{u}^h(x) \, \mathrm{d}x. \tag{2.6}$$

In many cases, it will be useful to work with the system governing the evolution of the spatial derivative of $u^h(x,t)$.

PROPOSITION 2.1 Let $\overline{u}^h \in PL(\mathrm{I})$ be a discrete initial datum for (2.5) satisfying Assumption 2.2. If $u^h(x,t)$ is a solution to the Cauchy problem (2.5), then $v^h := D^- u^h$ is a solution to the following system of ODEs:

$$\begin{aligned} \frac{\mathrm{d}v^h}{\mathrm{d}t} &= D^- D^+ W'(v^h) && \text{in } \mathrm{I} \times [0,T], \\ v^h(0,t) &= v^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T], \\ D^- v^h(0,t) &= D^- v^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T], \\ v^h(\cdot,0) &= D^- \overline{u}^h && \text{on } \mathrm{I} \times \{0\}. \end{aligned} \tag{2.7}$$

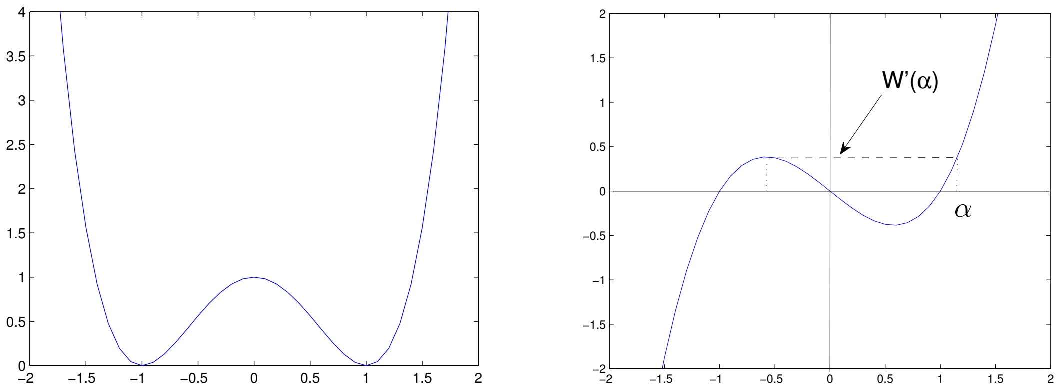

FIG. 1. Left: the graph of the potential W; right: the graph of its derivative.

2.1 *A priori estimates*

We denote by $\alpha > 1$ the real number such that $W'(\alpha) = \alpha^3 - \alpha = W'(-1/\sqrt{3})$; see Figure 1. Let us denote by $M(t) := \max_{i=1,\dots,N} v_i(t)$ and $m(t) := \min_{i=1,\dots,N} v_i(t)$ the maximum and minimum of the nodal values of v, respectively.

The following result will be needed in Proposition 2.2; the proof can be found in [2, Lemmas 5.1 and 5.2].

LEMMA 2.1 Let $v_1, \dots, v_N$ be real differentiable functions in an interval $[0,T)$. Define $M(t) := \max_{i=1,\dots,N} v_i(t)$. Then $M(t)$ is continuous, right-differentiable in $[0,T)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t^+} M(t) = \max_{i=1,\dots,N} \left\{ \frac{\mathrm{d}}{\mathrm{d}t^+} v_i(t) : v_i(t) = M(t) \right\} \quad \forall t \in [0,T).$$

PROPOSITION 2.2 (L^∞ estimate) Let $u^h(t)$ be solutions of the discrete Cauchy problem (2.5) with initial data $\overline{u}^h$ satisfying Assumption 2.2. Then

$$\|v^h(t)\|_{L^\infty(\mathrm{I})} \leqslant c \quad \forall t \in [0,\infty), \tag{2.8}$$

$$\|u^h(t)\|_{L^\infty(\mathrm{I})} \leqslant c \quad \forall t \in [0,\infty), \tag{2.9}$$

where the constant $c > 0$ is independent of h.

