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A phase-field approximation of the Willmore flow with volume constraint

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The well-posedness of a phase-field approximation to the Willmore flow with volume constraint is established. The existence proof relies on the underlying gradient flow structure of the problem: the time discrete approximation is solved by a variational minimization principle.

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1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $1 \leqslant N \leqslant 3$, with smooth boundary Γ. We are interested in the following evolution problem:

$$\partial_t v - \Delta\mu + W''(v)\,\mu - \overline{W''(v)\,\mu} = 0, \qquad (t,x) \in (0,\infty)\times\Omega, \tag{1}$$

$$\mu = -\Delta v + W'(v), \qquad (t,x) \in (0,\infty)\times\Omega, \tag{2}$$

$$\nabla v\cdot\nu = \nabla\mu\cdot\nu = 0, \qquad (t,x) \in (0,\infty)\times\Gamma, \tag{3}$$

$$v(0) = v_0, \qquad x \in \Omega, \tag{4}$$

where the nonlinearity W is a smooth double-well potential (for instance, $W(r) = (r^2-1)^2/4$), ν is the outward unit normal vector field to Γ, and $\bar{f}$ denotes the spatial mean value of an integrable function f, namely,

$$\bar{f} := \frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x \quad \text{for } f \in L^1(\Omega).$$

As one can easily realize from (1) and (3) by integrating over Ω, the mean value of v is conserved during the evolution, that is, $\overline{v}(t) = \overline{v_0}$.

The initial-boundary value problem (1)–(4) is a phase-field approximation of the Willmore flow (cf., in particular, [5, 6]), the Willmore flow belonging to a class of geometric evolutions of hypersurfaces involving nonlinear functions of the principal curvatures of the hypersurface. Recall that the Willmore flow *with volume constraint* for a family of (smooth) hypersurfaces $(\Sigma(t))_{t\geqslant 0}$ reads

$$\mathcal{V} = -\Delta_\Sigma H - \frac{H}{2}(H^2 - 4K) + \lambda, \tag{5}$$

where $\mathcal{V}$, H, K, and Δ_Σ denote the normal velocity of Σ, the sum of its principal curvatures (scalar mean curvature), the product of its principal curvatures (Gauß curvature), and the Laplace–Beltrami operator on Σ, respectively, while λ is the Lagrange multiplier accounting for the volume conservation

$$\int_\Sigma \mathcal{V}\,\mathrm{d}s = 0.$$

In addition, the Willmore flow is the L^2-gradient flow of the Willmore energy

$$\mathcal{E}_W(\Sigma) := \int_\Sigma H^2\,\mathrm{d}s. \tag{6}$$

Related geometric evolution flows involve more complicated energies such as the Helfrich energy and additional constraints, for instance on the area, and are found in the modelling of biological cell membranes. We refer, e.g., to [2–6, 9] and the references therein for a more detailed description of these flows and their applications. To our knowledge, the energetic phase-field approximation (1)–(4) has been introduced in [6] in order to describe the deformation of a vesicle membrane under the elastic bending energy, with prescribed bulk volume and surface area, a related model without constraints being considered in [7]. Here, we restrict our analysis to the case of only the volume constraint, leaving the more complex case of two constraints as in [6] to a subsequent investigation. A nice feature of (1)–(4) already reported in [6] is that it inherits the gradient flow structure of the Willmore flow and, for $\alpha \in \mathbb{R}$, it is actually a gradient flow in $L^2_\alpha(\Omega) := \{w \in L^2(\Omega) : \bar{f} = \alpha\}$ for the functional

$$E(v) := \frac{1}{2}\int_\Omega [-\Delta v(x) + W'(v(x))]^2\,\mathrm{d}x, \tag{7}$$

a property which is a cornerstone of the forthcoming analysis. The connection between the minimizers of the Willmore energy (6) and those of a suitably rescaled version of the energy (7) of the stationary phase-field model has been investigated in [4, 8, 9], and we refer to [5, 6, 11] for the analysis of the relationship between the phase-field approach (1)–(4) and the Willmore flow, with or without volume and surface constraints. However, the well-posedness of the phase-field approximation does not seem to have been considered so far, and the aim of this note is to show the well-posedness of (1)–(4) under suitable assumptions on the data: more precisely, we assume that there is $C_0 > 0$ such that

$$W \in \mathcal{C}^3(\mathbb{R}), \qquad W \geqslant 0, \tag{8}$$

$$W''(r) \geqslant -C_0 \quad \text{and} \quad rW'(r) \geqslant -C_0, \qquad r \in \mathbb{R}. \tag{9}$$

Next, owing to the already mentioned expected time invariance of the spatial mean value of solutions to (1)–(4), for $\alpha \in \mathbb{R}$ we define the function space

$$V := \{w \in H^2(\Omega) : \nabla w \cdot \nu = 0 \text{ on } \Gamma\} \quad \text{and its subset} \quad V_\alpha := \{w \in V : \overline{w} = \alpha\}. \tag{10}$$

The paper is devoted to the proof of the following existence and uniqueness result.

