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On a generalized Muskat–Brinkman type problem

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We show the solvability of a multidimensional Muskat type initial boundary value problem. The proposed mathematical model describing the transport phenomena of non-homogeneous flow in porous media, relies on a generalized formulation of the Brinkman equation.

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1. Introduction

The original Muskat problem was proposed in 1934 by Muskat [31] to study from Darcy's law the encroachment of water into an oil sand. Due to applications to oil reservoir, this problem obtains a great practical interest and also, in view of the mathematical difficulties, the *well-posedness* for the Muskat problem takes attention of many mathematicians.

In fact, many important results concerning the Muskat problem were obtained during the last 20 years. Most of existence and uniqueness results are related with the situation when there exists only one moving horizontal interface, that separate two different fluids. Two regimes were found for the Muskat problem: a stable regime, when this horizontal interface is stable under small deviations and an unstable one, which is to say, fingering occurs. The stable regime could be realized, if initially a horizontal interface separates the two fluids with a denser fluid from below and in the presence of gravity force. In the case of the stable regime we can mention the following results: Yi [41], Siegel, Caflisch, Howison [37] shown global-in–time existence for initial data, that is a small perturbation of a flat interface, which separate two fluids. Ambrose [3], Córdoba A., Córdoba D., Gancedo [17], Escher, Matioc [19] proved local-in-time existence and uniqueness of solutions for initial data in the Sobolev spaces. Further global well-posedness results were established by Constantin, Córdoba, Gancedo, Strain [15] for initial data smaller than an explicitly computable constant. Nevertheless the well-posedness of the Muskat problem for general initial data is not known. It is also interesting to mention the existence and non–uniqueness results of weak solutions for the Muskat problem obtained by Córdoba, Faraco, Gancedo [16] and Székelyhidi Jr. [39]. The number of constructed weak solutions, corresponding to the same initial data, is *infinite.* Indeed, the weak solutions considered have low regularity, and in particular, they do not satisfy a standard criterion of selection of the unique solution: the re-normalized criterion for the density equation (see Lemma 3.3). The literature concerning the Muskat problem is really huge nowadays, we do not pretend here to cover all of them. We address the reader to the references there in the above cited works, which seems to us very good to have an up to date scenario of the Muskat problem.

Whilst Darcy's equation continues to occupy a central place in the study of flow through porous media, it is only valid for a special class of flows. As in any fluid flow problem, the range of validity of Darcy's law may be expressed in terms of the Reynolds number Re, which is generally defined in terms of a characteristic length. In particular, for consolidated porous media Re is expressed in terms of mean porous size, on the other hand for unconsolidated porous media, it is in terms of grain size. In any case, it is claimed that Darcy's law is applicable for Re less than 10, where the viscous forces are predominant. Darcy's law may breakdown for many reasons, for instance when the Re number is bigger than 100, or applying for gases at low pressure, or if the mean porous diameter of the medium is comparable with the mean free path of the gas, etc. The reader is addressed to some empirical or almost–empirical modifications of Darcy's law studied by Scheidegger [36]. We also refer to the book of Nield, Bejan [32], where an excellent review of different approaches for modeling flows in the porous media is given.

It is important to observe that, in the original formulation of the Muskat problem given by the standard Darcy's law equation, the fluids are assumed to behave ideally with itself, that is to say, the viscosity of the fluids only takes place in the constitutive relation of the interaction forces, and do not in the Cauchy stress tensor, see for instance Appendix B in Rajagopal, Tao [34]. In this paper, we follow our original idea established in [13], perturbing the Darcy law equation with a positive viscosity term, that reduce to a commonly known Brinkman law equation. Then we formulate an initial boundary value problem assuming Dirichlet boundary data, and in this way, it is shown the solvability of a generalized Muskat type problem, that is presented with details in the next sections.

The Brinkman equation has the form

$$\frac{\mu}{\kappa}\,\mathbf{v} - \eta\,\Delta\mathbf{v} = -\nabla p,$$

where p is the pressure, $\mathbf{v}$ is the velocity field, κ is the permeability and η is an effective viscosity, which is not necessary equal to the dynamic viscosity μ, see for instance [32]. In fact, in many applications of the above equation, also in the original formulation due to Brinkman [10], it is used μ instead of η. Advances in homogenization theory made it possible rigorously to derive Darcy's and Brinkman's laws from Stokes' equations. As concluded in [2], there are three different limits depending on the size of the periodically arranged obstacles, which respectively lead to Darcy's, Brinkman's, and Stokes' equations as macroscopic (homogenized) relation. In the same direction, Sahimi [35] observes that, if the length scale where the fluctuations in the velocity field $\mathbf{v}$ are important is much larger than $O(\sqrt{\kappa})$, then the diffusive term the Brinkman's equation could be negligible and the Darcy's law is recovered. Moreover, when the porosity of the medium approaches the identity, it follows that $\eta/\mu \to 1$, also $\kappa \to \infty$, and the Brinkman's equation approaches the Stokes' equation. In fact, the ratio η/μ depends on the geometry, which is characterized by the porosity, ratio of the void space to the total volume of the medium, and tortuosity, which represents the hindrance to flow diffusions imposed by local boundaries or local viscosity of the porous medium. Therefore, the Brinkman equation could be interpreted as an interpolation between the macroscopic Darcy's law and the microscopic Stokes' equation. In the works Auriault [7], Nield, Bejan [32], Rajagopal [33] a more complete comparison study of Darcy's and Brinkman's laws have been developed.

One observes that, the application of the Brinkman law is the natural one in a high porous medium. Moreover, it has been recently employed to problems in automotive industry, design of thermal insulations, energy saving [12], biomedical hydrodynamics studies [25], transport flows in carbonate karst reservoirs [26], hydraulic fracturing (mining of shale gas) [29], ground water contamination, processes of vaporization-condensation [30], among others applications.

Finally, the numerical study of Stokes–Brinkman–Darcy type systems (stationary case) has been developed by Girault, Kanschat, Riviere [22], Ingram [23], Layton, Schieweck, Yotov [27]. The vanishing viscous limit for a Brinkman–Darcy type system has been studied by Kelliher, Temam, Wang [24]. The global well-posedness of the Cauchy problem associated to compressible Brinkman flow is shown in [1]. The Cahn–Hilliard–Brinkman system, modeling a diffuse interface of two-phase separation in porous medium, is investigated in the unpublished article [8]. Some theoretical results related with the problem studied here, can be found in the works [4] and [18], where it was studied the solvability for the Stokes and Navier–Stokes equations, describing the motion of two fluids.

2. Multidimensional Muskat type problem

In this section we establish the multidimensional Muskat type problem. The model presented here, follows the theory of Continuum Mechanics for Mixtures. The reader interested to more physical explanations is addressed to Atkin, Craine [5], also Rajagopal [33], Rajagopal, Tao [34], and the treatise of Truesdell, Topin [40]. In particular, for the study of the interaction terms, we address the reader to Massoudi [28].

Generally speaking, the Muskat problem is a piston-like displacement of two immiscible fluids in a (porous) media domain. A porous media shall be interpreted like a solid with holes (small ones, mostly interconnected) in it. Then, the fundamental statement of the Muskat problem is the existence of two different fluids in a domain $\Omega \subset \mathbb{R}^d$ (here Ω is assumed bounded), which immiscible fluids are separated by an unknown (free) surface S (connected, non-necessarily regular) of co–dimension one. Moreover, this surface S is not constant in time, and the main issue is to know if there exists an evolution in time of it.

Since the main motivation is related to planning operation of oil wells, which is to say, the interaction between water and oil, it will be used the subscripts o, w, to distinguish between each phase of the mixture. As usual we assume that the mixtures of each fluid and the porous medium is sufficiently dense, thus for each time–space point $(t, \mathbf{x})$, the properties of the mixture is well established, for instance the concept of seepage velocity of each fluid in the porous media.

Under the assumption that the fluids are immiscible, that is to say, remain separated for all times during the process of motion (piston–like), for each $t \geqslant 0$, the bounded smooth domain Ω is given by the union of two disjoint sets $\Omega_o(t)$ and $\Omega_w(t)$, with a common surface $S(t)$ of co-dimension one, where the surface is assumed Lipschitz. Then, we consider for each $t \geqslant 0$ and $\mathbf{x} \in \Omega$ the mixture density $\rho(t, \mathbf{x})$, given by

$$\rho(t, \mathbf{x}) = \rho_w(t, \mathbf{x})\, 1_{\Omega_w(t)}(\mathbf{x}) + \rho_o(t, \mathbf{x})\, 1_{\Omega_o(t)}(\mathbf{x}). \tag{1}$$

Moreover, the velocity $\mathbf{v}(t, \mathbf{x})$ of the mixture is

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{v}_w(t, \mathbf{x})\, 1_{\Omega_w(t)}(\mathbf{x}) + \mathbf{v}_o(t, \mathbf{x})\, 1_{\Omega_o(t)}(\mathbf{x}),$$

where $\mathbf{v}_\iota(t, \mathbf{x})$, $(\iota = o, w)$, is the velocity field of each component (also called phase) and is obtained from an average of the flow rate of the ι^{th}-phase divided by an unitary area.

