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On self-similar solutions to the surface diffusion flow equations with contact angle boundary conditions

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We consider the surface diffusion flow equation when the curve is given as the graph of a function $v(x,t)$ defined in a half line $\mathbf{R}^+ = \{x > 0\}$ under the boundary conditions $v_x = \tan\beta > 0$ and $v_{xxx} = 0$ at $x = 0$. We construct a unique (spatially bounded) self-similar solution when the angle β is sufficiently small. We further prove the stability of this self-similar solution. The problem stems from an equation proposed by W. W. Mullins (1957) to model formation of surface grooves on the grain boundaries, where the second boundary condition $v_{xxx} = 0$ is replaced by zero slope condition on the curvature of the graph.

For construction of a self-similar solution we solve the initial-boundary problem with homogeneous initial data. However, since the problem is quasilinear and initial data is not compatible with the boundary condition a simple application of an abstract theory for quasilinear parabolic equation is not enough for our purpose. We use a semi-divergence structure to construct a solution.

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1. Introduction

We consider the initial-boundary problem for the surface diffusion flow equation of the form

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{1}{(1+v_x^2)^{1/2}}\frac{\partial}{\partial x}\left(\frac{v_{xx}}{(1+v_x^2)^{3/2}}\right)\right], \quad x > 0, t > 0, \tag{1.1}$$

with the boundary condition

$$v_x = \tan\beta, \quad x = 0, t > 0, \tag{1.2}$$

$$v_{xxx} = 0, \quad x = 0, t > 0, \tag{1.3}$$

and the initial condition

$$v = a, \quad x > 0, t = 0, \tag{1.4}$$

where β is a nonnegative number and $v_x = \partial v/\partial x$, $v_{xxx} = \partial^3 v/\partial x^3$. We are interested in finding

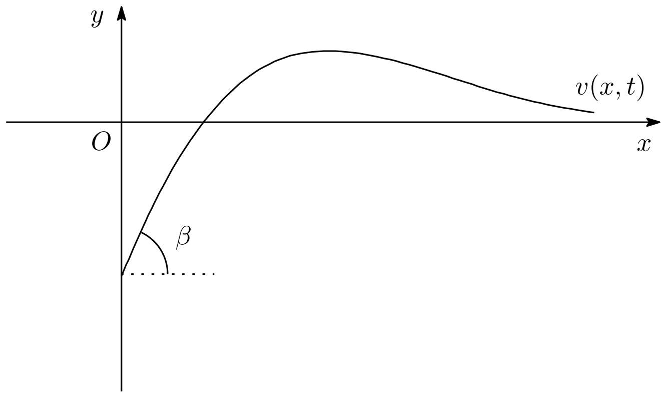

FIG. 1. Profile of thermal groove

a solution for small $\beta > 0$ and small bounded a. In particular, we shall find a bounded self-similar solution and discuss its stability. We say that a solution of (1.1)–(1.3) is self-similar if the rescaled function

$$v_\lambda(x,t) = \frac{1}{\lambda} v(\lambda x, \lambda^4 t)$$

satisfies $v_\lambda(x,t) = v(x,t)$ for all $\lambda > 0$. By definition, a self-similar solution v is of the form $v(x,t) = t^{1/4} Z(x/t^{1/4})$ with some function of one variable Z called a *profile function*. Evidently, $(\tan\beta)x$ is a trivial self-similar solution but is unbounded. If Z is bounded, we say the self-similar solution is (spatially) bounded. Note that a bounded self-similar solution corresponds to a solution of (1.1)–(1.3) with zero initial data, i.e., the case $a \equiv 0$. In this paper we prove that the system (1.1)–(1.4) is solvable globally-in-time and it asymptotically converges to a bounded self-similar solution for large time provided that a and β are small. In particular, we prove the unique existence of a bounded self-similar solution for small β.

This problem stems from a model describing the development of the surface groove proposed by W. W. Mullins [22]. There the condition (1.3) is replaced by no-flux condition $\kappa_s = 0$ where κ is the upward curvature of the graph curve $y = v(x,t)$ and s is the arc-length parameter. Moreover, a is assumed to be zero so that the initial surface is flat.