Proof. At time $t = 0$, the statement follows directly from the assumptions on the initial data.

Step 1. Let us first prove (2.8). We will show that $\max_i |v_i(t)|$ is nonincreasing whenever it is greater than α. We distinguish two cases.

Case 1: $\max_i |v_i(t)| = M(t) \geqslant \alpha$. As $M(t)$ is a solution to (2.7), by Lemma 2.1 we have

$$\frac{\mathrm{d}}{\mathrm{d}t^+} M(t) = \max_{i=1,\dots,N} \left\{ \frac{\mathrm{d}}{\mathrm{d}t^+} v_i(t) : v_i(t) = M(t) \right\} = \max_{i\,:\,v_i(t)=M(t)} D^- D^+ W'(v_i)$$
$$= \frac{1}{h^2} \max_{i\,:\,v_i(t)=M(t)} (W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})).$$

From $M \geqslant \alpha$ and $M \geqslant v_{i\pm1}$ we then get

$$W'(M(t)) \geqslant \max\{W'(v_{i-1}), W'(v_{i+1})\}.$$

Hence

$$\max_{i\,:\,v_i(t)=M(t)} \frac{\mathrm{d}v_i}{\mathrm{d}t^+} \leqslant 0, \tag{2.10}$$

which gives the upper bound

$$\max_{i=1,\dots,N} v_i(t) \leqslant \max_{i=1,\dots,N} \{\alpha, \max_i v_i(0)\}. \tag{2.11}$$

Case 2: $\max_i |v_i(t)| = -m(t) \geqslant \alpha$. Reasoning as above we obtain

$$\min_{i=1,\dots,N} v_i(t) \geqslant \min_{i=1,\dots,N} \{-\alpha, \min_i v_i(0)\}. \tag{2.12}$$

Putting together (2.11) and (2.12), we finally get

$$\|v^h(t)\|_{L^\infty(\mathrm{I})} \leqslant \max\{\alpha, \|D^- \overline{u}^h\|_{L^\infty(\mathrm{I})}\} \qquad \forall t \in [0, \infty),$$

which is (2.8).

Step 2. Estimate (2.9) now follows directly from (2.6) and (2.8). □

THEOREM 2.1 (Global existence of discrete solutions) Assume that the initial datum $\overline{u}^h \in PL(\mathrm{I})$ in (2.5) satisfies the periodic boundary conditions $\overline{u}^h(0) = \overline{u}^h(1)$. Then the Cauchy problem (2.5) admits a unique global solution $u^h \in C^\infty([0, +\infty), PL(\mathrm{I}))$.

Proof. As noted before, there exists a solution $u^h \in C^\infty([0, t_0], PL(\mathrm{I}))$, for some $t_0 > 0$. Proposition 2.2 guarantees L^∞ bounds on both $u^h(t)$ and its discrete derivative $v^h(t)$, which are uniform in time. As a consequence, the solution to the Cauchy problem can be extended for all times $t \in [0, +\infty)$. □

PROPOSITION 2.3 (Energy decreasing property) Let $u^h(x, t)$ be the solution of (2.5) with an initial datum $\overline{u}^h$ satisfying Assumption 2.2. Define the discrete energy

$$E^h(t) := E[u^h(\cdot, t)] := h \sum_{i=1}^{N} W(D^- u_i(t)). \tag{2.13}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} E[u^h(\cdot, t)] = -\|u_t^h\|^2_{L^2_h(\mathrm{I})} \leqslant 0. \tag{2.14}$$

Proof. Keeping in mind the periodic boundary conditions, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[u^h(\cdot,t)] =& h\sum_{i=1}^{N} W'(D^-u_i)\partial_t(D^-u_i) + W'(D^-u_N)\partial_t u_N - W'(D^-u_0)\partial_t u_0\\ =& -h\sum_{i=1}^{N} D^+W'(D^-u^h)_i\partial_t u_i = -\|u_t^h(\cdot,t)\|^2_{L_h^2(\mathrm{I})} \leqslant 0. \qquad \square\end{aligned}$$

As a consequence, we have

$$E^h(t) \leqslant E^h(0) \leqslant C \quad \forall t \geqslant 0, \tag{2.15}$$

as the discrete initial datum is bounded in $W^{1,\infty}(\mathrm{I})$ uniformly in h, by Assumption 2.2.