The balance of mass (usually called continuity equation in fluid dynamics) is given in distribution sense by

$$\partial_t \rho(t,\mathbf{x}) + \operatorname{div}\big(\rho(t,\mathbf{x})\ \mathbf{v}(t,\mathbf{x})\big) = 0. \tag{2}$$

Let us interpret at this point the evolution of the interface $S(t)$. Let $\mathbf{w}$ be the velocity of the interface $S(t)$. From the continuity equation (2), we have on $S(t)$

$$\rho_o(\mathbf{v}_o - \mathbf{w}) \cdot \mathbf{n}_S = \rho_w(\mathbf{v}_w - \mathbf{w}) \cdot \mathbf{n}_S =: m,$$

where $\mathbf{n}_S$ is the unitary normal field to $S(t)$. Since we assume that the fluids are immiscible, we have $m = 0$, i.e. the fluid's flow does not pass through the interface S, hence

$$\mathbf{w} \cdot \mathbf{n}_S = \mathbf{v}_o \cdot \mathbf{n}_S = \mathbf{v}_w \cdot \mathbf{n}_S.$$

This equality describes the evolution of the interface $S(t)$. From the continuity of the normal velocities of the fluids on $S(t)$ ($\mathbf{v}_o \cdot \mathbf{n}_S = \mathbf{v}_w \cdot \mathbf{n}_S$) and the assumption that the fluids are incompressible, that is

$$\operatorname{div} \mathbf{v}_\iota = 0 \quad \text{in } \Omega_\iota(t) \quad \text{for } \iota \in M \equiv \{o, w\},$$

it follows, in the sense of distributions, that

$$\operatorname{div} \mathbf{v}(t,\mathbf{x}) = 0. \tag{3}$$

Now, let us consider the Conservation of Linear Momentum. One remarks that, we are not considering that the fluids behave ideally with itself, like standard Darcy's law assumption. Consequently, we shall consider an effective viscosity term in the constitutive relation to the Cauchy stress tensor. Therefore, we consider the following constitutive relation of the Cauchy stress tensor for incompressible fluids

$$\mathbf{T}_\iota = -p_\iota\ \mathbf{I_d} + 2\ \eta_\iota\ \mathbf{D}\mathbf{v}_\iota, \tag{4}$$

where $p_\iota \geqslant 0, \eta_\iota > 0$ are respectively the scalar function called pressure and the effective viscosity of the ι^{th}-phase. Usually the effective viscosity $\eta = \mu/\phi(\mathbf{x})$, where $\phi > 0$ is the porosity of the medium and $\mu > 0$ is the dynamic viscosity. Here for simplicity of exposition, we take $\eta = \mu$. Moreover, one observes that, $\mathbf{I_d}$ is the identity map on $\mathbb{R}^d$, and $\mathbf{D}\mathbf{v}_\iota$ is the stretching tensor, given in components by

$$[\mathbf{D}\mathbf{v}_\iota]_{ij} = \frac{1}{2}\Big(\partial_{\mathbf{x}_j}[\mathbf{v}_\iota]_i + \partial_{\mathbf{x}_i}[\mathbf{v}_\iota]_j\Big).$$

that is, $\mathbf{D}$ is just the symmetric part of the gradient operator, instead of the standard notation in Continuum Physics, where $\mathbf{D}$ is the symmetric part of the gradient operator applied to velocity field. The constitutive relation for the interaction force terms of the fluids with the porous media, is given by a functional dependence in the difference of the velocity of the fluid and the medium. As usual, we assume linear dependence, and since the porous media velocity is zero, the interaction force is given by

$$\mathbf{f}_\iota = -h_\iota\ \mathbf{v}_\iota, \tag{5}$$

with a given non-negative scalar function $h_\iota = h(t,\mathbf{x},\mu_\iota)$ (which takes in account the properties of the porous medium). For instance, we have

$$h_\iota\big(t,\mathbf{x},\mu_\iota(t,\mathbf{x})\big) = \frac{\mu_\iota(t,\mathbf{x})}{k(\mathbf{x})}, \qquad (k(\mathbf{x}) \text{ is the permeability}).$$

Certainly, more general conditions on h_ι may be considered. For instance, we may assume that h_ι is a (positive defined) second order tensor, which takes in account the anisotropy of the medium. For simplicity, we just consider here the scalar case. We observe also that, $\mu_\iota(t,\mathbf{x}) = \rho_\iota(t,\mathbf{x})\,\nu_\iota(t,\mathbf{x})$, where ν_ι is the kinematic viscosity of the ι^{th}-phase. Once we denote by $\mathbf{g}$ the body force density, the generalized Darcy's law is given by

$$\mathbf{f}_\iota(t,\mathbf{x}) + \operatorname{div}\mathbf{T}_\iota(t,\mathbf{x}) + \rho_\iota(t,\mathbf{x})\,\mathbf{g}(t,\mathbf{x}) = 0, \tag{6}$$

that has been proposed by Brinkman [10] as a new phenomenological relation between the velocity and the pressure gradient for flows in highly porous media.

Due to the assumption of viscous motion of the fluid mixture it is natural to consider that the velocity $\mathbf{v}$ is continuous on the interface, that is

$$\mathbf{v}_o = \mathbf{v}_w \quad \text{on } S(t). \tag{7}$$

Finally, we admit the dynamic boundary condition on $S(t)$, that requires the continuity of the normal component of the Cauchy stress tensor on $S(t)$:

$$\mathbf{T}_o\mathbf{n}_S = \mathbf{T}_w\mathbf{n}_S. \tag{8}$$

One observes that, equation (8) gives on the interface the difference of the oil and water pressures in terms of the viscosity components. Therefore, could be considered as a corrected physical interpretation of the Laplace formula (experimental one), which gives the difference of the pressures on the interface in terms of the capillarity pressure.

From the above considerations (4)–(8) we conclude that the evolution of the mixture velocity $\mathbf{v}$ is described in distribution sense by

$$\begin{aligned} &h(t,\mathbf{x},\mu)\,\mathbf{v}(t,\mathbf{x}) - \operatorname{div}\big(\mu\,\mathbf{D}\mathbf{v}(t,\mathbf{x})\big) = -\nabla p(t,\mathbf{x}) + \rho(t,\mathbf{x})\,\mathbf{g}(t,\mathbf{x}),\\ &\mu(t,\mathbf{x}) = \rho(t,\mathbf{x})\,\nu(t,\mathbf{x}), \end{aligned} \tag{9}$$

where the kinematic viscosity of the fluid mixture

$$\nu(t,\mathbf{x}) = \nu_w(t,\mathbf{x})\,1_{\Omega_w(t)}(\mathbf{x}) + \nu_o(t,\mathbf{x})\,1_{\Omega_o(t)}(\mathbf{x})$$

is governed, analogously to the density, by the transport equation

$$\partial_t\nu(t,\mathbf{x}) + \operatorname{div}\big(\nu(t,\mathbf{x})\,\mathbf{v}(t,\mathbf{x})\big) = 0 \tag{10}$$

in the distribution sense. The functions $\mu \equiv \rho\nu$ and p are called the dynamic viscosity and the pressure of the fluid mixture, which are respectively positive and non-negative functions, defined by a similar formula to (1).

We recall that, the process is assumed isothermal, thus we do not consider the Conservation of Energy.

Let $T > 0$ be any fixed real number and $\Omega \subset \mathbb{R}^d$ (with $d = 2$ or 3) is an open and bounded domain having a C^2- smooth boundary Γ. We define by $\Omega_T := (0,T)\times\Omega, \quad \Gamma_T := (0,T)\times\Gamma$. Moreover, the outside unitary normal to Ω at $\mathbf{x}\in\Gamma$ is denoted by $\mathbf{n} = \mathbf{n}(\mathbf{x})$. Then, from equations (2), (3), (9) and (10), we formulate a generalized Muskat type initial-boundary value problem, denoted **GMP**:

For all $(t, \mathbf{x}) \in \Omega_T$, find $(\rho(t, \mathbf{x}), \nu(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x}))$ solution of

$$\begin{cases} \partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0, \qquad \partial_t \nu + \mathbf{v} \cdot \nabla \nu = 0, \\ h(t, \mathbf{x}, \rho\nu)\, \mathbf{v} - \operatorname{div}\big(\rho\nu\ \mathbf{D}\mathbf{v}\big) = -\nabla p + \rho\, \mathbf{G}, \quad \operatorname{div} \mathbf{v} = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho|_{\Gamma_T^-} = \rho_b, \quad \nu|_{t=0} = \nu_0, \quad \nu|_{\Gamma_T^-} = \nu_b, \quad \mathbf{v}|_{\Gamma_T} = \mathbf{b}, \end{cases}$$

where $\mathbf{g}$ is a given vector function, also ρ_0, ρ_b, ν_0, ν_b are given initial-boundary data for the density and effective viscosity respectively, $\mathbf{b}$ is the boundary data for the velocity field $\mathbf{v}$ and

$$\begin{aligned} \Gamma_T^- &:= \big\{(t, \mathbf{r}) \in \Gamma_T : (\mathbf{b} \cdot \mathbf{n})(t, \mathbf{r}) < 0\big\}, \\ \Gamma_T^+ &:= \big\{(t, \mathbf{r}) \in \Gamma_T : (\mathbf{b} \cdot \mathbf{n})(t, \mathbf{r}) > 0\big\} \end{aligned} \tag{11}$$

called respectively the in-flux, and out-flux boundary zones of the "oil-water" mixture.