The equation (1.1) is the surface diffusion law $V = -\kappa_{ss}$ for the graph curve $y = v(x,t)$ where V is the upward normal velocity. The condition (1.2) says that the contact angle of the curve $y = v(x,t)$ at the wall equals $\pi/2 - \beta$.

Let us explain the derivation of Mullins' system. Denote $\mu(\kappa)$ by the increase in chemical potential per atom. We consider the situation where $\mu(\kappa)$ is given by

$$\mu(\kappa) = \gamma\Omega\kappa,$$

where γ is the surface-free energy per unit area, Ω is the molecular volume. The gradient of chemical potential along the surface is obtained via the gradient of the curvature with respect to arc-length parameter s. Therefore, by the Nernst–Einstein relation, a drift of surface atoms R is

$$R = -\frac{D_s\gamma\Omega}{kT}\frac{\partial\kappa}{\partial s},$$

where D_s is the coefficient of surface diffusion, T is the temperature and k is the Boltzmann constant. The surface flux j is the product R by the number N of atoms per unit area,

$$j = -\frac{D_s \gamma \Omega N}{kT} \frac{\partial \kappa}{\partial s}. \tag{1.5}$$

One can obtain the speed of movement V of the surface element along its normal by multiplying Ω to the surface divergence of $-j$, that is,

$$V = \frac{D_s \kappa \Omega^2 N}{kT} \frac{\partial^2 \kappa}{\partial s^2}.$$

The resultant of the grain boundary tension and two surface tensions is assumed to vanish along the line of intersection. The equilibrium angle is $2\gamma_s \sin\beta = \gamma_b$, where γ_s and γ_b are the surface and boundary-free energies per unit area. The absolute value of all slopes is assumed to be small compared with unity. This asserts that $1 \gg \gamma_b/(2\gamma_s) = \sin\beta \simeq \tan\beta$, which is the first boundary condition (1.2). In addition to this, we require a vanishing current of atoms out of the grain boundary, that is, $j = 0$ at $x = 0$. Thus, we have the second boundary condition $\kappa_s = 0$ at $x = 0$. The small slope approximation of $\kappa_s = 0$ is exactly our second boundary condition (1.3).

Mullins [22] linearized the equation (1.1) and the boundary condition (1.3) around $v = 0$ and studied the linear problem of the form

$$\frac{\partial y}{\partial t} = -\frac{\partial^4 y}{\partial x^4}, \quad x > 0, t > 0, \tag{1.6}$$

with the boundary condition

$$y_x = \tan\beta, \quad x = 0, t > 0, \tag{1.7}$$

$$y_{xxx} = 0, \quad x = 0, t > 0, \tag{1.8}$$

and the initial condition

$$y = 0, \quad x > 0, t = 0. \tag{1.9}$$

The solution is again expected to be self-similar. Mullins applied the Laplace transform and derived the depth $y(0,t)$ which is proportional to $t^{1/4}$. Then he studied a profile function Z solving the ordinary differential equation of the form

$$Z'''' - \frac{1}{4} z Z' + \frac{1}{4} Z = 0.$$

Mullins assumed Z to be a power series $Z = \sum_{n=0}^{\infty} a_n z^n$. He showed that $\{a_n\}$ can be determined by a recursion relation. However, its convergence was not discussed.

P. A. Martin [21] improves and extends the results of Mullins. He studies the same problem (1.6)–(1.9). However, the technique developed by Martin is different. He uses the Fourier cosine transform with respect to x. By this technique, he obtains the explicit integral representation formula for the solution y. Based on this formula he proved that the solution decays exponentially at space infinity. In the latter half of [21], he studies multi-groove systems such as periodic surface profile case and two grooves case.

Note that for the original Mullins' system (1.1)–(1.2) with $\kappa_s = 0$, it is not known whether or not bounded self-similar solutions exist. In this paper we linearized the boundary condition $\kappa_s = 0$

to get (1.3) to prove the existence of a self-similar solution. Since (1.1) is quasilinear, such a result was not known even for our simplified problem.