THEOREM 1 Given $\alpha \in \mathbb{R}$ and $v_0 \in V_\alpha$, there is a unique solution v to (1)–(4) satisfying

$$v \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^\infty(0,T;V_\alpha) \quad \text{and} \quad \mu := -\Delta v + W'(v) \in L^2(0,T;V)$$

for all $T > 0$. In addition,

$$t \mapsto E(v(t)) := \frac{1}{2}\|\mu(t)\|_2^2 \quad \text{is a nonincreasing function,} \tag{11}$$

$$\int_0^\infty \|-\Delta\mu(t) + W''(v(t))\mu(t) - \overline{W''(v)\mu}(t)\|_2^2 \, dt \leqslant 2E(v_0). \tag{12}$$

Owing to the above mentioned gradient flow structure, a classical approach to existence is to use an implicit time scheme and solve a minimization problem at each step (see, e.g., [1] or [10, Chap. 8]). The existence of a minimizer to the corresponding stationary problem is discussed in Section 2, and Subsection 2.1 also collects some properties of the auxiliary variable μ. The time discretization is next implemented in Subsection 2.2 and convergence of the time discrete scheme is proved in Subsection 2.3 with the help of monotonicity and compactness properties. Finally, uniqueness is shown in Section 3 by a standard contraction argument.

2. Existence

2.1 *The energy functional*

Following [6], we define the functional E on V by

$$E(w) := \frac{1}{2}\int_\Omega [-\Delta w(x) + W'(w(x))]^2 \, dx. \tag{13}$$

Observe that E is well defined for any $w \in V$ thanks to the continuous embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$ and (8). Indeed, for $w \in V$, we have $w \in L^\infty(\Omega)$ and

$$|W'(w)| \leqslant |W'(0)| + \int_{-|w|}^{|w|} |W''(r)| \, dr \leqslant |W'(0)| + \sup_{[-\|w\|_\infty, \|w\|_\infty]} |W''| \cdot |w|.$$

Consequently, $W'(w) \in L^2(\Omega)$ and E is well defined. We gather some properties of E in the next lemma.

LEMMA 2 Given $\alpha \in \mathbb{R}$, there is $C_1(\alpha) > 0$ depending only on Ω, C_0 in (9), and α such that

$$\|w\|_{H^2} + \|W'(w)\|_2 \leqslant C_1(\alpha)\big(1 + \sqrt{E(w)}\big) \quad \text{for all } w \in V_\alpha. \tag{14}$$

Proof. Consider $w \in V_\alpha$ and put $\mu := -\Delta w + W'(w)$. Then $\mu \in L^2(\Omega)$ with $\|\mu\|_2^2 = 2E(w)$, and we infer from (9) that

$$\int_\Omega w\mu \, dx = \|\nabla w\|_2^2 + \int_\Omega wW'(w) \, dx \geqslant \|\nabla w\|_2^2 - C_0|\Omega|.$$

Combining the above inequality with the Poincaré–Wirtinger inequality

$$\|w - \overline{w}\|_2 \leqslant C_2\|\nabla w\|_2, \tag{15}$$

we obtain

$$\begin{aligned}
\|\nabla w\|_2^2 &\leqslant C_0|\Omega| + \int_\Omega w\mu \, dx \leqslant C_0|\Omega| + \|w\|_2\|\mu\|_2 \\
&\leqslant C_0|\Omega| + \sqrt{2E(w)}(\alpha|\Omega|^{1/2} + \|w - \alpha\|_2) \leqslant C_0|\Omega| + \sqrt{2E(w)}(\alpha|\Omega|^{1/2} + C_2\|\nabla w\|_2) \\
&\leqslant C_0|\Omega| + \alpha|\Omega|^{1/2}\sqrt{2E(w)} + \frac{1}{2}\|\nabla w\|_2^2 + C_2^2 E(w),
\end{aligned}$$