In the following sections we show that, the **GMP** is solvable and describes the motion of immiscible fluids. One of the main difficulties, to prove the solvability for **GMP**, is to show the strong convergence of an approximating sequence for the density function. Another important issue is the trace of the density function, once it is just assumed measurable and bounded, in fact, we follow the technical and important results proved by Boyer in [9]. Similar remarks are also posed for the kinematic viscosity.

3. Functional notation and auxiliary results

In this section we present the notations, the definitions of functional spaces and some useful results, used through the paper.

We will use the standard notations for the Lebesgue function space $L^p(\Omega)$, the Sobolev spaces $W^{s,p}(\Omega)$ and $H^s(\Omega) \equiv W^{s,2}(\Omega)$, where a real $s \geqslant 0$ is the smoothness indices and a real $p \geqslant 1$ is the integrability indices. The vector counterparts of these spaces are denoted by $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$ and $\mathbf{H}^s(\Omega) := (H^s(\Omega))^d$.

Let us consider the space of functions of L^2-bounded deformation

$$\mathbf{LD}_0^2(\Omega) = \Big\{\boldsymbol{\psi} \in \mathbf{L}^2(\Omega) :\ \mathbf{D}\boldsymbol{\psi} \in (\mathbf{L}^2(\Omega))^d,\ \ \boldsymbol{\psi} = 0 \text{ on } \Gamma\Big\},$$

endowed by the norm $||\boldsymbol{\psi}||_{\mathbf{LD}_0^2(\Omega)} = ||\boldsymbol{\psi}||_{\mathbf{L}^2(\Omega)} + ||\mathbf{D}\boldsymbol{\psi}||_{\mathbf{L}^2(\Omega)}$. Due to the Poincaré and Korn inequalities, that is

$$||\boldsymbol{\psi}||_{\mathbf{H}^1(\Omega)} \leqslant C||\mathbf{D}\boldsymbol{\psi}||_{\mathbf{L}^2(\Omega)}, \qquad \forall \boldsymbol{\psi} \in \mathbf{LD}_0^2(\Omega), \tag{12}$$

the space $\mathbf{LD}_0^2(\Omega)$ shall be considered equivalent to $\mathbf{H}_0^1(\Omega)$. We also consider the following spaces

$$\begin{aligned} \mathbf{V}^1(\Omega) &:= \big\{\boldsymbol{\psi} \in \mathbf{H}^1(\Omega) : \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in } \mathfrak{D}'(\Omega)\big\}, \\ \mathbf{V}^{1/2}(\Gamma) &:= \Big\{\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma) : \quad \int_\Gamma \boldsymbol{\psi} \cdot \mathbf{n}\, d\mathbf{x} = 0\Big\}. \end{aligned}$$

Let us formulate auxiliary results (Lemmas 3.1–3.3 and Corollary 3.4), used to prove the main result of our article (Theorem 4.2). First, we consider the Stokes type system

$$\begin{cases} -\operatorname{div}(\nu\mathbf{D}\mathbf{v}) + k(\mathbf{x})\mathbf{v} = -\nabla p + \mathbf{g}, \qquad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\ \mathbf{v} = \mathbf{b} \quad \text{on } \Gamma. \end{cases} \tag{13}$$

The solvability of (13) in the particular case, when ν is a constant, $k = 0$ and $\mathbf{b} = \mathbf{0}$, has been done by Cattabriga [11]. Let us show the existence result for system (13) in a general situation.

LEMMA 3.1 Let us assume that

$$\begin{aligned} &\mathbf{b} \in \mathbf{V}^{1/2}(\Gamma) \quad \text{and} \\ &\mathbf{g} \in \mathbf{L}^q(\Omega) \quad \text{for some } q > 1 \quad \text{if } d = 2; \quad q = \tfrac{6}{5} \quad \text{if } d = 3. \end{aligned} \tag{14}$$

Let $k(\mathbf{x}), \nu(\mathbf{x})$ be measurable positive functions, satisfying

$$\begin{aligned} k \in L^s(\Omega) \quad \text{for some } s > 1 \quad \text{if } d = 2; \quad s = 3/2 \quad \text{if } d = 3. \\ \nu(\mathbf{x}) \in [\overline{\nu}^0, \overline{\nu}^1] \quad \text{a.e. in } \Omega \end{aligned} \tag{15}$$

for some positive real numbers $\overline{\nu}^0 < \overline{\nu}^1$. Then, there exists a unique weak solution $\mathbf{v} \in \mathbf{V}^1(\Omega)$ of (13), satisfying

$$\|\mathbf{v}\|_{\mathbf{V}^1(\Omega)} \leqslant C(||\mathbf{b}||_{\mathbf{V}^{1/2}(\Gamma)} + \|\mathbf{g}\|_{\mathbf{L}^q(\Omega)}) \tag{16}$$

with a positive constant C depending only on Ω and the data k, ν.

Proof. By Lemma 2.2, p. 24 in [21], there exists an operator

$$\mathbf{a} \in \mathbf{V}^{1/2}(\Gamma) \longmapsto \mathbf{v}_a \in \mathbf{V}^1(\Omega),$$

such that $\mathbf{a}$ is the trace of $\mathbf{v}_a$ on Γ and this operator is *linear* and *bounded*, that is

$$\|\mathbf{v}_a\|_{\mathbf{V}^1(\Omega)} \leqslant C\|\mathbf{a}\|_{\mathbf{V}^{1/2}(\Gamma)} \tag{17}$$

with C depending only on Ω. See, also Lemma 3.2, p. 41, in [21], where the concrete operator is proposed, satisfying (17).

Let $\mathbf{z} := \mathbf{v} - \mathbf{v}_b$ be the solution of the system

$$\begin{cases} -\text{div}(\nu \mathbf{D}\mathbf{z}) + k(\mathbf{x})\,\mathbf{z} = -\nabla p + \mathbf{f}, \quad \text{div}\mathbf{z} = 0 \quad \text{in } \Omega, \\ \mathbf{z} = \mathbf{0} \quad \text{on } \Gamma \end{cases} \tag{18}$$

in the sense of the integral identity

$$\int_\Omega \{\nu\, \mathbf{D}\mathbf{z} : \mathbf{D}\boldsymbol{\psi} + k\,\mathbf{z}\boldsymbol{\psi}\}\, d\mathbf{x} = -\int_\Omega \{\nu\, \mathbf{D}\mathbf{v}_b : \mathbf{D}\boldsymbol{\psi} \, + k\mathbf{v}_b\boldsymbol{\psi} - \mathbf{g}\boldsymbol{\psi}\}\, d\mathbf{x}$$

for any $\boldsymbol{\psi} \in \mathbf{V}^1(\Omega)$, such that $\boldsymbol{\psi} = 0$ on Γ. Here $\mathbf{f} := \text{div}(\nu\mathbf{D}\mathbf{v}_b) - k\mathbf{v}_b + \mathbf{g}$. The solvability of (18) follows from the Lax–Milgram theorem (see Proposition 2.2, p.12 in [14]), the Korn inequality (12), assumptions (14)–(15) and the embedding theorem

$$W^{1,2}(\Omega) \subset L^r(\Omega) \text{ for } r < \infty \quad \text{if } d = 2; \quad r = 6 \quad \text{if } d = 3. \tag{19}$$

By a standard argument, we show that, the function $\mathbf{v} := \mathbf{z} + \mathbf{v}_b$ is the unique weak solution of (13).

Due to (12) the function $\mathbf{z}$ satisfies the following estimates

$$\begin{aligned}\overline{\nu}^0\|\mathbf{Dz}\|^2_{\mathbf{L}^2(\Omega)}+\int_\Omega k|\mathbf{z}|^2\,d\mathbf{x} &\leqslant \int_\Omega \nu|\mathbf{Dz}|^2\,d\mathbf{x}+\int_\Omega k|\mathbf{z}|^2\,d\mathbf{x}\\ &\leqslant \int_\Omega \nu|\mathbf{Dz}:\mathbf{Dv_b}|\,d\mathbf{x}+\int_\Omega k|\mathbf{z}||\mathbf{v}_b|\,d\mathbf{x}+\int_\Omega |\mathbf{z}||\mathbf{g}|d\mathbf{x}\\ &\leqslant \frac{\overline{\nu}^0}{2}\|\mathbf{Dz}\|^2_{\mathbf{L}^2(\Omega)}+\int_\Omega k|\mathbf{z}|^2\,d\mathbf{x}+C\,||\mathbf{v}_b||^2_{\mathbf{V}^1(\Omega)}\\ &\quad + C\int_\Omega k|\mathbf{v}_b|^2 d\mathbf{x}+C\,\|\mathbf{g}\|^2_{\mathbf{L}^q(\Omega)},\end{aligned}$$

where the constant C depends only on Ω and $\overline{\nu}^0, \overline{\nu}^1$. By (14)–(15), (17) and (19), we have

$$\|\mathbf{Dz}\|_{\mathbf{L}^2(\Omega)} \leqslant C(||\mathbf{b}||_{\mathbf{V}^{1/2}(\Gamma)}+\|\mathbf{g}\|_{\mathbf{L}^q(\Omega)}),$$

that implies (16). □

Next we give a time dependent generalization of Lemma 3.1.