COROLLARY 2.1 (Hölder continuity) Let $\overline{u}^h$ be initial data satisfying Assumption 2.2. Then the solutions u^h of (2.5) are uniformly bounded in $C^{1/2}([0,T); L_h^2(\mathrm{I}))$.

Proof. Let $0 \leqslant t_1 < t_2 < +\infty$. We have to show $\|u^h(t_1) - u^h(t_2)\|_{L_h^2} \leqslant c|t_2 - t_1|^{1/2}$ for some constant $c > 0$ independent of h. Using Hölder's inequality and (2.15), we get

$$\|u^h(t_1) - u^h(t_2)\|_{L_h^2(\mathrm{I})} \leqslant |t_2 - t_1|^{1/2}(E^h(t_1) - E^h(t_2))^{1/2} \leqslant \sqrt{E^h(0)}|t_2 - t_1|^{1/2}. \qquad \square$$

The following corollary will be an important ingredient in the convergence proof.

COROLLARY 2.2 Let $u^h(x,t)$ be solutions to the Cauchy problem (2.5) with initial data $\overline{u}^h$ satisfying Assumption 2.2. Then $\frac{\mathrm{d}}{\mathrm{d}t}u^h \in L^2([0,\infty), L_h^2(\mathrm{I}))$, i.e.

$$\int_0^\infty \left\|\frac{\mathrm{d}}{\mathrm{d}t}u^h(\cdot,t)\right\|^2_{L_h^2(\mathrm{I})} \mathrm{d}t \leqslant E^h(0) \leqslant c, \tag{2.16}$$

where the constant c is independent of h.

Proof. Recalling (2.14) we have

$$E^h(0) - E^h(t) = \int_0^t \|u_t^h(\cdot,\tau)\|^2_{L_h^2(\mathrm{I})}\,\mathrm{d}\tau \quad \forall t \geqslant 0.$$

As $E^h(0) \leqslant C$ by Assumption 2.2, the conclusion follows by letting $t \to +\infty$. $\square$

2.2 *The stability estimate*

We shall make another assumption on the initial data $\overline{u}^h$ which guarantees the stability of the solution to (2.5): we take initial data $\overline{u} \in W^{1,\infty}(\mathrm{I})$ as in Assumption 2.1 which further satisfy

$$1/\sqrt{3} \leqslant (-1)^{j+1}\overline{u}_x(x) \leqslant \alpha \quad \forall x \in (a_{j-1}, a_j),\ j \in \{1,\ldots,m\}. \tag{2.17}$$

Note that (2.17) implies in particular that m is even and $\overline{u}_x$ takes values only in the regions where the potential W is convex.

We point out that a similar assumption was made in [3] for the Perona–Malik equation.
We now formulate the discrete analog of (2.17).

ASSUMPTION 2.3 Let α be as above and let $\overline{u}^h \in PL(\mathrm{I})$ be discrete initial data satisfying Assumption 2.2, with $a_1^h, \dots, a_m^h$ the grid points corresponding to the nondifferentiability points of $\overline{u}$. We require that $\overline{u}^h$ satisfies

$$1/\sqrt{3} \leqslant (-1)^{j+1} D^- \overline{u}_i \leqslant \alpha \quad \forall ih \in (a_{j-1}^h, a_j^h],\ j \in \{1, \dots, m\}. \tag{2.18}$$

