

## Instability of gravity wetting fronts for Richards equations with hysteresis

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We study the evolution of saturation profiles in a porous medium. When there is a more saturated medium on top of a less saturated medium, the effect of gravity is a downward motion of the liquid. While in experiments the effect of fingering can be observed, i.e., an instability of the planar front solution, it has been verified in different settings that the Richards equation with gravity has stable planar fronts. In the present work we analyze the Richards equation coupled to a play-type hysteresis model in the capillary pressure relation. Our result is that, in a homogeneous medium, imposing appropriate initial and boundary conditions, the planar front solution is unstable. In particular, we find that the Richards equation with gravity and hysteresis does not define an  $L^1$ -contraction.

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### 1. Introduction

The standard model for the description of saturation distributions in porous media is the Richards equation. Denoting the fluid pressure by  $p$  and the volume fraction of pore space that is filled with fluid by  $s$  (saturation), the combination of mass conservation and Darcy's law for the velocities yields the Richards equation

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x]). \quad (1.1)$$

In this equation, a normalization of porosity, density, and gravity are performed, the acceleration of gravity is 1 and points in direction  $-e_x$ . The permeability  $k = k(s)$  is a given function  $k : \mathbb{R} \rightarrow [0, \infty)$ . The unknowns are pressure  $p$  and saturation  $s$ , two scalar variables which depend on time  $t \in [T_-, T_+]$  and position  $\tilde{x} \in \Omega$ , where  $\Omega \subset \mathbb{R}^{n+1}$  is the domain of the porous medium. We decompose the spatial variables according to the direction of gravity,  $\tilde{x} = (x, y) \in \Omega \subset \mathbb{R}^{n+1}$  for  $x = \tilde{x} \cdot e_x$  and  $y \in \mathbb{R}^n$ .

Equation (1.1) must be supplemented with a capillary pressure relation

$$p \in \mathcal{F}(s).$$

The simplest possibility (and the standard choice for the Richards equation) is a functional dependence,  $p = p_c(s)$ , where  $p_c$  is a monotone function. In applications, the capillary pressure function can have infinite slope and can even be multivalued,  $p_c = \mathcal{F} \subset \mathbb{R} \times \mathbb{R}$ . In this degenerate case (even without hysteresis) we must write the inclusion symbol in the above pressure relation. In physical variables, the saturation  $s$  has only values in  $[0, 1]$ , the permeability  $k$  is a function on

$[0, 1]$ , and  $p_c = \mathfrak{F}$  is a maximal monotone graph  $\mathfrak{F} \subset [0, 1] \times \mathbb{R}$ . Typically,  $k$  vanishes for small arguments and  $p_c$  is multi-valued in the end-points of its domain. Nevertheless, the instability result of the current work is shown for non-degenerate coefficient functions, defined on  $\mathbb{R}$ .

### 1.1 Gravity wetting fronts

We are interested in the situation that a more saturated medium is above a less saturated medium such that, under the influence of gravity, the saturation increases in the lower part of the medium. A question which receives considerable interest is the following: Does the penetration into the initially dryer medium always happen with a one-dimensional front, i.e., with negligible variations in the horizontal variable  $y$ , or can there also appear *fingers*, i.e., smaller structures with a higher saturation along which the fluid moves downwards? We refer to the overview in [16] for fingering effects in other physical systems.

In various experimental works, the development of fingers was verified for well-adapted initial and boundary conditions. Early works date into the 1960'ies. A more recent study is [29], where an experimental set-up with finger development is described. It contains the measurement of saturation profiles (non-monotonic in  $x$ ), and the observation that the time evolution of fingers is approximately given by a translation in  $x$ . Fingers are also observed in [4], where the influence of an increased initial saturation is studied. If the fluid enters a less dry medium, the finger widens and eventually disappears, and the saturation profile becomes monotonic in  $x$ . The importance of a very dry initial condition is also discussed in [19].

To model the experimentally observed fingering effect, the standard Richards equation with a fixed capillary pressure curve seems to be inadequate [14]. This observation coincides with the mathematical analysis of [31] which contains a stability result for one-dimensional front solutions under Richards equation with a *function*  $\mathfrak{F}$ . A stability result for the classical Richards equation can be derived also in degenerate cases, see [21] and [9].

As a consequence, modifications of the Richards equation have been introduced in order to capture gravity fingering. One of the most prominent models was introduced by Hassanizadeh and Gray [15]. Their suggestion is to replace the algebraic relation  $p = p_c(s)$  by a kinetic equation such as  $\tau \partial_t s = p - p_c(s)$  for some real parameter  $\tau$ . Such *non-equilibrium Richards equations* (NERE) are studied, e.g., in [13, 20], where a *low-frequency instability criterion* is introduced and used to predict an instability in the NERE model. Once more, a low initial saturation is important for a spatial instability. The same model is also analyzed in [23] with the result that non-monotonic one-dimensional profiles can be induced by the NERE model; two-dimensional numerical simulations shows a non-monotonic finger solution. The combined model with static hysteresis of play-type and with a dynamic effect ( $\tau > 0$ ) was treated analytically and numerically in [17]; also in this model, fingering can be observed.

Another possible modification of the Richards equation is to introduce a rate independent hysteresis in the form that different capillary pressure curves are used for imbibition and drainage. This most elementary model is actually closely related to the play-type hysteresis studied here. Numerically, gravity fingers for this model were observed in [18]. For a theoretical analysis of different hysteresis models we refer to [33] and [3]. Yet another model is used in [12], where a higher dimensional instability is observed numerically.

## 1.2 A rate independent hysteresis model

It is well known that porous media exhibit hysteresis effects [5]. Furthermore, the importance of hysteresis for the development of gravity fingers seems also to be evident. Much less clear is the choice of an appropriate hysteresis model. Beliaev and Hassanizadeh distinguish in [7] between static capillary pressure hysteresis and a dynamic variant. They give thermodynamic arguments in favor of the (static) play-type hysteresis model and are able to confirm the model to some extent by reported measurements. Furthermore, the model is expanded by the inclusion of dynamic effects in the spirit of [15].

In this work we discuss the play-type hysteresis model in its simplest form and study its possible effects in terms of the gravity fingers instability. We emphasize that the play-type model has many virtues: it gives a reasonable agreement with experimental data, it is rate-independent (as are most of the reported measurements), such that, in particular no additional time-scale ( $\tau$ ) is introduced. Furthermore, the play-type model is thermodynamically consistent, and it can, to some extent, be justified theoretically [24, 25]. The purely static play type model was analyzed with respect to homogenization in [26] and, in a two-phase flow situation, in [6], with respect to non-uniqueness in [8]. Let us emphasize that we do not doubt the presence of dynamic hysteresis effects in porous media – but in order to understand instability mechanisms, we analyze here the purely static hysteresis model. Our main result is a rigorous instability result for the Richards equation with play-type hysteresis.

We next describe this model in more detail. Mathematically, we interpret the operator  $\mathfrak{F}$  in the relation  $p \in \mathfrak{F}(s)$  not as an algebraic relation for every time instance, but rather as a relation between the evolution of pressure and saturation. With a parameter  $\gamma > 0$ , which is a measure for the difference in pressure between imbibition and drainage, we may specify  $\mathfrak{F}$  through the differential relation

$$p \in p_c(s) + \gamma \operatorname{sign}(\partial_t s), \quad (1.2)$$

where  $\operatorname{sign}(\xi) := [-1, 1]$  for  $\xi = 0$  and  $\operatorname{sign}(\xi) \in \{\pm 1\}$  for  $\xi \neq 0$ . Relation (1.2) demands that the pressure  $p$  is always in the  $s$ -dependent interval  $[p_c(s) - \gamma, p_c(s) + \gamma]$ . Furthermore, for  $p$  strictly between  $p_c(s) - \gamma$  and  $p_c(s) + \gamma$ , the time derivative  $\partial_t s$  necessarily vanishes.

The hysteresis relation can be made more general by demanding that, loosely speaking, the effect of different values of  $\gamma$  is averaged. The result is a Prandtl-Ishlinskii hysteresis relation. For a finite number of  $\gamma$ 's, the relation can be written as

$$p \in p_j(s_j) + \gamma_j \operatorname{sign}(\partial_t s_j) \quad \forall j = 1, \dots, N, \quad s = \sum_{j=1}^N c_j s_j. \quad (1.3)$$

Here,  $c_j$  and  $\gamma_j$  are given positive numbers for  $j = 1, \dots, N$ . We demand  $\sum_j c_j = 1$  such that the saturation is a convex combination of the different  $s_j$ , which can be thought of as the saturations in different materials that constitute the porous medium. Regarding physical units, the numbers  $\gamma_j$  are pressure variables. Finally, the functions  $p_j$  are monotone graphs. The general Prandtl-Ishlinskii hysteresis can be formulated equivalently, replacing the finite sums by integrals. A homogenization result is derived in [26] for linear laws  $p_j$ : a porous medium which consists of different materials that exhibit the play-type hysteresis (1.2) (with different parameters) can be described in its averaged behavior by a Prandtl-Ishlinskii relation.

Our result is an instability statement for problem (1.1)–(1.2). Since this simple model is a special case of the more complex models such as (1.3), it is clear that instability can occur also in the more complex models.

To conclude the description of our model we finally describe the boundary conditions. We consider a time domain  $t \in (T_-, T_+)$  containing  $t = 0$  and a spatial domain  $(x, y) \in [L_-, L_+] \times Y$ , with  $Y = [0, L_y]^n$  a rectangle in  $\mathbb{R}^n$  with periodically identified boundaries, the physically relevant cases are  $n = 1$  and  $n = 2$ . Initial values are given by  $s_0 : [L_-, L_+] \rightarrow \mathbb{R}$ , which we identify with  $s_0 : [L_-, L_+] \times Y \rightarrow \mathbb{R}$ , and boundary conditions  $\bar{p}_\pm : (T_-, T_+) \rightarrow \mathbb{R}$ . The initial and boundary conditions are chosen as

$$s(x, y, t = 0) = s_0(x) \quad \forall y \in Y, x \in (L_-, L_+), \quad (1.4)$$

$$p(x = L_\pm, y, t) = \bar{p}_\pm(t) \quad \forall y \in Y, t \in (T_-, T_+). \quad (1.5)$$

Wetting fronts appear when a more saturated medium is above a less saturated medium. Mathematically, we choose a constant initial saturation  $s_0$ , set  $\bar{p}_-(t) = p_c(s_0) + \gamma$  for all  $t$ , and  $\bar{p}_+(t) > p_c(s_0) + \gamma$ , at least for  $t \in (T_-, 0)$ .

### 1.3 Main result: Instability

Our result is that, for appropriate boundary conditions, the hysteresis system is unstable. We use the concept of instability that is made precise in Definition 1.2.

**THEOREM 1.1** (Instability of the hysteresis system) Let  $p_c, k \in C^2(\mathbb{R}, \mathbb{R})$  satisfy the positivity and monotonicity  $k, \partial_s k, \partial_s p_c > 0$  and let  $\gamma > 0$  be positive. Then the hysteresis system (1.1)–(1.2) possesses unstable planar front solutions in the sense of Definition 1.2.

**DEFINITION 1.2** (Instability) We say that (1.1)–(1.2) possesses unstable planar front solutions if the following holds. There exist domain parameters  $L_y, L_+, T_- > 0$ , initial data  $s_0 : (-\infty, L_+) \times Y \rightarrow \mathbb{R}$  which are constant on  $(-\infty, 0) \times Y$ , and boundary data  $\bar{p}_\pm : (T_-, \infty) \rightarrow \mathbb{R}$  which are constant on  $(0, \infty)$ , such that the following holds: for every deviation parameter  $\varepsilon > 0$  and arbitrary smallness restriction  $\rho > 0$ , there exist domain parameters  $L_-$  and  $T_+$  and a perturbation of the initial values  $w \in C_c^1((L_-, L_+) \times Y)$  such that, denoting by  $(s, p)$  and by  $(\tilde{s}, \tilde{p})$  the solutions to system (2.7)–(2.8) with  $s|_{t=T_-} = s_0$  and  $\tilde{s}|_{t=T_-} = s_0 + w$ , respectively, there holds

$$\|w\|_{L^1((L_-, L_+) \times Y)} \leq \rho, \quad \text{but} \quad \|\tilde{s}(\cdot, T_+) - s(\cdot, T_+)\|_{L^1((L_-, L_+) \times Y)} \geq \varepsilon. \quad (1.6)$$

Since the definition of instability is quite involved, we mention already here an immediate consequence of our main theorem. In the estimate (1.6),  $\varepsilon$  can be larger than  $\rho$ , therefore the system cannot be contractive in  $L^1$ .