There are two approaches to construct a self-similar solution. One is to solve an ordinary differential equation (ODE) for a profile function. For our problem this seem to be difficult since one has to solve a nonlinear equation of order 4 globally for $x > 0$. Another way is a partial differential equation (PDE) approach initiated by Giga and Miyakawa [15] and developed by Cazenave and Weissler [7]. The main idea is to solve (1.1)–(1.3) by imposing a homogeneous initial data (in our case we consider zero initial data). One advantage of PDE method over ODE is that it is easy to show the stability of a constructed self-similar solution.

Although there is a large literature for solvability of the surface diffusion equation (e.g., [2, 3, 10, 11]), there are a few papers discussing the boundary value problem (e.g., [12–14, 17, 18]). A further difficult point is that in our setting we have to handle initial data like $a = 0$ which is incompatible with the boundary condition. We first transform the problem with homogeneous boundary condition by subtracting a solution $y = U^L$ of the linearized problem (1.6)–(1.9). To solve $u = v - U^L$ we rearrange the equation

$$\partial_t u = -\partial_x^4 u - \partial_x\big(\Xi(u_x + U_x^L, (u + U^L)_{xx}, (u + U^L)_{xxx})\big), \quad x > 0, t > 0, \tag{1.10}$$

with the boundary condition

$$u_x = 0, \quad x = 0, t > 0, \tag{1.11}$$

$$u_{xxx} = 0, \quad x = 0, t > 0. \tag{1.12}$$

The highest order term in Ξ is linear in $(u+U^L)_{xxx}$ and its coefficient equals $(1+(u_x+U_x^L)^2)^{-2}-1$ which is very small when u_x and β are close to zero (so that U_x^L is also close to zero). We solve this equation in $BUC_{1-\alpha}(J, h^{3+\gamma}_{\text{even}}(\mathbf{R})) \cap L^\infty(J, BUC^1_{\text{even}}(\mathbf{R}))$ (see Section 2 for the definition) by adjusting an abstract method of Da Prato–Grisvard [9] and Angenent [1]. We study an integral equation corresponding to (1.10)–(1.12) for u and construct a solution by a fixed-point argument. The smallness condition is invoked to justify that terms in $\partial_x \Xi$ is small compared with other terms in (1.10), so that the contraction mapping principle works. Since our data may be incompatible, one cannot work in $h^{4+\gamma}$. This is a reason why a general theory on local existence for quasilinear equation [8] does not apply to our setting. Note that L^p type space is not suitable in handling this problem since we seek homogeneous functions so we use little Hölder spaces.

Recently, Hamamuki [16] studies the self-similar solutions to the evaporation-condensation problem which is of the form

$$\frac{\partial_t w}{(1 + w_x^2)^{1/2}} = 1 - e^{-\kappa}, \quad x > 0, t > 0, \tag{1.13}$$

with the boundary condition

$$\frac{\partial w}{\partial x} = \tan\beta, \quad x = 0, t > 0. \tag{1.14}$$

This problem (1.13)–(1.14) was also proposed as an evaporation-condensation model by Mullins [22]. The equation (1.13) is, of course, nonlinear. However, since the equation is of second order, he is able to apply the viscosity solution theory to study the problem (1.13)–(1.14). He proves that the solution becomes asymptotically self-similar as $t \to \infty$ without assuming that the angle $\beta > 0$ is small. His method is based on constructing suitable barriers. The groove depth is also studied. His approach is quite different from the approach we discuss in this paper. His technique seems not to be applicable to our problem (1.1)–(1.3) since our problem is of fourth order.