hence $\|\nabla w\|_2^2 \leqslant C(\alpha)(1+E(w))$. Using again (15), we conclude that

$$\|w\|_{H^1} \leqslant C(\alpha)\big(1+\sqrt{E(w)}\big). \tag{16}$$

Now, $w \in V$ solves

$$-\Delta w + W'(w) + C_0 w = \mu + C_0 w \tag{17}$$

and, since the function $r \mapsto W'(r)+C_0 r$ is nondecreasing by (9), a classical monotonicity argument shows that

$$\|\Delta w\|_2 + \|W'(w)+C_0 w\|_2 \leqslant \|\mu + C_0 w\|_2.$$

Indeed, it suffices to test equation (17) with $W'(w)+C_0 w$, observe that

$$\int_\Omega (-\Delta w)(W'(w)+C_0 w)\,\mathrm{d}x = \int_\Omega (W''(w)+C_0)|\nabla w|^2\,\mathrm{d}x \geqslant 0,$$

and compare the terms in (17) in order to get the above estimate. Consequently,

$$\|\Delta w\|_2 + \|W'(w)\|_2 \leqslant \|\mu\|_2 + 2C_0\|w\|_2,$$

which, together with (16) and $\|\mu\|_2 = \sqrt{2E(w)}$, gives (14). □

Next, given $\tau > 0$ and $f \in L^2(\Omega)$, we define the functional $F_{\tau,f}$ on V by

$$F_{\tau,f}(w) := \frac{1}{2}\|w-f\|_2^2 + \tau E(w), \qquad w \in V. \tag{18}$$

LEMMA 3 Given $\alpha \in \mathbb{R}$, the functional $F_{\tau,f}$ has (at least) a minimizer in V_α.

Proof. We set $F := F_{\tau,f}$ to simplify notation. Since E is nonnegative, F is obviously nonnegative and there is a minimizing sequence $(w_n)_{n\geqslant 1}$ in V_α such that

$$m_\alpha := \inf_{w\in V_\alpha} F(w) \leqslant F(w_n) \leqslant m_\alpha + 1/n, \quad n \geqslant 1. \tag{19}$$

Since $F(w_n) \geqslant \tau E(w_n)$, we readily infer from (19) that $(E(w_n))_{n\geqslant 1}$ is bounded, which in turn implies that $(w_n)_{n\geqslant 1}$ is bounded in $H^2(\Omega)$ by Lemma 2. Owing to the compactness of the embedding of $H^2(\Omega)$ in $\mathcal{C}(\bar{\Omega})$, we deduce that there are $w \in H^2(\Omega)$ and a subsequence of $(w_n)_{n\geqslant 1}$ (not relabeled) such that

$$w_n \to w \quad \text{in } \mathcal{C}(\bar{\Omega}) \quad \text{and} \quad w_n \rightharpoonup w \quad \text{in } H^2(\Omega). \tag{20}$$

Clearly, the first convergence implies that $(W'(w_n))_{n\geqslant 1}$ converges towards $W'(w)$ in $L^2(\Omega)$ and therefore

$$F(w) \leqslant \liminf_{n\to\infty} F(w_n) \leqslant m_\alpha.$$

As w obviously belongs to V_α by (20), we also have $F(w) \geqslant m_\alpha$ and w is a minimizer of F in V_α. □

We next derive an energy inequality and the Euler–Lagrange equation satisfied by minimizers of $F_{\tau,f}$ in V_α when $\bar{f} = \alpha$.

LEMMA 4 Let $\alpha \in \mathbb{R}$ and let w be a minimizer of $F_{\tau,f}$ in V_α. Assume further that $\bar{f} = \alpha$. Then $\mu := -\Delta w + W'(w)$ belongs to V,

$$\int_\Omega \left[\frac{w-f}{\tau} - \Delta\mu + W''(w)\mu - \overline{W''(w)\mu} \right] \psi \, \mathrm{d}x = 0 \quad \text{for all } \psi \in V, \tag{21}$$

and

$$\left\| -\Delta\mu + W''(w)\mu - \overline{W''(w)\mu} \right\|_2 \leqslant \|w - f\|_2/\tau. \tag{22}$$