PROPOSITION 2.4 (Stability estimate) Let u^h be solutions to (2.5) with initial data $\overline{u}^h$ satisfying Assumptions 2.2 and 2.3. Then u^h satisfies

$$1/\sqrt{3} \leqslant (-1)^{j+1} D^- u_i(t) \leqslant \alpha \quad \forall ih \in (a_{j-1}^h, a_j^h],\ j \in \{1, \dots, m\},\ t \geqslant 0. \tag{2.19}$$

Proof. Fix $j \in \{1, \dots, m\}$. Without loss of generality we can assume that $\overline{u}^h$ is increasing on $[a_j^h h, a_{j+1}^h h]$, that is, $v_i(0) = D^- \overline{u}_i \in [1/\sqrt{3}, \alpha]$ in $[a_j^h h, a_{j+1}^h h]$. We let

$$m(t) := \min_{i=1,\dots,N} v_i(t), \qquad M(t) := \max_{i=1,\dots,N} v_i(t),$$

and distinguish two cases:

Case 1: $M(t) = \alpha$ for some $t \geqslant 0$. By Lemma 2.1 and (2.7) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t^+} M(t) &= \max_{i:\, v_i(t)=M(t)} \frac{\mathrm{d}}{\mathrm{d}t^+} v_i(t) = \max_{i:\, v_i(t)=M(t)} D^- D^+ W'(v_i)\\ &= \frac{1}{h^2} \max_{i:\, v_i(t)=M(t)} (W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})) \leqslant 0,\end{aligned} \tag{2.20}$$

where we have used the fact that $W'(\alpha) \geqslant W'(x)$ for all $x \leqslant \alpha$.

Case 2: $m(t) = 1/\sqrt{3}$ for some $t \geqslant 0$. As above, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t^+} m(t) &= \min_{i:\, v_i(t)=m(t)} \frac{\mathrm{d}}{\mathrm{d}t^+} v_i(t) = \min_{i:\, v_i(t)=m(t)} D^- D^+ W'(v_i)\\ &= \frac{1}{h^2} \min_{i:\, v_i(t)=m(t)} (W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})) \geqslant 0,\end{aligned} \tag{2.21}$$

where we have used the fact that $W'(1/\sqrt{3}) \leqslant W'(x)$ for all $x \geqslant -\alpha$.

The conclusion follows from (2.20) and (2.21). □

3. Convergence of the scheme

PROPOSITION 3.1 Let the initial data $\overline{u}^h$ satisfy Assumption 2.2. Then the solutions u^h converge, up to a subsequence as $h \to 0$, to a limit function $u \in C(\mathrm{I} \times [0, +\infty))$, uniformly on compact subsets of $\mathrm{I} \times [0, +\infty)$.

Proof. By (2.1), Proposition 2.2 and Corollary 2.2 we know that the solutions u^h are uniformly bounded in $X_T := H^1([0, T], L^2(\mathrm{I})) \cap L^\infty([0, T], W^{1,\infty}(\mathrm{I}))$ for all $T > 0$. The conclusion follows from the compact embedding of X_T into $C(\mathrm{I} \times [0, T])$ [3]. □

Recalling Proposition 2.4 and reasoning exactly as in [12, Proposition 3.3], we obtain the following estimate.

LEMMA 3.1 Let $u^h(t)$ be a solution to the Cauchy problem (2.5) with initial data $\overline{u}^h$ satisfying Assumptions 2.2 and 2.3. Then, for every open set $\mathrm{I}_1 \subset\subset \mathrm{I} \setminus \{a_1, \ldots, a_m\}$, there exists a constant $c = c(\mathrm{I}_1)$ such that for h small enough

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} u^h(t) \right\|^2_{L^2_h(\mathrm{I}_1)} \leqslant E^h(0)\left(\frac{1}{t} + c\right) \qquad \forall t > 0. \tag{3.1}$$