LEMMA 3.2 Let us assume that

$$\mathbf{b}\in L^2\big(0,T;\mathbf{V}^{1/2}(\Gamma)\big) \quad \text{and}$$
$$\mathbf{g}\in L^2\big(0,T;\mathbf{L}^q(\Omega)\big) \quad \text{for } q>1 \quad \text{if } d=2; \quad q=\frac{6}{5} \quad \text{if } d=3.$$

Let $k(t,\mathbf{x}), \nu(t,\mathbf{x})$ be measurable positive functions and

$$k\in L^\infty(0,T;L^s(\Omega)) \quad \text{for } s>1 \quad \text{if } d=2; \quad s=3/2 \quad \text{if } d=3,$$
$$\nu\in[\overline{\nu}^0,\overline{\nu}^1] \quad \text{a.e. in } \Omega_T$$

for some positive real numbers $\overline{\nu}^0 < \overline{\nu}^1$. Then, there exists a unique weak solution $\mathbf{v}\in L^2(0,T;\mathbf{V}^1(\Omega))$ of (13) for a.e. $t\in(0,T)$, satisfying

$$\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))}\leqslant C \tag{20}$$

with a positive constant C depending only on Ω and the data $k, \nu, \mathbf{b}, \mathbf{g}$.

Proof. Since $\mathbf{b}\in L^2(0,T;\mathbf{V}^{1/2}(\Gamma))$, then there exists a sequence of *simple* functions $\{\mathbf{b}_n=\mathbf{b}_n(t)\in\mathbf{V}^{1/2}(\Gamma): t\in(0,T)\}_{n=1}^{\infty}$, such that

$$\begin{aligned}&||\mathbf{b}_n(t)-\mathbf{b}(t)||_{\mathbf{V}^{1/2}(\Gamma)}\to 0 \qquad \text{for a.e. } t\in[0,T],\\ &\|\mathbf{b}_n-\mathbf{b}\|_{L^2(0,T;\mathbf{V}^{1/2}(\Gamma))}\to 0,\end{aligned} \tag{21}$$

in accordance with the definitions, given in [20], p. 649–650 and the proof, given in Theorem 1, p. 133 [42] (see, *The "if" part*). Applying (17), there exists a sequence of *simple* functions $\{\mathbf{v}_{b,n}(t)\in\mathbf{V}^1(\Omega): t\in(0,T)\}_{n=1}^{\infty}$, such that $\mathbf{v}_{b,n}(t)=\mathbf{b}_n(t)$ on Γ for $t\in(0,T)$, satisfying

$$\begin{aligned}\|\mathbf{v}_{b,n}(t)\|_{\mathbf{V}^1(\Omega)} &\leqslant C\|\mathbf{b}_n(t)\|_{\mathbf{V}^{1/2}(\Gamma)},\\ \|\mathbf{v}_{b,n}(t)-\mathbf{v}_{b,n'}(t)\|_{\mathbf{V}^1(\Omega)} &\leqslant C\|\mathbf{b}_n(t)-\mathbf{b}_{n'}(t)\|_{\mathbf{V}^{1/2}(\Gamma)}\end{aligned} \tag{22}$$

for any $\forall n, n' = 1, 2, \ldots$ Here C is a positive constant depending only on Ω. Therefore the sequence of *simple* functions $\{\mathbf{v}_{b,n}\}_{n=1}^{\infty}$ is the Cauchy sequence in $L^2(0,T;\mathbf{V}^1(\Omega))$ by (21)–(22). Since $L^2(0,T;\mathbf{V}^1(\Omega))$ is a Banach space, there exists a *measurable* $\mathbf{v}_b \in L^2(0,T;\mathbf{V}^1(\Omega))$, such that $\mathbf{v}_b(t) = \mathbf{b}(t)$ on Γ for a.e. $t \in (0,T)$ and

$$\mathbf{v}_{b,n} \to \mathbf{v}_b \qquad \text{in } L^2(0,T;\mathbf{V}^1(\Omega)) \qquad \text{as } n \to \infty. \tag{23}$$

Let $\mathbf{z} := \mathbf{v} - \mathbf{v}_b$ be the solution of the system

$$\begin{cases} -\mathrm{div}(\nu \mathbf{D}\mathbf{z}) + k(\mathbf{x})\,\mathbf{z} = -\nabla p + \mathbf{f}, \quad \mathrm{div}\mathbf{z} = 0 \quad \text{in } \Omega_T, \\ \mathbf{z}(t) = \mathbf{0} \quad \text{on } \Gamma \qquad \text{for a.e. } t \in [0,T], \end{cases} \tag{24}$$

in the sense of the integral identity

$$\int_{\Omega_T} \{\nu\, \mathbf{D}\mathbf{z} : \mathbf{D}\boldsymbol{\psi} + k\,\mathbf{z}\boldsymbol{\psi}\}\; d\mathbf{x}dt = -\int_{\Omega_T} \{\nu\, \mathbf{D}\mathbf{v}_b : \mathbf{D}\boldsymbol{\psi} \; + k\,\mathbf{v}_b\boldsymbol{\psi} - \mathbf{g}\boldsymbol{\psi}\}\; d\mathbf{x}dt$$

for any $\boldsymbol{\psi} \in L^2(0,T;\mathbf{V}^1(\Omega))$, such that $\boldsymbol{\psi}(t) = \mathbf{0}$ on Γ for a.e. $t \in (0,T)$. Here $\mathbf{f} := \mathrm{div}(\nu\,\mathbf{D}\mathbf{v}_b) - k\,\mathbf{v}_b + \mathbf{g}$. Due to the Lax–Milgram theorem there exists a solution $\mathbf{z} \in L^2(0,T;\mathbf{V}^1(\Omega))$ of (24). Obviously, $\mathbf{v} = \mathbf{z} + \mathbf{v}_b$ is the unique weak solution of (13) for a.e. $t \in (0,T)$, satisfying (20), that can be shown by the same way as (16). ☐

Let us consider the linear transport equation in bounded domains, that is

$$\begin{cases} \partial_t \rho + \mathrm{div}\big(\mathbf{v}\rho\big) = 0 \quad \text{in} \quad \Omega_T, \\ \rho|_{t=0} = \rho_0 \qquad\qquad \text{in} \quad \Omega \quad \text{and} \quad \rho = \rho_b \quad \text{on } \Gamma_T^-, \end{cases} \tag{25}$$

where Γ_T^- is defined by (11). The proof of the second auxiliary result, which we follow here, has been obtained by Boyer [9], see Theorem 4.1. For convenience, we denote by $d\mathbf{r}dt$ the induced surface measure on Γ_T, and also, we introduce the measure $d\mu := (\mathbf{b}\cdot\mathbf{n})\,d\mathbf{x}dt$ on Γ_T. From Jordan's decomposition, we have $\mu = \mu^+ - \mu^-$, where $d\mu^{\pm} := (\mathbf{b}\cdot\mathbf{n})^{\pm}\,d\mathbf{x}dt$. Then, we have the following

LEMMA 3.3 Assume that $\rho_0 \in L^\infty(\Omega)$, $\rho_b \in L^\infty(\Gamma_T;\mu^-)$, $\mathbf{v} \in L^2(0,T;\mathbf{V}^1(\Omega))$ and $\mathbf{b} \in L^2(0,T;\mathbf{V}^{1/2}(\Gamma))$ are given functions, such that $\mathbf{v}|_{\Gamma_T} = \mathbf{b}$. Then, there exists a unique pair

$$(\rho,\rho^o) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_T;\mu^+),$$

which is a weak solution of (25), satisfying the integral equality

$$\iint_{\Omega_T} \rho\,\Big(\phi_t + (\mathbf{v}\cdot\nabla)\phi\Big)\; d\mathbf{x}dt + \int_\Omega \rho_0\;\phi(0)\; d\mathbf{x} = \iint_{\Gamma_T} \rho^o\,\phi\; d\mu^+ - \iint_{\Gamma_T} \rho_b\,\phi\; d\mu^- \tag{26}$$

for each test function $\phi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$. Moreover, the pair (ρ,ρ^o) satisfies

$$\|\rho\|_{L^\infty(\Omega_T)}, \|\rho^o\|_{L^\infty(\Gamma_T;\mu^+)} \leqslant \max\{\|\rho_0\|_{L^\infty(\Omega)}, \|\rho_b\|_{L^\infty(\Gamma_T;\mu^-)}\}.$$

The pair (ρ,ρ^o) is a re-normalized solution of (25), which means that for any function $\beta \in C^1(\mathbb{R})$, the pair

$$\alpha = (\beta(\rho) \quad \text{in } \Omega_T,\; \beta(\rho^o) \quad \text{on } \Gamma_T^+)$$

is the unique solution of the system

$$\begin{cases} \partial_t \alpha + \operatorname{div}\big(\mathbf{v}\alpha\big) = 0 & \text{in } \Omega_T \quad \text{and} \quad \alpha = \beta(\rho^o) \quad \text{on } \Gamma_T^+, \\ \alpha|_{t=0} = \beta(\rho_0) & \text{in } \Omega \quad \text{and} \quad \alpha = \beta(\rho_b) \quad \text{on } \Gamma_T^- \end{cases} \tag{27}$$

in the distributional sense, i.e. α satisfies the equality

$$\iint_{\Omega_T} \alpha \,\Big(\phi_t + (\mathbf{v}\cdot\nabla)\phi\Big)\, d\mathbf{x}dt + \int_\Omega \beta(\rho_0)\,\phi(0)\,d\mathbf{x} = \iint_{\Gamma_T} \alpha\,\phi\,d\mu^+ - \iint_{\Gamma_T} \beta(\rho_b)\,\phi\,d\mu^- \tag{28}$$

for each test function $\phi \in C_c^\infty((-\infty, T)\times\mathbb{R}^d)$.