**COROLLARY 1.3** Let  $p_c, k \in C^2(\mathbb{R}, \mathbb{R})$  and  $\gamma > 0$  be as in Theorem 1.1. Then the hysteresis system (1.1)–(1.2) does not define an  $L^1$ -contraction.

We note that, in our main result, we impose positivity and monotonicity assumptions on  $p_c$  and  $k$  that are natural in the context of the Richards equation, but we restrict ourself to the non-degenerate case. In particular, we show that the instability of front solutions is not a consequence of degenerate coefficients.

Our result can be seen as a weak instability statement in the sense that the numbers  $L_-$  and  $T_+$  must be chosen in dependence of  $\rho$ . This means that visible deviation from the original solution may

appear only at late observation times, and the front may already be at a position far from the upper boundary. In fingering experiments, this means that the fingers may be very wide and that they may develop only at large times. On the one hand, this limitation is consequence of our method of proof. On the other hand, it is not clear if an arbitrarily small perturbation can create fingers of finite size in the non-degenerate setting. Based on the description in [4] concerning the finger widening, we expect that, for a stronger instability result, degenerate coefficient functions must be studied.

**Method of proof.** We study a switch in the pressure boundary condition on the upper boundary. Until time  $t = 0$ , a large pressure on the upper boundary generates an imbibition process, water invades a medium with low saturation from the top. At time  $t = 0$ , the pressure on the upper boundary jumps to a lower value. This induces a decrease of pressure in a region near the upper boundary, while a (gravity driven) imbibition process continues in the lower part of the domain. This setting is in accordance with experiments.

The switch at time  $t = 0$  effectively means that a first evolution process is considered until time 0, while a second evolution process runs after time 0. Both evolution processes are stable – but the combined process is unstable: a small perturbation of initial data at time  $t = T_-$  results in a small perturbation at time  $t = 0$ , but this perturbation changes the second process for all later times. If  $T_+$  is large enough, the perturbation at time  $t = T_+$  is large.

The proof of the theorem rests entirely on the analysis of the one-dimensional system, i.e. the system with one spatial variable  $x \in (L_-, L_+)$  and a time variable  $t \in (T_-, T_+)$ . We describe solutions of this system after the switching time with a free boundary problem. The qualitative properties of this free boundary problem can be analyzed, see Figure 1 for an illustration. In particular, there exists a flux parameter  $q(t)$  which decreases in time, but does not vanish in the limit of large times. This implies that the front continues to proceed with a finite speed. Since the limiting front speed depends on the saturations at time  $t = 0$ , this implies that a small perturbation at time  $t = 0$  can result in a large perturbation at time  $t = T_+$ .

**Further literature.** For degenerate Richards equations without hysteresis, existence statements [1, 2, 27] and uniqueness results [1, 9, 21] are available. Concerning the case that hysteresis is included, we are not aware of any result in the degenerate case. In the one-dimensional case, the oil-trapping effect [28] shows that the degeneracy can change qualitative properties of solutions.

Positive results on the stability of planar fronts are available for many systems. In comparison, instability results are rare. As in our approach, long-wave perturbations are considered in [11] to show the (linearized) instability of planar fronts in a reaction diffusion system. Other instability results for planar fronts appear in [10, 30].

The remainder of this text is organized as follows. In Section 2 we recall fundamental facts about the play-type hysteresis model and introduce one-dimensional front solutions. We present stability results for the hysteresis model in special situations, e.g. the stability of the system for  $s$ -independent permeabilities in Theorem 2.3. Section 3 is devoted to the thorough analysis of the one-dimensional hysteresis system for finite times. We restrict this analysis to special initial and boundary conditions of physical relevance. The main results regard existence and monotonicity of solutions to the one-dimensional problem. Section 4 is devoted to the analysis of these solutions for large times. We determine the limiting flux for large times in Lemmas 4.1 and 4.3. We conclude Section 4 with the proof of the main instability result, Theorem 1.1.

## 2. Preliminaries and stability

In this section we collect known properties of the system (1.1)–(1.2). In Section 2.1 we recall existence results of [26]. In Section 2.2 we define our concept of stability. Section 2.3 collects some positive stability results.

### 2.1 Existence result for a system with hysteresis

Existence properties of the hysteresis system were studied in [26] for  $s$ -independent permeability and affine capillary pressure, neglecting gravity. The emphasis in that existence result was to generalize relation (1.2) to a Prandtl-Ishlinskii hysteresis relation in order to treat the system which is obtained after homogenization. In order to specify the results of [26] to our context, we set  $\Gamma(x, \cdot) \equiv \delta_\gamma(\cdot)$  and  $p_c(\sigma) \equiv a\sigma + b$ , and read the results for  $s(x, t) := u(x, t)$ ,  $a := a^*$ ,  $b := b^*$ , and  $k := K^*$ , where the letters used in [26] appear on the right hand side of the four settings. Equations (1.8)–(1.10) of [26] with  $w(x, t) \equiv w(x, \gamma, t)$  then read

$$w(x, t) = p_c(s(x, t)), \quad (2.1)$$

$$\partial_t s = \nabla \cdot (k \nabla p), \quad (2.2)$$

$$p(x) \in w(x, t) + \gamma \operatorname{sign}(\partial_t w), \quad (2.3)$$

and coincide with our system. Theorem 3.2 and case (ii) of Corollary 3.3 in [26] provide the following existence result. The uniqueness is observed in Remark 3.4 of the same article, where the boundary condition is imposed as  $p = g$  on  $\partial\Omega \times (0, T)$ .

**THEOREM 2.1** (Existence for an hysteresis system, [26]) Let  $\Omega \subset \mathbb{R}^n$  be a rectangle,  $T > 0$ ,  $s \mapsto p_c(s)$  strictly monotone affine and  $k(x, s) = K^*(x)$  piecewise constant. Let initial and boundary values be given by  $s_0 \in L^2(\Omega)$  and  $g \in C^1([0, T], H^2(\Omega, \mathbb{R}))$ . Then there exists a unique pair  $(s, p)$  with

$$s, \partial_t s \in L^\infty(0, T; L^2(\Omega)), \quad p \in H^1(0, T; H^1(\Omega)),$$

such that relations (2.1)–(2.3) are satisfied in the sense of distributions and almost everywhere in  $\Omega \times (0, T)$ , and the boundary conditions are satisfied in the sense of traces.

Theorem 2.1 was shown with an approximation procedure. A discretization of  $\Omega$  with triangles of maximal diameter  $h$  replaces the system by an ordinary differential inclusion equation with independent variable  $t$ . This equation still contains the inclusion of (2.3). One can treat this degeneracy by replacing the inverse of the sign-function  $\operatorname{sign}_\gamma := \gamma \operatorname{sign}$  by the Lipschitz function  $\psi_\delta^\gamma : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\psi_\delta^\gamma(r) := \begin{cases} \delta r & \text{for } r \in [-\gamma, \gamma], \\ \gamma\delta + \frac{1}{\delta}(r - \gamma) & \text{for } r > \gamma, \\ -\gamma\delta + \frac{1}{\delta}(r + \gamma) & \text{for } r < -\gamma. \end{cases} \quad (2.4)$$

More specifically, for  $\delta > 0$ , we solve an ordinary differential equation, which we write as

$$\begin{aligned} \partial_t s_h &= \psi_\delta^\gamma(p_h - p_c(s_h)), \\ \nabla^h \left( k(s_h) \nabla^h p_h \right) &= \psi_\delta^\gamma(p_h - p_c(s_h)). \end{aligned}$$

To make the the method precise, the operator  $\nabla^h$  is expressed with a finite element method, see (2.3)–(2.6) of [26]. We emphasize that, in the existence result, it is important to send first  $\delta \rightarrow 0$  with discretization parameter  $h > 0$  fixed, and then send  $h \rightarrow 0$ .

With Theorem 2.1 we have an existence and uniqueness result for system (1.1)–(1.2) in the case that  $\Omega$  is a rectangle, that  $k(x, s) = K^*$  is independent of  $s$ , that  $p_c(\cdot)$  is affine function, and that gravity is neglected. We expect that these assumptions can be relaxed for the existence proof, and that the same existence result can be obtained for smooth strictly positive  $k$ , and smooth and strictly monotonically increasing  $p_c$ . From now on, we concentrate on stability aspects and skip the further discussion of existence results.

## 2.2 One-dimensional system and stability property

The one-dimensional system. Let us now consider one-dimensional solutions to (1.1)–(1.2), i.e., solutions  $s(x, y, t) = s(x, t)$  and  $p(x, y, t) = p(x, t)$ . With  $x$  as the only spatial variable and gravity pointing in the negative  $x$ -direction, the system for  $s(x, t)$  and  $p(x, t)$  with  $(x, t) \in (L_-, L_+) \times (T_-, T_+)$  reads

$$\partial_t s = \partial_x (k(s)[\partial_x p + 1]), \quad (2.5)$$

$$p \in p_c(s) + \gamma \operatorname{sign}(\partial_t s). \quad (2.6)$$

We recall that we always demand that  $\partial_x p$  and  $\partial_t s$  are functions in  $L^2((L_-, L_+) \times (T_-, T_+))$ , that (2.5) holds in the sense of distributions and that (2.6) holds almost everywhere. We note that we must assume for a planar solution that the initial and boundary conditions are  $y$ -independent,  $s(x, y, t = T_-) = s_0(x)$  and  $p(x = L_\pm, y, t) = \bar{p}_\pm(t)$ .

Every solution to this one-dimensional problem is a solution to the higher dimensional problem if we identify  $s$  and  $p$  with their trivial extensions in the periodic variable  $y \in Y = [0, L_y]^n$ . We call a solution of problem (2.5)–(2.6) a planar solution of (1.1)–(1.2).

The planar front in higher dimension. Our interest is the stability of a planar solution. We will actually show the instability of the planar solution with a study of purely one-dimensional perturbations: we will show that a small perturbation of the initial values can result in a different asymptotic propagation speed of the front. With this difference, the perturbed front moves arbitrarily far away from the original front.

In fingering, one is interested in a slightly different instability process. In fingering experiments, one observes that at some points  $y \in Y = [0, L_y]^n$ , the front moves faster than in other points  $y \in Y$ . This relates to a *spatial instability*.

In the analysis of further instability effects, one has to consider the system with an additional (small) source term in the conservation law. One then studies, for  $f : (L_-, L_+) \times Y \times (T_-, T_+) \rightarrow \mathbb{R}$ , the system

$$\partial_t s = \nabla \cdot (k(s)[\nabla p + e_x]) + f, \quad (2.7)$$

$$p \in p_c(s) + \gamma \operatorname{sign}(\partial_t s). \quad (2.8)$$

## 2.3 Stability results

The system without hysteresis has an  $L^1$ -contraction property. We recall here this well-known result (see [1, 34]) and present the proof for the simplest case, namely for strong solutions to the system



with non-degenerate coefficient functions. The  $L^1$ -contraction property derived in Theorem 2.2 and Theorem 2.3 implies, in particular, that the system (1.1)–(1.2) does not have the instability property of Definition 1.2. In particular, the system without hysteresis and the system with constant permeability are both stable.

In the case of strong solutions the result is readily obtained by considering two solutions and by testing the difference of the equations with the sign of the solution difference. The interest in more recent uniqueness studies is to have the same result for degenerate equations, when the distributional derivative  $\partial_t s$  is not necessarily an integrable function. In this case, the proof of the contraction property can be performed with the technique of doubling the variables. For this interesting field we refer to [9, 21].

**THEOREM 2.2** (Stability in absence of hysteresis) We consider (2.7)–(2.8) in the case without hysteresis, i.e., for  $\gamma = 0$ ,

$$\partial_t s = \nabla \cdot (k(s)[\nabla(p_c(s)) + e_x]) + f. \quad (2.9)$$

Let  $k = k(s)$  and  $p_c = p_c(s)$  be smooth, independent of  $x$ , with  $k, k'$ , and  $p'_c$  strictly positive. Then an  $L^1$ -contraction property holds. More precisely, for two solutions  $s_1$  and  $s_2$  with  $\partial_t s_i \in L^2(\Omega \times (T_-, T_+))$  to the same boundary conditions and with the right hand sides  $f_i \in L^1(\Omega \times (T_-, T_+))$ , there holds

$$\int_{\Omega} |s_1 - s_2|(x, t_2) dx \leq \int_{\Omega} |s_1 - s_2|(x, t_1) dx + \int_{t_1}^{t_2} \int_{\Omega} |f_1 - f_2|(x, t) dx dt \quad (2.10)$$

for all  $t_2 > t_1$ .

*Proof.* We note that the non-degenerate problem without hysteresis (2.9) is a standard parabolic problem and existence results are classical. We consider strong solutions  $s_1$  and  $s_2$  to sources  $f_1$  and  $f_2$ .

