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Longtime behavior of a diffuse interface model for binary fluid mixtures with shear dependent viscosity

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In memoriam Andrea Maffio (1962–2009)

We consider a system which describes the behavior of a binary mixture of immiscible incompressible fluids with shear dependent viscosity by means of the diffuse interface approach. This system consists of Navier–Stokes type equations, characterized by a nonlinear stress-strain law, which are nonlinearly coupled with a convective Cahn–Hilliard equation for the order parameter. We analyze the corresponding dynamical system and, by means of the short trajectory method, we prove the existence of global and exponential attractors. We also discuss the dependence of an upper bound of the fractal dimension on the physical parameters of the system.

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1. Introduction

The mathematical treatment of sharp interface problems is rather complicated also from the numerical viewpoint. Therefore, it is particularly convenient to introduce models where the interface is diffused (i.e., it has some small thickness ε). This is done by means of a convenient order parameter whose evolution is governed by a gradient flow type equation (see, e.g., [3]). For instance, if we want to describe the motion of a (homogeneous) incompressible, isothermal and immiscible binary fluid mixture (e.g., oil and water), a typical model is the so-called Cahn–Hilliard-Navier–Stokes system for the (mean) fluid velocity $\mathbf{u}$ and the order parameter ϕ (i.e., the relative concentration of one phase). This system reads as follows (with unit density):

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\boldsymbol{\tau}(\phi, \mathbf{e}(\mathbf{u})) + \nabla\pi = k\mu\nabla\phi + \mathbf{g}, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = \mathbf{0}, \tag{1.2}$$

$$\partial_t\phi + \mathbf{u}\cdot\nabla\phi - m\Delta\mu = 0, \tag{1.3}$$

$$\mu = -\varepsilon\Delta\phi + \alpha F'(\phi), \tag{1.4}$$

in $\Omega \times (0, T)$, $\Omega \subseteq \mathbb{R}^N$, $N = 2, 3$, $T > 0$. Here $k, \varepsilon, m, \alpha$ are given positive constants, π denotes the pressure, $\mathbf{g}$ is a given (time-independent) external force and F is a given double-well potential. The stress tensor $\boldsymbol{\tau}$ is defined by the constitutive relation

$$\boldsymbol{\tau}(\phi, \mathbf{e}(\mathbf{u})) = \nu(\phi)\mathbf{e}(\mathbf{u}), \tag{1.5}$$

where ν is a strictly positive function and $\mathbf{e}$ is the symmetric velocity gradient, namely,

$$\mathbf{e}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{tr}). \tag{1.6}$$

System (1.1)–(1.5) is known as model H and was proposed in [19] (see also [29]) and then rigorously justified in [15]. From the mathematical viewpoint, this system has been first studied in [31] for $\Omega = \mathbb{R}^2$. Then, in the case of bounded domains, a careful analysis has been carried out in [6] (see also [7]). More recently, the case of logarithmic potentials and constant mobility m has been considered in [1], where, in particular, the convergence of solutions to a single equilibrium has been established in absence of nongradient external forces. This issue has also been investigated in [33] for smooth potentials. A rather complete picture of the longtime behavior in the case $N = 2$ on a bounded domain can be found in [13]. In the case $N = 3$, existence of trajectory attractors has been demonstrated in [14] with time-dependent external forces (see also [2] for an alternative approach in the case $\mathbf{g} \equiv \mathbf{0}$). Regarding the numerical analysis of Cahn–Hilliard–Navier–Stokes systems we refer the reader to, e.g., [4, 12, 16, 17, 21, 28] and their references.

Here we want to consider a nontrivial generalization of this model which accounts for a shear dependent viscosity. More precisely, instead of (1.5), we assume the following stress-strain relationship:

$$\boldsymbol{\tau}(\phi, \mathbf{e}(\mathbf{u})) = (\nu_1(\phi) + \nu_2(\phi)|\mathbf{e}(\mathbf{u})|^{p-2})\mathbf{e}(\mathbf{u}). \tag{1.7}$$

Here ν_i are strictly positive given functions and $p > 1$. Of course, in the case $p = 2$ we obtain the previous model.

In the case of single fluids, assumption (1.7) is known as the Ladyzhenskaya model (see [20]), while the particular case $p = 3$ is the Smagorinsky model of turbulence (cf. [30]). We recall that, in the case $N = 3$, the main features of the corresponding generalized Navier–Stokes equations endowed with, say, no-slip or periodic boundary conditions are the uniqueness of weak solutions if $p \geqslant 5/2$ and the existence of a (unique) strong solution if $p \geqslant 11/5$. This problem has been widely and deeply investigated in recent years by several people (see [5, 11, 23, 25] and references therein). Regarding the asymptotic behavior, in the seminal paper [22], the authors introduced a new method to prove the existence of a global attractor of finite fractal dimension. This approach, now known as the *short trajectory method*, has been refined in [24] (see also [11, 7.4]) to prove the existence of exponential attractors as well. This method is actually rather flexible and it has been applied to many other dynamical systems so far (see, e.g., [26, Rem. 3.8] and references therein).

System (1.1)–(1.4) with a generalized version of (1.7) has been considered first in [18]. There, some existence, uniqueness and regularity results have been proven on bounded domains and for periodic or no-slip and no-flux boundary conditions for $\mathbf{u}$ and ϕ, μ, respectively. The case of singular potentials has also been considered, proving the existence of a measure-valued solution. Here, in the case of smooth potentials and periodic boundary conditions, we want to investigate the asymptotic behavior of (1.1)–(1.4), (1.7) along the lines of [24]. More precisely, using the short trajectory method, we prove the existence of a global attractor and of an exponential attractor. In addition, we observe that the upper bound of the fractal dimension of these attractors grows at most polynomially with the data. We will essentially focus on the case $N = 3$ with $p \geqslant 11/5$. The degenerate case $p \in (1, 2)$ and $N = 2$ will be treated elsewhere. Boundary conditions like no-slip for $\mathbf{u}$ and no-flux for ϕ and μ might also be considered possibly with further restrictions, e.g., on p (see [11, 7.4] and references therein for simple fluids, cf. also Remark 2.7).

The present analysis requires combining some different existing, albeit nontrivial, techniques and we think that it is a further significant application of the short trajectory approach. Moreover, this is a first step towards the possibility of considering similar models with more challenging features like, say, singular potentials and degenerate mobility coefficients.

Equations (1.1)–(1.4) and (1.7) are given in $\mathbb{R}^N \times (0, \infty)$, and $\mathbf{u}$, ϕ and the chemical potential μ are supposed to be L-periodic with respect to each variable x_k, $k = 1, \dots, N$. The system is also endowed with initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \phi(0) = \phi_0. \tag{1.8}$$

The plan of the paper is the following. In Section 2 we introduce the basic assumptions and we define the notion of weak solution. Then we state and prove some existence, regularity and continuous dependence results. Section 3 will be devoted to the main results about the asymptotic behavior, that is, the existence of global and exponential attractors.

2. Well-posedness and smoothness

The aim of this section is to establish main results concerning the properties of solutions to our system. Namely, we prove the compactness of the set of weak solutions (Theorem 2.4), which is tantamount to the (global) existence of weak solutions. A key ingredient of the subsequent analysis is the regularity result (Theorem 2.6), which ensures the uniqueness of solutions (Theorem 2.8) under sufficiently general conditions.

Let us begin by stating our assumptions on the potential F:

$$\begin{aligned} & F \in C^3(\mathbb{R}; \mathbb{R}), \\ & \liminf_{|y|\to\infty} F''(y) > 0, \\ & |F^{(3)}(y)| \leqslant C_F(1 + |y|^{r-1}), \qquad r \in \begin{cases} [1, 3], & N = 3, \\ [1, \infty), & N = 2. \end{cases} \end{aligned} \tag{2.1}$$

Moreover, for the sake of simplicity, we can assume $F \geqslant 0$.

Regarding ν_i, $i = 1, 2$, we suppose

$$\begin{aligned} & \nu_1, \nu_2 \in C^1(\mathbb{R}; \mathbb{R}), \\ & \min_{y\in\mathbb{R}} \nu_i(y) \geqslant \nu_*^i, \qquad \max_{y\in\mathbb{R}} \nu_i(y) \leqslant \nu_i^*, \qquad \max_{y\in\mathbb{R}} |\nu_i'(y)| \leqslant \nu_i^\sharp, \end{aligned} \tag{2.2}$$

for some positive ν_*^i, ν_i^*, and $\nu_i^\sharp$.

Also, we take $m = 1$ in (1.3) and we observe that the stress tensor $\boldsymbol{\tau}$ has the following properties:

$$\begin{aligned} &\boldsymbol{\tau}(\phi, \mathbf{0}) = \mathbf{0}, \\ &|\boldsymbol{\tau}(\phi, \mathbf{e}) - \boldsymbol{\tau}(\phi, \tilde{\mathbf{e}})| \leqslant c_1 \nu^* (1 + |\mathbf{e}| + |\tilde{\mathbf{e}}|)^{p-2} |\mathbf{e} - \tilde{\mathbf{e}}|, \\ &|\boldsymbol{\tau}(\phi, \mathbf{e}) - \boldsymbol{\tau}(\tilde{\phi}, \mathbf{e})| \leqslant c_2 \nu^\sharp (1 + |\mathbf{e}|)^{p-1} |\phi - \tilde{\phi}|, \\ &(\boldsymbol{\tau}(\phi, \mathbf{e}) - \boldsymbol{\tau}(\phi, \tilde{\mathbf{e}})) : (\mathbf{e} - \tilde{\mathbf{e}}) \geqslant c_3 \nu_* (1 + |\mathbf{e}| + |\tilde{\mathbf{e}}|)^{p-2} |\mathbf{e} - \tilde{\mathbf{e}}|^2, \end{aligned} \tag{2.3}$$

where

$$\nu^* := \max\{\nu_1^*, \nu_2^*\}, \qquad \nu_* := \min\{\nu_*^1, \nu_*^2\}, \qquad \nu^\sharp := \max\{\nu_1^\sharp, \nu_2^\sharp\}, \tag{2.4}$$

for some suitable p such that

$$p \geqslant \frac{3N+2}{N+2}. \tag{2.5}$$

All the functions here considered are L-periodic in space with respect to each variable and the reference domain is $\Omega = (0, L)^N$. By $H^a_{per}(\Omega)$, $a > 0$, we denote the Banach space of L-periodic functions which belong to $W^{a,2}(\Omega)$ whose norm is defined by $\| \cdot \|_{H^a_{per}(\Omega)} = \| \cdot \|_{W^{a,2}(\Omega)}$. The notation L^2_{div} and $W^{a,p}_{div}$, $a > 0$, is reserved for the vector-valued functions of $L^2(\Omega; \mathbb{R}^N)$ and $W^{a,p}(\Omega; \mathbb{R}^N)$ which are divergence-free and have zero mean value. The latter canonical norms are indicated as $\|\cdot\|_2$ and $\|\cdot\|_{a,p}$, respectively. An equivalent norm $\|\nabla \cdot\|_p$ is often used as well. In addition, if V is a Banach space, then V^* stands for its dual.