PROPOSITION 3.2 Let $\overline{u}^h$ be initial data satisfying Assumptions 2.2 and 2.3, and let u^h be the corresponding solutions to the Cauchy problem (2.5). Then, for any compact subset K of $\mathrm{I} \setminus \{a_1, \ldots, a_m\}$ and for every $t > 0$, there exists a function $\psi \in H^1(K)$ such that

$$W'(v^h) \to \psi \qquad \text{uniformly on } K \text{ (up to a subsequence).}$$

Proof. As $W'(v^h)$ is uniformly bounded in $L^\infty(\mathrm{I})$ by Assumption 2.3, up to a suitable subsequence we have

$$W'(D^- u^h) \to \psi \qquad \text{weakly* in } L^\infty(K).$$

Moreover, by (2.1) and Lemma 3.1, $\frac{\mathrm{d}}{\mathrm{d}t} u^h = D^+ W'(D^- u^h)$ is uniformly bounded in $L^2(\mathrm{I})$. The conclusion then follows from the Arzelà–Ascoli Theorem. □

Proposition 3.2 allows us to obtain the strong convergence of $D^- u^h$, which is needed to pass to the limit in the nonlinear problem (2.5).

PROPOSITION 3.3 Let $u^h(t)$ be solutions to the Cauchy problem (2.5) with initial data $\overline{u}^h$ satisfying Assumptions 2.2 and 2.3. Then, up to a subsequence as $h \to 0$,

$$D^- u^h \to u_x \qquad \text{a.e. on } \mathrm{I} \times [0, +\infty) \tag{3.2}$$

and

$$W'(D^- u^h) \to W'(u_x) \qquad \text{in } L^2_{\mathrm{loc}}(\mathrm{I} \times [0, +\infty)). \tag{3.3}$$

Proof. By Propositions 2.2 and 3.1 we have

$$D^- u^h \to u_x \qquad \text{weakly* in } L^\infty(\mathrm{I}) \text{ for every } t \geqslant 0. \tag{3.4}$$

Let K be as in Proposition 3.2. As W' is invertible on $[-\alpha, -1/\sqrt{3}]$ and $[1/\sqrt{3}, \alpha]$, Proposition 3.2 implies

$$D^- u^h(t) = (W')^{-1}(W'(D^- u^h(t))) \to u_x(t) \qquad \text{uniformly on } K$$

for all $t > 0$, which gives (3.2). Claim (3.3) then follows from (3.2) and Lebesgue's Theorem. □

3.1 *The limit problem*

THEOREM 3.1 Let $\overline{u} \in W^{1,\infty}(\mathrm{I})$ be an initial datum satisfying Assumptions 2.1 and (2.17). Let $\overline{u}^h$ be finite element discretizations of $\overline{u}$ satisfying Assumptions 2.2 and 2.3, let u^h be the corresponding solutions to (2.5), and let $u \in C(\mathrm{I} \times [0, +\infty))$ be the limit of u^h, as $h \to 0$, given by Proposition 3.1. Then u is the unique solution to the following PDE:

$$\begin{array}{lll}
\text{(i)} & u_t = W'(u_x)_x & \text{in } (\mathrm{I} \setminus \{a_1, \ldots, a_m\}) \times [0, +\infty), \\
\text{(ii)} & W'(u_x^-) = W'(u_x^+) & \text{on } \{a_1, \ldots, a_m\} \times [0, +\infty), \\
\text{(iii)} & u^- = u^+ & \text{on } \{a_1, \ldots, a_m\} \times [0, +\infty), \\
\text{(iv)} & u(0) = \overline{u} & \text{at } \mathrm{I} \times \{0\},
\end{array} \tag{3.5}$$

where we set

$$u^{\pm} := \lim_{x\to a_j^{\pm}} u(x), \qquad u_x^{\pm} := \lim_{x\to a_j^{\pm}} u_x(x).$$

In particular $W'(u_x) \in C(\mathrm{I}\times[0,+\infty))$ and $u \in C^{\infty}((\mathrm{I}\setminus\{a_1,\dots,a_m\})\times(0,+\infty))$.