We use a Kirchoff transformation. Choosing a function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  with  $\Phi'(s) = k(s)p'_c(s)$ , we use the generalized pressure  $u = \Phi(s)$  as a new dependent variable. Because of  $\nabla u = \Phi'(s)\nabla s = k(s)p'_c(s)\nabla s = k(s)\nabla(p_c(s))$ , the equations for  $s_1$  and  $s_2$  transform into

$$\begin{aligned} \partial_t s_1 &= \nabla \cdot (\nabla u_1 + k(s_1)e_x) + f_1, & u_1 &= \Phi(s_1), \\ \partial_t s_2 &= \nabla \cdot (\nabla u_2 + k(s_2)e_x) + f_2, & u_2 &= \Phi(s_2). \end{aligned}$$

Let  $H_\eta$  be a family of uniformly bounded smooth functions  $H_\eta : \mathbb{R} \rightarrow \mathbb{R}$  that are odd and strictly increasing. Using  $H_\eta(u_1 - u_2) = -H_\eta(u_2 - u_1)$  as a test-function in the equation for  $s_1$  and  $H_\eta(u_2 - u_1)$  as a test-function in the equation for  $s_2$ , adding the equations and integrating yields

$$\begin{aligned} &\int_{\Omega} \partial_t (s_2 - s_1) H_\eta(u_2 - u_1) + \int_{\Omega} \nabla(u_2 - u_1) \nabla [H_\eta(u_2 - u_1)] \\ &\quad - \int_{\Omega} \partial_x [k(s_2) - k(s_1)] [H_\eta(u_2 - u_1)] = - \int_{\Omega} (f_1 - f_2) H_\eta(u_2 - u_1). \end{aligned} \quad (2.11)$$

We choose for  $H_\eta$  uniformly bounded and odd approximations of the sign function,  $H_\eta(\xi) \rightarrow \text{sign}(\xi)$  for every  $\xi$  in the limit  $\eta \rightarrow 0$ . In the limit  $\eta \rightarrow 0$ , since the sign of  $u_2 - u_1$  is identical to the sign of  $s_2 - s_1$ , the first integrand converges to  $\partial_t |s_1 - s_2|$ . At this point we exploit that  $\partial_t s_i$  are integrable functions. The second integrand of (2.11) is non-negative for every  $\eta > 0$ .



It remains to consider the third integral, which we expand by adding and subtracting the same term,

$$\begin{aligned} I_3^\eta &= - \int_{\Omega_t} \partial_x [k(s_2) - k(s_1)] \{H_\eta(\Phi(s_2) - \Phi(s_1)) - H_\eta(k(s_2) - k(s_1))\} \\ &\quad - \int_{\Omega_t} \partial_x [k(s_2) - k(s_1)] H_\eta(k(s_2) - k(s_1)) =: I_{31}^\eta + I_{32}^\eta. \end{aligned} \quad (2.12)$$

Let  $\hat{H}_\eta$  be a primitive of  $H_\eta$  with  $\hat{H}_\eta(0) = 0$ . Then the integrand in the last integral is  $\partial_x \hat{H}_\eta(k(s_2) - k(s_1))$ . The identical boundary conditions for  $p_1$  and  $p_2$  (and thus for  $s_1$  and  $s_2$ ) imply that

$$I_{32}^\eta = - \int_{\Omega_t} \partial_x [k(s_2) - k(s_1)] H_\eta(k(s_2) - k(s_1)) = - \int_{\Omega_t} \partial_x [\hat{H}_\eta(k(s_2) - k(s_1))] = 0.$$

Concerning  $I_{31}^\eta$  we note that the factor  $H_\eta(\Phi(s_2) - \Phi(s_1)) - H_\eta(k(s_2) - k(s_1))$  is uniformly bounded and converges to 0 pointwise in  $(x, t)$ . By Lebesgue's convergence theorem,  $I_{31}^\eta$  vanishes in the limit  $\eta \rightarrow 0$ . In the limit  $\eta \rightarrow 0$  we thus obtain

$$\int_{\Omega} \partial_t |s_2 - s_1| \leq - \int_{\Omega} (f_1 - f_2) \text{sign}(u_2 - u_1) \leq \int_{\Omega} |f_1 - f_2|.$$

An integration from  $t_1$  to  $t_2$  yields the desired estimate (2.10).  $\square$

Spatial stability under perturbations in weighted  $L^2$ -spaces are analyzed in [31]. The result of that article is another stability result for solutions of the system without hysteresis. The following result contains the warning that hysteresis does not automatically lead to instability.

**THEOREM 2.3** (Stability of the hysteresis system for constant  $k$ ) We consider problem (2.7)–(2.8) with general  $\gamma \geq 0$ , strictly monotone and smooth  $p_c$ , and  $k$  independent of  $s$ . Then the system has an  $L^1$ -contraction property. In particular, in the class of strong solutions, every planar solution is stable.

*Proof.* We follow the proof of Theorem 2.2. Let  $(s_1, p_1)$  and  $(s_2, p_2)$  two solutions of (2.7)–(2.8) with  $p_i \in p_c(s_i) + \gamma \text{sign}(\partial_t s_i)$  for  $i = 1, 2$ . Assuming, for notational convenience,  $k = 1$ , we consider the difference of the two equations  $\partial_t s_i = \Delta p_i + f_i$  and multiply with  $H_\eta(p_1 - p_2)$ . This results in

$$\begin{aligned} 0 &\geq - \int_{\Omega} \nabla(p_1 - p_2) \nabla(H_\eta(p_1 - p_2)) \\ &= \int_{\Omega} \partial_t (s_1 - s_2) H_\eta(p_1 - p_2) - \int_{\Omega} (f_1 - f_2) H_\eta(p_1 - p_2) \\ &= \int_{\Omega} \partial_t (s_1 - s_2) H_\eta(p_c(s_1) - p_c(s_2)) - \int_{\Omega} (f_1 - f_2) H_\eta(p_1 - p_2) \\ &\quad + \int_{\Omega} \partial_t (s_1 - s_2) [H_\eta(p_1 - p_2) - H_\eta(p_c(s_1) - p_c(s_2))]. \end{aligned}$$

We choose again a sequence of functions  $H_\eta$  that are odd and bounded approximations of the sign-function. With this choice, by monotonicity of  $p_c$ , there holds  $H_\eta(p_c(s_1) - p_c(s_2)) \rightarrow$

$\text{sign}(p_c(s_1) - p_c(s_2)) = \text{sign}(s_1 - s_2)$ . In particular, the first integrand converges to  $\partial_t |s_1 - s_2|$  as in the last proof. For the other integral we have a positivity property. Indeed, for  $\partial_t s_1 \geq \partial_t s_2$ , there holds

$$p_c(s_1) - p_c(s_2) \in (p_1 - p_2) - \gamma(\text{sign}(\partial_t s_1) - \text{sign}(\partial_t s_2)) \leq p_1 - p_2,$$

and therefore, by monotonicity of  $H_\eta$ ,

$$\partial_t(s_1 - s_2) [H_\eta(p_1 - p_2) - H_\eta(p_c(s_1) - p_c(s_2))] \geq 0.$$

We find the non-negativity of the integrand. The same argument can be repeated for negative  $\partial_t(s_1 - s_2)$ . By taking the limit  $\eta \rightarrow 0$  we find the same contraction result as in Theorem 2.2.  $\square$

Our instability result of Theorem 1.1 implies that the hysteresis system with an  $s$ -dependent permeability  $k$  does not possess the  $L^1$ -contraction property.

We mention here that we do expect a stability property in another special situation. We conjecture that (even for an  $s$ -dependent positive  $k$ ) a strictly monotonically increasing (in time) planar solution is stable for the hysteresis system. We note that a strictly increasing planar solution  $(s, p)$  satisfies  $\partial_t s > 0$  everywhere and hence a system where the hysteresis relation is replaced by  $p = p_c(s) + \gamma$ . Nevertheless, such a stability result needs a deep analysis since the comparison solution  $(\bar{s}, \bar{p})$  will, in general, only satisfy the law (1.2).

### 3. The one-dimensional free boundary problem

In this section we consider once more  $x \in (L_-, L_+)$  as the only spatial variable and analyze the hysteresis system (1.1)–(1.2) in its one-dimensional version (2.5)–(2.6). We restrict our analysis to  $x$ -independent initial values and piecewise constant boundary conditions. Let the equations be specified by a number  $\gamma > 0$  and coefficient functions

$$k, p_c \in C^2(\mathbb{R}, \mathbb{R}) \text{ with } k, k', p_c' > 0 \text{ on } \mathbb{R}. \quad (3.1)$$

In order to specify boundary conditions we use four real numbers  $\bar{s}_0, p_-, p_+^0$ , and  $p_+$ , satisfying

$$\bar{s}_0 \in \mathbb{R}, \quad p_- := p_c(\bar{s}_0) + \gamma, \quad p_+^0 > p_c(\bar{s}_0) + \gamma, \quad p_+ < p_+^0. \quad (3.2)$$

The boundary conditions (1.4)–(1.5) are specified to the following piecewise constant initial saturation and boundary pressures.

$$s(x, t = 0) = \bar{s}_0 \text{ for all } x \in (L_-, L_+), \quad (3.3)$$

$$p(x = L_-, t) = \bar{p}_-(t) := p_- \text{ for all } t \in (T_-, T_+), \quad (3.4)$$

$$p(x = L_+, t) = \bar{p}_+(t) := \begin{cases} p_+^0 & \text{for } t \in [T_-, 0), \\ p_+ & \text{for } t \in [0, T_+]. \end{cases} \quad (3.5)$$

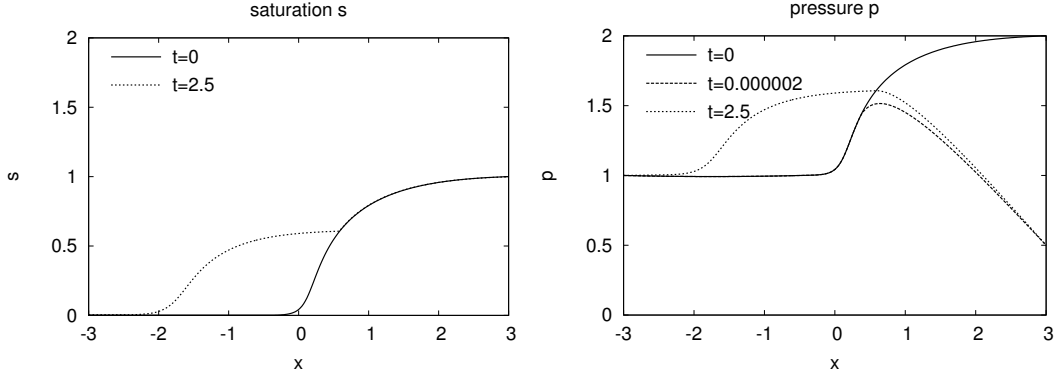


FIG. 1. Numerical solutions obtained with the scheme indicated in (2.4). The parameter functions are  $p_c(s) = s$  and  $k(s) = 0.1 + s^2$ , the parameters are  $\gamma = 1$ ,  $s_0 = 0$ ,  $p_+^0 = 2$ ,  $p_+ = 0.5$ , with the regularization parameter  $\delta = 10^{-3}$ . The saturation at time 0.000002 is indistinguishable from the saturation at time 0. The large change in the pressure at the switching time  $t = 0$  is clearly visible. The saturation continues to increase in the left part of the domain despite the low imposed pressure  $p_+$ . We observe a small, but non-vanishing front speed for large times, as derived in Section 4.1. The numerical results are obtained by Andreas Rätz, TU Dortmund.

### 3.1 Behavior of solutions for $t \in [T_-, 0]$ .

The boundary conditions are chosen in order to create an imbibition process on the time interval  $(T_-, 0]$ . The constant function  $p \equiv p_-$  together with  $s \equiv s_0$  is a solution of (2.5)–(2.6) satisfying the initial condition and the left boundary condition, but the increased pressure  $p_+ > p_-$  on the upper boundary initiates the wetting process. During this imbibition process, relation (2.6) reduces to  $p = p_c(s) + \gamma$ .

LEMMA 3.1 (Solution properties on  $[T_-, 0)$ ) We consider (2.5)–(2.6) with conditions (3.1)–(3.5). There exists a solution  $(s, p)$  of this system satisfying

$$\partial_t s > 0 \quad \text{and} \quad p = p_c(s) + \gamma, \quad (3.6)$$

$$p_- \leq p \leq p_+^0 \quad \text{and} \quad \partial_x s > 0, \quad (3.7)$$

for all  $t \in (T_-, 0)$  and all  $x \in (L_-, L_+)$ .