As a consequence of Lemma 3.3 we have the next result, which is similar to that one established in [9] (see Lemma 4.2) for approximated solutions. We present here the proof of this result, since the same argument will be used in the sequel.

COROLLARY 3.4 Let the data ρ_0, ρ_b, $\mathbf{b}$, $\mathbf{v}$ fulfill the assumptions of Lemma 3.3. Moreover we assume that

$$\rho_0(\mathbf{x}) \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Omega, \qquad \rho_b(t,\mathbf{x}) \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Gamma_T^-$$

for some real numbers $\overline{\rho}^0 < \overline{\rho}^1$. Then the unique weak solution $(\rho, \rho^o) \in L^\infty(\Omega_T)\times L^\infty(\Gamma_T;\mu^+)$ of (25) satisfies

$$\rho \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Omega_T \quad \text{and} \quad \rho^o \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Gamma_T^+. \tag{29}$$

Proof. In view of Lemma 3.3, the unique solution (ρ, ρ^o) of (25) is a re-normalized solution, which means that for any positive function $\beta \in C^1(\mathbb{R})$ the pair $\alpha = (\beta(\rho), \beta(\rho^o))$ is the unique weak solution of system (27). In particular if we choose β with $\operatorname{supp}(\beta) \subset \overline{\mathbb{R}\setminus[\overline{\rho}^0, \overline{\rho}^1]}$, then α solves the system

$$\begin{cases} \partial_t \alpha + \operatorname{div}(\mathbf{v}\alpha) = 0 & \text{in } \Omega_T \quad \text{and} \quad \alpha = \beta(\rho^o) \quad \text{on } \Gamma_T^+, \\ \alpha|_{t=0} = \beta(\rho_0) \equiv 0 & \text{in } \Omega \quad \text{and} \quad \alpha = \beta(\rho_b) \equiv 0 \quad \text{on } \Gamma_T^-. \end{cases}$$

Since the zero is the unique solution of this system, we conclude that

$$\alpha = \big(\beta(\rho), \beta(\rho^o)\big) \equiv 0,$$

which implies (29). □

4. Single-phase filtration. Solvability of GMP

Let us first consider one non–homogeneous fluid, which motion is described by the system **GMP**, and assume that the data $\rho_0, \rho_b,\ \nu_0, \nu_b,$ **b**, **g**, the function h satisfy the following properties

$$\begin{aligned} &\mathbf{b} \in L^2\big(0,T;\mathbf{V}^{1/2}(\Gamma)\big), \\ &\mathbf{g} \in L^2\big(0,T;\mathbf{L}^q(\Omega)\big) \quad \text{for } q>1 \quad \text{if } d=2; \quad q=\tfrac{6}{5} \quad \text{if } d=3, \\ &\rho_0(\mathbf{x}) \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Omega, \qquad \rho_b(t,\mathbf{x}) \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Gamma_T^-, \\ &\nu_0(\mathbf{x}) \in [\overline{\nu}^0, \overline{\nu}^1] \quad \text{a.e. in } \Omega, \qquad \nu_b(t,\mathbf{x}) \in [\overline{\nu}^0, \overline{\nu}^1] \quad \text{a.e. in } \Gamma_T^- \end{aligned} \tag{30}$$

for some positive real numbers $\overline{\rho}^0 \leqslant \overline{\rho}^1, \overline{\nu}^0 \leqslant \overline{\nu}^1$ and, h is a Carathéodory function, more precisely for each $r \geqslant 0$, $h(\cdot,\cdot,r)$ is a measurable function, and for almost all $(t,\mathbf{x}) \in \Omega_T$, the map $0 \leqslant r \mapsto h(t,\mathbf{x},r)$ is continuous. Moreover, we assume for almost all $(t,\mathbf{x}) \in \Omega_T$, and each $r \geqslant 0$,

$$\begin{gathered} 0 \leqslant h(t,\mathbf{x},r) \leqslant h_0(t,\mathbf{x})\, r^m \quad \text{for some} \quad m \in \mathbb{R}^+, \\ h_0 \in L^\infty\big(0,T;L^s(\Omega)\big) \quad \text{for } s > 1 \quad \text{if } d = 2; \quad s = 3/2 \quad \text{if } d = 3. \end{gathered} \tag{31}$$

The following definition tells us in which sense we consider that a triple is a weak solution to the problem **GMP**.

DEFINITION 4.1 The triple $((\rho,\rho^o),\ (\nu,\nu^o),\ \mathbf{v})$ is called a weak solution to the problem **GMP**, if $(\rho,\rho^o),\ (\nu,\nu^o) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_T;\mu^+)$, $\mathbf{v} \in L^2(0,T;\mathbf{V}^1(\Omega))$, which satisfy the integral identities

$$\begin{gathered} \iint_{\Omega_T} \rho\,\Big(\phi_t + (\mathbf{v}\cdot\nabla)\phi\Big)\,d\mathbf{x}dt + \int_\Omega \rho_0\,\phi(0)\,d\mathbf{x} = \iint_{\Gamma_T} \rho^o\,\phi\,d\mu^+ - \iint_{\Gamma_T} \rho_b\,\phi\,d\mu^-, \\ \iint_{\Omega_T} \nu\,\Big(\phi_t + (\mathbf{v}\cdot\nabla)\phi\Big)\,d\mathbf{x}dt + \int_\Omega \nu_0\,\phi(0)\,d\mathbf{x} = \iint_{\Gamma_T} \nu^o\,\phi\,d\mu^+ - \iint_{\Gamma_T} \nu_b\,\phi\,d\mu^-, \\ \int_\Omega \Big(h(t,\mathbf{x},\nu\rho)\,\mathbf{v}\cdot\boldsymbol{\psi} + \nu\rho\,\mathbf{Dv}:\mathbf{D}\boldsymbol{\psi}\Big)\,d\mathbf{x} = \int_\Omega \rho\,\mathbf{g}\cdot\boldsymbol{\psi}\,d\mathbf{x} \quad \text{for a.a. } t \in (0,T), \end{gathered}$$

for each test functions $\phi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$ and $\boldsymbol{\psi} \in \mathbf{V}^1(\Omega)$, such that $\boldsymbol{\psi} = 0$ on Γ. Moreover the trace of $\mathbf{v}$ is equal to $\mathbf{b}$ on Γ_T.

THEOREM 4.2 Under assumptions (30), (31) the problem **GMP** has a weak solution.

In the following Section 4.1, we prove this theorem.

4.1 *Schauder's fixed point argument*

We apply the Schauder fixed point argument to show the existence result. To begin, we consider the closed convex subset

$$\mathcal{Z} = \{(\rho,\nu) \in L^2(\Omega_T)^2 : \rho \in [\overline{\rho}^0,\overline{\rho}^1], \quad \nu \in [\overline{\nu}^0,\overline{\nu}^1] \quad \text{a.e. in } \Omega_T\} \tag{32}$$

of the Banach space $L^2(\Omega_T)^2$, with the norm

$$||(\rho,\nu)||_{L^2(\Omega_T)^2} := ||\rho||_{L^2(\Omega_T)} + ||\nu||_{L^2(\Omega_T)}.$$

Let $(\overline{\rho},\overline{\nu})$ be an arbitrary fixed element of $\mathcal{Z}$, and consider the coupled system

$$\begin{cases} -\mathrm{div}(\overline{\rho}\,\overline{\nu}\mathbf{Dv}) + h(t,\mathbf{x},\overline{\rho}\,\overline{\nu})\mathbf{v} = -\nabla p + \overline{\rho}\mathbf{g}, \quad \mathrm{div}\mathbf{v} = 0 \quad \text{in } \Omega_T, \\ \mathbf{v} = \mathbf{b} \quad \text{on } \Gamma_T, \end{cases} \tag{33}$$

$$\begin{cases} \partial_t\rho + \mathrm{div}(\mathbf{v}\rho) = 0 \quad \text{in } \Omega_T, \\ \rho = \rho^o \quad \text{on } \Gamma_T^+, \\ \rho|_{t=0} = \rho_0 \quad \text{in } \Omega, \\ \rho = \rho_b \quad \text{on } \Gamma_T^-, \end{cases} \qquad \begin{cases} \partial_t\nu + \mathrm{div}(\mathbf{v}\nu) = 0 \quad \text{in } \Omega_T, \\ \nu = \nu^o \quad \text{on } \Gamma_T^+, \\ \nu|_{t=0} = \nu_0 \quad \text{in } \Omega, \\ \nu = \nu_b \quad \text{on } \Gamma_T^-. \end{cases} \tag{34}$$

The solvability result for this system is presented in the following lemma.