Proof. Let $\varepsilon \in (0,1)$ and $\varphi \in V_0$. As $w + \varepsilon\varphi$ belongs to V_α, we have $F_{\tau,f}(w) \leqslant F_{\tau,f}(w+\varepsilon\varphi)$, from which we deduce by classical arguments (after passing to the limit as $\varepsilon \to 0$) that

$$\frac{1}{\tau} \int_\Omega (w-f)\varphi \, \mathrm{d}x + \int_\Omega \mu(-\Delta\varphi + W''(w)\varphi) \, \mathrm{d}x \geqslant 0.$$

Since the above inequality is valid for φ and $-\varphi$, we actually have the identity

$$\frac{1}{\tau} \int_\Omega (w-f)\varphi \, \mathrm{d}x + \int_\Omega \mu(-\Delta\varphi + W''(w)\varphi) \, \mathrm{d}x = 0 \tag{23}$$

for all $\varphi \in V_0$. Now, if $\psi \in V$, the function $\psi - \overline{\psi}$ belongs to V_0 and it follows from (23) that

$$\frac{1}{\tau} \int_\Omega (w-f)\psi \, \mathrm{d}x + \int_\Omega \mu(-\Delta\psi + W''(w)\psi) \, \mathrm{d}x = \overline{W''(w)\mu} \int_\Omega \psi \, \mathrm{d}x, \tag{24}$$

since w and f have the same mean value α. Since $\mu \in L^2(\Omega)$ solves the variational equality (24) for all test functions $\psi \in V$, we deduce that μ is in V and satisfies (21).

Next, for $\eta \in (0,1)$, let φ_η be the unique solution in V_0 to

$$\varphi_\eta - \eta\Delta\varphi_\eta = -\Delta\mu + W''(w)\mu - \overline{W''(w)\mu} \quad \text{in } \Omega,$$

the right-hand side of the previous equation being in $L^2(\Omega)$ since $\mu \in V$ and $w \in H^2(\Omega)$ is bounded. Also, the right-hand side of the equation has a zero mean value so that $\varphi_\eta \in V_0$. Taking $\psi = \varphi_\eta$ in (21), we realize that

$$\int_\Omega \left[\frac{w-f}{\tau} + \varphi_\eta - \eta\Delta\varphi_\eta \right] \varphi_\eta \, \mathrm{d}x = 0,$$

from which we deduce that

$$\|\varphi_\eta\|_2^2 \leqslant \|\varphi_\eta\|_2^2 + \eta\|\nabla\varphi_\eta\|_2^2 = -\int_\Omega \frac{w-f}{\tau}\varphi_\eta \, \mathrm{d}x \leqslant \frac{\|w-f\|_2}{\tau}\|\varphi_\eta\|_2,$$

whence

$$\|\varphi_\eta\|_2 \leqslant \|w-f\|_2/\tau.$$

Since $(\varphi_\eta)_\eta$ converges toward $-\Delta\mu + W''(w)\mu - \overline{W''(w)\mu}$ in $L^2(\Omega)$ as $\eta \to 0$, (22) follows from the above inequality. □

2.2 *Time discretization*

Let $\alpha \in \mathbb{R}$ and take an initial condition $v_0 \in V_\alpha$. We consider a positive time step $\tau \in (0,1)$ and define a sequence $(v_n^\tau)_{n\geqslant 1}$ inductively as follows:

$$v_0^\tau := v_0, \tag{25}$$

$$v_{n+1}^\tau \text{ is a minimizer of } F_{\tau,v_n^\tau} \text{ in } V_\alpha, \qquad n \geqslant 0, \tag{26}$$

the functional F_{τ,v_n^τ} being defined in (18). Setting

$$\mu_n^\tau := -\Delta v_n^\tau + W'(v_n^\tau) \quad \text{and} \quad M_n^\tau := \overline{W''(v_n^\tau)\mu_n^\tau}, \tag{27}$$

we define three piecewise constant time-dependent functions v^τ, μ^τ, and M^τ by

$$(v^\tau(t), \mu^\tau(t), M^\tau(t)) := (v_n^\tau, \mu_n^\tau, M_n^\tau) \quad \text{for } t \in [n\tau,(n+1)\tau) \text{ and } n \geqslant 0. \tag{28}$$