We next mention several works related to our study. Broadbridge and Tritscher [6] try to solve the grain boundary problem using a nonlinear model equation of the type

$$\frac{\partial y}{\partial t} = -B\frac{\partial}{\partial x}\left\{f(y_x)\frac{\partial}{\partial x}\Big[f(y_x)y_{xx}\{[f'(y_x)]^2 + [y_x f'(y_x) + f(y_x)]^2\}^{1/2}\Big]\right\}, \tag{1.15}$$

with the boundary conditions. Here, (1.15) corresponds to the linear model when $f \equiv 1$ and the nonlinear Mullins' system when $f(y_x) = f_0(y_x) = (1 + y_x^2)^{-1/2}$. In [6] they search for a linearizable form, which is in this case $f(y_x) = \alpha/(\beta + y_x)$ (α and β are constants). For this purpose, they apply the linearizing transformation (which is called as *Storm transformation*) to simplify the boundary conditions. By assuming a similarity solution of the form $y_x = g(\xi)$ with $\xi = x(B_1 t)^{-1/4}$ ($B_1 = B\alpha^{-1}(1 + \beta^2)^{1/2}$), they reduce the equation to the linear ODE. The linear ODE is then solved by the Frobenius power series method. Finally they compare the linearizable model with which they are treating in [6] and the Mullins' system. In particular they compare the groove depth $y(0, t)$ at the origin. They observe that the small-slope approximation is valid for most metals in inert gases. However, in surface-active environments, grain boundary slopes taking large values, the error differences in the grooves depth become large between the linear model and the nonlinear model. Note that their results do not yield self-similar solution to (1.1)–(1.3).

Kanel, Novick-Cohen and Vilenkin [19] find travelling wave solutions which describe grain boundary motion in a bicrystal which has a triple junction. The triple junction separates the surface in three phases, that is, grain 1, grain 2 and an outside. The boundary between grain 1 and grain 2 is called a grain boundary. The boundary between grains and outside is called an exterior surface. In this situation, the grain boundary evolves according to motion by mean curvature. Away from it, the evolution of the exterior surface is governed by the surface diffusion. Thus, the motion is coupled with mean curvature and surface diffusion. This problem has already been propounded by Mullins [23] in 1958. After expressing the problem via an angle formulation, they show the existence of a solution based on the theory of stable and unstable manifolds and integral formulations using the Green functions. It seems that their approach does not apply to our setting since their initial data is compatible.

Zhu [26] studies the existence of the stationary solution to the equation (1.1) in the open interval $I = (a, b) \subset \mathbf{R}$ with zero boundary conditions, i.e.,

$$y_x = y_{xxx} = 0 \quad \text{on } \partial I,$$

and the initial data

$$y|_{t=0} = y_0 \quad \text{on } \overline{I}.$$

He shows the existence of a stationary solution. He also proves the stationary solution is asymptotically stable in a suitable norm as time goes to infinity. He establishes the energy estimate of Schauder type for the solution, then applies the Leray–Schauder fixed-point theorem. Since he discusses compatible data, his approach does not apply to our study.

This paper is organized as follows. In Section 2, we study the linearized equation and recall a result of P. A. Martin. We also give the definitions of some function spaces and show that the bi-Laplacian operator $-\partial_x^4$ generates the non C_0-bounded analytic semigroup on L^∞. In Section 3, we construct the mild solution of the problem (1.1)–(1.4). Finally, we prove the stability of self-similar solution.

A self-similar solution is constructed in a similar way in [4] but for the differential form of (1.1). The spatial derivative of the self-similar solution we construct in this paper is actually the solution of [4]. However, in [4] it is not clear the self-similar solution in [4] is bounded. Also, stability of the self-similar solution is not discussed in [4].

2. Linear equation with boundary conditions

2.1 *Explicit formula for the linear problem*

Before turning to a closer examination of our nonlinear problem (1.1)–(1.4), we must draw attention to the linear problem (1.6)–(1.9). As described in the Introduction, there are several available results on the linear problem. In this paper, we recall the result of the paper by P. A. Martin [21].

LEMMA 2.1 There is a solution for the problem (1.6)–(1.9) of the form

$$U^L(x,t) = -\frac{2\tan\beta}{\pi}\int_0^\infty \left(1 - e^{-k^4 t}\right)\frac{\cos kx}{k^2}\,dk,$$

which decays exponentially as $x \to \infty$.