We note that the uniqueness for system (2.5)–(2.6) is assured by Theorem 2.1 only for constant  $k$ , affine  $p_c$ , and the case without gravity. For this reason we cannot formulate in the above lemma that every solution satisfies the monotonicity properties (3.6)–(3.7).

*Proof.* We analyze (2.5) with (2.6) replaced by  $p = p_c(s) + \gamma$ , i.e. the equation

$$\partial_t s = \partial_x (k(s) [\partial_x (p_c(s)) + 1]). \quad (3.8)$$

This equation is non-degenerate parabolic and has therefore a classical solution on  $(T_-, 0] \times [L_-, L_+]$ . Furthermore, solutions of equation (3.8) satisfy a maximum principle. This implies that  $p = p_c(s) + \gamma$  satisfies the bounds  $p_- \leq p \leq p_+^0$  of (3.7) and the corresponding lower bound  $s \geq \bar{s}_0$ .

Differentiating (3.8) with respect to  $t$  shows that the time-derivative  $v = \partial_t s$  satisfies

$$\partial_t v = \partial_x \left( k'(s)v[\partial_x(p_c(s) + 1)] \right) + \partial_x \left( k(s)[\partial_x(p'_c(s)v)] \right). \quad (3.9)$$

We consider  $s$  as a given function that determines the coefficients of this linear equation for  $v$ . Because of the uniform positivity  $k > 0$  and  $p'_c > 0$ , equation (3.9) for  $v$  is again a non-degenerate parabolic equation. It is supplemented with the boundary condition  $v(L_\pm, t) = 0$ . The idea is now to apply the strong maximum principle to  $v$  in order to conclude  $v \geq 0$  and thus (3.6).

The argument can be made rigorous with a regularization of the boundary condition in  $L_+$ . We use, for  $\varepsilon_j \searrow 0$ , a sequence of  $C^\infty$ -functions  $p_+^\varepsilon : [T_-, 0] \rightarrow \mathbb{R}$  that are monotonically increasing and satisfy  $p_+^\varepsilon \equiv p_-$  in the interval  $[T_-, T_- + \varepsilon]$ . The regularized boundary condition is now  $p(x = L_+, t) = p_+^\varepsilon(t)$ . The corresponding solutions  $(s^\varepsilon, p^\varepsilon)$  and  $v^\varepsilon = \partial_t s^\varepsilon$  satisfy (3.8) and (3.9), the boundary conditions  $v(L_+, t) = 0$  and  $v(L_+, t) = \partial_t p_+^\varepsilon(t) \geq 0$ . Furthermore,  $v^\varepsilon = 0$  holds in  $t = T_-$ . The maximum principle for smooth functions yields the non-negativity of  $v^\varepsilon$  in the whole domain. For a sequence of functions  $p_+^\varepsilon(t)$  that approximate  $p_+^0$  for  $\varepsilon \rightarrow 0$ , the solutions of the  $\varepsilon$ -problem converge to a solution of the original problem. Monotonicity properties of solutions remain valid in the limit and show  $\partial_t s \geq 0$ . The non-negative function  $\partial_t s$  can not vanish identically because of the initial and boundary conditions. Then the strong maximum principle implies the strict inequality (3.6).

In order to conclude the second inequality of (3.7), we repeat the argument with the differentiated equation, this time considering  $v = \partial_x s$ . The non-negativity of  $v = \partial_x s$  on the boundaries  $x = L_\pm$  follows from the a priori bounds on  $s$ , expressed in  $p_- \leq p_c(s) + \gamma \leq p_+^0$ . A regularization argument as above yields  $\partial_x s \geq 0$ . The strong parabolic maximum principle provides  $\partial_x s(L_+, t) > 0$  for all  $t \in (T_-, 0]$ , since  $s$  assumes its maximum at the right boundary (it is not constant by the left boundary condition). This implies also the strict inequality  $\partial_x s > 0$  in the interior.  $\square$

We emphasize that, during the time-span  $(T_-, 0)$ , the boundary conditions imply a pure wetting process for equations (2.5)–(2.6). The monotonicity  $\partial_t s \geq 0$  together with  $p = p_c(s) + \gamma$  is consistent with (2.6). The hysteresis relation has no effect in the time-span  $(T_-, 0)$ .

The switching time  $t = 0$ . Our instability result for the hysteresis system is a consequence of properties of the evolution equation (2.5)–(2.6) on the whole interval  $(T_-, T_+)$ . On the interval  $(T_-, 0)$ , hysteresis was not relevant. Instead, due to a decreased pressure boundary condition  $p_+ < p_+^0$  on the upper boundary (see (3.2)), the hysteresis relation (2.6) will be relevant for  $t > 0$ .

From now on, our analysis concerns the hysteresis system (2.5)–(2.6) on the time interval  $[0, T_+)$ . The boundary conditions for the pressure and the initial condition for the saturation are given by

$$p(L_\pm, t) = p_\pm, \quad s|_{t=0} = s_1. \quad (3.10)$$

Here, the initial saturation  $s_1$  is given as  $s_1(x) := s(x, 0)$ , where  $s$  is the solution of the system on  $(T_-, 0]$ . From Lemma 3.1 we know that  $s_1 \in C^2([L_-, L_+], \mathbb{R})$  is strictly increasing in  $x$ .

For the subsequent analysis, we perform a refined study of the system in the time instance  $t = 0$ . We recall that for  $t > 0$ , the pressure value  $p_+$  at the right end point  $L_+$  is below the value  $p_c(s_1(L_+)) + \gamma = p_+^0$ . Assuming that  $\gamma$  is sufficiently large, the hysteresis relation (2.6) allows that the pressure jumps to a lower value with an unchanged saturation, i.e.  $s(x, t) = s_1(x)$

for  $t > 0$  sufficiently small and  $x < L_+$  sufficiently large. Our next aim is to construct a function  $p_1 : [L_-, L_+] \rightarrow \mathbb{R}$  which describes initial values for the pressure in the sense that  $p(t) \rightarrow p_1$  for  $t \searrow 0$ .

For a given monotone saturation function  $s_1 : [L_-, L_+] \rightarrow \mathbb{R}$  we consider the following system of equations. The unknowns are the point  $x_1 \in (L_-, L_+)$ , the flux parameter  $q_1 > 0$ , and a pressure function  $p_1 : [L_-, L_+] \rightarrow \mathbb{R}$ .

$$p_1 = p_c(s_1) + \gamma \quad \text{on } (L_-, x_1) \quad (3.11)$$

$$k(s_1)[\partial_x p_1 + 1] = q_1 \quad \text{on } (x_1, L_+) \quad (3.12)$$

$$p_1(L_+) = p_+ \quad (3.13)$$

$$p_1(x_1 + 0) = p_c(s_1(x_1)) + \gamma \quad (3.14)$$

$$q_1 = (k(s_1)[\partial_x(p_c(s_1)) + 1]) \Big|_{x_1-0} \quad (3.15)$$

In these equations, (3.12) is the evolution equation with  $\partial_t s$  set to zero, while (3.14) and (3.15) express the continuity of pressure and flux across  $x = x_1$ .

We note that, on the left interval  $(L_-, x_1)$ , the pressure  $p_1$  is determined by the algebraic relation (3.11). Once that  $q_1 > 0$  is given, the ordinary differential equation (3.12) together with the initial condition (3.13) determines  $p_1$  on the right interval  $(x_1, L_+)$ . The two free parameters  $q_1$  and  $x_1$  must be determined from the continuity relations (3.14) and (3.15).

For the subsequent construction we introduce the number  $q_{\text{ref}} > 0$  as the flux at the outflow boundary for  $t = 0$ ,

$$q_{\text{ref}} := k(s_1)[\partial_x(p_c(s_1)) + 1] \Big|_{x=L_-} . \quad (3.16)$$

Due to Lemma 3.1, the reference flux  $q_{\text{ref}}$  is positive. Furthermore, again by Lemma 3.1, the differential equation  $\partial_x(k(s)[\partial_x(p_c(s)) + 1]) = \partial_t s \geq 0$  implies that, at time  $t = 0$ , the flux is monotonically increasing in  $x$  and hence satisfies  $k(s_1)[\partial_x(p_c(s_1)) + 1] > q_{\text{ref}}$  on  $(L_-, L_+)$ . We define additionally a reference pressure function  $p_{\text{ref}} : [L_-, L_+] \rightarrow \mathbb{R}$  as the solution of the ordinary differential equation

$$k(s_1)[\partial_x p_{\text{ref}} + 1] = q_{\text{ref}} \quad \text{on } (L_-, L_+) \quad \text{with } p_{\text{ref}}(x = L_-) = p_- . \quad (3.17)$$

The monotonicity of the flux function implies  $p_{\text{ref}} \leq p_c(s_1) + \gamma$  and  $p_{\text{ref}}(L_+) < p_c(s_1(L_+)) + \gamma = p_+^0$ .

**LEMMA 3.2 (Pressure system for  $t = 0$ )** Let the initial saturation  $s_1 \in C^2([L_-, L_+], \mathbb{R})$  satisfy  $\partial_x s_1 > 0$  on  $[L_-, L_+]$  and  $\partial_x(k(s_1)[\partial_x(p_c(s_1)) + 1]) > 0$  on  $(L_-, L_+)$ . Let the pressure boundary value  $p_+$  satisfy  $p_{\text{ref}}(L_+) < p_+ < p_+^0$ . Then problem (3.11)–(3.15) has a unique solution  $p_1 \in C^1([L_-, L_+], \mathbb{R})$ ,  $x_1 \in (L_-, L_+)$ ,  $q_1 > 0$ . There holds  $p_1 \leq p_c(s_1) + \gamma$ .

*Proof.* We consider the map

$$\mathcal{Q}_1 : (q_{\text{ref}}, q_{\text{max}}) \ni q \mapsto p \in C^1([L_-, L_+], \mathbb{R}), \quad (3.18)$$

where  $p$  is defined as the solution  $p = p_1$  of (3.12) and (3.13) to  $q_1 = q$ . The number  $q_{\text{max}} > q_{\text{ref}}$  is defined below. We note already here that, because of  $p_{\text{ref}}(L_+) < p_+$ , for  $q = q_{\text{ref}}$ , there holds  $p > p_{\text{ref}}$  on  $[L_-, L_+]$ . We furthermore define the map

$$\mathcal{Q}_2 : (q_{\text{ref}}, q_{\text{max}}) \ni q \mapsto \xi_1 \in [L_-, L_+], \quad (3.19)$$

where  $\xi_1$  is the largest intersection point of the two pressure functions as in (3.11), i.e. a point with  $p(\xi_1) = p_c(s_1(\xi_1)) + \gamma$  for  $p = \mathcal{Q}_1(q)$ . For  $q = q_{\text{ref}}$  we have, on the left boundary,  $p(L_-) > p_{\text{ref}}(L_-) = p_- = p_c(s_1(\xi_1)) + \gamma$ , and on the right boundary we have  $p(L_+) = p_+ < p_+^0 = p_c(s_1(L_+)) + \gamma$ . Therefore, an intersection point  $\xi_1 = \mathcal{Q}_2(q)$  exists for  $q = q_{\text{ref}}$ . By continuity of the above constructions, there exists a maximal interval  $(q_{\text{ref}}, q_{\text{max}})$  such that the intersection point  $\mathcal{Q}_2(q)$  exists for all  $q \in (q_{\text{ref}}, q_{\text{max}})$ .

The value of  $q \in (q_{\text{ref}}, q_{\text{max}})$  is now chosen in such a way that (3.15) is satisfied for  $q_1 = q$ ,  $p = \mathcal{Q}_1(q)$  and  $x_1 = \mathcal{Q}_2(q)$ . We verify the existence of such a value  $q$  by analyzing relation (3.15) in the limits  $q \searrow q_{\text{ref}}$  and  $q \nearrow q_{\text{max}}$ . As a preparation we note the following monotonicity property. Increasing the parameter  $q$  increases the values of  $\partial_x p$  and hence decreases the values of  $p$  for  $p = \mathcal{Q}_1(q)$  on the whole interval  $[L_-, L_+]$ . Since  $p_c(s_1(\cdot)) + \gamma$  is monotonically increasing in  $x \in [L_-, L_+]$ , the largest intersection point of the two graphs moves to the left: the map  $q \mapsto \xi_1 = \mathcal{Q}_2(q)$  is monotonically decreasing.

The limit  $q \searrow q_{\text{ref}}$  is easily analyzed. The left hand side of (3.15) tends to  $q_{\text{ref}}$  while the right hand side is strictly above that value (we exploit that the point  $x_1$  moves to the right when  $q$  is decreasing). In particular, for  $q$  close to  $q_{\text{ref}}$ , the left hand side of (3.15) is smaller than the right hand side.