LEMMA 4.3 For each $(\overline{\rho}, \overline{\nu}) \in \mathcal{Z}$, there exists a unique solution $(\rho, \nu, \mathbf{v})$ of system (33)–(34), such that

$$(\rho, \nu) \in \mathcal{Z}, \qquad \|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} \leqslant C. \tag{35}$$

Hereupon, C is a positive constant depending only on data (30), (31).

Proof. Due to (30), (31) and Lemma 3.2, the Stokes type system (33) has a unique solution $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ in $L^2(0, T; \mathbf{V}^1(\Omega))$, such that

$$\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} \leqslant C,$$

where the constant C depends only on Ω, $\mathbf{b}$, $\mathbf{g}$, $\overline{\nu}^i$, $\overline{\rho}^i$, $i = 0, 1$. The last one permits to apply Lemma 3.3 and Corollary 3.4 with the help of (30). Hence systems (34) have unique solutions (ρ, ρ^o), $(\nu, \nu^o) \in L^\infty(\Omega_T) \times L^\infty(\Gamma_T; \mu^+)$, satisfying the estimates

$$\rho \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Omega_T \quad \text{and} \quad \rho^o \in [\overline{\rho}^0, \overline{\rho}^1] \quad \text{a.e. in } \Gamma_T^+,$$
$$\nu \in [\overline{\nu}^0, \overline{\nu}^1] \quad \text{a.e. in } \Omega_T \quad \text{and} \quad \nu^o \in [\overline{\nu}^0, \overline{\nu}^1] \quad \text{a.e. in } \Gamma_T^+.$$

Therefore the solvability of coupled systems (33)–(34) is shown and $(\rho, \nu, \mathbf{v})$ satisfies (35). □

Recall that, solving (33)–(34) we have constructed the operator

$$P : \mathcal{Z} \to \mathcal{Z}, \quad (\rho, \nu) = P(\overline{\rho}, \overline{\nu}), \qquad \forall\, (\overline{\rho}, \overline{\nu}) \in \mathcal{Z}.$$

To find a fixed point of P (by Schauder's theorem), which will be a solution of the system **GMP**, it is enough to show that $P(\mathcal{Z})$ is a relatively compact subset of the Banach space $L^2(\Omega_T)^2$, and also P is a continuous operator with respect to the norm $||(\cdot, \cdot)||_{L^2(\Omega_T)^2}$. First, let us show the following lemma.

LEMMA 4.4 The set $P(\mathcal{Z})$ is relatively compact in $L^2(\Omega_T)^2$.

Proof. Let us consider an arbitrary sequence $\{(\overline{\rho}^n, \overline{\nu}^n) \in \mathcal{Z}\}_{n=0}^\infty$. For each n, due to Lemma 4.3, there exists a unique solution $(\rho^n, \rho^{o,n})$, $(\nu^n, \nu^{o,n})$, $\mathbf{v}^n$ of system (33)–(34), which satisfies (35). We have $(\rho^n, \nu^n) = P(\overline{\rho}^n, \overline{\nu}^n) \in \mathcal{Z}$. Since the set $\mathcal{Z}$ is bounded in $L^\infty(\Omega_T) \subset L^2(\Omega_T)$, there exists a suitable sub-sequence, which we continue to denote by the same index "n", just for convenience of reading. Then,

$$\rho^n,\ \nu^n \rightharpoonup \rho,\ \nu \qquad \star\text{-weakly in } L^\infty(\Omega_T) \quad \text{and} \quad \text{weakly in } L^2(\Omega_T),$$
$$\mathbf{v}^n \rightharpoonup \mathbf{v} \qquad \text{weakly in } L^2(0, T; \mathbf{V}^1(\Omega)) \tag{36}$$

with $\rho, \nu, \mathbf{v}$, satisfying (35).

In the sequel the issue is to prove that, the weak convergences (36) implies the strong convergence in $L^2(\Omega_T)$ for a suitable sub-sequence of $\{\rho^n\}_{n=0}^\infty$. The strong convergence for a sub-sequence of $\{\nu^n\}_{n=0}^\infty$ can be shown analogously. First, there exists a sub-sequence of $\{\rho^n, (\rho^n)^2\}_{n=1}^\infty$, such that

$$\rho^n,\ (\rho^n)^2 \rightharpoonup \rho,\ \beta^* \qquad \star\text{-weakly in } L^\infty(\Omega_T) \text{ and weakly in } L^2(\Omega_T).$$

Also, we have

$$\rho^{o,n},\ (\rho^{o,n})^2 \rightharpoonup \rho^o,\ \beta^{o,*} \qquad \star\text{-weakly in } L^\infty(\Gamma_T; \mu^+).$$

By construction the pair $(\rho^n, \mathbf{v}^n)$ fulfills equality (26), that yields the estimate

$$\left| \iint\limits_{\Omega_T} \rho^n \, \phi_t \, d\mathbf{x}dt \right| \leqslant C \, \|\phi\|_{L^2(0,T;H_0^1(\Omega))}, \tag{37}$$

which is valid for an arbitrary function $\phi \in C_0^\infty(0,T;H_0^1(\Omega))$. Also, applying Lemma 3.3 the functions $\alpha = (\rho^n)^2$ solves system (27) (in the integral sense (28)), hence

$$\left| \iint\limits_{\Omega_T} (\rho^n)^2 \, \phi_t \, d\mathbf{x}dt \right| \leqslant C \, \|\phi\|_{L^2(0,T;H_0^1(\Omega))}, \quad \forall \phi \in C_0^\infty(0,T;H_0^1(\Omega)). \tag{38}$$

From the compact embedding of $L^2(\Omega)$ in $H^{-1}(\Omega)$, (37), (38), and the well known compactness results of [6] , [38], we have

$$\rho^n, \ (\rho^n)^2 \to \rho, \ \beta^* \quad \text{strongly in } L^2\big(0,T;H^{-1}(\Omega)\big). \tag{39}$$

Since $\mathbf{v}^n - \mathbf{v}_b \in L^2(0,T;\mathbf{H}_0^1(\Omega))$ (see (23)), due to (36), (39), we obtain

$$\iint_{\Omega_T} \rho^n \left((\mathbf{v}^n - \mathbf{v}_b)\cdot\nabla\right)\phi \, d\mathbf{x}dt \to \iint_{\Omega_T} \rho\left((\mathbf{v} - \mathbf{v}_b)\cdot\nabla\right)\phi \, d\mathbf{x}dt$$

for each test functions $\phi \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$, that is to say

$$\iint_{\Omega_T} \rho^n(\mathbf{v}^n\cdot\nabla)\phi \, d\mathbf{x}dt \to \iint_{\Omega_T} \rho(\mathbf{v}\cdot\nabla)\phi \, d\mathbf{x}dt. \tag{40}$$

Similarly, applying the same idea, we deduce

$$\iint_{\Omega_T} (\rho^n)^2(\mathbf{v}^n\cdot\nabla)\phi \, d\mathbf{x}dt \to \iint_{\Omega_T} \beta^*(\mathbf{v}\cdot\nabla)\phi \, d\mathbf{x}dt. \tag{41}$$

Let us pass to the limit $n \to \infty$ in equalities (26) and (28), written for $(\rho^n, \rho^{o,n})$, $\mathbf{v}^n$ and $\alpha = ((\rho^n)^2, (\rho^{o,n})^2)$, $\mathbf{v}^n$, respectively. Thus with the help of (40), (41), we obtain that the triples (ρ, ρ^o), $\mathbf{v}$ and $\alpha = (\beta^*, \beta^{o,*})$, $\mathbf{v}$ fulfill equalities (26) and (28). Applying Lemma 3.3 to (ρ, ρ^o), $\mathbf{v}$, we have that, the triple $(\rho^2, (\rho^o)^2)$, $\mathbf{v}$ satisfies (28) too. In view of the uniqueness of solution for system (27), we obtain

$$\beta^* \equiv \rho^2 \quad \text{a.e. in } \Omega_T.$$

Hence $\rho^n \to \rho$ strongly in $L^2(\Omega_T)$. Therefore we derive that there exists a suitable sequence of $\{(\rho^n, \nu^n) = P(\overline{\rho}^n, \overline{\nu}^n) : \ (\overline{\rho}^n, \overline{\nu}^n) \in \mathcal{Z}\}_{n=0}^\infty$, such that

$$(\rho^n, \nu^n) \to (\rho, \nu) \quad \text{strongly in } \ L^2(\Omega_T)^2 \quad \text{and} \quad (\rho, \nu) \in \mathcal{Z}.$$

Consequently, P is a compact operator on the set $\mathcal{Z}$. □

Now we proof the following lemma.