Proof. See [21, Section 2], where β is denoted by θ_{eq}. □

2.2 *Function spaces*

Now we turn our attention to the nonlinear problem (1.1)–(1.4). In this paper, we consider our problem on the half line $\mathbf{R}^+ = (0,\infty)$. However, in the sequel, we extend the solution as an even function on the whole line $\mathbf{R}$. This extension as an even function is natural because our homogeneous linear problem can be reduced to be a whole space problem by even extension. Thus, we shall use the function spaces of even functions. We first recall the space of bounded functions and Hölder continuous functions defined on $\mathbf{R}$. For a measurable function φ in $\mathbf{R}$ we denote the L^∞-norm by $|\varphi|_\infty$, i.e.,

$$|\varphi|_\infty := \operatorname{ess.sup}_{x\in\mathbf{R}}|\varphi(x)|.$$

For $\nu \in (0,1)$ we define its ν-Hölder quotient at $x, y \in \mathbf{R}$ by

$$[\varphi]_{\nu,x,y} := \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\nu}, \quad x \neq y.$$

For $\varphi \in L^\infty(\mathbf{R})$ we define its ν-Hölder seminorm by

$$[\varphi]_\nu := \operatorname{ess.sup}\left\{[\varphi]_{\nu,x,y}; x, y \in \mathbf{R}, x \neq y\right\}.$$

We recall several basic Banach spaces. We use the same notation as in [20, Chapter 0].

DEFINITION 2.2 (i) Let $L^\infty(\mathbf{R})$ be the space of all real valued essentially bounded measurable functions on $\mathbf{R}$. This space is a Banach space equipped with the norm $|\varphi|_\infty$. Let $BUC(\mathbf{R})$ be the space of all bounded and uniformly continuous functions on $\mathbf{R}$. This is a closed subspace of $L^\infty(\mathbf{R})$.

(ii) For $k = 1, 2, \dots$, let $W^{k,\infty}(\mathbf{R})$ be the Sobolev space such that

$$W^{k,\infty}(\mathbf{R}) := \{\varphi \in L^\infty(\mathbf{R}); k\text{-th distributional derivative } \partial_x^k \varphi \text{ is in } L^\infty(\mathbf{R})\}.$$

It is the Banach space equipped with the norm

$$\|\varphi\|_{k,\infty} = \sum_{l=0}^{k} |\partial_x^l \varphi|_\infty$$

which is equivalent to $|\|\varphi\||_{k,\infty} := |\varphi|_\infty + |\partial_x^k \varphi|_\infty$. We often denote $|\partial_x^k \varphi|_\infty$ by $|\varphi|_{k,\infty}$.
Let $BUC^k(\mathbf{R})$ be the closed subspace of $W^{k,\infty}(\mathbf{R})$ defined by

$$BUC^k(\mathbf{R}) := \left\{\varphi \in BUC(\mathbf{R});\ \begin{array}{l} \varphi \text{ is } k\text{-times continuously differentiable} \\ \text{and } \varphi^{(l)} \in BUC(\mathbf{R}) \text{ for } l = 0, 1, \dots, k \end{array}\right\}.$$

(iii) For $\nu \in (0, 1)$ let $C^\nu(\mathbf{R})$ be the space of all bounded ν-Hölder continuous functions on $\mathbf{R}$, i.e.,

$$C^\nu(\mathbf{R}) := \{\varphi \in C(\mathbf{R}); \|\varphi\|_\nu := |\varphi|_\infty + [\varphi]_\nu < \infty\}.$$

This is a Banach space equipped with the norm $\|\varphi\|_\nu$.
Unfortunately, the space of bounded smooth function $BUC^\infty(\mathbf{R}) = \bigcap_{k\geqslant 0} BUC^k(\mathbf{R}) \subset C^\nu(\mathbf{R})$ is not dense in $C^\nu(\mathbf{R})$. One defines the closure of $BUC^\infty(\mathbf{R})$ in $C^\nu(\mathbf{R})$ by $h^\nu(\mathbf{R})$ which is characterized as

$$h^\nu(\mathbf{R}) = \{\varphi \in C^\nu(\mathbf{R}); \lim_{y\to x} [\varphi]_{\nu,x,y} = 0\}.$$

This space is called a little Hölder space.