Regarding the limit  $q \nearrow q_{\text{max}}$  we have to distinguish two cases. Let us assume that  $q_{\text{max}}$  is finite. Since the intersection point  $\mathcal{Q}_2(q)$  ceases to exist at  $q = q_{\text{max}}$ , by monotonicity of  $\mathcal{Q}_2$  we can conclude that  $\mathcal{Q}_2(q) \rightarrow L_-$  for  $q \nearrow q_{\text{max}}$ . In this case, the left hand side of (3.15) tends to a value larger than  $q_{\text{ref}}$ , while the right hand side tends to  $q_{\text{ref}}$ . In particular, relation (3.15) is satisfied for some  $q \in (q_{\text{ref}}, q_{\text{max}})$ . On the other hand, in the case  $q_{\text{max}} = \infty$ , the left hand side of (3.15) tends to infinity while the right hand side remains bounded. Therefore, also in this case we find  $q \in (q_{\text{ref}}, q_{\text{max}})$  such that (3.15) holds. In both cases we find the desired solution by setting  $q_1 = q$ ,  $x_1 = \mathcal{Q}_2(q)$ , and, for  $p = \mathcal{Q}_1(q)$ ,

$$p_1(x) := \begin{cases} p_c(s_1) + \gamma & \text{for } x < x_1, \\ p(x) & \text{for } x \geq x_1. \end{cases}$$

The inequality  $p_1 \leq p_c(s_1) + \gamma$  holds as an equality for  $x \leq x_1$ . By construction, we have (3.12) and (3.15) satisfied, i.e.  $k(s_1)[\partial_x p_1 + 1] = q_1 = (k(s_1)[\partial_x (p_c(s_1)) + 1])(x_1)$ . The last expression was assumed to be monotonically increasing in  $x$ . This implies that  $\partial_x p_1 < \partial_x (p_c(s_1))$  on  $(x_1, L_+)$  and thus  $p_1 \leq p_c(s_1) + \gamma$  on  $[L_-, L_+]$ . The regularity of the solution can be read off from the ordinary differential equation.  $\square$

### 3.2 The free boundary problem for $t \in [0, T_+)$

We now study the evolution equations (2.5)–(2.6) for  $t > 0$ . We expect the following qualitative behavior of solutions. Due to the low pressure boundary condition on the right end point  $L_+$  we expect that, on some interval  $(X(t), L_+)$ , the hysteresis relation (2.6) is satisfied with  $p < p_c(s_1) + \gamma$  and  $\partial_t s = 0$ . On the left interval  $(L_-, X(t))$  we expect further imbibition, i.e., that (2.6) is satisfied with  $p = p_c(s) + \gamma$  and  $\partial_t s > 0$ .

Under these assumptions, by the evolution equation (2.5), the flux  $k(s_1)[\partial_x p + 1]$  is constant on the right interval. The equations are

$$k(s_1)[\partial_x p + 1] = q \quad \text{on } \{(x, t) : X(t) < x < L_+\}, \quad (3.20)$$

$$p(X(t) + 0, t) = p_c(s_1(X(t))) + \gamma, \quad (3.21)$$

$$p(L_+, t) = p_+. \quad (3.22)$$

We emphasize that the boundary values for  $p$  in (3.21) and (3.22) are known (once that  $X(t)$  is known). Equation (3.20) can be written as  $\partial_x p = -1 + q/k(s_1)$  and integrated for every  $q \in \mathbb{R}$ . The monotonicity in  $q$  shows that, for given  $X(t) := \xi$ , the equations (3.20)–(3.22) can be solved with some appropriate parameter  $q =: Q(\xi)$ .

We note that  $Q$  is non-increasing in  $\xi$ . We show this by considering two values  $Q(\xi_1) = q_1 > q_2 = Q(\xi_2)$ ; our aim is to show  $\xi_1 \leq \xi_2$ . The two corresponding solutions  $p_1$  and  $p_2$  of the ordinary differential equation (3.20) with initial condition (3.22) satisfy the comparison principle  $p_2 \geq p_1$ . Since  $\xi_j$  is the intersection point of the functions  $p_j$  and the increasing function  $p_c \circ s_1$ , we find  $\xi_1 \leq \xi_2$ .

On the left domain  $(L_-, X(t))$  we demand  $p = p_c(s) + \gamma$  and the equations

$$\partial_t s = \partial_x (k(s)[\partial_x p + 1]) \quad \text{on } \{(x, t) : x < X(t)\}, \quad (3.23)$$

$$p(X(t) - 0, t) = p_c(s_1(X(t))) + \gamma, \quad p(L_-, t) = p_-, \quad (3.24)$$

$$(k(s)[\partial_x p + 1])(X(t) - 0, t) = Q(X(t)). \quad (3.25)$$

Equation (3.24) imposes the continuity of the pressure, see (3.21). Equation (3.25) imposes the continuity of the flux, see the definition of  $Q$  with (3.20). We may regard (3.24) as the boundary conditions for  $p$  on the left domain, and (3.25) as a transmission condition that determines the free boundary  $X(\cdot)$ .

Initially, there holds

$$X(t = 0) = x_1, \quad s(t = 0) = s_1, \quad p(t = 0) = p_1, \quad (3.26)$$

with  $x_1, s_1$ , and  $p_1$  given by Lemma 3.2. We emphasize that only the initial condition for  $s$  must be prescribed for the further evolution.

**LEMMA 3.3 (Existence and monotonicity properties)** There exists a time horizon  $T_+ > 0$  such that the one-dimensional free boundary problem (3.20)–(3.25) with initial values  $s_1 \in C^2((L_-, L_+))$  as constructed above, has a solution  $(s, p)$  with the regularity

$$\partial_x s, \partial_x p \in L^2(0, T_+; L^2(L_-, L_+)), \quad X \in C^0([0, T_+], (L_-, L_+)). \quad (3.27)$$

The solution satisfies, for numbers  $s_{\min}, s_{\max}$  that depend on initial and boundary conditions and on  $p_c$ , but not on  $T_+$ , the maximum principle  $s_{\min} \leq s \leq s_{\max}$  for all times. Furthermore, for all  $t \in (0, T_+)$ , we have the monotonicity properties

$$\begin{aligned} \partial_t s &\geq 0, \quad \partial_x s \geq 0, \\ t \mapsto X(t) &\text{ is monotonically non-decreasing,} \end{aligned} \quad (3.28)$$

where  $\partial_t s$  is understood in the distributional sense.

*Proof.* We interpret the problem as the free boundary problem (3.23)–(3.25) on the left domain. The coupling to the right domain is expressed by the Neumann condition (3.25) involving  $Q(\xi)$ .



*Step 1: Domain transformation.* The boundary value problem (3.23)–(3.25) can be transformed onto a fixed domain. Using a new independent variable  $y \in (-1, 0)$  we can set

$$y := \frac{x - X(t)}{X(t) - L_-}, \quad u(y, t) := p(x, t) = p(y(X(t) - L_-) + X(t), t).$$

For the new unknown  $u : (-1, 0) \times (0, T_+) \rightarrow \mathbb{R}$ , derivatives are calculated as

$$\partial_y u = (X(t) - L_-) \partial_x p, \quad \partial_t u = \partial_t p + R(y, t) \partial_x p \text{ for } R(y, t) = (1 + y) \partial_t X(t).$$

These rules allow to transform the parabolic problem on a variable domain (3.23) into a parabolic problem for  $u$  on the fixed domain  $(-1, 0)$ . The transformed problem is strictly parabolic on a fixed domain, with an additional unknown  $X(t)$ . A Dirichlet condition is imposed on the left boundary point  $y = -1$ . On the right boundary point  $y = 0$ , a Dirichlet condition and a Neumann condition is imposed. This additional relation determines  $X(t)$ .

*Step 2: Space discretization.* We want to replace the system by a spatially discretize system. Using a large number  $M \in \mathbb{N}$ , we approximate the parabolic free boundary problem by an ordinary differential equation for  $M + 1$  real variables. In order to avoid moving grid-points, we discretize the transformed system. Selecting uniformly distributed  $M + 1$  points  $-1 = y_0 < \dots < y_M = 0$ , we want to approximate the solution values  $u(y_m, t)$  by the  $M + 1$  unknowns  $u_m(t)$ ,  $m = 0, \dots, M$ . The position  $X(t)$  of the free boundary is approximated by the additional unknown  $X_0(t)$ . The parabolic equation (3.23) is discretized replacing spatial derivatives by finite differences. The left boundary condition in (3.24) determines  $u_0$ . In order to express the time derivative  $\partial_t u_M(t)$  of the right end-point, we use the Neumann condition (3.25),  $(k(s)[\partial_x p + 1])(X(t), t) = Q(X(t))$ . Given  $X(t) = X_0(t)$  and  $s(X(t), t) = s_M(t)$ , this relation provides the flux to the right of the free boundary point and we can express  $\partial_t s_M(t)$  with the discretized evolution equation (3.23). Finally, the Dirichlet condition (3.24) is used to calculate  $X_0(t)$ . More precisely, the value  $X_0(t)$  is defined, in every time instance  $t$ , as the unique point with  $s_1(X_0(t)) = s_M(t)$ . This discretization process defines an ordinary differential equation for the unknowns  $s_m(t)$ , which can be solved until a maximal time of existence  $T_+^M \in (0, \infty) \cup \{+\infty\}$ . In particular, we have  $X_0(t) < L_+$  for  $t < T_+^M$ .

Once we can derive  $M$ -independent estimates and monotonicity properties for the discrete solutions, we obtain a solution to the original system by performing the limit  $M \rightarrow \infty$ . The monotonicity properties carry over to the limit function. Once we furthermore show that  $T_+^M \nearrow 0$  for  $M \rightarrow \infty$  and the continuity of the limit function  $X$ , the proof of the lemma is complete.

In the following, we want to avoid the clumsy notation required for the space discrete solutions. We therefore present the monotonicity properties and the uniform estimates for spatially continuous solutions. This allows us also to make calculations on the time-dependent domain. The calculations are formal in the sense that we assume the existence of solutions and certain regularity properties of solutions. As sketched before, in the rigorous proof, estimates and monotonicities must be calculated for the space discrete equations.

*Step 3: Energy estimate.* To simplify notation, we assume that the left Dirichlet condition is given as  $s(L_-) = 0$ . Using the solution  $s$  as a test-function in (3.23) then provides

$$\int_{L_-}^{X(t)} s \partial_t s + \int_{L_-}^{X(t)} k(s)[\partial_x p + 1] \partial_x s = Q(X(t)) s(X(t)).$$

The left hand side contains the positive term  $k(s)p'_c(s)|\partial_x s|^2$ . Collecting positive terms on the left hand side we write

$$\begin{aligned} \partial_t \int_{L_-}^{X(t)} \frac{1}{2} |s|^2 + \int_{L_-}^{X(t)} k(s)p'_c(s)|\partial_x s|^2 \\ = \frac{1}{2} |s|^2(X(t)) \partial_t X(t) + Q(X(t))s(X(t)) - \int_{L_-}^{X(t)} k(s)\partial_x s. \end{aligned}$$

The monotonicity of  $s$  in  $x$  implies that  $s$  is bounded by its value in the right end-point, hence by  $\|s_1\|_\infty$ . Using additionally  $X(t) < L_+$  and the monotonicity of  $X$  in  $t$ , we obtain that the time integral of the right hand side is bounded. This provides the  $L^2$ -boundedness of  $\partial_x s$ . The corresponding  $L^2$ -boundedness of  $\partial_x p$  is a consequence of  $p = p_c(s) + \gamma$  and the  $C^1$ -property of the coefficient function  $p_c(\cdot)$ .

*Step 4: Boundedness and monotonicity results for (3.23)–(3.25).* With  $s_{\min} := s_1(L_-)$  and  $s_{\max} := s_1(L_+)$ , the maximum principle for  $s$  is a consequence of the monotonicities. We will verify all monotonicity properties simultaneously with a contradiction argument. To this end, we define the time instance  $t_* > 0$  as the first time instance in which one of the three monotonicity properties fails to hold. More precisely, we set

$$t_* := \sup \{t^* \in [0, T_+] : \text{on the interval } (0, t^*) \text{ the properties (3.28) hold}\}.$$

We start by showing  $t_* > 0$  for the discrete system. In  $t = 0$ , the saturation is strictly monotonically increasing in  $x$  by  $\partial_x s = \partial_x s_1 > 0$ . Assuming that  $\partial_x s_1$  is continuous in  $t$  (which is true for spatially discrete solutions), we obtain  $\partial_x s \geq 0$  for a small time interval  $(0, \varepsilon_1)$ . Similarly, the formal time derivative of the initial values is strictly positive in  $t = 0$  by the property  $\partial_t s > 0$  of the system on the time interval  $(T_-, 0)$ . For a contradiction argument, let us assume that the saturation  $s_M$  of the right boundary point is decreasing. In that case,  $X_0$  is decreasing in  $t$ , since  $s_1$  is increasing in  $x$ . This implies that  $Q(X_0(t))$  is increasing. The saturation is therefore decreasing in the free boundary point, non-decreasing in the left neighboring point, and the flux towards the right neighboring point is increasing. We conclude that the convexity in the free boundary point is increasing, hence  $s_M$  should be increasing, contradicting our assumption. The monotonicity of  $s$  in  $t$  implies that also  $X(\cdot)$  is monotonically non-decreasing in  $t$ .