Proof. Multiplying by $\varphi \in C_0^1(\mathrm{I}\times[0,+\infty))$ the first equation in (2.5), after an integration by parts we get

$$\int_0^{\infty}\int_{\mathrm{I}} u^h\varphi_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\infty}\int_{\mathrm{I}} W'(D^-u^h)D^-\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{3.6}$$

As $u^h \to u$ locally uniformly on $\mathrm{I}\times[0,+\infty)$ by Proposition 3.1, and $W'(D^-u^h)D^-\varphi \to W'(u_x)\varphi_x$ in $L^2(\mathrm{I}\times[0,+\infty))$ by Proposition 3.3, we can pass to the limit in (3.6):

$$\int_0^{\infty}\int_{\mathrm{I}} u\varphi_t\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\infty}\int_{\mathrm{I}} W'(u_x)\varphi_x\,\mathrm{d}x\,\mathrm{d}t. \tag{3.7}$$

Since $u_t \in L^2(\mathrm{I}\times[0,+\infty))$, (3.7) implies $W'(u_x) \in L^2([0,+\infty), H^1(\mathrm{I}))$, so that we can integrate by parts and obtain

$$\int_0^{\infty}\int_{\mathrm{I}} u_t\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\infty}\int_{\mathrm{I}} W'(u_x)_x\varphi\,\mathrm{d}x\,\mathrm{d}t, \tag{3.8}$$

which proves statement (i). Equalities (ii) and (iii) follow from the continuity of $W'(u_x)$ and u, respectively. □

REMARK 3.1 Problem (3.9) is equivalent to the limit problem derived in [13, 8] for the regularization (1.8). On the other hand, due to the numerical simulations performed in [1], it is expected to be different from the limit problem corresponding to the Cahn–Hilliard regularization (1.7) discussed in [4, 14].

COROLLARY 3.1 If u satisfies (3.5), then $v = u_x = \lim_{h\to 0} v^h$ is the unique solution to the following PDE:

$$\begin{aligned}
v_t &= W'(v)_{xx} && \text{in } (\mathrm{I}\setminus\{a_1,\dots,a_m\})\times[0,+\infty),\\
W'(v^-) &= W'(v^+) && \text{on } \{a_1,\dots,a_m\}\times[0,+\infty),\\
W'(v)_x^- &= W'(v)_x^+ && \text{on } \{a_1,\dots,a_m\}\times[0,+\infty),\\
v(0) &= \overline{u}_x && \text{on } (\mathrm{I}\setminus\{a_1,\dots,a_m\})\times\{0\}.
\end{aligned} \tag{3.9}$$

Passing to the limit in (2.16) as $h \to 0$, we obtain an integral estimate on the time derivative of u.

PROPOSITION 3.4 Let u be as in Theorem 3.1. We have $u_t \in L^2(\mathrm{I}\times(0,+\infty))$ and

$$\int_{\mathrm{I}\times(0,\infty)} \left(\frac{\mathrm{d}u}{\mathrm{d}t}(x,t)\right)^2 \mathrm{d}x\,\mathrm{d}t = E[\overline{u}].$$

Let us briefly discuss the new phenomena which may occur if one tries to weaken Assumption (2.17). We first observe that the energy balance condition (3.5)(ii) must be satisfied by any limit of the semidiscrete scheme.

1. If the lower bound $|\bar{u}_x| \geqslant 1/\sqrt{3}$ is violated, there are intervals of the domain I where the derivative of the solution belongs to the nonconvex region of the potential W. This leads to instability and one expects the onset of a microstructure, due to rapid oscillations of the derivative. However, in contrast to other regularizations, such oscillations do not seem able to stop the evolution [10].
2. If the upper bound $|\bar{u}_x| \leqslant \alpha$ is violated, then also the lower bound cannot hold for positive times, due to the energy balance condition (3.5)(ii). In this case one expects that the jump points of the derivative a_j move in time and a hysteresis phenomenon occurs, as discussed in [8].