(iv) For $k = 1, 2, \dots$ and $\nu \in (0, 1)$ let $C^{k+\nu}(\mathbf{R})$ be the space of BUC^k functions having ν-Hölder continuous k-th derivatives, i.e.,

$$C^{k+\nu}(\mathbf{R}) = \{\varphi \in BUC^k(\mathbf{R}); \partial_x^k \varphi \in C^\nu(\mathbf{R})\}.$$

This space is a Banach space equipped with the norm

$$\|\varphi\|_{k+\nu} = \|\varphi\|_{k,\infty} + [\partial_x^k \varphi]_\nu.$$

To simplify the notation we often denote the seminorm $[\partial_x^k \varphi]_\nu$ by $[\varphi]_{k+\nu}$.
The closure of $BUC^\infty(\mathbf{R})$ in $C^{k+\nu}(\mathbf{R})$ is denoted by $h^{k+\nu}(\mathbf{R})$. It is characterized as

$$h^{k+\nu}(\mathbf{R}) = \{\varphi \in C^{k+\nu}(\mathbf{R}); \lim_{y\to x} [\partial_x^k \varphi]_{\nu,x,y} = 0\}.$$

To develop the semigroup theory we often need to consider complexified space, which are spaces of complex-valued functions. In this case the resulting Banach space is a complex Banach spaces. We do not distinguish real and complex Banach space to simplify the notation.

We shall give notation of the space of even functions.

DEFINITION 2.3 Let X be a space of measurable functions defined on $\mathbf{R}$. Let X_{even} denote its subspace of even functions in X, i.e.,

$$X_{\text{even}} = \{\varphi \in X; \varphi(x) = \varphi(-x), \quad \text{a.e. } x\}.$$

For example,

$$L^\infty_{\text{even}}(\mathbf{R}) = \{\varphi \in L^\infty(\mathbf{R}); \varphi(x) = \varphi(-x), \quad \text{a.e. } x\},$$
$$BUC_{\text{even}}(\mathbf{R}) = \{\varphi \in BUC(\mathbf{R}); \varphi(x) = \varphi(-x), \quad \text{for all } x\}.$$

Note that $\partial_x\varphi(0) = 0$ for $\varphi \in BUC^1_{\text{even}}(\mathbf{R})$. We also note that $L^\infty_{\text{even}}(\mathbf{R})$ is a closed subspace of $L^\infty(\mathbf{R})$. Similar statements hold for $BUC^k_{\text{even}}(\mathbf{R})$, $C^\nu_{\text{even}}(\mathbf{R})$, $C^{k+\nu}_{\text{even}}(\mathbf{R})$, $W^{k,\infty}_{\text{even}}(\mathbf{R})$.

We occasionally use a function space on a half line $\overline{\mathbf{R}^+} = \{x \geqslant 0\}$, for example $h^\gamma(\overline{\mathbf{R}^+})$, which is defined as $h^\gamma(\mathbf{R})$ by replacing $\mathbf{R}$ by $\overline{\mathbf{R}^+}$.

In order to construct the solution of the problem (1.1)–(1.4) via the analytic semigroup theory, we shall use the weighted continuous function spaces in time with values in a Banach space. Such spaces are often used in the analytic semigroup theory especially to analyze the singularity as time goes to zero. The reader is referred to [8, Section 2] and [20, Subsection 4.3.2] for more details.

DEFINITION 2.4 For $T > 0$ set $J = [0,T]$, $\dot{J} = J \setminus \{0\}$. Let $0 < \mu < 1$ be fixed.

$$BUC_{1-\mu}(J,E) := \{u \in C(\dot{J},E); [t \mapsto t^{1-\mu}u] \in BUC(\dot{J},E), \lim_{t\to 0+} t^{1-\mu}\|u(t)\|_E = 0\},$$

where E is a (real or complex) Banach space.