LEMMA 4.5 The operator P is continuous in the norm of $L^2(\Omega_T)^2$.

Proof. Let $\{(\overline{\rho}^n, \overline{v}^n) \in \mathcal{Z}\}_{n=0}^{\infty}$ be a sequence converging to $(\overline{\rho}, \overline{v}) \in \mathcal{Z}$ in $L^2(\Omega_T)^2$, that is

$$||(\overline{\rho}^n, \overline{v}^n) - (\overline{\rho}, \overline{v})||_{L^2(\Omega_T)^2} \to 0, \quad n \to \infty. \tag{42}$$

Let $(\rho^n, \nu^n) = P(\overline{\rho}^n, \overline{v}^n) \in \mathcal{Z}$, $\mathbf{v}^n \in L^2(0, T; \mathbf{V}^1(\Omega))$ and $(\rho, \nu) = P(\overline{\rho}, \overline{v}) \in \mathcal{Z}$, $\mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$ be the solutions of (33) and (34), respectively. The triples $(\rho^n, \rho^{o,n})$, $(\nu^n, \nu^{o,n})$, $\mathbf{v}^n$ and (ρ, ρ^o), (ν, ν^o), $\mathbf{v}$ satisfy (35). Let us consider $\mathbf{z}^n := \mathbf{v}^n - \mathbf{v}$, $P^n := p^n - p$, which satisfy the system

$$\begin{cases} -\mathrm{div}(\overline{\rho}^n \overline{v}^n \mathbf{D}\mathbf{z}^n) + h(t, x, \overline{\rho}^n \overline{v}^n)\, \mathbf{z}^n = -\nabla P^n + \mathbf{f}^n, \quad div \mathbf{z}^n = 0 \quad \text{in } \Omega_T, \\ \mathbf{z}^n = \mathbf{0} \quad \text{on } \Gamma_T, \end{cases}$$

with $\mathbf{f}^n := (\overline{\rho}^n - \overline{\rho})\mathbf{g} + \mathrm{div}((\overline{\rho}^n \overline{v}^n - \overline{\rho v})\mathbf{D}(\mathbf{v})) - s_n \mathbf{v}$. Here, we define

$$s_n := h(t, \mathbf{x}, \overline{\rho}^n \overline{v}^n) - h(t, \mathbf{x}, \overline{\rho}\, \overline{v}) \quad \text{and} \quad l_n := |\overline{\rho}^n - \overline{\rho}\,| + |\overline{v}^n - \overline{v}|.$$

Due to (31) and (35) the function $\mathbf{z}^n(t)$ satisfies for a.a. $t \in (0, T)$ the following

$$\begin{aligned} \overline{\rho}^0 \overline{v}^0 \|\mathbf{D}\mathbf{z}^n\|^2_{\mathbf{L}^2(\Omega)} &\leqslant \int_\Omega l_n |\mathbf{g}||\mathbf{z}^n|\, d\mathbf{x} + C \int_\Omega l_n |\mathbf{D}\mathbf{v}||\mathbf{D}\mathbf{z}^n|\, d\mathbf{x} + \int_\Omega |s_n||\mathbf{v}||\mathbf{z}^n|\, d\mathbf{x} \\ &\leqslant C \left(||l_n \mathbf{g}||^2_{\mathbf{L}^q(\Omega)} + ||l_n \mathbf{D}\mathbf{v}||^2_{\mathbf{L}^2(\Omega)} \right) + C||s_n||^2_{\mathbf{L}^s(\Omega)} ||\mathbf{v}||^2_{\mathbf{L}^r(\Omega)} + \frac{\overline{\rho}^0 \overline{v}^0}{4} \|\mathbf{D}\mathbf{z}^n\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

where $\frac{1}{s} + \frac{2}{r} = 1$ for s given by (31), and the constant C depend only on Ω, $\overline{v}^i$, $\overline{\rho}^i$, $i = 0, 1$. Here we have used Holder's and Young's inequalities, inequality (12), and the embedding (19). Therefore, we have

$$\|\mathbf{D}\mathbf{z}^n\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant C(I_n + J_n),$$

where

$$I_n = ||l_n \mathbf{g}||^2_{L^2(0,T;\mathbf{L}^q(\Omega))} + ||l_n \mathbf{D}\mathbf{v}||^2_{\mathbf{L}^2(\Omega_T)}, \quad J_n = \int_0^T ||s_n||^2_{\mathbf{L}^s(\Omega)} ||\mathbf{v}||^2_{\mathbf{H}^1(\Omega)} dt.$$

Now, let us define the cut-off function $\phi_L(t, \mathbf{x}) = \min\ \{\phi(t, \mathbf{x}), L\}$ for $L > 0$. We have

$$\begin{aligned} I_n \leqslant CL^2 ||(\overline{\rho}^n, \overline{v}^n) - (\overline{\rho}, \overline{v})||^2_{L^2(\Omega_T)^2} + C \int_0^T \left(\int_\Omega ||\mathbf{g}|^q - |\mathbf{g}|^q_L|\, d\mathbf{x} \right)^{2/q} dt \\ + C \iint_{\Omega_T} ||\mathbf{D}\mathbf{v}|^2 - |\mathbf{D}\mathbf{v}|^2_L|\, d\mathbf{x} dt. \end{aligned}$$

Choosing $L := ||(\overline{\rho}^n, \overline{v}^n) - (\overline{\rho}, \overline{v})||^{-1/2}_{L^2(\Omega_T)^2}$, we derive $I_n \to 0$ as $n \to \infty$.

Also we have

$$J_n \to 0 \quad \text{as } n \to \infty,$$

which can be proved by contradiction. Indeed, let us assume that there exists a sub-sequence of $\{J_n\}_{n=0}^{\infty}$, which is not convergent to zero as $n \to \infty$:

$$J_{n'} \geqslant const > 0, \quad \forall n'. \tag{43}$$

But, using (42), there exists a sub-sequence $\{s_{n''}\}_{n''=0}^{\infty}$ of $\{s_{n'}\}_{n'=0}^{\infty}$, such that $s_{n''} \to 0$ a.e. in Ω_T, since h is a Carathéodory function (see (31)). Applying the uniform L^{∞}-boundness of $\{(\overline{\rho}^{n''}, \overline{\nu}^{n''})\}_{n''=0}^{\infty}$, $(\overline{\rho}, \overline{\nu})$ and (31), we get

$$\int_0^T ||s_{n''}||_{\mathbf{L}^s(\Omega)}(t)dt \to 0$$

by the dominated convergence theorem. Moreover Fubini's theorem, the relation between the convergence in L^1-norm and the point-wise convergence give the existence of a suitable sub-sequence $\{s_{n'''}\}_{n'''=0}^{\infty}$ of $\{s_{n''}\}_{n''=0}^{\infty}$, such that $||s_{n'''}||_{\mathbf{L}^s(\Omega)}(t) \to 0$ a.e. in $(0, T)$. Therefore the uniform L^{∞}-boundness of $\{||s_{n'''}||_{\mathbf{L}^s(\Omega)}(t)\}_{n''=0}^{\infty}$ (see (31)) and the dominated convergence theorem imply

$$J_{n'''} \to 0 \quad \text{as } n''' \to \infty,$$

which is a contradiction with (43). Therefore we conclude

$$\|\mathbf{v}^n - \mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} \to 0 \quad \text{as } n \to 0. \tag{44}$$

Now, using (44), we can prove

$$r_n := ||(\rho^n, \nu^n) - (\rho, \nu)||_{L^2(\Omega_T)^2} \underset{n\to\infty}{\to} 0$$

by contradiction. Let us assume that there exists a sub-sequence of $\{r_n\}_{n=0}^{\infty}$, which is not convergent to zero as $n \to \infty$:

$$r_{n'} \geqslant const > 0, \quad \forall n'. \tag{45}$$

Although, there exists a sub-sequence

$$\begin{aligned}
&\rho^{n''},\ \nu^{n''} \rightharpoonup \widetilde{\rho},\ \widetilde{\nu} && \star\text{-weakly in } L^{\infty}(\Omega_T) \text{ and weakly in } L^2(\Omega_T),\\
&\rho^{o,n''},\ \nu^{o,n''} \rightharpoonup \widetilde{\rho}^o,\ \widetilde{\nu}^o && \star\text{-weakly in } L^{\infty}(\Gamma_T; \mu^+),\\
&\mathbf{v}^{n''} \to \mathbf{v} && \text{strongly in } L^2(0, T; \mathbf{V}^1(\Omega)),
\end{aligned}$$

where $(\widetilde{\rho}, \widetilde{\rho}^o)$, $(\widetilde{\nu}, \widetilde{\nu}^o)$, $\mathbf{v}$ satisfy (35) and solve systems (34), respectively. In view of the uniqueness of solution for the two transport systems in (34) (uniqueness is an important point here), for the given $\mathbf{v}$, we conclude that $\widetilde{\rho} \equiv \rho$, $\widetilde{\nu} \equiv \nu$. Now we can argue as in Lemma 4.4 and show the existence of a sub-sequence

$$\rho^{n'''},\ \nu^{n'''} \to \rho,\ \nu \quad \text{strongly in } L^2(\Omega_T),$$

which is a contradiction with our assumption (45). Consequently, we have shown the continuity of P. □

The thesis of Theorem 4.2 follows combining Lemmas 4.3, 4.4 and 4.5.