DEFINITION 2.1 Assume $(\mathbf{u}_0, \phi_0) \in L^2_{div} \times H^1_{per}$ and $\mathbf{g} \in (W^{1,2}_{div})^*$. For any given $T > 0$, the pair $(\mathbf{u}, \phi)$ is called a *weak solution* provided that

$$\mathbf{u} \in L^\infty(0, T; L^2_{div}) \cap L^p(0, T; W^{1,p}_{div}), \tag{2.6}$$

$$\phi \in L^\infty(0, T; H^1_{per}) \cap L^2(0, T; H^3_{per}), \tag{2.7}$$

and (1.1)–(1.4) hold in the sense of distributions in $\Omega \times (0, T)$ with (1.7) and (1.8). Concerning (1.1), only test functions with zero (spatial) divergence are considered. In addition, the spatial average of ϕ is conserved, that is,

$$\overline{\phi}(t) \equiv \overline{\phi}_0, \qquad \forall t \geqslant 0, \tag{2.8}$$

where $\overline{\phi} := |\Omega|^{-1} \int_\Omega \phi \, dx$.

REMARK 2.2 Note that (2.8) follows by taking 1 as test function in (1.3). Moreover, due to our assumptions (2.1) and (2.5) all the nonlinearities are integrable, and for finite $T > 0$,

$$\partial_t \mathbf{u} \in L^{p'}(0, T; (W^{1,p}_{div})^*), \tag{2.9}$$

$$\partial_t \phi \in L^2(0, T; (H^1_{per})^*). \tag{2.10}$$

Consequently, any weak solution has a representative

$$(\mathbf{u}, \phi) \in C([0, T]; L^2_{div} \times H^1_{per}), \tag{2.11}$$

so that (1.8) makes sense. Note that $\mathbf{u}$, ϕ and $\Delta\phi$ are admissible test functions for (1.1) and (1.3), respectively. Finally, we note that the pressure π is excluded from the subsequent analysis thanks to the fact we only work with divergence free test functions for (1.1). In the current setting of periodic boundary conditions, the existence of a function π such that (1.1) holds in the sense of distributions can be established a posteriori (see, e.g., [32, Ch. 3, Prop. 1.1]).

REMARK 2.3 We will observe throughout our analysis that, as one might expect, the most difficult term to handle is the convective one in (1.1), which gives rise to the lower bound (2.5) (see [23, Ch. 5, Lemma 2.44]).

For future reference, it will be useful to write down the abstract weak formulation of (1.1) explicitly, that is,

$$\langle \partial_t \mathbf{u} + N(\phi, \mathbf{e}(\mathbf{u})) + \mathbf{u}\cdot\nabla\mathbf{u} - k\mu\nabla\phi - \mathbf{g}, \mathbf{v}\rangle = 0, \qquad \forall \mathbf{v} \in L^p(0,T;W^{1,p}_{div}), \tag{2.12}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality between $L^{p'}(0,T;(W^{1,p}_{div})^*)$ and $L^p(0,T;W^{1,p}_{div})$, and

$$\langle N(\phi,\mathbf{e}(\mathbf{u})), \mathbf{v}\rangle = \int_{\Omega\times(0,T)} \boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t. \tag{2.13}$$

THEOREM 2.4 Let $(\mathbf{u}^n, \phi^n)$ be a sequence of weak solutions such that $(\mathbf{u}^n(0), \phi^n(0)) \to (\mathbf{u}_0, \phi_0)$ in $L^2_{div} \times H^1_{per}$. Then, modulo a subsequence, $\mathbf{u}^n \to \mathbf{u}$, $\phi^n \to \phi$ in the spaces specified in (2.17) and (2.18) below, and $(\mathbf{u}, \phi)$ is again a weak solution.

Proof. We confine ourselves to the case $N = 3$. The two-dimensional case can be treated in the same way with a more general growth condition on F (cf. (2.1)). Also, from now on, C_F will denote a positive constant which controls the growth of F and depends on F only, but it may change from line to line. Multiply (1.1) by $k^{-1}\mathbf{u}^n$, (1.3) by μ, and add the resulting equations. The convective term in (1.1) disappears (cf. [23, Ch. 5, Lemma 2.9]). The convective term in (1.3) cancels out with the coupling term on the right-hand side of (1.1). Using (2.3) and Korn's inequality, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\mathbf{u}^n,\phi^n] + c_1\nu_*\|\nabla\mathbf{u}^n\|_2^2 + c_2\nu_*\|\nabla\mathbf{u}^n\|_p^p + c_3\|\nabla\mu\|_2^2 \leqslant c_4\|\mathbf{g}\|^2_{(W^{1,2}_{div})^*}, \tag{2.14}$$

where

$$E[\mathbf{u},\phi] := \frac{1}{k}\|\mathbf{u}\|_2^2 + \varepsilon\|\nabla\phi\|_2^2 + 2\alpha\int_\Omega F(\phi)\,\mathrm{d}x. \tag{2.15}$$

We readily deduce that

$$\{\mathbf{u}^n\} \text{ is bounded in } L^\infty(0,T;L^2_{div}) \cap L^p(0,T;W^{1,p}_{div}).$$

By Poincaré's inequality, since $\overline{\phi^n} \equiv \overline{\phi^n}(0)$ is bounded, we further see that

$$\{\phi^n\} \text{ is bounded in } L^\infty(0,T;H^1_{per}).$$

Similarly, if $N = 3$, we have

$$\|\mu^n\|_{H^1_{per}} \leqslant c(\|\nabla\mu^n\|_2^2 + |\overline{F'(\phi^n)}|),$$
$$|\overline{F'(\phi^n)}| \leqslant C_F\int_\Omega (1+|\phi^n|^4)\,\mathrm{d}x \leqslant C_F(1+\|\phi^n\|^4_{H^1_{per}}).$$

Hence we find that

$$\{\mu^n\} \text{ is bounded in } L^2(0,T;H^1_{per}).$$

On the other hand, we have

$$c^{-1}\|\phi^n\|_{H^3_{per}} \leqslant \|\nabla\mu^n\|_2 + \|F''(\phi^n)\nabla\phi^n\|_2 + |\overline{\phi^n}|,$$
$$\|F''(\phi^n)\nabla\phi^n\|_2 \leqslant \|F''(\phi^n)\|_2\|\nabla\phi^n\|_\infty \leqslant C_F(1+\|\phi^n\|_6^3)\|\phi^n\|^{1/4}_{H^1_{per}}\|\phi^n\|^{3/4}_{H^3_{per}}.$$

Here (and below), we frequently use the interpolation inequalities

$$\|\varphi\|_\infty \leqslant c\|\varphi\|^{3/4}_{H^1_{per}}\|\varphi\|^{1/4}_{H^3_{per}}, \qquad \|\nabla\varphi\|_\infty \leqslant c\|\varphi\|^{1/4}_{H^1_{per}}\|\varphi\|^{3/4}_{H^3_{per}}, \tag{2.16}$$

as well as the embedding $H^1_{per} \hookrightarrow L^6$. We eventually find that

$$\{\phi^n\} \text{ is bounded in } L^2(0,T;H^3_{per}).$$

With this information at hand, we come to a subsequence (not relabelled) such that

$$\begin{aligned}
\mathbf{u}^n &\to \mathbf{u} && \text{weakly in } L^p(0,T;W^{1,p}_{div}),\\
\phi^n &\to \phi && \text{weakly in } L^2(0,T;H^3_{per}),\\
\mu^n &\to \mu && \text{weakly in } L^2(0,T;H^1_{per}),\\
\partial_t\mathbf{u}^n &\to \partial_t\mathbf{u} && \text{weakly in } L^{p'}(0,T;(W^{1,p}_{div})^*),\\
\partial_t\phi^n &\to \partial_t\phi && \text{weakly in } L^2(0,T;(H^1_{per})^*).
\end{aligned} \tag{2.17}$$

In virtue of the Aubin–Lions lemma, we also have

$$\begin{aligned}
\mathbf{u}^n &\to \mathbf{u} && \text{strongly in } L^p(0,T;L^s)\ \forall s\in[1,3p/(3-p)),\\
\mathbf{u}^n &\to \mathbf{u} && \text{in } C([0,T];(L^2_{div})_{\text{weak}}),\\
\phi^n &\to \phi && \text{strongly in } L^2(0,T;H^{3-\epsilon}_{per}) \quad \forall\epsilon>0.
\end{aligned} \tag{2.18}$$

Here the first convergence is for all $s \geqslant 1$ if $p \geqslant 3$. It is standard to obtain the limit in all the lower order nonlinearities; thus we will only treat the stress tensor. From (2.3) and (2.13) it follows that

$$\langle N(\phi^n,\mathbf{e}(\mathbf{v})) - N(\phi^n,\mathbf{e}(\mathbf{u}^n)), \mathbf{v}-\mathbf{u}^n\rangle \geqslant 0.$$

Then we obtain (cf. (2.12))

$$\langle \partial_t\mathbf{u}^n + N(\phi^n,\mathbf{e}(\mathbf{v})) + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - k\mu^n\nabla\phi^n - \mathbf{g}, \mathbf{v}-\mathbf{u}^n\rangle \geqslant 0$$

for an arbitrary smooth function $\mathbf{v}$. Thus we deduce

$$\begin{aligned}
&\langle \partial_t\mathbf{u}^n + N(\phi^n,\mathbf{e}(\mathbf{v})) + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - k\mu^n\nabla\phi^n - \mathbf{g}, \mathbf{v}\rangle\\
&\qquad \geqslant \frac{1}{2}\|u^n(T)\|_2^2 - \frac{1}{2}\|u^n(0)\|_2^2 + \langle N(\phi^n,\mathbf{e}(\mathbf{v})) - k\mu^n\nabla\phi^n - \mathbf{g}, \mathbf{u}^n\rangle.
\end{aligned}$$

Taking the lower limit, we have

$$\begin{aligned}
&\langle \partial_t\mathbf{u} + N(\phi,\mathbf{e}(\mathbf{v})) + (\mathbf{u}\cdot\nabla)\mathbf{u} - k\mu\nabla\phi - \mathbf{g}, \mathbf{v}\rangle\\
&\qquad \geqslant \frac{1}{2}\|u(T)\|_2^2 - \frac{1}{2}\|u(0)\|_2^2 + \langle N(\phi,\mathbf{e}(\mathbf{v})) - k\mu\nabla\phi - \mathbf{g}, \mathbf{u}\rangle
\end{aligned}$$

or

$$\langle \partial_t \mathbf{u} + N(\phi, \mathbf{e}(\mathbf{v})) + (\mathbf{u}\cdot\nabla)\mathbf{u} - k\mu\nabla\phi - \mathbf{g}, \mathbf{v}-\mathbf{u}\rangle \geqslant 0.$$

At this stage, $\mathbf{v}$ can be replaced by an arbitrary function in $L^p(0,T;W^{1,p}_{div})$; using a standard argument (see, e.g., [27, Lemma 9.43]), we deduce that $\mathbf{u}$ is also solution to (1.1). □

REMARK 2.5 Using a suitable approximating scheme, the above result readily gives the global existence of a weak solution (see also [18, Thm. 1]). For such a result, assumptions (2.1) can be weakened by taking, e.g., $F \in C^2(\mathbb{R};\mathbb{R})$.

It is well known that weak solutions—that is, having only regularity (2.6)—of the Ladyzhenskaya model are unique only if $p \geqslant (N+2)/2$ (see [18, Thm. 3]), which is stronger than (2.5) for $N = 3$. Thus, if we want to get uniqueness under (2.5), we have to improve the regularity of solutions (see [18, Sec. 6] for $N = 2$).