We next use a contradiction argument to show  $t_* = T_+$ . Let us therefore assume  $t_* < T_+$  and study of the properties of solutions in the time instance  $t = t_*$ . We distinguish three cases, all will provide a contradiction.

*Case 1.*  $\partial_x s(x, t_*) = 0$  for some  $L_- \leq x \leq X(t_*)$ . We consider the function  $v = \partial_x s$  as in Lemma 3.1. The function  $v$  is non-negative on  $(0, t_*)$  by construction of  $t_*$ . Furthermore, the function  $v$  satisfies the parabolic equation (3.9), and hence satisfies a maximum principle. We emphasize that, in one space dimension, the maximum principle holds also for spatially discrete solutions. In particular, the minimum  $x$  must lie on the boundary. The case  $x = L_-$  is excluded by the fact that the saturation is non-decreasing on  $(0, t_*)$  and the left boundary condition is constant in time. We therefore have the minimum in  $x = X(t_*)$ .

We now read (3.23) as an elliptic equation for  $p$  with the non-negative right hand side  $\partial_t s \geq 0$ . The saturation (and, hence, the pressure  $p = p_c(s) + \gamma$ ) is maximal at the right boundary  $x = X(t_*)$  by construction of  $t_*$ . The Hopf Lemma for elliptic inequalities (which holds in one space

dimension also for spatially discrete solutions) then implies  $\partial_x p(x, t_*) = p'_c(s(x, t_*)) \partial_x s(x, t_*) > 0$ , a contradiction.

*Case 2.*  $\partial_t X(t_*) = 0$ . We differentiate the Dirichlet condition (3.24) with respect to time, noting that we kept the relation in the discretization scheme. We obtain  $\partial_t s(X(t), t) + \partial_x s_1(X(t)) \partial_t X(t) = \partial_x s_1(X(t)) \partial_t X(t)$ . Because of  $\partial_t X(t_*) = 0$ , the material derivative of  $s$  coincides with the partial derivative and we obtain  $\partial_t s(X(t), t) = 0$ .

We can now consider the non-negative function  $v = \partial_t s$ , which solves a parabolic equation. We observed that this function has a minimum  $v = 0$  in  $X(t_*)$ . The Hopf lemma for parabolic equations implies negativity of the space derivative,  $\partial_x v(X(t_*), t_*) < 0$ . Once more, we note that this holds in one space dimension also for discrete solutions to parabolic problems.

On the other hand, the time derivative of the Neumann condition implies (3.25) implies, for  $\partial_t X(t_*) = 0$ ,

$$\begin{aligned} 0 &= \frac{d}{dt} [(k(s)[\partial_x p + 1]) (X(t), t)] \\ &= k'(s) \partial_t s [\partial_x p + 1] + k(s) p''_c(s) \partial_t s \partial_x s + k(s) p'_c(s) \partial_t \partial_x s \\ &= k(s) p'_c(s) \partial_t \partial_x s, \end{aligned}$$

and thus  $\partial_t \partial_x s(X(t_*), t_*) = 0$ . This provides the desired contradiction in the second case.

*Case 3.*  $\partial_t s(x, t_*) = 0$  for some  $L_- < x \leq X(t)$ . Arguing as above for the non-negative function  $v = \partial_t s$ , we know that the minimum with value 0 is necessarily attained at  $x = X(t_*)$ . We use once more the Dirichlet condition (3.23), differentiated with respect to time. We obtain  $\partial_t s(X(t), t) + \partial_x s(X(t)) \partial_t X(t) = \partial_x s_1(X(t)) \partial_t X(t)$ . The condition  $\partial_t s(X(t), t) = 0$  implies

$$[\partial_x s_1(X(t)) - \partial_x s(X(t))] \partial_t X(t) = 0.$$

Since  $s$  is strictly increasing in  $t$  inside the domain, and the saturation values of  $s$  and  $s_1$  coincide in  $X(t)$ , the squared bracket is positive. This provides  $\partial_t X(t_*) = 0$ , which we already excluded in Case 2. This provides the desired contradiction in the third and last case.

*Step 5: Continuity of  $X$  and positivity of  $T_+ > 0$ .* The discrete system is an ordinary differential equation with differentiable solution. In particular, the non-decreasing function  $t \mapsto X_0(t) = X_0^M(t)$  has no jumps. It remains to exclude the case that the monotone limit function  $X = \lim_{M \rightarrow \infty} X_0^M$  has jumps. Once that jumps of  $X$  are excluded, we also infer the positivity of  $T_+$ , since the limiting solution cannot jump to the right end-point  $x = L_+$  in arbitrarily short time. Regarding the maximal interval of existence we note that solutions cannot cease to exist because of a blow-up of saturation values, since the latter are bounded.

Let us assume for a contradiction that at a time instance  $t_j \in [0, T_+)$  a jump of  $X$  occurs:  $X(t) \rightarrow x_l$  for  $t \nearrow t_j$  and  $X(t) \rightarrow x_r$  for  $t \searrow t_j$  for  $x_l \leq x_l < x_r \leq L_+$ . With this hypothesis we find, due to the Dirichlet condition (3.24), in the limit  $M \rightarrow \infty$  a solution  $s$  with  $s(x, t_j) = s_1(x)$  for all  $x \in (x_l, x_r)$ . This solution satisfies  $\partial_x s(x, t_j) = \partial_x s_1(x)$ , hence the flux in a point  $(x, t_j)$  satisfies the Neumann condition  $Q(x) = k(s_1)[\partial_x(p_c(s_1)) + 1] \geq Q(x_1) > Q(x)$ , the first inequality by monotonicity of the flux of  $s_1$  in  $x$ , the strict inequality since  $Q$  is strictly decreasing. This provides the desired contradiction.  $\square$

Consistency. Our next aim is to verify, that the solution of the free boundary problem as constructed in Lemma 3.3 solves indeed the one-dimensional hysteresis system. In order to verify this fact, we have to reconstruct the saturation values by setting

$$s(x, t) := \begin{cases} p_c^{-1}(p(x, t) - \gamma) & \text{if } t \leq 0 \text{ or } t > 0, x < X(t), \\ s_1(x) & \text{else.} \end{cases} \quad (3.29)$$

We can now establish the link between the one-dimensional free boundary problem and the original hysteresis problem.

LEMMA 3.4 (Consistency) Let initial values  $s_1 = s_{t=0}$  be given by a solution  $s$  of the hysteresis system on  $(T_-, 0)$ . On the time interval  $(0, T_+)$  let  $s$  be a solution of the one-dimensional free boundary problem (3.20)–(3.26) with initial values  $s_1$ . We extend  $s$  to the right of the free boundary point as in (3.29). Then the constructed saturation function has the regularity

$$\partial_x s \in L^2(T_-, T_+; L^2((L_-, L_+), \mathbb{R})), \quad (3.30)$$

and  $\partial_t s$  is a distribution of class  $\partial_t s \in L^2(T_-, T_+; H^{-1}((L_-, L_+), \mathbb{R}))$ . If  $\gamma > 0$  is sufficiently large, the pair  $(p, s)$  is a solution to the one-dimensional hysteresis system (2.5)–(2.6) on  $(T_-, T_+)$ .

*Proof.* The regularity of  $\partial_x s$  follows on  $(T_-, 0)$  from the fact that  $s$  solves a parabolic problem on that time interval. On the interval  $(0, T_+)$ , we obtained the regularity in Lemma 3.4. We note additionally that, by choice of the initial values  $s_1$  in the time instance  $t = 0$ , the distributional derivative  $\partial_t s$  has no singular contribution in  $t = 0$ . From the equations for  $t < 0$  and  $t > 0$  and from the fact that the flux function  $k(s)[\partial_x p + 1]$  has no jump in  $x = X(t)$  for all  $t > 0$ , we obtain  $\partial_t s \in L^2(T_-, T_+; H^{-1}((L_-, L_+), \mathbb{R}))$ .

*Hysteresis equations.* It remains to verify that (2.5) and (2.6) hold almost everywhere. For  $t < 0$  and for  $t > 0$  with  $x < X(t)$ , the evolution equation is imposed explicitly with  $p = p_c(s) + \gamma$ . Because of  $\partial_t s \geq 0$ , both relations (2.5) and (2.6) are satisfied in these regions. In the domain  $t > 0$  and  $x > X(t)$ , the saturation is  $s = s_1$  such that the time derivative is  $\partial_t s = 0$ . Relation (3.20) then implies (2.5) on that domain. Regarding relation (2.6) we only have to check that  $p_c(s_1) - \gamma \leq p \leq p_c(s_1) + \gamma$ . Since the solutions are bounded, the lower bound  $p_c(s_1) - \gamma \leq p$  is satisfied for  $\gamma$  sufficiently large. It remains to verify the upper bound  $p \leq p_c(s_1) + \gamma$ . There holds  $p(X(t)) = p_c(s_1(X_1(t))) + \gamma$  by (3.21). Using (3.20) and (3.25) we conclude  $(k(s_1)[\partial_x p + 1])(X(t) + 0, t) = q = (k(s)[\partial_x p + 1])(X(t) - 0, t) \leq (k(s_1)[\partial_x(p_c(s_1)) + 1])(X(t))$ , where the last inequality is a consequence of  $p \geq p_c(s_1) + \gamma$  for  $x < X(t)$ , which is due to  $\partial_t p = p'_c(s)\partial_t s \geq 0$ . The expression  $k(s_1)[\partial_x(p_c(s_1)) + 1]$  is monotonically increasing in  $x > X(t)$  such that the  $\partial_x p \leq \partial_x(p_c(s_1))$  holds in this domain. This implies  $p \leq p_c(s_1) + \gamma$  for  $x > X(t)$ , which concludes the proof.  $\square$

We note that the continuity of fluxes of (3.25), together with the continuity of the saturation, implies that  $p(\cdot, t) \in H^2((L_-, L_+))$  can be expected. On the other hand, the saturation  $s$  will not have this regularity, since, in general,  $\partial_x s$  has a jump across  $x = X(t)$ . This jump is visible in Figure 1.

#### 4. Long-time behavior and instability

##### 4.1 Solutions of the free boundary problem for large times

For all times  $t > 0$ , the saturation on  $x < X(t)$  continues to increase by Lemma 3.4. The relevant question is whether the flux at the free boundary point  $X(t)$  (which is related to the front-speed of

the wetting front) tends to zero for  $t \rightarrow \infty$ , or if it remains finite. The latter case would correspond to a front that continues to proceed with finite speed. Our first aim is now to collect equations that determine the behavior of solutions for large times.

In the free boundary problem, the equations for the right domain suggest a limit problem. If  $x_\infty$  denotes the limiting position of the free boundary  $X(t)$  (we recall that  $X$  is monotonically increasing and bounded by  $L_+$ ), we can expect that the limiting profile satisfies (3.20) with the right boundary condition (3.22) and the left boundary condition (3.21), which is formulated in (4.1) and (4.2) below.

It remains to formulate a last relation that determines the limiting position  $x_\infty$  of the free boundary. We expect that the increasing saturation on the left domain leads to an almost vanishing slope  $\partial_x s(X(t) - 0, t)$  and hence to an almost vanishing gradient  $\partial_x p$  in the limit of large times. By the continuity (3.25) we therefore expect that the flux  $q$  coincides with the permeability  $k(s_1)$  for large times, which expresses that, in the free boundary point, the flow is purely gravity driven. The limit relation is expressed with (4.3) if we set  $\zeta = 0$ . For flexibility in the subsequent analysis, we allow also perturbations of the flux condition and include the parameter  $\zeta \in \mathbb{R}$ . The next lemma is devoted to the limit system which determines  $q_\infty > 0$ ,  $x_\infty < L_+$ , and  $p : (x_\infty, L_+) \rightarrow \mathbb{R}$ .

$$k(s_1)[\partial_x p + 1] = q_\infty \quad \text{on } x \in (x_\infty, L_+), \quad (4.1)$$

$$p(x = L_+) = p_+, \quad p(x_\infty) = p_c(s_1(x_\infty)) + \gamma, \quad (4.2)$$

$$k(s_1(x_\infty)) = q_\infty + \zeta. \quad (4.3)$$

LEMMA 4.1 (Solvability of system (4.1)–(4.3) describing large times) Let initial values  $s_1$  be as in Lemma 3.2, and let the boundary condition satisfy  $p_{\text{ref}}(L_+) < p_+ < p_+^0$ . Then there exists  $\zeta_1 > 0$  such that system (4.1)–(4.3) possesses, for all  $\zeta \in (-\zeta_1, \zeta_1)$ , a unique solution

$$x_\infty \in (L_-, L_+), \quad q_\infty > 0, \quad p \in H^2((x_\infty, L_+), \mathbb{R}).$$

The solution depends continuously on  $\zeta$ .