2.3 *Analytic semigroup generated by the bi-Laplace operator*

In this section we shall give a proof that the bi-Laplace operator $-\partial_x^4$ generates non C_0-bounded and bounded analytic semigroup in L^∞ type spaces. The analyticity result is essentially known; see, e.g., [20, Theorem 3.2.4]. However, the *bounded* analyticity is not written in [20]. We give a complete proof for the reader's convenience.

Let us consider the resolvent equation

$$(\lambda + \partial_x^4)u = f, \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\}, |\arg\lambda| < \pi$$

in a formal way. We take the Fourier transform of the both sides to get

$$\hat{u}(\xi) = (\lambda + |\xi|^4)^{-1}\hat{f}(\xi).$$

(This calculation is justified when u and f are Schwartz' tempered distributions). Applying the inverse Fourier transformation one obtains

$$u(x) = K^\lambda * f(x)$$

with

$$K^\lambda(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{\lambda + \xi^4}\,d\xi, \qquad (i = \sqrt{-1}). \tag{2.1}$$

LEMMA 2.5 For a given $\theta_0 \in (0,\pi)$, there is a constant C_{θ_0} such that

$$\|K^\lambda * f\|_\infty \leqslant C_{\theta_0}|\lambda|^{-1}\|f\|_\infty$$

for all $f \in L^\infty(\mathbf{R}), \lambda \in \Sigma_{\theta_0} := \{\lambda \in \mathbb{C}\setminus\{0\}; |\arg\lambda| \leqslant \theta_0\}$.

Proof. To calculate (2.1), we first calculate

$$K_\theta(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{e^{i\theta}+\xi^4}\,d\xi, \qquad \text{for } 0<\theta\leqslant\theta_0<\pi. \tag{2.2}$$

Thus, our concern is the roots of $e^{i\theta}+\xi^4$. We set

$$\zeta = \exp(i\theta/4), \quad \omega = \exp(i\pi/4). \tag{2.3}$$

Then the roots of $e^{i\theta}+\xi^4$ are $\zeta\omega, i\zeta\omega, i^2\zeta\omega$ and $i^3\zeta\omega$. Hereafter, we denote the residue of f at the point a by $\mathrm{Res}(f,a)$. By residue theorem, we have

$$\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{e^{i\theta}+\xi^4}\,d\xi = \begin{cases} 2\pi i\left[\mathrm{Res}\left(\dfrac{e^{ix\xi}}{e^{i\theta}+\xi^4},\zeta\omega\right)+\mathrm{Res}\left(\dfrac{e^{ix\xi}}{e^{i\theta}+\xi^4},i\zeta\omega\right)\right], & \text{if } x>0,\\ -2\pi i\left[\mathrm{Res}\left(\dfrac{e^{ix\xi}}{e^{i\theta}+\xi^4},i^2\zeta\omega\right)+\mathrm{Res}\left(\dfrac{e^{ix\xi}}{e^{i\theta}+\xi^4},i^3\zeta\omega\right)\right], & \text{if } x<0. \end{cases} \tag{2.4}$$

We calculate the residues in (2.4) respectively.

$$\begin{aligned}\mathrm{Res}\left(\frac{e^{ix\xi}}{e^{i\theta}+\xi^4},\zeta\omega\right) &= \frac{e^{i\zeta\omega x}}{(\zeta\omega-i\zeta\omega)(\zeta\omega-i^2\zeta\omega)(\zeta\omega-i^3\zeta\omega)}\\ &= \frac{\exp(-\zeta x/\sqrt{2})\cdot\exp(i\zeta x/\sqrt{2})}{4\zeta^3\omega^3},\end{aligned} \tag{2.5}$$

$$\mathrm{Res}\left(\frac{e^{ix\xi}}{e^{i\theta}+\xi^4},i\zeta\omega\right) = -\frac{\exp(-\zeta x/\sqrt{2})\cdot\exp(-i\zeta x/\sqrt{2})}{4i\zeta^3\omega^3}, \tag{2.6}$$