THEOREM 2.6 Let $\mathbf{g} \in L^2_{div}$ and $(\mathbf{u}_0, \phi_0) \in W^{1,2}_{div} \times H^2_{per}$. Then there exists a weak solution with regularity

$$\mathbf{u} \in L^\infty(0,T;W^{1,2}_{div}) \cap L^p(0,T;W^{1,3p}_{div}) \cap L^2(0,T;W^{2,2}_{div}), \tag{2.19}$$

$$\phi \in L^\infty(0,T;H^2_{per}) \cap L^2(0,T;H^4_{per}). \tag{2.20}$$

Proof. Let us assume $N = 3$. We again give a formal proof, which can be made rigorous at the level of suitable approximation. Rewrite (1.3)–(1.4) as

$$\phi_t + \varepsilon\Delta^2\phi = -\mathbf{u}\cdot\nabla\phi + \alpha F^{(3)}(\phi)|\nabla\phi|^2 + \alpha F''(\phi)\Delta\phi =: R_1 + R_2 + R_3.$$

Testing with $\Delta^2\phi$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\phi\|_2^2 + \varepsilon\|\Delta^2\phi\|_2^2 \leqslant c\varepsilon^{-1}\int_\Omega (R_1^2 + R_2^2 + R_3^2)\,\mathrm{d}x.$$

We begin with the estimate

$$\int_\Omega R_1^2\,\mathrm{d}x = \int_\Omega |\mathbf{u}|^2|\nabla\phi|^2\,\mathrm{d}x \leqslant \|\mathbf{u}\|_3^2\|\nabla\phi\|_6^2 \leqslant h_1(t)\|\phi\|^2_{H^2_{per}},$$

where

$$h_1(t) := c\|\nabla\mathbf{u}\|_p^2.$$

Furthermore, we have

$$\int_\Omega R_2^2\,\mathrm{d}x = \alpha^2\int_\Omega |F^{(3)}(\phi)|^2|\nabla\phi|^4\,\mathrm{d}x \leqslant C_F\alpha^2(1+\|\phi\|_{12}^4)\|\nabla\phi\|_6^4.$$

Using now the estimates

$$\|\phi\|_{12} \leqslant c\|\phi\|^{7/8}_{H^1_{per}}\|\phi\|^{1/8}_{H^3_{per}}, \qquad \|\nabla\phi\|_6 \leqslant c\|\phi\|_{H^2_{per}} \leqslant \tilde{c}\|\phi\|^{1/2}_{H^1_{per}}\|\phi\|^{1/2}_{H^3_{per}},$$

we obtain

$$\int_\Omega R_2^2\,\mathrm{d}x \leqslant h_2(t)\|\phi\|^2_{H^2_{per}},$$

where

$$h_2(t) := C_F\alpha^2(1 + \|\phi\|_{H^1_{per}}^{9/2}\|\phi\|_{H^3_{per}}^{3/2}).$$

Finally, we observe that

$$\int_\Omega R_3^2\,\mathrm{d}x = \alpha^2\int_\Omega |F''(\phi)|^2|\Delta\phi|^2\,\mathrm{d}x \leqslant C_F\alpha^2(1+\|\phi\|_\infty^6)\|\Delta\phi\|_2^2 \leqslant h_3(t)\|\phi\|_{H^2_{per}}^2,$$

where

$$h_3(t) := C_F\alpha^2(1 + \|\phi\|_{H^1_{per}}^{9/2}\|\phi\|_{H^3_{per}}^{3/2}).$$

Recalling Poincaré's inequality and the fact that $\overline{\phi}$ is a constant of motion, we eventually have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{H^2_{per}}^2 + \varepsilon\|\phi\|_{H^4_{per}}^2 \leqslant H_1(t)\|\phi\|_{H^2_{per}}^2, \tag{2.21}$$

where $H_1(t) := c\varepsilon(h_1(t) + h_2(t) + h_3(t) + 1)$. Hence (2.20) follows by Gronwall's lemma.

Further, arguing formally, we test (1.1) with $-\Delta\mathbf{u}$. More precisely, we test $\partial^2\mathbf{u}/\partial x_k^2$ and sum the resulting equation over k. Obviously,

$$\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right) = \frac{\partial}{\partial x_k}\mathbf{e}(\mathbf{u}).$$

Hence, from the stress tensor term, integrating by parts, we obtain

$$\begin{aligned} I(\phi,\mathbf{e}(\mathbf{u})) &:= \int_\Omega \boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u})) : \mathbf{e}(-\Delta\mathbf{u})\,\mathrm{d}x = -\int_\Omega \boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u})) : \frac{\partial}{\partial x_k}\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\mathrm{d}x \\ &= \int_\Omega \partial_{\mathbf{e}(\mathbf{u})}\boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u})) : \left(\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\otimes\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\right)\mathrm{d}x + \int_\Omega\left(\partial_\phi\boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u}))\frac{\partial}{\partial x_k}\phi\right) : \mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\mathrm{d}x. \end{aligned}$$

In the former integral, with a slight abuse of notation, : stands for the product of two fourth-order tensors. Writing everything in terms of components, one sees that

$$\partial_{\mathbf{e}(\mathbf{u})}\boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u})) : (\mathbb{A}\otimes\mathbb{A}) = \partial_{\mathbf{e}(\mathbf{u})_{ij}}\boldsymbol{\tau}(\phi,\mathbf{e}(\mathbf{u}))_{mn}\mathbb{A}_{ij}\mathbb{A}_{mn} \geqslant c(\nu_1(\phi_1) + \nu_2(\phi)|\mathbf{e}(\mathbf{u})|^{p-2})|\mathbb{A}|^2$$

for any symmetric tensor $\mathbb{A}\in\mathbb{R}^{3\times3}$. Thus we have

$$I(\phi,\mathbf{e}(\mathbf{u})) \geqslant \int_\Omega \mathbf{J}_p(\phi,\mathbf{u})\,\mathrm{d}x - c\nu^\sharp\int_\Omega(1+|\mathbf{e}(\mathbf{u})|)^{p-1}\left|\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\right||\nabla\phi|\,\mathrm{d}x,$$

where

$$\mathbf{J}_p(\phi,\mathbf{u}) := (\nu_1(\phi_1) + \nu_2(\phi)|\mathbf{e}(\mathbf{u})|^{p-2})\left|\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\right|^2,$$

with the implicit summation over k. We recall that the following estimate holds (see [23, Ch. 5, Lemma 3.24]):

$$\int_\Omega \mathbf{J}_p(\phi,\mathbf{u})\,\mathrm{d}x \geqslant c\nu_*\int_\Omega(1+|\mathbf{e}(\mathbf{u})|^{p-2})\left|\mathbf{e}\left(\frac{\partial}{\partial x_k}\mathbf{u}\right)\right|^2\mathrm{d}x \geqslant c_1\nu_*(\|\mathbf{u}\|_{W^{2,2}}^2 + \|\nabla\mathbf{u}\|_{3p}^p). \tag{2.22}$$

On the other hand, using the well-known orthogonality of the convective term,

$$\int_\Omega \frac{\partial}{\partial x_k}[(\mathbf{u}\cdot\nabla)\mathbf{u}] : \frac{\partial}{\partial x_k}\mathbf{u}\,\mathrm{d}x = \int_\Omega \left(\frac{\partial}{\partial x_k}\mathbf{u}\cdot\nabla\right)\mathbf{u} : \frac{\partial}{\partial x_k}\mathbf{u}\,\mathrm{d}x \leqslant \|\nabla\mathbf{u}\|_3^3,$$

while the right-hand side is estimated simply as

$$\left|\int_\Omega \mathbf{g}\cdot(-\Delta\mathbf{u})\,\mathrm{d}x\right| \leqslant \frac{c_1}{2}\nu_*\|\mathbf{u}\|^2_{W^{2,2}} + c\nu_*^{-1}\|\mathbf{g}\|_2^2.$$

Altogether, we have deduced the following inequality:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_2^2 + \mathbf{J}_p(\mathbf{u}) + c_1\nu_*\|\mathbf{u}\|^2_{W^{2,2}} + c_1\nu_*\|\nabla\mathbf{u}\|^p_{3p}\\
&\quad\leqslant c_2\nu^\sharp\int_\Omega(1+|\mathbf{e}(\mathbf{u})|)^{p-1}|\mathbf{e}(\nabla\mathbf{u})|\,|\nabla\phi|\,\mathrm{d}x + \|\nabla\mathbf{u}\|_3^3 + \int_\Omega|\mu|\,|\phi|\,|\Delta\mathbf{u}|\,\mathrm{d}x + c\nu_*^{-1}\|\mathbf{g}\|_2^2\\
&\quad=: P_1 + P_2 + P_3 + c\nu_*^{-1}\|\mathbf{g}\|_2^2.
\end{aligned}$$

Observe that

$$P_1 \leqslant \frac{1}{2}\int_\Omega \mathbf{J}_p(\mathbf{u})\,\mathrm{d}x + c\nu_*^{-1}\nu^{\sharp 2}\int_\Omega(1+|\mathbf{e}(\mathbf{u})|)^p|\nabla\phi|^2\,\mathrm{d}x,$$

where the last integral is estimated by

$$\|\nabla\phi\|_6^2(1+\|\nabla\mathbf{u}\|^p_{3p/2}) \leqslant c\|\phi\|^2_{H^2_{per}}(1+\|\nabla\mathbf{u}\|_p^{p/2}\|\nabla\mathbf{u}\|_{3p}^{p/2}) \leqslant \frac{c_1}{2}\nu_*\|\nabla\mathbf{u}\|^p_{3p} + h_4(t),$$

with

$$h_4(t) := c\nu_*^{-2}\nu^{\sharp 4}\|\phi\|^4_{H^2_{per}}(1+\|\nabla\mathbf{u}\|_p^p).$$

Similarly, we have

$$P_3 \leqslant \frac{c_1}{2}\nu_*\|\mathbf{u}\|^2_{W^{2,2}} + c\nu_*^{-1}\int_\Omega|\mu|^2|\nabla\phi|^2\,\mathrm{d}x,$$

where the integral is estimated by

$$c\nu_*^{-1}\|\mu\|_4^2\|\nabla\phi\|_4^2 \leqslant c\nu_*^{-1}\|\mu\|^2_{H^1_{per}}\|\phi\|^2_{H^2_{per}} =: h_5(t).$$

The most difficult term to handle is P_2, coming from the convective term. Here we slightly modify the technique of [23, Chap. 5, proof of Thm.3.4].