LEMMA 5 For $\tau \in (0,1)$, $t_1 \geqslant 0$, and $t_2 > t_1$, we have

$$E(v^\tau(t_2)) \leqslant E(v^\tau(t_1)) \leqslant E(v_0), \tag{29}$$

$$\|v^\tau(t_2) - v^\tau(t_1)\|_2^2 \leqslant 2E(v_0)(\tau + t_2 - t_1), \tag{30}$$

$$\int_\tau^\infty \|-\Delta\mu^\tau(t) + W''(v^\tau(t))\mu^\tau(t) - M^\tau(t)\|_2^2 \, dt \leqslant 2E(v_0). \tag{31}$$

Proof. Let $n \geqslant 0$. Since $v_n^\tau \in V_\alpha$, we infer from (26) that $F_{\tau,v_n^\tau}(v_{n+1}^\tau) \leqslant F_{\tau,v_n^\tau}(v_n^\tau)$, that is,

$$\frac{1}{2\tau}\|v_{n+1}^\tau - v_n^\tau\|_2^2 + E(v_{n+1}^\tau) \leqslant E(v_n^\tau). \tag{32}$$

Let $t_2 > t_1 \geqslant 0$ and put $n_i := [t_i/\tau]$ (the integer part of t_i/τ), $i = 1,2$. On the one hand, $n_2 \geqslant n_1$ and it readily follows from (32) by induction that

$$E(v^\tau(t_2)) = E(v_{n_2}^\tau) \leqslant E(v_{n_1}^\tau) = E(v^\tau(t_1)),$$

whence (29). In particular, we have

$$\frac{1}{2}\sup_{t\geqslant 0}\|\mu^\tau(t)\|_2^2 = \sup_{t\geqslant 0} E(v^\tau(t)) = \sup_{n\geqslant 0} E(v_n^\tau) \leqslant E(v_0^\tau) = E(v_0). \tag{33}$$

On the other hand, summing (32) over $n \in \mathbb{N}$ gives

$$\frac{1}{2\tau}\sum_{n=0}^\infty \|v_{n+1}^\tau - v_n^\tau\|_2^2 \leqslant E(v_0^\tau) = E(v_0), \tag{34}$$

from which we deduce that

$$\|v^\tau(t_2) - v^\tau(t_1)\|_2 = \|v_{n_2}^\tau - v_{n_1}^\tau\|_2 \leqslant \sum_{n=n_1}^{n_2-1}\|v_{n+1}^\tau - v_n^\tau\|_2 \leqslant (n_2-n_1)^{1/2}\Big(\sum_{n=n_1}^{n_2-1}\|v_{n+1}^\tau - v_n^\tau\|_2^2\Big)^{1/2}$$

$$\leqslant \left(1 + \frac{t_2 - t_1}{\tau}\right)^{1/2}(2\tau E(v_0))^{1/2} \leqslant \sqrt{2E(v_0)}(\tau + (t_2 - t_1))^{1/2},$$

and thus (30). Finally, for $n \geqslant 0$, we have $\overline{v_{n+1}^\tau} = \overline{v_n^\tau} = \alpha$ by (26) and we infer from (22) that

$$\|-\Delta\mu_{n+1}^\tau + W''(v_{n+1}^\tau)\mu_{n+1}^\tau - M_{n+1}^\tau\|_2 \leqslant \|v_{n+1}^\tau - v_n^\tau\|_2/\tau.$$

Combining (34) and the previous inequality gives

$$\int_\tau^\infty \|-\Delta\mu^\tau(t) + W''(v^\tau(t))\mu^\tau(t) - M^\tau(t)\|_2^2\,\mathrm{d}t$$
$$\leqslant \sum_{n=0}^\infty \int_{(n+1)\tau}^{(n+2)\tau} \|-\Delta\mu_{n+1}^\tau + W''(v_{n+1}^\tau)\mu_{n+1}^\tau - M_{n+1}^\tau\|_2^2\,\mathrm{d}t \leqslant \sum_{n=0}^\infty \|v_{n+1}^\tau - v_n^\tau\|_2^2/\tau \leqslant 2E(v_0),$$

and the proof is complete. □

Useful bounds on $(v^\tau)_\tau$ and $(\mu^\tau)_\tau$ follow from Lemma 5.