3.2 *Long-time behavior*

THEOREM 3.2 Let u be a solution of (3.5). Then there exists a unique limit

$$u_\infty(x) := \lim_{t\to+\infty} u(t,x), \qquad x \in \mathrm{I},$$

which is given by the piecewise linear solution to

$$\begin{aligned} &\text{(i)} && W'((u_\infty)_x)_x = 0 && \text{in } \mathrm{I}\setminus\{a_1,\dots,a_m\},\\ &\text{(ii)} && W'((u_\infty)_x^-) = W'((u_\infty)_x^+) && \text{on } \{a_1,\dots,a_m\},\\ &\text{(iii)} && u_\infty^- = u_\infty^+ && \text{on } \{a_1,\dots,a_m\}. \end{aligned} \tag{3.10}$$

Proof. We divide the proof into three steps.

Step 1 (Existence of u_∞). By Proposition 3.4, there exists a sequence of times $t_n \to +\infty$ such that

$$\int_{t_n}^{t_n+1} \|u_t\|^2_{L^2(\mathrm{I})}\,\mathrm{d}t \to 0. \tag{3.11}$$

We now define a sequence w^n of solutions to (3.5) in the following way:

$$w^n(x,t) := u(x, t_n + t), \qquad t \in [0,1].$$

From (3.11) we have

$$\int_0^1 \|w^n_t\|^2_{L^2(\mathrm{I})}\,\mathrm{d}t \xrightarrow[n\to\infty]{} 0, \tag{3.12}$$

whence $w^n \to w \in H^1([0,1], L^2(\mathrm{I})) \cap L^\infty([0,1], W^{1,\infty}(\mathrm{I}))$ with $w_t \equiv 0$, that is, the limit function $w = u_\infty$ does not depend on t.

Step 2 (Limit equation). As every w^n solves (3.5), from (3.7) we get

$$\int_0^1 \int_{\mathrm{I}} W'(w^n_x)\varphi_x\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all test functions $\varphi \in C^1(\mathrm{I})$ independent of t. Passing to the limit as $n \to \infty$ and recalling (3.12), we get (i) and (ii), while (iii) follows from the Lipschitz continuity of w.

We now show that w_x is a piecewise constant function which assumes exactly two values, p^- and p^+. Indeed, (3.10)(i) implies that, for all $j \in \{1,\dots,m\}$, there exists $p_j \in [-\alpha, -1/\sqrt{3}] \cup [1/\sqrt{3}, \alpha]$ such that $W'(w_x) \equiv p_j$. Moreover, from condition (ii) we know that

$$W'(p_i) = W'(p_j) \qquad \forall i, j \in \{1,\dots,m\}. \tag{3.13}$$

Since W' is monotone in the intervals $[-\alpha, -1/\sqrt{3}]$ and $[1/\sqrt{3}, \alpha]$, for all $p \in [1/\sqrt{3}, \alpha]$ there exists only one value $\tilde{p} \in [-\alpha, -1/\sqrt{3}]$ such that

$$W'(p) = W'(\tilde{p}). \tag{3.14}$$

The claim then follows from (3.13) and (3.14).