Let us assume $p < 3$ first. Using the interpolation inequalities

$$\begin{aligned}
\|v\|_3 &\leqslant \|v\|_p^\alpha\|v\|_{3p}^{1-\alpha}, \qquad \alpha = \frac{p-1}{2},\\
\|v\|_3 &\leqslant \|v\|_2^\beta\|v\|_{3p}^{1-\beta}, \qquad \beta = \frac{2(p-1)}{3p-2},
\end{aligned}$$

we can write

$$P_2 \leqslant \|\nabla\mathbf{u}\|_3^{3(1-a)}\|\nabla\mathbf{u}\|_3^{3a} \leqslant \|\nabla\mathbf{u}\|_2^{Q_1}\|\nabla\mathbf{u}\|_p^{Q_2}\|\nabla\mathbf{u}\|_{3p}^{Q_3},$$

where

$$Q_1 = \frac{6(1-a)(p-1)}{3p-2}, \qquad Q_2 = \frac{3a(p-1)}{2}, \qquad Q_3 = 3\left[\frac{(1-a)p}{3p-2} + \frac{a(3-p)}{2}\right].$$

We now apply Young's inequality with

$$\frac{1}{\delta} + \frac{1}{\delta'} = 1, \qquad \delta Q_3 = p, \qquad \delta' Q_1 = 2.$$

These conditions determine the choice of a and δ, namely,

$$a = \frac{4p}{3(p-1)(5p-6)}, \qquad \delta = \frac{5p-6}{3}.$$

We finally deduce

$$P_2 \leqslant \frac{c_1}{4}\nu_* \|\nabla \mathbf{u}\|_{3p}^p + H_3(t)\|\nabla \mathbf{u}\|_2^2,$$

where

$$H_3(t) := c\nu_*^{-\frac{3}{5p-9}} \|\nabla \mathbf{u}\|_p^{\frac{2p}{5p-9}}. \tag{2.23}$$

Observe that $2p/(5p-9) \leqslant p$ is equivalent to $p \geqslant 11/5$, and thus H_3 is integrable in time. Summing up, recalling (2.22), we come to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathbf{u}\|_2^2 + c_2\nu_*\|\mathbf{u}\|_{2,2}^2 + c_3\nu_*\|\nabla \mathbf{u}\|_{3p}^p \leqslant H_2(t) + H_3(t)\|\nabla \mathbf{u}\|_2^2, \tag{2.24}$$

where $H_2(t) := h_4(t) + h_5(t)$. Thus (2.19) follows by Gronwall's lemma.

The case $p \geqslant 3$ is simpler, since now $\|\nabla \mathbf{u}\|_3^3$ is integrable (see (2.14)). Hence (2.24) holds with $H_2 = h_4 + h_5 + \|\nabla \mathbf{u}\|_3^3$ and $H_3 \equiv 0$. □

REMARK 2.7 In the case of simple fluids, following [8], if $N = 3$ and

$$p > 12/5, \tag{2.25}$$

the regularity of solutions can be proven in the case $\mathbf{g} \in (W^{1,2}_{div})^*$. This result is obtained by taking $\mathbf{u}_t$ in place of $-\Delta\mathbf{u}$ as a test function. It should be possible to extend [8, Thm. 3.3] to system (1.1)–(1.4) provided that ν_1 and ν_2 are constants. If so, all the following results would hold under a more general (nongradient) external force. We recall that the longterm dynamics is strongly affected by the presence of such a force (see Remark 3.2 below). We also refer to [9] for an alternative approach to improving regularity of three-dimensional non-Newtonian fluids, based on estimates of fractional time differences, which works for $p > 11/5$ and is independent of boundary conditions.

THEOREM 2.8 Any solution satisfying (2.19) and (2.20) is unique in the class of weak solutions. More precisely: given two solutions $(\mathbf{u}_1, \phi_1)$, $(\mathbf{u}_2, \phi_2)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{w}\|_2^2 + \|\psi\|^2_{H^1_{per}}) + \varepsilon\|\psi\|^2_{H^3_{per}} + c_1\nu_*\|\nabla\mathbf{w}\|_2^2 + c_1\nu_*\int_\Omega \mathbf{I}_p^2(\mathbf{e}(\mathbf{u}_1), \mathbf{e}(\mathbf{u}_2))\,\mathrm{d}x \leqslant H_4(t)(\|\mathbf{w}\|_2^2 + \|\psi\|^2_{H^1_{per}}), \tag{2.26}$$

where $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, $\psi = \phi_1 - \phi_2$, and

$$\mathbf{I}_p(\mathbf{e}(\mathbf{u}_1), \mathbf{e}(\mathbf{u}_2)) := (1 + |\mathbf{e}(\mathbf{u}_1)| + |\mathbf{e}(\mathbf{u}_2)|)^{p/2-1}|\mathbf{e}(\mathbf{w})|. \tag{2.27}$$

Here $H_4 \in L^1(0,T)$ denotes a function only depending on the norms of $(\mathbf{u}_2, \phi_2)$ in the spaces specified in (2.6) and (2.7) and on the norms of $(\mathbf{u}_1, \phi_1)$ in the spaces specified in (2.19) and (2.20).

Proof. We start with the equation for ψ, that is,

$$\partial_t \psi + (\mathbf{u}_1 \cdot \nabla\phi_1 - \mathbf{u}_2 \cdot \nabla\phi_2) + \varepsilon\Delta^2\psi - \alpha\Delta(F'(\phi_1) - F'(\phi_2)) = 0.$$

Multiplying it by $-\Delta\psi$, which is an admissible test function, we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\psi\|_2^2 + \varepsilon\|\nabla\Delta\psi\|_2^2 \leqslant \int_\Omega (\mathbf{u}_2\phi_2 - \mathbf{u}_1\phi_1)\cdot\nabla\Delta\psi\,\mathrm{d}x + \alpha\int_\Omega \nabla(F'(\phi_1) - F'(\phi_2))\cdot\nabla\Delta\psi\,\mathrm{d}x =: I_1 + I_2.$$

The first term on the right-hand side is estimated as follows:

$$I_1 \leqslant \frac{\varepsilon}{8}\|\nabla\Delta\psi\|_2^2 + c\varepsilon^{-1}\int_\Omega (|\mathbf{w}|^2|\phi_1|^2 + |\mathbf{u}_1|^2|\psi|^2)\,\mathrm{d}x,$$

the integral being estimated by

$$c\varepsilon^{-1}(\|\mathbf{w}\|_2^2\|\phi_2\|_\infty^2 + \|\mathbf{u}_1\|_3^2\|\psi\|_6^2) \leqslant h_6(t)(\|\mathbf{w}\|_2^2 + \|\psi\|_{H^1_{per}}^2),$$

where

$$h_6(t) := c\varepsilon^{-1}(\|\phi_2\|_{H^1_{per}}^{3/2}\|\phi_2\|_{H^3_{per}}^{1/2} + \|\nabla\mathbf{u}_1\|_p^2).$$

Furthermore, we have

$$I_2 \leqslant \frac{\varepsilon}{8}\|\nabla\Delta\psi\|_2^2 + c\varepsilon^{-1}\int_\Omega |\nabla(F'(\phi_1) - F'(\phi_2))|^2\,\mathrm{d}x.$$

Observe now that we can write

$$F'(\phi_1) - F'(\phi_2) = \int_0^1 F''((1-s)\phi_1 + s\phi_2)\,\mathrm{d}s\,\psi,$$
$$\nabla(F'(\phi_1) - F'(\phi_2)) = \int_0^1 F^{(3)}((1-s)\phi_1 + s\phi_2)\nabla((1-s)\phi_1 + s\phi_2)\,\mathrm{d}s\,\psi + \int_0^1 F''((1-s)\phi_1 + s\phi_2)\,\mathrm{d}s\,\nabla\psi.$$

Consequently, recalling (2.1), we have

$$\int_\Omega |\nabla(F'(\phi_1) - F'(\phi_2))|^2\,\mathrm{d}x \leqslant C_F\int_\Omega (1 + |\phi_1| + |\phi_2|)^4(|\nabla\phi_1| + |\nabla\phi_2|)^2|\psi|^2\,\mathrm{d}x + C_F\int_\Omega (1 + |\phi_1| + |\phi_2|)^6|\nabla\psi|^2\,\mathrm{d}x =: J_1 + J_2. \tag{2.28}$$

In view of (2.16), these integrals are then estimated as follows:

$$\begin{aligned} J_1 &\leqslant C_F(1+\|\phi_1\|_\infty^4+\|\phi_2\|_\infty^4)(\|\nabla\phi_1\|_2^2+\|\nabla\phi_2\|_2^2)\|\psi\|_\infty^2 \\ &\leqslant C_F(1+\|\phi_1\|_{H^1_{per}}^5+\|\phi_2\|_{H^1_{per}}^5)(\|\phi_1\|_{H^3_{per}}+\|\phi_2\|_{H^3_{per}})\|\psi\|_{H^1_{per}}^{3/2}\|\psi\|_{H^3_{per}}^{1/2} \\ &\leqslant \frac{\varepsilon}{8}\|\psi\|_{H^3_{per}}^2+C_F\varepsilon^{-4/3}(1+\|\phi_1\|_{H^1_{per}}^{20/3}+\|\phi_2\|_{H^1_{per}}^{20/3})(\|\phi_2\|_{H^3_{per}}^{4/3}+\|\phi_2\|_{H^3_{per}}^{4/3})\|\psi\|_{H^1_{per}}^2, \\ J_2 &\leqslant C_F(1+\|\phi_1\|_6^6+\|\phi_2\|_6^6)\|\nabla\psi\|_\infty^2 \leqslant C_F(1+\|\phi_1\|_{H^1_{per}}^6+\|\phi_2\|_{H^1_{per}}^6)\|\psi\|_{H^1_{per}}^{1/2}\|\psi\|_{H^3_{per}}^{3/2} \\ &\leqslant \frac{\varepsilon}{8}\|\psi\|_{H^3_{per}}^2+C_F\varepsilon^{-4}(1+\|\phi_1\|_{H^1_{per}}^{24}+\|\phi_2\|_{H^1_{per}}^{24})\|\psi\|_{H^1_{per}}^2. \end{aligned}$$

To summarize, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{H^1_{per}}^2+\varepsilon\|\psi\|_{H^3_{per}}^2 \leqslant (h_6(t)+h_7(t))(\|\mathbf{w}\|_2^2+\|\psi\|_{H^1_{per}}^2), \tag{2.29}$$

where

$$\begin{aligned} h_7(t) := {}& C_F\varepsilon^{-4/3}(1+\|\phi_1\|_{H^1_{per}}^{20/3}+\|\phi_2\|_{H^1_{per}}^{20/3})(\|\phi_2\|_{H^3_{per}}^{4/3}+\|\phi_2\|_{H^3_{per}}^{4/3}) \\ &+C_F\varepsilon^{-4}(1+\|\phi_1\|_{H^1_{per}}^{24}+\|\phi_2\|_{H^1_{per}}^{24}). \end{aligned}$$

We proceed to handle the equation for w which can be written as follows:

$$\begin{aligned} \partial_t\mathbf{w}+\mathbf{u}_1\cdot\nabla\mathbf{u}_1-\mathbf{u}_2\cdot\nabla\mathbf{u}_2-\nabla\cdot\big\{\boldsymbol{\tau}(\phi_1,\mathbf{e}(\mathbf{u}_1))-\boldsymbol{\tau}(\phi_2,\mathbf{e}(\mathbf{u}_2))\big\}+\nabla(\pi_1-\pi_2) \\ = k\mu_1\nabla\phi_1-k\mu_2\nabla\phi_2. \end{aligned}$$

Multiply it by $\mathbf{w}$ (an admissible test function due to (2.5)) to obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_2^2+\underbrace{\int_\Omega\{\boldsymbol{\tau}(\phi_2,\mathbf{e}(\mathbf{u}_1))-\boldsymbol{\tau}(\phi_2,\mathbf{e}(\mathbf{u}_2))\}:\mathbf{e}(\mathbf{w})\,\mathrm{d}x}_{K_1} \\ = \int_\Omega\{\boldsymbol{\tau}(\phi_1,\mathbf{e}(\mathbf{u}_1))-\boldsymbol{\tau}(\phi_2,\mathbf{e}(\mathbf{u}_1))\}:\mathbf{e}(\mathbf{w})\,\mathrm{d}x+\int_\Omega(\mathbf{u}_2\cdot\nabla\mathbf{u}_2-\mathbf{u}_1\cdot\nabla\mathbf{u}_1)\cdot\mathbf{w}\,\mathrm{d}x \\ +k\int_\Omega(\mu_2\nabla\phi_2-\mu_1\nabla\phi_1)\cdot\mathbf{w}\,\mathrm{d}x =: K_2+K_3+K_4. \end{aligned}$$