COROLLARY 6 For all $T > 0$, there is $C_3(T) > 0$ depending only on α, v_0, W, and T such that, for $\tau \in (0,1) \cap (0,T)$,

$$\sup_{t\in[0,T]} \|v^\tau(t)\|_{H^2} \leqslant C_3(T), \tag{35}$$

$$\int_\tau^T (\|\mu^\tau(t)\|_{H^1}^4 + \|\mu^\tau(t)\|_{H^2}^2)\,\mathrm{d}t \leqslant C_3(T). \tag{36}$$

Proof. The boundedness (35) of $(v^\tau)_\tau$ is a straightforward consequence of (14) and (33). Next, owing to the continuous embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$ and (35), the family $(W''(v^\tau))_\tau$ is bounded in $L^\infty((0,T)\times\Omega)$, which, together with (33), implies that

$$(W''(v^\tau)\mu^\tau)_\tau \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{37}$$

Setting $f^\tau := -\Delta\mu^\tau + W''(v^\tau)\mu^\tau - M^\tau$, it follows from (31) and (37) that

$$\left(\int_\tau^T \|\Delta\mu^\tau(t)\|_2^2\,\mathrm{d}t\right)^{1/2} = \left(\int_\tau^T \|W''(v^\tau(t))\mu^\tau(t) - M^\tau(t) - f^\tau(t)\|_2^2\,\mathrm{d}t\right)^{1/2}$$
$$\leqslant 2\left(\int_\tau^T \|W''(v^\tau(t))\mu^\tau(t)\|_2^2\,\mathrm{d}t\right)^{1/2} + \left(\int_\tau^T \|f^\tau(t)\|_2^2\,\mathrm{d}t\right)^{1/2} \leqslant C(T),$$

which gives the boundedness of $(\mu^\tau)_\tau$ in $L^2(\tau,T;H^2(\Omega))$ with the help of (33). Finally, μ^τ is in V and solves

$$-\Delta\mu^\tau + (W''(v^\tau) + C_0)\mu^\tau = f^\tau + C_0\mu^\tau + M^\tau \quad \text{in } \Omega.$$

Taking the scalar product in $L^2(\Omega)$ of the previous equation with μ^τ and using the nonnegativity (9) of $W'' + C_0$, we obtain

$$\|\nabla\mu^\tau\|_2^2 \leqslant \|\nabla\mu^\tau\|_2^2 + \int_\Omega (W''(v^\tau) + C_0)(\mu^\tau)^2\,\mathrm{d}x \leqslant \|f^\tau\|_2\|\mu^\tau\|_2 + C_0\|\mu^\tau\|_2^2 + |M^\tau|\,\|\mu^\tau\|_2.$$

We next deduce from (33) and (37) that

$$\|\nabla\mu^\tau\|_2^2 \leqslant C(T)(1 + \|f^\tau\|_2),$$

and the boundedness of the right-hand side of the above inequality in $L^2(\tau,T)$ follows at once from (31). □

2.3 *Convergence*

Owing to (30), (35), and the compactness of the embedding of $H^2(\Omega)$ in $\mathcal{C}(\bar{\Omega})$, a refined version of the Ascoli–Arzelà theorem (in the spirit of [1, Prop. 3.3.1]) ensures that $(v^\tau)_\tau$ is relatively compact in $\mathcal{C}([0,T]\times\bar{\Omega})$ for all $T>0$. Consequently, there are three functions v, μ, and M and a subsequence $(v^{\tau_k})_{k\geqslant 1}$ of $(v^\tau)_\tau$ such that, for all $T>0$,

$$v\in\mathcal{C}([0,T]\times\bar{\Omega})\cap L^\infty(0,T;H^2(\Omega)),\qquad \mu\in L^\infty(0,T;L^2(\Omega)),\qquad M\in L^\infty(0,T),$$

and

$$v^{\tau_k}(t)\to v(t)\qquad \text{in } \mathcal{C}(\bar{\Omega}) \text{ for all } t\in[0,T], \tag{38}$$

$$v^{\tau_k}\overset{*}{\rightharpoonup} v\qquad \text{in } L^\infty(0,T;H^2(\Omega)), \tag{39}$$

$$\mu^{\tau_k}\overset{*}{\rightharpoonup} \mu\qquad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{40}$$

$$M^{\tau_k}\overset{*}{\rightharpoonup} M\qquad \text{in } L^\infty(0,T). \tag{41}$$

Thanks to the smoothness (8) of W and the convergences (38)–(41), it is straightforward to pass to the limit in (27) and conclude that

$$\mu=-\Delta v+W'(v)\quad\text{and}\quad M=\overline{W''(v)\mu}. \tag{42}$$

In addition, (36), (40), and a lower semicontinuity argument guarantee that

$$\mu\in L^4(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\qquad\text{for all } T>0. \tag{43}$$