4.2 *Two-phase filtration*

In this section we show that the system **GMP** describes the motion of *immiscible* fluids.

We assume that the fluids "1" and "2" occupy two measurable disjoint sets Ω_1 and Ω_2 at the initial moment $t = 0$, such that $\Omega_1 \cup \Omega_2 = \Omega$. Moreover the fluids "1" and "2" enter inside of the domain Ω through two separated boundary zones Γ_1^- and Γ_2^-, such that $\Gamma_1^- \cup \Gamma_2^- = \Gamma_T^-$. Let the data ρ_0, ρ_b and ν_0, ν_b satisfy the natural assumptions for $i = 1, 2$:

$$(\rho_0, \nu_0) := (\rho_{i,0}, \nu_{i,0}) \quad \text{in } \Omega_i \quad \text{and} \quad (\rho_b, \nu_b) := (\rho_{i,b}, \nu_{i,b}) \quad \text{on } \Gamma_i^- \tag{46}$$

with

$$\rho_{i,0} \in [\overline{\rho}_i^0, \overline{\rho}_i^1], \quad \nu_{i,0} \in [\overline{\nu}_i^0, \overline{\nu}_i^1] \quad \text{a.e. in } \Omega_i(0),$$
$$\rho_{i,b} \in [\overline{\rho}_i^0, \overline{\rho}_i^1], \quad \nu_{i,b} \in [\overline{\nu}_i^0, \overline{\nu}_i^1] \quad \text{a.e. on } \Gamma_i^-,$$

for positive real numbers $\overline{\rho}_1^0 \leqslant \overline{\rho}_1^1 < \overline{\rho}_2^0 \leqslant \overline{\rho}_2^1$ and $\overline{\nu}_1^0 \leqslant \overline{\nu}_1^1 < \overline{\nu}_2^0 \leqslant \overline{\nu}_2^1$.

In the following, we demonstrate that the process of motion is *immiscible.*

COROLLARY 4.6 Let the data $\rho_0, \nu_0, \rho_b, \nu_b$, and $\mathbf{b}$ fulfill the assumptions of Theorem 4.2. If $\rho_0, \nu_0, \rho_b, \nu_b$ satisfy the additional assumptions (46), then the weak solution (ρ, ρ^o), (ν, ν^o), $\mathbf{v}$ of **GMP** describes the behaviour of two *immiscible* fluids "1" and "2" in the porous media, that is, there exist two measurable disjoint sets F_1, F_2 with $F_1 \cup F_2 = \Omega_T$ and two measurable disjoint boundary zones Γ_1^+, Γ_2^+ with $\Gamma_1^+ \cup \Gamma_2^+ = \Gamma_T^+$, such that

$$\rho \in [\overline{\rho}_i^0, \overline{\rho}_i^1], \quad \nu \in [\overline{\nu}_i^0, \overline{\nu}_i^1] \quad \text{a.e. in } F_i,$$
$$\rho^o \in [\overline{\rho}_i^0, \overline{\rho}_i^1], \quad \nu^o \in [\overline{\nu}_i^0, \overline{\nu}_i^1] \quad \text{a.e. on } \Gamma_i^+$$

for $i = 1, 2$, respectively.

Proof. 1. First, let us take an arbitrary non-negative function $\beta \in C^1(\mathbb{R})$ with $\text{supp}(\beta) = \overline{\mathbb{R} \setminus \cup_{i=1,2}[\overline{\rho}_i^0, \overline{\rho}_i^1]}$. By the same arguments as in proof of Lemma 3.4, we obtain that $\alpha = (\beta(\rho), \beta(\rho^o))$ is the unique weak solution of system (27) and $\alpha \equiv 0$. Therefore, the functions ρ, ρ^o take values just in $\cup_{i=1,2}[\overline{\rho}_i^0, \overline{\rho}_i^1]$.

2. Now, taking an arbitrary non-negative function $\beta_1 \in C^1(\mathbb{R})$ with $\text{supp}(\beta_1) = [\overline{\rho}_1^0, \overline{\rho}_1^1]$, we consider the measurable sets

$$F_1^\rho := \{(t, \mathbf{x}) \in \Omega_T : \beta_1(\rho(t, \mathbf{x})) > 0\}, \quad F_2^\rho := \Omega_T \backslash F_1^\rho,$$
$$\Gamma_1^{\rho^o} := \left\{(t, \mathbf{x}) \in \Gamma_T^+ : \beta_1(\rho^o(t, \mathbf{x})) > 0\right\}, \quad \Gamma_2^{\rho^o} := \Gamma_T^+ \backslash \Gamma_1^{\rho^0},$$

which satisfy

$$\rho \in [\overline{\rho}_i^0, \overline{\rho}_i^1] \quad \text{a.e. in } F_i^\rho \quad \text{and} \quad \rho^o \in [\overline{\rho}_i^0, \overline{\rho}_i^1] \quad \text{a.e. on } \Gamma_i^{\rho^o}, \quad i = 1, 2.$$

Analogously, we can also define the sets $F_1^\nu := \{(t, \mathbf{x}) \in \Omega_T : \beta_2(\nu(t, \mathbf{x})) > 0\}$, $F_2^\nu := \Omega_T \backslash F_1^\nu$ and $\Gamma_1^{\nu^o} := \left\{(t, \mathbf{x}) \in \Gamma_T^+ : \beta_2(\nu^o(t, \mathbf{x})) > 0\right\}$, $\Gamma_2^{\nu^o} := \Gamma_T^+ \backslash \Gamma_1^{\nu^o}$, such that

$$\nu \in [\overline{\nu}_i^0, \overline{\nu}_i^1] \quad \text{a.e. in } F_i^\nu \quad \text{and} \quad \nu^o \in [\overline{\nu}_i^0, \overline{\nu}_i^1] \quad \text{a.e. on } \Gamma_i^{\nu^o}, \quad i = 1, 2,$$

for an arbitrary non–negative function $\beta_2 \in C^1(\mathbb{R})$ with $\mathrm{supp}(\beta_2) = [\overline{\nu}_1^0, \overline{\nu}_1^1]$.

By Proposition 4.2 in [9], we see that the $\alpha = (\beta_1(\rho)(1-\beta_2(\nu)),\ \beta_1(\rho^o)(1-\beta_2(\nu^o)))$ and $\alpha = ((1-\beta_1(\rho))\beta_2(\nu),\ (1-\beta_1(\rho^o))\beta_2(\nu^o))$ satisfy the following system (in the distributional sense)

$$\partial_t \alpha + \mathrm{div}(\mathbf{v}\alpha) = 0 \quad \text{in } \Omega_T,$$
$$\alpha|_{t=0} = 0 \quad \text{in } \Omega \quad \text{and} \quad \alpha|_{\Gamma_T^-} = 0.$$

Moreover, from the uniqueness of solution of this system, we have $\alpha \equiv 0$. Therefore, we have

$$F_i^\rho \equiv F_i^\nu =: F_i \quad \text{and} \quad \Gamma_i^{\rho^o} \equiv \Gamma_i^{\nu^o} =: \Gamma_i^+, \quad i = 1, 2.$$

The sets F_i and Γ_i^+, $i = 1, 2$, defined above, do not depend on the choose of β_1, β_2. Indeed, it is enough to apply one more time Proposition 4.2 in [9]. □

Finally, as a particular result we have the following corollary.

COROLLARY 4.7 Let the data $\rho_0, \nu_0, \rho_b, \nu_b$, and $\mathbf{b}$ fulfill the assumptions of Lemma 4.6. Let $\rho_i := \overline{\rho}_i^0 \equiv \overline{\rho}_i^1$ and $\nu_i := \overline{\nu}_i^0 \equiv \overline{\nu}_i^1$, $i = 1, 2$, then there exist measurable disjoint sets F_1, F_2 with $F_1 \cup F_2 = \Omega_T$ and two measurable disjoint boundary zones Γ_1^+, Γ_2^+ with $\Gamma_1^+ \cup \Gamma_2^+ = \Gamma_T^+$, such that

$$(\rho, \nu) = (\rho_i, \nu_i) \quad \text{a.e. in } F_i,$$
$$(\rho^o, \nu^o) = (\rho_i, \nu_i) \quad \text{a.e. on } \Gamma_i^+ \quad \text{for } i = 1, 2.$$

Open problems

1. Let us remark that the uniqueness result for the problem **GMP** is not shown.
2. The investigation of the regularity of the interface $S(t)$, which separates the two fluids during the motion, is an interesting open problem. Let us point that such problem was studied in the article [17] for the interface $S(t)$ between two incompressible 2-D fluids, where the evolution equation for $S(t)$ was obtained from *Darcy's law*.

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