Thanks to (2.3) and Korn's inequality, we have

$$K_1 \geqslant c_1\nu_*\|\nabla\mathbf{w}\|_2^2+c_1\nu_*\int_\Omega \mathbf{I}_p^2(\mathbf{e}(\mathbf{u}_1),\mathbf{e}(\mathbf{u}_2))\,\mathrm{d}x.$$

On the other hand (cf. (2.3) once more)

$$\begin{aligned} K_2 &\leqslant c_2\nu^\sharp\int_\Omega(1+|\mathbf{e}(\mathbf{u}_1)|)^{p-1}|\mathbf{e}(\mathbf{w})|\,|\psi|\,\mathrm{d}x \leqslant \tilde{c}_2\nu^\sharp\int_\Omega \mathbf{I}_p(\mathbf{e}(\mathbf{u}_1),\mathbf{e}(\mathbf{u}_2))(1+|\mathbf{e}(\mathbf{u}_1)|)^{p/2}|\psi|\,\mathrm{d}x \\ &\leqslant \frac{c_1\nu_*}{4}\int_\Omega \mathbf{I}_p^2(\mathbf{e}(\mathbf{u}_1),\mathbf{e}(\mathbf{u}_2))\,\mathrm{d}x+c_3\nu_*^{-1}(\nu^\sharp)^2\int_\Omega(1+|\nabla\mathbf{u}_1|^p)|\psi|^2\,\mathrm{d}x. \end{aligned}$$

The second term is estimated as follows:

$$c\nu_*^{-1}(\nu^\sharp)^2(1+\|\nabla\mathbf{u}_1\|_{3p/2}^p)\|\psi\|_6^2 \leqslant h_8(t)\|\psi\|_{H^1_{per}}^2,$$

where

$$h_8(t) := c\nu_*^{-1}(\nu^\sharp)^2(1+\|\nabla\mathbf{u}_1\|_2^{2p/(3p-2)}\|\nabla\mathbf{u}_1\|_{3p}^{(3p^2-4p)/(3p-2)}).$$

Here we have used the interpolation inequality

$$\|v\|_{3p/2} \leqslant \|v\|_2^{2/(3p-2)}\|v\|_{3p}^{(3p-4)/(3p-2)}.$$

Furthermore, we insert $\pm\mathbf{u}_2\cdot\nabla\mathbf{u}_1$ and use again the orthogonality of convective terms to deduce

$$K_3 = -\int_\Omega (\mathbf{w}\cdot\nabla\mathbf{u}_1)\cdot\mathbf{w}\,\mathrm{d}x \leqslant \|\nabla\mathbf{u}_1\|_{3p}\|\mathbf{w}\|_{6p/(3p-1)}^2.$$

Here we have used the inequalities

$$\|v\|_{6p/(3p-1)} \leqslant \|v\|_2^a\|v\|_6^{1-a} \leqslant c\|v\|_2^a\|\nabla v\|_2^{1-a}, \qquad a = \frac{2p-1}{2p}.$$

Therefore, we obtain

$$K_3 \leqslant c_4\|\nabla\mathbf{u}_1\|_{3p}\|\mathbf{w}\|_2^{(2p-1)/p}\|\nabla\mathbf{w}\|_2^{1/p} \leqslant \frac{c_1\nu_*}{4}\|\nabla\mathbf{w}\|_2^2 + h_9(t)\|\mathbf{w}\|_2^2,$$

where

$$h_9(t) := c_5\nu_*^{-1/(2p-1)}\|\nabla\mathbf{u}_1\|_{3p}^{2p/(2p-1)}.$$

Note that the exponent $2p/(2p-1)$ is strictly smaller than p. We now use integration by parts and recall that $\mathbf{w}$ is divergence-free, to write

$$K_4 = k\int_\Omega(\nabla\mu_1\phi_1 - \nabla\mu_2\phi_2)\cdot\mathbf{w}\,\mathrm{d}x = -k\int_\Omega \eta\nabla\phi_2\cdot\mathbf{w}\,\mathrm{d}x - k\int_\Omega \mu_1\nabla\psi\cdot\mathbf{w}\,\mathrm{d}x =: K_{4a}+K_{4b},$$

where we have set $\eta := \mu_1-\mu_2$. Then, we observe that

$$\begin{aligned}|\eta| &\leqslant \varepsilon|\Delta\psi| + \alpha C_F(1+|\phi_1|+|\phi_2|)^3|\psi|,\\ \|\eta\|_2 &\leqslant \varepsilon\|\psi\|_{H^2_{per}} + \alpha C_F(1+\|\phi_1\|_6^3+\|\phi_2\|_6^3)\|\psi\|_\infty \\ &\leqslant c(\varepsilon+\alpha C_F(1+\|\phi_1\|_{H^1_{per}}^3+\|\phi_2\|_{H^1_{per}}^3))\|\psi\|_{H^2_{per}}.\end{aligned} \tag{2.30}$$

We thus have

$$\begin{aligned}K_{4a} &\leqslant ck(\varepsilon+\alpha C_F(1+\|\phi_1\|_{H^1_{per}}^3+\|\phi_1\|_{H^1_{per}}^3))\|\psi\|_{H^2_{per}}\|\nabla\phi_2\|_\infty\|\mathbf{w}\|_2\\ &\leqslant ck(\varepsilon+\alpha C_F(1+\|\phi_1\|_{H^1_{per}}^3+\|\phi_1\|_{H^1_{per}}^3))\|\psi\|_{H^1_{per}}^{1/2}\|\psi\|_{H^3_{per}}^{1/2}\|\phi_2\|_{H^1_{per}}^{1/4}\|\phi_2\|_{H^3_{per}}^{3/4}\|\mathbf{w}\|_2\\ &\leqslant \frac{\varepsilon}{8}\|\psi\|_{H^3_{per}}^2 + h_{10}(t)(\|\psi\|_{H^1_{per}}^2+\|\mathbf{w}\|_2^2),\end{aligned}$$

where

$$h_{10}(t) := ck^{4/3}\varepsilon^{-1/3}(\varepsilon + \alpha C_F(1 + \|\phi_1\|^3_{H^1_{per}} + \|\phi_1\|^3_{H^1_{per}}))^{4/3}\|\phi_2\|_{H^3_{per}}.$$

Finally, we note that

$$|\mu| \leqslant \varepsilon|\Delta\phi| + \alpha C_F(1 + |\phi|^4), \qquad \|\mu\|_2 \leqslant \varepsilon\|\phi\|_{H^2_{per}} + \alpha C_F(1 + \|\phi\|^4_8).$$

Using the interpolation inequality

$$\|\phi\|_8 \leqslant c\|\phi\|_{H^{5/4}_{per}} \leqslant \tilde{c}\|\phi\|^{3/4}_{H^1_{per}}\|\phi\|^{1/4}_{H^2_{per}},$$

we deduce

$$\|\mu_1\|_2 \leqslant (\varepsilon + \alpha C_F(1 + \|\phi_1\|^3_{H^1_{per}}))\|\phi_1\|_{H^2_{per}}.$$

Hence, we find

$$\begin{aligned} K_{4b} &\leqslant k\|\mu_1\|_2\|\nabla\psi\|_\infty\|\mathbf{w}\|_2 \leqslant k\|\mu_1\|_2\|\psi\|^{1/4}_{H^1_{per}}\|\psi\|^{3/4}_{H^3_{per}}\|\mathbf{w}\|_2 \\ &\leqslant \frac{\varepsilon}{8}\|\psi\|^2_{H^3_{per}} + h_{11}(t)(\|\mathbf{w}\|^2_2 + \|\psi\|^2_{H^1_{per}}), \end{aligned}$$

where

$$h_{11}(t) := c\varepsilon^{-3/5}(\varepsilon + \alpha C_F(1 + \|\phi_1\|^3_{H^1_{per}}))^{8/5}\|\phi_1\|^{4/5}_{H^1_{per}}\|\phi_1\|^{4/5}_{H^3_{per}}.$$

Collecting the above estimates and invoking (2.29), we see that (2.26) holds with $H_4(t) := \sum_{j=6}^{11} h_j(t)$, and this completes the proof. □

A consequence of Theorem 2.8 is the following

COROLLARY 2.9 Let $(\mathbf{u}_i, \phi_i)$, $i = 1, 2$, be two weak solutions and suppose that $(\mathbf{u}_1, \phi_1)$ satisfies (2.19) and (2.20). Let $0 < 2\ell \leqslant T$ be given. Then

$$\|\mathbf{w}(t)\|^2 + \|\psi(t)\|^2_{H^1_{per}} \leqslant \lambda_1(\|\mathbf{w}(s)\|^2 + \|\psi(s)\|^2_{H^1_{per}}), \qquad 0 \leqslant s \leqslant t \leqslant T, \tag{2.31}$$

$$\begin{aligned} &\|\mathbf{w}\|^2_{L^2(\ell,2\ell;W^{1,2}_{div})} + \|\psi\|^2_{L^2(\ell,2\ell;H^3_{per})} + \|\mathbf{I}_p(\mathbf{e}(\mathbf{u}_1), \mathbf{e}(\mathbf{u}_2))\|_{L^2(\ell,2\ell;L^2)} \\ &\qquad \leqslant \lambda_2(\|\mathbf{w}\|^2_{L^2(0,\ell;L^2_{div})} + \|\psi\|^2_{L^2(0,\ell;H^1_{per})}), \end{aligned} \tag{2.32}$$

where λ_j, $j = 1, 2$, depend on T and ℓ, on the norms of $(\mathbf{u}_2, \phi_2)$ in the spaces specified in (2.6) and (2.7), on the norms of $(\mathbf{u}_1, \phi_1)$ in the spaces specified in (2.19) and (2.20) and on the other structural constants of the system.

Proof. For a fixed $s \in [0, T]$ we set

$$\begin{aligned} Y_s(t) &= \|\mathbf{w}(t)\|^2_2 + \|\psi(t)\|^2_{H^1_{per}} \\ &\quad + \int_s^t (\varepsilon\|\psi\|^2_{H^3_{per}} + c_1\nu_*\|\nabla\mathbf{w}\|^2_2) + c_1\nu_*\int_s^t\int_\Omega \mathbf{I}^2_p(\mathbf{e}(\mathbf{u}_1), \mathbf{e}(\mathbf{u}_2))\,\mathrm{d}x \end{aligned}$$

for all $t \in [s, T]$. Now (2.26) can be written as $\frac{\mathrm{d}}{\mathrm{d}t}Y_s(t) \leqslant H_4(t)Y_s(t)$, which yields

$$Y_s(t) \leqslant Y_s(s)\exp\left(\int_s^t H_4(\tau)\,\mathrm{d}\tau\right). \tag{2.33}$$

Thus (2.31) follows with $\lambda_1 = \exp(\int_0^T H_4)$. In addition, if $s \in (0, \ell)$ and $t = 2\ell$, then we get

$$\int_s^{2\ell} (\varepsilon \|\psi\|^2_{H^3_{per}} + c_1 \nu_* \|\nabla \mathbf{w}\|_2^2) + c_1 \nu_* \int_s^{2\ell} \int_\Omega \mathbf{I}_p^2(\mathbf{e}(\mathbf{u}_1), \mathbf{e}(\mathbf{u}_2)) \, dx \leqslant (\|\mathbf{w}(s)\|_2^2 + \|\psi(s)\|^2_{H^1_{per}}) \exp\left(\int_s^{2\ell} H_4(\tau) \, d\tau\right).$$

Integrating over $s \in (0, \ell)$ immediately gives (2.32) with $\lambda_2 = \ell^{-1} \exp(\int_0^{2\ell} H_4)$. □

We also need the following Lipschitz estimate:

LEMMA 2.10 Let $0 < \ell \leqslant T$ be fixed and let $(\mathbf{u}_1, \phi_1)$, $(\mathbf{u}_2, \phi_2)$ be two weak solutions satisfying (2.19), (2.20). Set $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ and $\psi = \phi_1 - \phi_2$. Then, for any $p \geqslant 11/5$, we have

$$\|\partial_t \mathbf{w}\|_{L^{p'}(0,\ell;(W^{1,p}_{div})^*)} + \|\partial_t \psi\|_{L^2(0,\ell;(H^1_{per})^*)} \leqslant \Lambda\{\|\mathbf{w}\|_{L^2(0,\ell;W^{1,2}_{div})} + \|\mathbf{I}_p(\mathbf{e}(\mathbf{u}_1), \mathbf{e}(\mathbf{u}_2))\|_{L^2(0,\ell;L^2)} + \|\psi\|_{L^2(0,\ell;H^3_{per})} + \|\psi\|_{L^\infty(0,\ell;H^1_{per})}\}, \tag{2.34}$$

where Λ is a computable constant, depending on the norms of the solutions in the spaces (2.19), (2.20).