Step 3 (Uniqueness). Once we know that w_x assumes precisely two values $p^- < p^+$, with $p^- \in [-\alpha, -1/\sqrt{3}]$ and $p^+ \in [1/\sqrt{3}, \alpha]$, the uniqueness of such values follows by direct integration. More precisely, assuming without loss of generality $w_x = p^+ > 0$ on $[0, a_1]$ and recalling (3.10)(iii), we have

$$0 = w(1) - w(0) = \sigma(p^+), \tag{3.15}$$

where

$$\sigma(p) := p \sum_{\ell=0}^{m/2-1} (a_{2\ell+1} - a_{2\ell}) + \tilde{p} \sum_{k=1}^{m/2} (a_{2k} - a_{2k-1}), \qquad p \in [1/\sqrt{3}, \alpha].$$

Since σ is strictly increasing on $[1/\sqrt{3}, \alpha]$, equation (3.15) uniquely determines the value of p^+, and consequently of p^-. □

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References

1. Bellettini, G., Fusco, G., & Guglielmi, N. A concept of solution and numerical experiments for forward-backward diffusion equations. *Discrete Contin. Dynam. Systems* **16** (2006), 783–842. Zbl 1105.35007 MR 2257160
2. Bellettini, G., Novaga, M., & Paolini, E. Global solutions to the gradient flow equation of a nonconvex functional. *SIAM J. Math. Anal.* **37** (2006), 1657–1687. Zbl 1109.35050 MR 2215602
3. Bellettini, G., Novaga, M., Paolini, M., & Tornese, C. Convergence of discrete schemes for the Perona–Malik equation. *J. Differential Equations* **245** (2008), 892–924. Zbl 1155.35002 MR 2427400
4. De Giorgi, E. Conjectures concerning some evolution problems. *Duke Math. J.* **81** (1996), 255–268. Zbl 0874.35027 MR 1395405
5. Demoulini, S. Young measure solutions for a nonlinear parabolic equation of forward-backward type. *SIAM J. Math. Anal.* **27** (1996), 376–403. Zbl 0851.35066 MR 1377480
6. Elliott, C. M., & Songmu, Z. On the Cahn–Hilliard equation. *Arch. Ration. Mech. Anal.* **96** (1986), 339–357. Zbl 0624.35048 MR 0855754
7. Esedoḡlu, S. An analysis of the Perona–Malik scheme. *Comm. Pure Appl. Math.* **54** (2001), 1442–1487. Zbl 1031.68133 MR 1852979
8. Evans, L. C., & Portilheiro, M. Irreversibility and hysteresis for a forward-backward diffusion equation. *Math. Models Methods Appl. Sci.* **14** (2004), 1599–1620. Zbl 1064.35091 MR 2103092
9. Feng, X., & Prohl, A. Numerical analysis of the Cahn–Hilliard equation and approximation of the Hele-Shaw problem. *Interfaces Free Bound.* **7** (2005), 1–28. Zbl 1072.35150 MR 2126141

10. FIERRO, F., GOGLIONE, R., & PAOLINI, M. Numerical simulations of mean curvature flow in presence of a nonconvex anisotropy. *Math. Models Methods Appl. Sci.* **4** (1998), 573–601. Zbl 0946.58014 MR 1634826

11. GHISI, M., & GOBBINO, M. A class of local classical solutions for the one-dimensional Perona–Malik equation. *Trans. Amer. Math. Soc.* **361** (2009), 6429–6446. Zbl 1213.35320 MR 2538599

12. GHISI, M., & GOBBINO, M. Gradient estimates for the Perona–Malik equation. *Math. Ann.* **337** (2007), 557–590. Zbl 1130.35081 MR 2274543

13. PLOTNIKOV, P. I. Passage to the limit with respect to viscosity in an equation with a variable direction of parabolicity. *Differentsial′nye Uravneniya* **30** (1994), 665–674, 734 (in Russian). Zbl 0824.35100 MR 1299852

14. SLEMROD, M. Dynamics of measure valued solutions to a backward-forward heat equation. *J. Dynam. Differential Equations* **3** (1991), 1–28. Zbl 0747.35013 MR 1094722

15. WITELSKI, T. P., SCHAEFFER, D. G., & SHEARER, M. A discrete model for an ill-posed nonlinear parabolic PDE. *Phys. D* **160** (2001), 189–221. Zbl 0989.65111 MR 1872040