It remains to derive the equation solved by v. Let $\psi\in V$, $t>0$, $n=[t/\tau]$, and $m\in\{0,\ldots,n-1\}$. Using the definition of v^τ_{m+1} and Lemma 4, we are led to

$$\int_\Omega\left[\frac{v^\tau_{m+1}-v^\tau_m}{\tau}-\Delta\mu^\tau_{m+1}+W''(v^\tau_{m+1})\mu^\tau_{m+1}-M^\tau_{m+1}\right]\psi\,\mathrm{d}x=0,$$

which also reads

$$\int_\Omega(v^\tau_{m+1}-v^\tau_m)\psi\,\mathrm{d}x=\int_{(m+1)\tau}^{(m+2)\tau}\int_\Omega[\Delta\mu^\tau(s)-W''(v^\tau(s))\mu^\tau(s)+M^\tau(s)]\psi\,\mathrm{d}x\,\mathrm{d}s.$$

Summing the above identities over $m\in\{0,\ldots,n-1\}$ and recalling (28), we obtain

$$\int_\Omega(v^\tau(t)-v_0)\psi\,\mathrm{d}x=\int_\tau^{(n+1)\tau}\int_\Omega[\Delta\mu^\tau(s)-W''(v^\tau(s))\mu^\tau(s)+M^\tau(s)]\psi\,\mathrm{d}x\,\mathrm{d}s.$$

Noticing that $t\leqslant(n+1)\tau\leqslant t+\tau$, we may take $\tau=\tau_k$ in the above identity and pass to the limit as $k\to\infty$ with the help of (38)–(41) to obtain

$$\int_\Omega(v(t)-v_0)\psi\,\mathrm{d}x=\int_0^t\int_\Omega[\Delta\mu(s)-W''(v(s))\mu(s)+M(s)]\psi\,\mathrm{d}x\,\mathrm{d}s. \tag{44}$$

Collecting (42)–(44) completes the proof of the existence part of Theorem 1. The properties (11) and (12) next follow from (29), (31), and the convergences (38)–(41).

REMARK 7 It is not difficult to check that the above proof actually only requires W to be $\mathcal{C}^2$-smooth, so that the existence statement of Theorem 1 is also true under this weaker assumption.

3. Uniqueness

Let v_1 and v_2 be two solutions to (1)–(4) with $\mu_i := -\Delta v_i + W'(v_i)$ and $M_i := \overline{W''(v_i)\mu_i}$, $i = 1, 2$. Fix $T > 0$. Since $H^2(\Omega)$ is continuously embedded in $L^\infty(\Omega)$, the regularity properties of v_1, v_2, μ_1, and μ_2 listed in Theorem 1 ensure that there is $K > 0$ depending on T such that

$$\sup_{t\in[0,T]} (\|v_1(t)\|_\infty + \|v_2(t)\|_\infty + \|\mu_1(t)\|_2 + \|\mu_2(t)\|_2) + \int_0^T (\|\mu_1(s)\|_\infty^2 + \|\mu_2(s)\|_\infty^2)\,\mathrm{d}s \leqslant K. \quad (45)$$

It then follows from (45) and the smoothness (8) of W that

$$\begin{aligned} |W''(v_1)\mu_1 - W''(v_2)\mu_2| &\leqslant |W''(v_1) - W''(v_2)|\,|\mu_1| + |W''(v_2)|\,|\mu_1 - \mu_2| \\ &\leqslant \|W'''\|_{L^\infty(-K,K)}|v_1 - v_2|\,|\mu_1| + \|W''\|_{L^\infty(-K,K)}|\mu_1 - \mu_2| \\ &\leqslant C(|\mu_1|\,|v_1 - v_2| + |\mu_1 - \mu_2|), \end{aligned} \quad (46)$$

from which we deduce that

$$\begin{aligned} |M_1 - M_2| &\leqslant \frac{1}{|\Omega|}\int_\Omega |W''(v_1)\mu_1 - W''(v_2)\mu_2|\,\mathrm{d}x \leqslant C\int_\Omega (|\mu_1|\,|v_1 - v_2| + |\mu_1 - \mu_2|)\,\mathrm{d}x \\ &\leqslant C(\|\mu_1\|_2\|v_1 - v_2\|_2 + \|\mu_1 - \mu_2\|_2). \end{aligned} \quad (47)$$