Proof. We use the equation and a duality argument. First, recall that

$$\|\partial_t \mathbf{w}\|_{L^{p'}(0,\ell;(W^{1,p}_{div})^*)} = \sup_{\mathbf{z}} \left\{ \int_0^\ell \langle \partial_t \mathbf{w}, \mathbf{z} \rangle \, dt : \|\mathbf{z}\|_{L^p(0,\ell;W^{1,p}_{div})} \leqslant 1 \right\}.$$

Then, from (1.1) it follows that

$$\begin{aligned} \int_0^\ell \langle \partial_t \mathbf{w}, \mathbf{z} \rangle \, dt &= \int_{\Omega \times (0,\ell)} ((\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 - (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1) \cdot \mathbf{z} \, dx \, dt \\ &+ \int_{\Omega \times (0,\ell)} (\boldsymbol{\tau}(\phi_2, \mathbf{e}(\mathbf{u}_2)) - \boldsymbol{\tau}(\phi_1, \mathbf{e}(\mathbf{u}_1))) : \nabla \mathbf{z} \, dx \, dt \\ &+ k \int_{\Omega \times (0,\ell)} (\mu_1 \nabla \phi_1 - \mu_2 \nabla \phi_2) \cdot \mathbf{z} \, dx \, dt =: P_1 + P_2 + P_3. \end{aligned}$$

Integrating by parts and using the fact that $\mathbf{u}$ is divergence-free, we have

$$\begin{aligned} P_1 &= \int_{\Omega \times (0,\ell)} (\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2) : \nabla \mathbf{z} \, dx \, dt \leqslant \int_{\Omega \times (0,\ell)} (|\mathbf{u}_1| + |\mathbf{u}_2|)|\mathbf{w}| \, |\nabla \mathbf{z}| \, dx \, dt \\ &\leqslant \int_0^\ell (\|\mathbf{u}_1\|_{6p/(5p-6)} + \|\mathbf{u}_2\|_{6p/(5p-6)}) \|\mathbf{w}\|_6 \|\nabla \mathbf{z}\|_p \, dt \leqslant L_1 \left(\int_0^\ell \|\nabla \mathbf{w}\|_2^2 \right)^{1/2}, \end{aligned}$$

where

$$L_1 := c\ell^{(p-2)/(2p)} \sup_{t \in (0,\ell)} (\|\mathbf{u}_1(t)\|_{6p/(5p-6)} + \|\mathbf{u}_2(t)\|_{6p/(5p-6)}).$$

This is indeed finite as $6p/(5p-6) \leqslant 6$, in view of (2.19). Here we have also used the estimate

$$\left(\int_0^\ell \|\nabla \mathbf{z}\|_p^2\right)^{1/2} \leqslant c\ell^{(p-2)/(2p)}. \tag{2.35}$$

Furthermore, invoking (2.3), we get

$$\begin{aligned} P_2 &\leqslant c\nu^* \int_{\Omega\times(0,\ell)} (1+|\mathbf{e}(\mathbf{u}_1)|+|\mathbf{e}(\mathbf{u}_2)|)^{p-2}|\mathbf{e}(\mathbf{w})|\,|\nabla \mathbf{z}|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + c\nu^\sharp \int_{\Omega\times(0,\ell)} (1+|\mathbf{e}(\mathbf{u}_1)|)^{p-1}|\psi|\,|\nabla \mathbf{z}|\,\mathrm{d}x\,\mathrm{d}t =: P_{2a}+P_{2b}. \end{aligned}$$

Observe now that

$$\begin{aligned} P_{2a} &\leqslant c\nu^* \int_{\Omega\times(0,\ell)} \mathbf{I}_p(\mathbf{e}(\mathbf{u}_1),\mathbf{e}(\mathbf{u}_2))(1+|\mathbf{e}(\mathbf{u}_1)|+|\mathbf{e}(\mathbf{u}_2)|)^{p/2-1}|\nabla \mathbf{z}|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant c\nu^* \int_0^\ell \|\mathbf{I}_p(\mathbf{e}(\mathbf{u}_1),\mathbf{e}(\mathbf{u}_2))\|_2(1+\|\nabla \mathbf{u}_1\|_p+\|\nabla \mathbf{u}_2\|_p)^{p/2-1}\|\nabla \mathbf{z}\|_p\,\mathrm{d}t \\ &\leqslant L_{2a}\left(\int_0^\ell \|\mathbf{I}_p(\mathbf{e}(\mathbf{u}_1),\mathbf{e}(\mathbf{u}_2))\|_2^2\right)^{1/2}, \end{aligned}$$

where

$$L_{2a} = c\nu^*\left(\int_0^\ell (1+\|\nabla \mathbf{u}_1\|_p^p+\|\nabla \mathbf{u}_2\|_p^p)\,\mathrm{d}t\right)^{(p-2)/2p}.$$

Also, setting $\rho := 6p(p-1)/(5p-6)$, we have

$$P_{2b} \leqslant c\nu^\sharp \int_0^\ell (1+\|\nabla \mathbf{u}_1\|_\rho)^{p-1}\|\psi\|_6\|\nabla \mathbf{z}\|_p\,\mathrm{d}t \leqslant L_{2b} \sup_{t\in(0,\ell)} \|\psi(t)\|_{H^1_{per}},$$

where

$$L_{2b} = c\nu^\sharp\left(\int_0^\ell (1+\|\nabla \mathbf{u}_1(t)\|_\rho^p)\,\mathrm{d}t\right)^{(p-1)/p},$$

and this quantity is finite, thanks to (2.19) and $\rho \leqslant 3p$. The last term can be estimated as follows:

$$P_3 \leqslant k\int_{\Omega\times(0,\ell)} |\mu_1-\mu_2|\,|\nabla\phi_1|\,|\mathbf{z}|\,\mathrm{d}x\,\mathrm{d}t + k\int_{\Omega\times(0,\ell)} |\mu_2|\,|\nabla\psi|\,|\mathbf{z}|\,\mathrm{d}x\,\mathrm{d}t := P_{3a}+P_{3b}.$$

In this case, we obtain

$$P_{3a} \leqslant k\int_0^\ell \|\mu_1-\mu_2\|_2\|\nabla\phi_1\|_3\|\mathbf{z}\|_6\,\mathrm{d}t \leqslant L_{3a}\left(\int_0^\ell \|\psi\|^2_{H^3_{per}}\right)^{1/2},$$

where, in view of (2.30) and (2.35), we have set

$$L_{3a} = ck\ell^{(p-2)/(2p)} \sup_{t\in(0,\ell)} [(\varepsilon+\alpha C_F(\|\phi_1(t)\|^3_{H^1_{per}}+\|\phi_2(t)\|^3_{H^1_{per}}))\|\nabla\phi_1(t)\|_3].$$

On the other hand, we get

$$P_{3b} \leqslant k \int_0^\ell \|\mu_2\|_{6/5} \|\nabla\psi\|_\infty \|\mathbf{z}\|_6 \,\mathrm{d}t \leqslant L_{3b} \biggl(\int_0^\ell \|\psi\|^2_{H^3_{per}} \biggr)^{1/2},$$

where

$$L_{3b} = ck\ell^{(p-2)/(2p)} \sup_{t\in(0,\ell)} \|\mu_2(t)\|_{6/5}.$$

Consider now the second term on the left-hand side of (2.34), that is,

$$\|\partial_t\psi\|_{L^2(0,\ell;(H^1_{per})^*)} = \sup_\omega \biggl\{ \int_0^\ell \langle \partial_t\psi, \omega\rangle \,\mathrm{d}t : \|\omega\|_{L^2(0,\ell;H^1_{per})} \leqslant 1 \biggr\},$$

and observe that, by (1.3),

$$\int_0^\ell \langle \partial_t\psi, \omega\rangle \,\mathrm{d}t = \int_{\Omega\times(0,\ell)} (\mathbf{u}_2\phi_2 - \mathbf{u}_1\phi_1)\cdot\nabla\omega \,\mathrm{d}x\,\mathrm{d}t - \varepsilon \int_{\Omega\times(0,\ell)} \nabla(\Delta\psi)\cdot\nabla\omega \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \alpha \int_{\Omega\times(0,\ell)} \nabla(F'(\phi_1) - F'(\phi_2))\cdot\nabla\omega \,\mathrm{d}x\,\mathrm{d}t =: R_1 + R_2 + R_3.$$

Here we have

$$R_1 \leqslant \int_{\Omega\times(0,\ell)} (|\mathbf{w}|\,|\phi_1| + |\mathbf{u}_2|\,|\psi|)|\nabla\omega| \,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^\ell (\|\mathbf{w}\|_6\|\phi_1\|_3 + \|\mathbf{u}_2\|_2\|\psi\|_\infty)\|\nabla\omega\|_2 \,\mathrm{d}t$$
$$\leqslant L_4 \biggl\{ \biggl(\int_0^\ell \|\mathbf{w}\|^2_{W^{1,2}_{div}} \,\mathrm{d}t \biggr)^{1/2} + \biggl(\int_0^\ell \|\psi\|^2_{H^3_{per}} \,\mathrm{d}t \biggr)^{1/2} \biggr\},$$

where

$$L_4 = c \sup_{t\in(0,\ell)} (\|\phi_1(t)\|_{H^1_{per}} + \|\mathbf{u}_2(t)\|_2).$$

Similarly, we find

$$R_2 \leqslant \varepsilon \int_{\Omega\times(0,\ell)} |\nabla^3\psi|\,|\nabla\omega| \,\mathrm{d}x\,\mathrm{d}t \leqslant c\varepsilon \biggl(\int_0^\ell \|\psi\|^2_{H^3_{per}} \,\mathrm{d}t \biggr)^{1/2}.$$