*Proof.* The construction is very similar to the one of Lemma 3.2. Indeed, (3.12)–(3.14) coincide with (4.1)–(4.2). The only difference is the modified flux condition in the free boundary point.

We use  $q \in [0, q_1]$  as a parameter, where  $q_1$  is the flux value of (3.12) with corresponding free boundary point  $x_1$ . Once more, we denote by  $p =: \mathcal{Q}_1(q)$  the solution of (4.1) to  $q_\infty = q$ , with the right boundary condition  $p(x = L_+) = p_+$ . The rightmost intersection point  $\xi \in [L_-, L_+]$  of the graphs of  $p = \mathcal{Q}_1(q)$  and  $p_c(s_1(\cdot)) + \gamma$  is denoted by  $\mathcal{Q}_2(q) := \xi$ . The definitions are exactly as in Lemma 3.2. In particular, the map  $q \mapsto \mathcal{Q}_2(q)$  is again monotonically non-increasing.

We claim that the map  $\mathcal{Q}_2$  is well-defined on  $[0, q_1]$ . Indeed, for  $q \leq q_1$ , there holds  $\partial_x p \leq \partial_x p_1$  and therefore, since the same values are assumed in  $L_+$ , the comparison result  $p \geq p_1$ . The function  $p_1$  has an intersection point with  $p_c \circ s_1 + \gamma$ , namely  $x_1$ . Since on the right boundary  $p(L_+) = p_+ < p_+^0 = p_c(s_1(L_+)) + \gamma$ , also  $p$  and  $p_c \circ s_1 + \gamma$  have an intersection point  $\xi \geq x_1$ .

It remains to find  $q = q_\infty$  such that also (4.3) is satisfied. We only have to evaluate the two sides of (4.3) in the end-points of the  $q$ -interval. For  $q = 0$ , the number  $k(s_1(\mathcal{Q}_2(0)))$  is positive, hence greater than  $q$ . Instead, for  $q = q_1$ , there holds  $\mathcal{Q}_2(q_1) = x_1$  and hence

$$k(s_1(\mathcal{Q}_2(q_1))) = k(s_1(x_1)) < (k(s_1)[\partial_x(p_c(s_1)) + 1])(x_1) = q_1$$

by (3.15). The continuity of the involved maps and the intermediate value theorem provide the existence of  $q_\infty$ , such that (4.3) holds as an equality.

Uniqueness of solutions. Let  $q_\infty$  and  $x_\infty$  define a solution  $p$  of the system. We observe that for no other value  $x > x_\infty$  the function  $p$  has an intersection with  $p_c \circ s_1 + \gamma$ . This follows immediately from

$$k(s_1)[\partial_x p + 1] = q_\infty = k(s_1(x_\infty)) \leq k(s_1) \text{ and hence } \partial_x p \leq 0,$$

while  $\partial_x(p_c \circ s_1) > 0$ . Therefore all solutions of (4.1)–(4.3) are obtained as solutions of  $x_\infty = \mathcal{R}_2(q_\infty)$  by the above construction, which used the rightmost intersection point in  $\mathcal{R}_2$ . But  $k(s_1(\mathcal{R}_2(q)))$  is non-increasing in  $q$ , while the identity map  $q \mapsto q$  is strictly increasing in  $q$ . Therefore, (4.3) has at most one solution  $q_\infty$ .

Continuous dependence. All the constructed maps are continuous in  $s_1$  and  $\zeta$ . Additionally, the identity  $q \mapsto q$  has the derivative 1, which is bounded from below. We conclude that the zero  $q_\infty$  depends continuously on  $\zeta$ .  $\square$

Our interest is the limit behavior of solutions to (3.20)–(3.25) for large times. We claim that, in the limit of large  $t$ , the flux  $q(t) = Q(X(t))$  approaches the limit flux  $q_\infty$  of the above lemma (for  $\zeta = 0$ ). Indeed, for increasing  $t$ , by Lemma 3.4, the point  $X(t)$  is increasing, the flux  $q(t) = Q(X(t))$  of (3.20) is decreasing, and, accordingly, the solution  $p$  on the right interval is increasing. By boundedness of these quantities, it follows that there exist limits

$$X(t) \nearrow \tilde{x}_\infty, \quad q(t) \searrow \tilde{q}_\infty, \quad p(\cdot, t) \nearrow \tilde{p}(\cdot) \text{ uniformly on } [\tilde{x}_\infty, L_+],$$

for  $t \nearrow \infty$ . Furthermore, (3.20)–(3.22) imply equations (4.1) and (4.2) for the limit functions. If we can additionally verify (4.3) with  $\zeta = 0$ , by the uniqueness statement of Lemma 4.1, we have  $\tilde{x}_\infty = x_\infty$  and  $\tilde{q}_\infty = q_\infty$ , and hence the convergence to the limit determined by (4.1)–(4.3).

The true proof is more involved, since we have to deal with the dependence on the left endpoint  $L_-$ . Also the uniform convergence for  $t \rightarrow \infty$  is not obvious. Our result will be based on the following claim on the stabilization for monotone ordinary differential equations. We use once more the Kirchhoff transformation and the monotone function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  with  $\Phi'(s) = k(s)p'_c(s)$ .

**CLAIM 4.2** Let numbers  $s_{\min} < s_{\max}$  and  $q_{\min} < q_{\max}$  be given and let  $\Phi \in C^1(\mathbb{R})$  with  $\Phi'$  bounded from below by a positive number. We consider solutions  $s : (-L_0, 0) \rightarrow [s_{\min}, s_{\max}]$  of the ordinary differential equation

$$\partial_x(\Phi(s)) + k(s) = q + f, \tag{4.4}$$

where the right hand side contains a flux value  $q \in \mathbb{R}$  and a perturbation  $f : (-L_0, 0) \rightarrow \mathbb{R}$ . Then the following statement holds. For every error  $\varepsilon_1 > 0$  there exist parameters  $L_0, \varepsilon_2 > 0$  such that solutions  $s$  of (4.4) satisfy

$$|k(s(0)) - q| < \varepsilon_1, \tag{4.5}$$

for all  $q \in [q_{\min}, q_{\max}]$  and all  $f$  with  $\|f\|_{L^\infty} < \varepsilon_2$ .

*Proof.* Given  $\varepsilon_1 > 0$ , we first want to choose  $L_0$  large enough, such that every solution of the differential equation  $\partial_x(\Phi(s)) + k(s) = q$  on  $(-L_0, 0)$  solves  $|k(s(0)) - q| < \varepsilon_1/2$ . This is possible for fixed initial value  $s(-L_0)$  and fixed  $q$  since the differential equation provides exponential convergence to the solution  $s_0 \in \mathbb{R}$  of  $k(s_0) = q$ . The lower bound  $L_0$  for the interval length can be chosen with continuous dependence on  $q$  and on  $s(-L_0)$ . Since the initial values  $s(-L_0)$  and the values of  $q$  are chosen in a compact interval, there exists  $L_0 > 0$  satisfying the uniform estimate.

Let us now assume that (4.5) does not hold for all  $f$ . Then we find a sequence  $f_j \rightarrow 0$  in  $L^\infty((-L_0, 0))$ ,  $q_j \rightarrow q$ , and solutions  $s_j$  to  $\partial_x(\Phi(s_j)) + k(s_j) = q_j + f_j$  with  $|k(s_j(0)) - q_j| \geq \varepsilon_1$ . But then, by the compactness of Arzela-Ascoli, for a subsequence, the solutions  $s_j$  converge uniformly to a solution  $s$  of the limit problem, which satisfies  $|k(s(0)) - q| < \varepsilon_1/2$  by the first step. We find the desired contradiction.  $\square$

**LEMMA 4.3 (Behavior for large times)** We consider a solution of the free boundary problem (3.20)–(3.25) on  $(0, T)$ , with position  $X(t)$  and flux constant  $q(t)$  as in Lemma 3.3. Let furthermore limiting values  $x_\infty$  and  $q_\infty$  be as in Lemma 4.1. Then, for every  $\varepsilon > 0$ , there exist  $L_0 > 0$  and  $m_* > 0$  independent of  $T$  such that, for all  $L_- \leq -L_0$ ,

$$\exists N \subset [0, T] \text{ with one-dimensional Lebesgue-measure } |N| < m_*, \quad (4.6)$$

such that

$$|X(t) - x_\infty| + |q(t) - q_\infty| < \varepsilon \quad \forall t \in [0, T] \setminus N. \quad (4.7)$$

*Proof.* We study solutions of the free boundary problem for various  $L_- < 0$ . We have to compare the elliptic equations (3.20)–(3.22) on the right domain in a time instance  $t$  with (4.1)–(4.2). The continuous dependence on  $\zeta$  in Lemma 4.1 provides the existence of  $\varepsilon_1 > 0$  (depending on  $\varepsilon$ , but not on  $L_-$  and  $t$ ) such that

$$|k(s_1(X(t))) - q(t)| < \varepsilon_1 \quad \Rightarrow \quad |X(t) - x_\infty| + |q(t) - q_\infty| < \varepsilon. \quad (4.8)$$

We can satisfy the smallness requirement of the left hand side with the help of Claim 4.2. That claim provides  $L_0 > 0$  and  $\varepsilon_2 > 0$  (both depending on  $\varepsilon_1$ , but not on  $\delta$  and  $t$ ) such that for

$$\|\partial_x(\Phi(s)) + k(s) - q(t)\|_{L^\infty((-L_0, X(t)))} < \varepsilon_2 \quad (4.9)$$

the condition  $|k(s_1(X(t))) - q(t)| < \varepsilon_1$  is satisfied by (4.5). We therefore define

$$N := \left\{ t \in [0, T] : \|\partial_x(\partial_x(\Phi(s)) + k(s))\|_{L^1((-L_0, X(t)))} \geq \varepsilon_2 \right\}. \quad (4.10)$$

For all  $t \notin N$  holds (4.9), since  $q(t)$  is the value of  $\partial_x(\Phi(s)) + k(s)$  in the free boundary point  $X(t)$ . By this construction, we have the assertion (4.7) satisfied with the set  $N$ .

It remains to study the measure of  $N$ . We calculate, using the evolution equation and  $\partial_t s \geq 0$ ,

$$\varepsilon_2 |N| \leq \int_0^T \int_{-L_0}^{X(t)} |\partial_t s| \leq (L_+ + L_0)(s_{\max} - s_{\min}).$$

We therefore find

$$|N| \leq \frac{(L_+ + L_0)(s_{\max} - s_{\min})}{\varepsilon_2} =: m_*(\varepsilon).$$

Since  $m_*$  is independent of  $T$ , this concludes the proof.  $\square$



#### 4.2 Proof of the main instability result

We can now give a proof of main result, Theorem 1.1. We choose the domain parameter  $L_y = 1$ , the parameters  $L_+, T_- > 0$ , initial data  $s_0$  and boundary data  $\bar{p}_\pm : (T_-, \infty) \rightarrow \mathbb{R}$  as in Section 3. According to Definition 1.2, we have to find, for arbitrary deviation parameter  $\varepsilon > 0$  and smallness restriction  $\rho > 0$ , domain parameters  $L_-$  and  $T_+$  and a perturbation of the initial values  $w \in C^1((L_-, L_+) \times Y)$ , such that (1.6) holds.

For fronts as constructed above, we have seen in Lemma 4.1 that the front solution has a limiting flux value  $q_\infty$  for large times. We have furthermore derived in Lemma 4.3 a quantitative statement showing that the limiting flux value is approximately realized for large times. Our aim here is therefore to consider perturbations of the initial values of the form  $s_0 + \delta w$  and to analyze the corresponding limiting fluxes  $q_\infty^\delta$ ,  $\delta = 0$  and  $\delta = 1$ . The result in the next statement is that the fluxes  $q^\delta$  differ considerably for  $\delta = 0$  and  $\delta = 1$ .