Since $v_1 - v_2$ solves

$$\partial_t(v_1 - v_2) - \Delta(\mu_1 - \mu_2) = M_1 - M_2 - W''(v_1)\mu_1 + W''(v_2)\mu_2$$

and $v_1 - v_2$, $\mu_1 - \mu_2$ both belong to V, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_1 - v_2\|_2^2 &= \int_\Omega (\mu_1 - \mu_2)\Delta(v_1 - v_2)\,\mathrm{d}x + \int_\Omega (M_1 - M_2)(v_1 - v_2)\,\mathrm{d}x \\ &\quad - \int_\Omega [W''(v_1)\mu_1 - W''(v_2)\mu_2](v_1 - v_2)\,\mathrm{d}x. \end{aligned}$$

We deduce from (2), (45), (46), and (47) that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_1 - v_2\|_2^2 &= \int_\Omega (\mu_1 - \mu_2)[W'(v_1) - W'(v_2) - (\mu_1 - \mu_2)]\,\mathrm{d}x + \int_\Omega (M_1 - M_2)(v_1 - v_2)\,\mathrm{d}x \\ &\quad - \int_\Omega [W''(v_1)\mu_1 - W''(v_2)\mu_2](v_1 - v_2)\,\mathrm{d}x \\ &\leqslant \|W''\|_{L^\infty(-K,K)}\|\mu_1 - \mu_2\|_2\|v_1 - v_2\|_2 - \|\mu_1 - \mu_2\|_2^2 \\ &\quad + C(\|\mu_1\|_2\|v_1 - v_2\|_2 + \|\mu_1 - \mu_2\|_2)\|v_1 - v_2\|_2 \\ &\quad + C\int_\Omega (|\mu_1|\,|v_1 - v_2| + |\mu_1 - \mu_2|)|v_1 - v_2|\,\mathrm{d}x \\ &\leqslant C\|\mu_1 - \mu_2\|_2\|v_1 - v_2\|_2 - \|\mu_1 - \mu_2\|_2^2 + C(1 + \|\mu_1\|_\infty)\|v_1 - v_2\|_2^2 \\ &\leqslant C(1 + \|\mu_1\|_\infty)\|v_1 - v_2\|_2^2. \end{aligned}$$

Therefore, recalling (45),

$$\|(v_1 - v_2)(t)\|_2^2 \leqslant \|(v_1 - v_2)(0)\|_2^2 \exp\left(C\int_0^t (1 + \|\mu_1(s)\|_\infty)\,\mathrm{d}s\right) \leqslant C\|(v_1 - v_2)(0)\|_2^2$$

for $t \in [0, T]$, and the uniqueness assertion follows.

REMARK 8 Using the same kind of estimate, one can prove that, in the case when $f \in V_\alpha$, the minimizer w given by Lemma 3 is unique provided τ is sufficiently small. Indeed, assume for contradiction that there are two different minimizers w_1 and w_2 and introduce the related quantities $\mu_i = -\Delta w_i + W'(w_i)$, $i = 1, 2$, as in Lemma 4. Since $f \in V_\alpha$ and w_i is a minimizer of $F_{\tau,f}$ (cf. (18)), one has

$$\frac{1}{2\tau}\|w_i - f\|_2^2 + E(w_i) \leqslant E(f), \qquad i = 1, 2,$$

and consequently, on account of (22) as well, it turns out that estimates independent of τ can be found for $\|w_i\|_\infty$ and for $\|\tau^{1/2}\mu_i\|_\infty$, $i = 1, 2$. At this point, we take the difference of the two equations (21) written for w_1, μ_1 and w_2, μ_2 respectively, and choose $\psi = w_1 - w_2$. Then, arguing as above, it is not difficult to deduce that

$$\frac{1}{\tau}\|w_1 - w_2\|_2^2 \leqslant C(1 + \|\mu_1\|_\infty)\|w_1 - w_2\|_2^2,$$

whence $\|w_1 - w_2\|_2^2 \leqslant C\tau^{1/2}\|w_1 - w_2\|_2^2$. This means that, for τ small enough, $\|w_1 - w_2\|_2 = 0$ and uniqueness of the minimizer follows. Observe that this property is significant for the time discretization of Subsection 2.2 as, in view of (25) and the assumption $v_0 \in V_\alpha$, uniqueness is ensured for our time discrete solution as soon as the time step is smaller than a suitable one.

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