Also, recalling (2.28), we have

$$\int_\Omega \bigl|\nabla(F'(\phi_1) - F'(\phi_2))\bigr|^2 \,\mathrm{d}x \leqslant C_F(1 + \|\phi_1\|^4_\infty + \|\phi_2\|^4_\infty)(\|\nabla\phi_1\|^2_2 + \|\nabla\phi_2\|^2_2)\|\psi\|^2_\infty$$
$$+ C_F(1 + \|\phi_1\|^6_6 + \|\phi_2\|^6_6)\|\nabla\psi\|^2_\infty,$$

so that

$$R_3 \leqslant \int_0^\ell \|\nabla(F'(\phi_1) - F'(\phi_2))\|_2\|\nabla\omega\|_2 \,\mathrm{d}t \leqslant L_5 \biggl(\int_0^\ell \|\psi\|^2_{H^3_{per}} \,\mathrm{d}t \biggr)^{1/2},$$

where

$$L_5 = \sup_{t\in(0,\ell)} \{C_F(1 + \|\phi_1(t)\|^2_{H^1_{per}} + \|\phi_2(t)\|^2_{H^1_{per}})(\|\phi_1(t)\|_{H^2_{per}} + \|\phi_2(t)\|_{H^2_{per}})$$
$$+ C_F(1 + \|\phi_1(t)\|^{3/2}_{H^1_{per}} + \|\phi_2(t)\|^{3/2}_{H^1_{per}})\}.$$

We have thus proven (2.34) with

$$\Lambda := L_1 + L_{2a} + L_{2b} + L_{3a} + L_{3b} + L_4 + c\varepsilon + L_5. \qquad \square$$

3. Longtime behavior

In this section we investigate the longtime behavior of the dynamics of our system. First of all, we prove a dissipative estimate (see Theorem 3.1). We then employ the short trajectory approach (cf. [22, 24]) to establish the existence of global attractor (see Theorem 3.3). This will be a simple consequence of the compactness result proven in Theorem 2.4 above. We also show that the dynamics of trajectories has a smoothing property; consequently, there exists an exponential attractor. Finally, we discuss how an upper bound on its fractal dimension depends on the physical parameters of the system.

THEOREM 3.1 Let (2.1) and (2.2) hold. Then any weak solution $(\mathbf{u}, \phi)$ satisfies the dissipative estimate, for all $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\mathbf{u},\phi] + \tilde{c}_1\kappa E[\mathbf{u},\phi] + \tilde{c}_1\left(\frac{\nu_*^1}{k}\|\nabla\mathbf{u}\|_2^2 + \frac{\nu_*^2}{k}\|\nabla\mathbf{u}\|_p^p + \|\nabla\mu\|_2^2\right) \leqslant \frac{\tilde{c}_2}{k\nu_*^1}\|\mathbf{g}\|_{-1,2}^2 + \alpha\varepsilon\tilde{c}_3, \tag{3.1}$$

where

$$\kappa := \min\{\nu_*^1, \varepsilon\},$$

and $\tilde{c}_i$, $i = 1, 2$, are positive constants independent of the data, while $\tilde{c}_3 > 0$ depends on $\overline{\phi}$.

Proof. Let $(\mathbf{u}, \phi)$ be an arbitrary weak solution. Testing (1.1) with $k^{-1}\mathbf{u}$ and (1.3) with μ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E[\mathbf{u},\phi] + k^{-1}\int_\Omega \boldsymbol{\tau}(\phi, \mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{u})\,\mathrm{d}x + \|\nabla\mu\|_2^2 \leqslant k^{-1}|\langle\mathbf{g},\mathbf{u}\rangle|.$$

By (2.3) and Poincaré's inequality, we find

$$\int_\Omega \boldsymbol{\tau}(\phi, \mathbf{e}(\mathbf{u})) : \mathbf{e}(\mathbf{u})\,\mathrm{d}x \geqslant c_1(\nu_*^1\|\nabla\mathbf{u}\|_2^2 + \nu_*^2\|\nabla\mathbf{u}\|_p^p).$$

On the other hand,

$$|\langle\mathbf{g},\mathbf{u}\rangle| \leqslant \|\mathbf{g}\|_{-1,2}\|\nabla\mathbf{u}\|_2 \leqslant \frac{c_1\nu_*^1}{2}\|\nabla\mathbf{u}\|_2^2 + \frac{1}{2c_1\nu_*^1}\|\mathbf{g}\|_{-1,2}^2.$$

Altogether we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\mathbf{u},\phi] + c_1\left(\frac{\nu_*^1}{k}\|\nabla\mathbf{u}\|_2^2 + \frac{\nu_*^2}{k}\|\nabla\mathbf{u}\|_p^p\right) + \|\nabla\mu\|_2^2 \leqslant \frac{1}{c_1k\nu_*^1}\|\mathbf{g}\|_{-1,2}^2. \tag{3.2}$$

Testing now (1.4) with $\phi - \overline{\phi}$ gives

$$\langle\mu, \phi - \overline{\phi}\rangle = \varepsilon\|\nabla\phi\|_2^2 + \alpha\int_\Omega F'(\phi)(\phi - \overline{\phi})\,\mathrm{d}x. \tag{3.3}$$

Using Poincaré's inequality we get

$$\langle \mu, \phi - \overline{\phi} \rangle = \langle \mu - \overline{\mu}, \phi - \overline{\phi} \rangle \leqslant \|\mu - \overline{\mu}\|_2 \|\phi - \overline{\phi}\|_2$$
$$\leqslant c_2 \|\nabla \mu\|_2 \|\nabla \phi\|_2 \leqslant \frac{\varepsilon}{2} \|\nabla \phi\|_2^2 + \frac{c_2^2}{2\varepsilon} \|\nabla \mu\|_2^2. \tag{3.4}$$

Moreover, we deduce from (2.1) that

$$c_3 \int_\Omega F(\phi)\,\mathrm{d}x \leqslant \int_\Omega F'(\phi)(\phi - \overline{\phi})\,\mathrm{d}x + c_4 \tag{3.5}$$

for some $c_3 \in (0, 1)$ and for some $c_4 > 0$ depending on $\overline{\phi}$ (see (2.8)). Finally, combining (3.3)–(3.5) yields

$$c_5 \varepsilon \left(\varepsilon \|\nabla \phi\|_2^2 + 2\alpha \int_\Omega F(\phi)\,\mathrm{d}x \right) \leqslant \|\nabla \mu\|_2^2 + c_6 \alpha \varepsilon \tag{3.6}$$

for some $c_5 > 0$ and for some $c_6 > 0$ depending on $\overline{\phi}$. Then, substituting into (3.2), one deduces (3.1). □

REMARK 3.2 Note that, in the case of simple fluids (more generally, for convex $F(\cdot)$ where c_4 can be taken 0 in (3.5)), when there is no (nongradient) external force (i.e., $\mathbf{g} = \mathbf{0}$), inequality (3.1) implies that $\mathbf{0}$ is globally asymptotically stable. This still holds when $\|\mathbf{g}\|_2$ is small enough (cf. [10, Ch. II, Prop. 5.3] for the 2D Navier–Stokes equations), where $\mathbf{0}$ is replaced by the unique solution to the corresponding stationary problem.

We now have all the ingredients to construct a dynamical system and prove that it has a global attractor and an exponential attractor. We will also observe that the attractor's dimension can be estimated in a completely explicit way. We will only sketch the outline of such a computation, being satisfied with the (nontrivial) observation that all the estimates grow at most polynomially with the data. We recall that the standard method based on the estimate of the Lyapunov exponents cannot be used in this case since the differentiability of the semigroup solution is not known (cf. [8]).

Recall that by data we mean the constants k, ε and α, occurring in (1.1)–(1.4); the viscosity bounds ν_*, ν^* and $\nu^\sharp$ (see (2.2)–(2.4)); the norm of the external force $\|\mathbf{g}\|_2$ and a generic positive constant C_F related to the growth of the potential F. We also recall that, throughout the paper, we have used c and c_i to denote absolute (i.e., independent of the data) positive constants which may change from line to line.

From now on we set $N = 3$ and (rescaling the space variable) $\Omega = (0, 1)^3$. Thus the embedding constants are also absolute. As is well known, the embedding constants typically only depend on the size and the shape of domain. They can blow up, however, close to critical exponents, as for example with $W^{1,3} \subset L^q$, $N = 3$ and $q \to \infty$. We avoid such situations in our paper. We also point out that the following results can be reformulated for the case $N = 2$ with less restrictions.

We will now follow an abstract scheme of the "method of trajectories", as presented in Section 2 of [24]. In particular, we need to verify the assumptions (A1–A10), which will imply the desired results. For the reader's convenience, the statement of (A1–A10) will be recalled, with the notation slightly adapted to the present setting. We will also invoke some general abstract results from [11, Ch. 2].

Let $\mathbf{g} \in L^2_{div}$ and set

$$X := L^2_{div} \times H^1_{per}, \quad X^M := \{(\mathbf{v}, \psi) \in L^2_{div} \times H^1_{per} : |\bar{\psi}| \leqslant M\}, \tag{3.7}$$

for some given $M \geqslant 0$. This is a complete metric space with respect to the metric induced by the X-norm. Then, for any fixed $\tau > 0$, we introduce the spaces

$$X_\tau := L^2(0,\tau;X), \qquad X_\tau^M := L^2(0,\tau;X^M), \tag{3.8}$$

$$Y_\tau^M := \{(\mathbf{v},\psi) \in X_\tau^M \cap [L^p(0,\tau;W^{1,p}_{div})) \times L^2(0,\tau;H^3_{per})] : \\ (\mathbf{v}_t,\psi_t) \in L^{p'}(0,\tau;(W^{1,p}_{div})^*) \times L^2(0,\tau;(H^1_{per})^*)\}, \tag{3.9}$$

$$W_\tau^M := \big\{(\mathbf{v},\psi) \in X_\tau^M \cap [(L^2(0,\tau;W^{1,2}_{div})) \times L^2(0,\tau;H^3_{per})] : \\ (\mathbf{v}_t,\psi_t) \in L^{p'}(0,\tau;(W^{1,p}_{div})^*) \times L^2(0,\tau;(H^1_{per})^*)\}. \tag{3.10}$$

We clearly have $Y_\tau^M \hookrightarrow\hookrightarrow X_\tau^M$ and $W_\tau^M \hookrightarrow\hookrightarrow X_\tau^M$.

Recalling Theorem 2.4, Remark 2.5 and (2.11), we deduce that for an arbitrary initial condition in X^M and arbitrary $T \in (0,\infty)$, there exists at least one solution belonging to $C([0,T];X^M_{\text{weak}}) \cap Y_T^M$. In other words, the assumption [24, (A1)] is satisfied. Moreover, by virtue of Theorem 3.1, the set

$$B_0^M = \{(\mathbf{v},\psi) \in X^M : E[\mathbf{v},\psi] \leqslant R_0\}$$

is uniformly absorbing and positively invariant, provided that $R_0 > 0$ is sufficiently large. This means that the assumption [24, (A2)] is satisfied as well. Note that B_0^M is a closed set in X^M.

For any solution starting from B_0^M, arguing as in the proof of Theorem 2.4, one deduces

$$\sup_{t\in[0,T]} (\|\mathbf{u}(t)\|_2^2 + \|\phi(t)\|^2_{H^1_{per}}) + \int_0^T (\|\mathbf{u}\|^p_{1,p} + \|\phi\|^2_{H^3_{per}} + \|\mu\|^2_{H^1_{per}})\,\mathrm{d}t \\ \leqslant \Pi_1 = \Pi_1(M)(T+1). \tag{3.11}$$

Here and in what follows, by Π_k we denote an explicitly computable upper bound, depending polynomially on the data, values of Π_l for $l < k$ and, possibly, on other quantities which will be pointed out.