**LEMMA 4.4 (Flux variation in the one-dimensional problem)** Let  $w \in C_c^1((0, L_+) \times Y)$  satisfy  $0 \neq w \geq 0$  and let, for  $\delta \in \{0, 1\}$ , the pair  $(s^\delta, p^\delta)$  be a solution of the free boundary problem to initial values  $s_0(x) + \delta w(x)$ . Let  $q^\delta(t)$  denote the corresponding flux. Then there exists  $\varepsilon_q > 0$  such that, for any volume fraction  $0 < \Theta < 1$ , there exist constants  $L_0, T_0 > 0$ , such that, for all  $T > T_0$  and all  $L_- < -L_0$ , there exists a set  $M \subset [0, T]$  of measure  $|M| > \Theta T$  such that

$$q^1(t) - q^0(t) \geq \varepsilon_q, \quad \forall t \in M. \quad (4.11)$$

*Proof.* We start with the construction of  $\varepsilon_q > 0$ . Loosely speaking, we only have to make sure that  $q_\infty^1 - q_\infty^0 \geq 2\varepsilon_q$ .

*Step 1. Limiting fluxes.* We analyze the system (4.1)–(4.3), which determines the limiting speed  $q_\infty$  for arbitrary values of the one-dimensional initial saturation  $s_0$ . For the initial values  $s_0 + \delta w$  we denote the corresponding limiting flux by  $q_\infty^\delta$ . The comparison principle for the parabolic system on the time interval  $(T_-, 0)$  implies that the values of the saturation in  $t = 0$  are ordered.

$$s_1^1 := s^1(t = 0) > s^0(t = 0) = s_1^0.$$

We claim that this implies also

$$q_\infty^1 > q_\infty^0. \quad (4.12)$$

In order to show (4.12) we argue by contradiction and assume that  $q_\infty^1 \leq q_\infty^0$ . We first exploit (4.3) with  $\zeta = 0$ . The relation  $s_1^1 \geq s_1^0$  and the monotonicity of  $k$  imply

$$k(s_1^0(x_\infty^1)) \leq k(s_1^1(x_\infty^1)) = q_\infty^1 \leq q_\infty^0 = k(s_1^0(x_\infty^0)).$$

The monotonicity of  $k \circ s_1^0$  in  $x$  then yields  $x_\infty^1 \leq x_\infty^0$ . The differential equation (4.1) now implies for the limiting pressure functions  $p^0$  and  $p^1$  on  $(x_\infty^0, L_+)$

$$\partial_x p^1 = \frac{q_\infty^1}{k(s_1^1)} - 1 \leq \frac{q_\infty^0}{k(s_1^1)} - 1 < \frac{q_\infty^0}{k(s_1^0)} - 1 = \partial_x p^0.$$

The identical boundary conditions  $p^0(x = L_+) = p_+ = p^1(x = L_+)$  imply the strict inequality  $p^1 > p^0$  on  $(x_\infty^0, L_+)$ . The point  $x_\infty^1$  is defined as the maximum of  $p^1$ , and  $p^1$  lies above  $p^0$ , hence we conclude  $p^1(x_\infty^1) > p^0(x_\infty^0)$ . In particular, exploiting (4.3) and (4.2), we find

$$\begin{aligned} q_\infty^1 &= k(s_1^1(x_\infty^1)) = (k \circ (p_c + \gamma)^{-1})(p^1(x_\infty^1)) > (k \circ (p_c + \gamma)^{-1})(p^0(x_\infty^0)) \\ &= k(s_1^0(x_\infty^0)) = q_\infty^0. \end{aligned}$$

This is in contradiction with our assumption  $q_\infty^1 \leq q_\infty^0$ , hence (4.12) is verified.

*Step 2. Choice of  $\varepsilon_q$ ,  $T_0$  and  $L_0$ .* We set  $\varepsilon_q = (q_\infty^1 - q_\infty^0)/2$ , which is positive by inequality (4.12). Let now  $\Theta < 1$  be an arbitrarily large given volume fraction. Our aim is to find numbers  $T_0, L_0 > 0$  with the desired property. We apply Lemma 4.3 twice with  $\varepsilon := \varepsilon_q/2$ , with initial values  $s_1^0$  and with initial values  $s_1^1$ . Lemma 4.3 provides numbers  $L_0 > 0$  and  $m_* > 0$ . We use the number  $L_0$  and it remains to choose  $T_0$  based on the given values of  $m_*$  and  $\Theta$ . For all  $t \in [0, T] \setminus N$  we find, by (4.7) and the triangle inequality,

$$|q^1(t) - q^0(t)| \geq |q_\infty^1 - q_\infty^0| - |q^0(t) - q_\infty^0| - |q^1(t) - q_\infty^1| \geq 2\varepsilon_q - \varepsilon_q/2 - \varepsilon_q/2 = \varepsilon_q.$$

Choosing  $T_0$  large enough we achieve that the portion of  $N$  (which has measure at most  $m_*$ , independent of  $T$ ) in the set  $[0, T]$  is smaller than  $1 - \Theta$ . Setting  $M = [0, T] \setminus N$  concludes the proof.  $\square$

*Conclusion of the main theorem.* We can now prove the main instability result. We only have to verify that a difference in the fluxes leads, after a sufficiently large time, to a large difference between the solutions.

*Proof of Theorem 1.1. Choice of  $L_y$ ,  $L_+$ ,  $T_-$ , and  $w$ .* We choose  $L_y = 1$ ,  $T_- = -1$ , and choose  $L_+$  large enough to satisfy  $x_1 > 0$ . Let  $\varepsilon > 0$  and  $\rho > 0$  be given. We have to prove (1.6). Our aim is to find  $T_+ > 0$ ,  $L_- < 0$ , and a perturbation  $w$ . We pick an arbitrary function  $w \in C_c^1((0, L_+) \times Y)$  with  $0 \neq w \geq 0$  with the smallness required in (1.6). We emphasize that, since  $w$  is supported on  $(0, L_+)$  and independent of time, the smallness of  $w$  is independent of the choice of  $L_-$  and  $T_+$  below.

*Choice of  $T_+$  and  $L_-$ .* As in the last proof, let  $s^\delta$  be the solution to  $s_0 + \delta w$ , for  $\delta \in \{0, 1\}$ . Our aim is to select  $T$  and  $L_-$  in order to satisfy

$$\|s^1(\cdot, T) - s^0(\cdot, T)\|_{L^1((L_-, L_+) \times Y)} \geq \varepsilon. \quad (4.13)$$

Lemma 4.4 provides a positive number  $\varepsilon_q$ , which measures variations of the flux. We now choose the number  $\Theta < 1$  large enough to have

$$\Theta \varepsilon_q > 2(1 - \Theta)(q_{\max} - q_{\min}). \quad (4.14)$$

With this choice of  $\Theta$ , we select  $T_0$  and  $L_0$  according to Lemma 4.4.

In order to verify (4.13), we consider the total mass on the left domain. For notational convenience and without loss of generality we assume the initial saturation to be  $s_0 = 0$ , such that also  $s_{\min} = 0$  such that we have to study

$$m^\delta(t) := \int_{L_-}^{X^\delta(t)} s^\delta(x, t) dx.$$

The time increment is calculated with the conservation law (3.23),

$$\begin{aligned} \frac{d}{dt} m^\delta(t) &= s^\delta(X^\delta(t), t) \partial_t X^\delta(t) + \int_{L_-}^{X^\delta(t)} \partial_t s^\delta(x, t) dx \\ &= s^\delta(X^\delta(t), t) \partial_t X^\delta(t) + \left( k(s^\delta) [\partial_x p^\delta + 1] \right) \Big|_{L_-}^{X^\delta(t)}. \end{aligned} \quad (4.15)$$

We abbreviate the outflow at  $x = L_-$  by  $q_-^\delta(t) := (k(s^\delta)[\partial_x p^\delta + 1])(L_-)$ . Integrating (4.15) over  $[0, T]$  yields, for  $\delta = 0$ ,

$$m^0(T) - m^0(0) \leq s_{\max} L_+ + \int_0^T q^0(t) dt - \int_0^T q_-^0(t) dt. \quad (4.16)$$

On the other hand, we can derive a lower bound for  $m^1(T)$ . We use the estimate  $q^1(t) - q^0(t) \geq \varepsilon_q$  of (4.11) on the large set  $M \subset [0, T]$  (depending on  $T$ , but with a controlled volume fraction), and the estimate  $q^1(t) - q^0(t) \geq -(q_{\max} - q_{\min})$  on the remainder. We integrate (4.15) over  $[0, T]$  and exploit the positivity of  $s \partial_t X$ . Using subsequently (4.11) and (4.16) yields

$$\begin{aligned} m^1(T) - m^1(0) &\geq \int_0^T q^1(t) dt - \int_0^T q_-^1(t) dt \\ &\geq \int_0^T q^0(t) dt + T \Theta \varepsilon_q - T(1 - \Theta)(q_{\max} - q_{\min}) - \int_0^T q_-^1(t) dt \\ &\geq m^0(T) - m^0(0) - s_{\max} L_+ + \int_0^T (q_-^0 - q_-^1)(t) dt + \frac{1}{2} T \Theta \varepsilon_q. \end{aligned}$$

We can therefore compare the total mass at time  $T$  for  $\delta = 1$  and for  $\delta = 0$ . We use that the initial mass  $m^\delta(0)$  is bounded, independent of  $\delta$ , and that the total outflow can be bounded by an arbitrary positive number by enlarging  $L_-$  (e.g. by 1, see below). We find, with  $C$  independent of  $T$ ,

$$m^1(T) - m^0(T) \geq -C + \frac{1}{2} T \Theta \varepsilon_q.$$

We now transform this lower bound for a mass difference into a lower bound for the  $L^1$ -norm of  $(s^1 - s^0)(T)$  as required for (4.13). We calculate

$$\|s^1(\cdot, T) - s^0(\cdot, T)\|_{L^1((L_-, L_+) \times Y)} \geq \int_{L_-}^{L_+} (s^1 - s^0)(T) dx \geq -C + \frac{1}{2} T \Theta \varepsilon_q.$$

We can choose  $T$  large in order to have the right hand side large. In particular, we can achieve that (4.13) holds.

*Boundedness of  $q_-^\delta(t)$ .* In order to make the proof complete, we finally also verify the boundedness of the fluxes, a fact that is easy to believe. We claim that, for fixed  $T$  and with the constant  $C = 1$ , we can achieve the bound

$$\int_0^T |q_-^\delta(t)| dt \leq C \quad (4.17)$$

by imposing a large lower bound on  $|L_-|$ . This can be seen as follows. With  $p = p_c(s) + \gamma$ , the saturation  $s$  solves the parabolic problem (3.23) on  $(L_-, 0)$ , with constant Dirichlet data on the left boundary and constant (and matching) initial values. The Dirichlet data on the right boundary are bounded,  $s_{\min} \leq s \leq s_{\max}$ .

Transformed solutions  $\hat{s}(x) = s(|L_-|x)$  solve the same quasilinear parabolic problem on a fixed spatial domain  $(-1, 0)$  with a new time variable. The transformed solution satisfies an upper bound for the Neumann data on the left,  $|\partial_x s^\delta(x = -1)| \leq C_N$  for all times. This implies  $|\partial_x s(x = L_-)| \leq C_N/|L_-|$ . Choosing  $|L_-|$  large (in dependence of  $s_{\min}$ ,  $s_{\max}$ , and  $T$ ) provides (4.17) and concludes the proof.  $\square$

## 5. Conclusions

We studied the Richards equation with gravity. It is well known that the classical Richards equation (without hysteresis) defines an  $L^1$ -contraction and hence a stable evolution. Even in the “unstable” situation that a more saturated medium is above a less saturated medium, the classical Richards equation will therefore not show an instability; the model predicts a stable planar wetting front, in contrast to experiments.

We have therefore included a play-type hysteresis relation between pressure and saturation as suggested and discussed in [7]. Our rigorous analysis shows that this modified Richards equation does not define an  $L^1$ -contraction. Instead, for appropriate boundary data, we have shown that an arbitrarily small perturbation of the initial values can lead to the development of fingers.

Our results are obtained for non-degenerate coefficient functions. In this setting, the instability can be shown only on large domains, i.e., with wide fingers. In order to observe fingers in an arbitrary finite domain, we believe that degenerate coefficients must be considered. This is in accordance with experiments, where authors report that very dry sand must be used in order to obtain fingering effects.

The proof of our instability result was based on the analysis of a one-dimensional free boundary problem. The obtained saturation profile in the fingers is monotone in our setting. In order to obtain the experimentally observed non-monotone profiles, it might be necessary to include additionally dynamic hysteresis.

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