We now fix $\ell > 0$ and we consider the set $\mathcal{X}_\ell$ of ℓ-*trajectories*, that is, of all weak solutions $(\mathbf{u},\phi)$ given by Theorem 2.4 on the time interval $[0,\ell]$ such that $|\bar\phi(t)| \leqslant M$ for all $t \in [0,\ell]$. We endow $\mathcal{X}_\ell$ with the topology of X_ℓ. Of course, each trajectory makes sense pointwise because of (2.11). Note, however, that $\mathcal{X}_\ell$ is not a complete metric space.

For the sake of studying large time dynamics, we shall focus on the set

$$\mathcal{B}^0_\ell := \{(\mathbf{u},\phi) \in \mathcal{X}_\ell : (\mathbf{u}(0),\phi(0)) \in B_0^M\}, \tag{3.12}$$

consisting of all trajectories starting from B_0^M. Clearly, for each of such trajectories, there exists $\tau \in (0,\ell/2)$ such that

$$\|\mathbf{u}(\tau)\|^2_{1,2} + \|\phi(\tau)\|^2_{H^2_{per}} \leqslant 2\Pi_1(M)(1+\ell^{-1}) =: \Pi_2.$$

We are ready to apply Theorem 2.6. The key observation, however, is that the functions H_1 and H_3 (which only depend on the norms being controlled in (3.11)) are integrable with some power $\sigma > 1$, depending on $p > 11/5$ (cf. (2.23)). Hence, by taking ℓ small enough, namely $\ell \leqslant \Pi_3$, we ensure that

$$\int_0^{2\ell} (H_1 + H_3)\,\mathrm{d}t \leqslant 1 \tag{3.13}$$

for any solution taking values in B_0^M. Therefore, when applying Gronwall's lemma to (2.21) and (2.24), the exponential terms are uniformly bounded by exp(1). Eliminating thus possible exponential dependence on the data, we obtain

$$\sup_{t\in[\tau,\tau+2\ell]}(\|\mathbf{u}(t)\|_{1,2}^2+\|\phi(t)\|_{H^2_{per}}^2)+\int_\tau^{\tau+2\ell}(\|\mathbf{u}\|_{1,3p}^p+\|\mathbf{u}\|_{2,2}^2+\|\phi\|_{H^4_{per}}^2)\,\mathrm{d}t\leqslant \Pi_4.$$

By Theorem 2.8, this regularity also entails unique continuation of the trajectory after $t\geqslant\tau$, more precisely, for any $(\mathbf{v},\psi)\in\mathcal{B}_\ell^0$ and any $T\geqslant\ell$, there exists only one solution on $[0,T]$ such that $(\mathbf{u},\phi)|_{[0,\ell]}=(\mathbf{v},\psi)$. This just means that [24, (A3)] holds.

Consequently, one can introduce a semigroup $L_t:\mathcal{B}_\ell^0\to\mathcal{B}_\ell^0$ by setting

$$\{L_t(\mathbf{v},\psi)\}(s):=(\mathbf{u}(t+s),\phi(t+s)),\qquad s\in[0,\ell].$$

We proceed to show that L_t is Lipschitz continuous, which is the assumption [24, (A4)]. Indeed, if $(\mathbf{w},\psi)$ is a difference of two solutions starting from B_0^M, then due to Corollary 2.9 and the above higher regularity on $[\tau,2\ell]$, we deduce that

$$\|\mathbf{w}(s+\ell/2)\|_2^2+\|\psi(s+\ell/2)\|_{H^1_{per}}^2\leqslant\lambda_1(\|\mathbf{w}(s)\|_2^2+\|\psi(s)\|_{H^1_{per}}^2),\qquad\forall s\in(\ell/2,\ell).$$

Integrating over s implies that L_t is even uniformly Lipschitz continuous on $\mathcal{B}_\ell^0$ with respect to $t\in[0,T]$.

Furthermore, we claim that

$$\overline{\mathcal{B}_\ell^0}^{X_\ell}=\mathcal{B}_\ell^0. \tag{3.14}$$

Since $\mathcal{B}_\ell^0$ is positively invariant with respect to the (continuous) operators L_t, (3.14) implies that $\overline{L_t\mathcal{B}_\ell^0}^{X_\ell}\subset\mathcal{B}_\ell^0$, and consequently [24, (A5)] is satisfied. Setting $\mathcal{B}_\ell^1=\overline{L_\tau(\mathcal{B}_\ell^0)}^{X_\ell}$ for some $\tau\in[\ell/2,\ell]$ we obtain a compact invariant absorbing set.

Let us verify that (3.14) holds. Let $(\mathbf{v}_n,\psi_n)\in\mathcal{B}_\ell^0$, and let $(\mathbf{v}_n,\psi_n)\to(\mathbf{v},\psi)$ strongly in X_ℓ. Extracting a subsequence, we can assume that $(\mathbf{v}_n(t),\psi_n(t))\to(\mathbf{v}(t),\psi(t))$ for almost any t, strongly in X. It follows by Theorem 2.4 that $(\mathbf{v},\psi)$ is a weak solution, at least on $(\tau,\ell]$ for arbitrary $\tau>0$. Observe, furthermore, that $(\mathbf{v}_n,\psi_n)$ and hence $(\mathbf{v},\psi)$ are bounded in the spaces (2.6–2.10), with $T=\ell$, and thus $(\mathbf{v},\psi)$ is clearly a weak solution on $[0,\ell]$. Finally, from the continuity of $(\mathbf{v},\psi)$ and the closedness and positive invariance of B_0^M, one concludes that $(\mathbf{v}(0),\psi(0))\in B_0^M$.

Now, the assumptions (A1–A5) imply that the dynamical system of trajectories $(\mathcal{X}_\ell,L_t)$ has a global attractor $\mathcal{A}_\ell$ (see [24, Theorem 2.1]).

We will now deal with the attractor's fractal dimension. Due to Corollary 2.9 and Lemma 2.10, $L_\tau:\mathcal{X}_\ell\to W_\ell^M$ is Lipschitz continuous on $\mathcal{B}_\ell^1$, which is the assumption [24, (A6)]. Note that W_ℓ^M is compactly embedded in X_ℓ; in other words, L_τ has the so-called smoothing property. Moreover, since ℓ and τ were chosen small enough, we control the Lipschitz constant of L_τ, i.e.,

$$\mathcal{L}:=\mathrm{Lip}(L_\tau|\mathcal{B}_\ell^1;X_\ell^M,W_\ell^M)\leqslant\Pi_5.$$

It follows that the global attractor $\mathcal{A}_\ell$ has finite fractal dimension in X_ℓ^M. See [24, Theorem 2.2]. More precisely, we have an explicit estimate (cf. [11, Theorem 2.18], for example)

$$\dim_f^{X_\ell^M}(\mathcal{A}_\ell)\leqslant\log\mathcal{K}/\log 2,$$

where

$$\mathcal{K} := N_{X_\ell^M}(B_{W_\ell^M}(0;1), 1/4\mathcal{L}).$$

Here $\mathcal{K}$ denotes the minimal number of balls in X_ℓ^M that are needed to cover a unit ball in W_ℓ^M. Finally, the number $\mathcal{K}$ can also be explicitly calculated (see [11, Appendix]) so that

$$\log \mathcal{K} \leqslant \Pi_6(\mathcal{L}).$$

To bring the results back to the phase space X^M, we now consider the mapping $e : \mathcal{X}_\ell \to X^M$, $(\mathbf{u}, \phi) \mapsto (\mathbf{u}(\ell), \phi(\ell))$. Recalling Corollary 2.9 once more and arguing as in the proof of continuity of L_t above (cf. also [24, Lemma 2.1(ii)]), we deduce that e is Lipschitz continuous on $\mathcal{B}_\ell^0$. This tells us that the assumption [24, (A8)] is fulfilled. Therefore, invoking [24, Theorems 2.3, 2.4] we deduce that

$$\mathcal{A} := e(\mathcal{A}_\ell)$$

is the global attractor of the dynamical system $(e(\mathcal{B}_\ell^1), S_t)$, where S_t is the solution operator defined by $S_t(\mathbf{u}_0, \phi_0) := (\mathbf{u}(t), \phi(t))$ for all $t \geqslant 0$. Moreover, we have the estimate

$$\dim_f^{X^M}(\mathcal{A}) \leqslant \dim_f^{X_\ell^M}(\mathcal{A}_\ell).$$

Finally, we address the problem of an exponential attractor. The smoothing property, i.e., the Lipschitz continuity L_τ on $\mathcal{B}_\ell^1$ with respect to X_ℓ^M and W_ℓ^M, implies the existence of an exponential attractor $\mathcal{E}_\ell^*$ for the *discrete* dynamical system $(\mathcal{B}_\ell^1, L_{n\tau})$; moreover, the dimension of $\mathcal{E}_\ell^*$ admits the same upper bound as the global attractor $\mathcal{A}_\ell$. See [11, Proposition 2.26]. To extend the results to the full semigroup L_t, we need to estimate the modulus of continuity of $L_t\chi$ both with respect to t and χ.

We have already observed above that $L_t : \mathcal{B}_\ell^1 \to \mathcal{B}_\ell^1$ is Lipschitz continuous, uniformly with respect to $t \in [0, \tau]$. Thus [24, (A9)] holds true. Furthermore, if $(\mathbf{u}, \phi)$ is the continuation of the trajectory from $\mathcal{B}_\ell^1$, it follows from the regularity of Theorem 2.6 and the equation that $(\partial_t \mathbf{u}, \partial_t \phi)$ is bounded in $L^{p'}(0, \tau + \ell; (W_{div}^{1,p})^*) \times L^2(0, \tau + \ell; L^2)$. This entails that $t \mapsto L_t\chi$ is Hölder continuous with the target space $L^{p'}(0, \ell; (W_{div}^{1,p})^*) \times L^2(0, \ell; L^2)$ (see, e.g., [24, Lemma 2.2]). Invoking finally the pointwise boundedness of trajectories in $W_{div}^{2,2} \times H_{per}^2$ and a suitable interpolation, we conclude that $t \mapsto L_t\chi$ is Hölder continuous into the desired target space X_ℓ^M. Thus the assumption [24, (A10)] holds. This, in turn, gives exponential attractors for the dynamical systems $(\mathcal{X}_\ell, L_t)$ and eventually, $(e(\mathcal{B}_\ell^1), S_t)$; see [24, Theorems 2.5, 2.6].

These attractors admit the same (up to multiplying by an absolute constant) explicit estimate of fractal dimension as was the case for the global attractor. Summing up, we have proven the following

THEOREM 3.3 Let $\mathbf{g} \in L_{div}^2$. The dynamical system generated by (1.1)–(1.4) on X^M has an exponential attractor and the global attractor of finite fractal dimension. Moreover, if $p > 11/5$, then the dimension can be estimated by an explicitly computable constant which depends polynomially on the data.

REMARK 3.4 Recalling (2.23) and (3.13), we deduce that the length of the trajectory tends to 0 as p approaches the critical value $p = 11/5$. Therefore, in this case, we are not able to estimate the dimension in such a way that the dependence on the data is of polynomial type.

REMARK 3.5 We recall that the global attractor $\mathcal{A}$ attracts the dynamics originating from the whole space X^M in the following sense (see [24, Rem. 2.1]): for any bounded set $B \subset X^M$, indicating by B_t the set of all values of all solutions to (1.1)–(1.4) emanating from B at time t, we have $\mathrm{dist}_X(B_t, \mathcal{A}) \to 0$ as t goes to ∞.

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