$$\mathrm{Res}\left(\frac{e^{ix\xi}}{e^{i\theta}+\xi^4},i^2\zeta\omega\right) = -\frac{\exp(\zeta x/\sqrt{2})\cdot\exp(-i\zeta x/\sqrt{2})}{4\zeta^3\omega^3}, \tag{2.7}$$

$$\mathrm{Res}\left(\frac{e^{ix\xi}}{e^{i\theta}+\xi^4},i^3\zeta\omega\right) = \frac{\exp(\zeta x/\sqrt{2})\cdot\exp(i\zeta x/\sqrt{2})}{4i\zeta^3\omega^3}. \tag{2.8}$$

Thus from (2.4)–(2.6), we have when $x>0$

$$\int_{-\infty}^{\infty}\frac{e^{ix\xi}}{e^{i\theta}+\xi^4}\,d\xi = \frac{\pi}{\sqrt{2}\zeta^3}\left(\sin\frac{\zeta x}{\sqrt{2}}+\cos\frac{\zeta x}{\sqrt{2}}\right)\exp\left(-\frac{\zeta x}{\sqrt{2}}\right). \tag{2.9}$$

Similarly, from (2.4), (2.7) and (2.8), we have when $x < 0$

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{e^{i\theta} + \xi^4}\, d\xi = \frac{\pi}{\sqrt{2}\,\zeta^3} \left(-\sin \frac{\zeta x}{\sqrt{2}} + \cos \frac{\zeta x}{\sqrt{2}} \right) \exp\left(\frac{\zeta x}{\sqrt{2}} \right). \tag{2.10}$$

From (2.9) and (2.10) we can conclude that

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{e^{i\theta} + \xi^4}\, d\xi = \frac{\pi}{\sqrt{2}\,\zeta^3} \left(\sin \frac{\zeta |x|}{\sqrt{2}} + \cos \frac{\zeta |x|}{\sqrt{2}} \right) \exp\left(-\frac{\zeta |x|}{\sqrt{2}} \right). \tag{2.11}$$

To show that $K_\theta(x)$ is integrable for $0 < \theta \leqslant \theta_0 < \pi$, we have to compute the terms

$$\sin\left(\frac{\zeta|x|}{\sqrt{2}}\right) \exp\left(-\frac{\zeta|x|}{\sqrt{2}}\right) \quad \text{and} \quad \cos\left(\frac{\zeta|x|}{\sqrt{2}}\right) \exp\left(-\frac{\zeta|x|}{\sqrt{2}}\right)$$

in (2.11) respectively.

$$\begin{aligned}
&\sin\left(\frac{\zeta|x|}{\sqrt{2}}\right) \exp\left(-\frac{\zeta|x|}{\sqrt{2}}\right) \\
&\quad = \frac{\exp\left(\frac{i\zeta|x|}{\sqrt{2}}\right) - \exp\left(\frac{-i\zeta|x|}{\sqrt{2}}\right)}{2i} \cdot \exp\left(-\frac{\zeta|x|}{\sqrt{2}}\right) \\
&\quad = \frac{1}{2i}\left[\exp\left(\frac{(-1+i)\zeta|x|}{\sqrt{2}}\right) - \exp\left(\frac{(-1-i)\zeta|x|}{\sqrt{2}}\right) \right] \\
&\quad = \frac{1}{2i}\Bigg[\exp\left(-\frac{|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} + \sin\frac{\theta}{4}\right)\right) \cdot \exp\left(\frac{i|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} - \sin\frac{\theta}{4}\right)\right) \\
&\qquad - \exp\left(-\frac{|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} - \sin\frac{\theta}{4}\right)\right) \cdot \exp\left(-\frac{i|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} + \sin\frac{\theta}{4}\right)\right) \Bigg],
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\cos\left(\frac{\zeta|x|}{\sqrt{2}}\right) \exp\left(-\frac{\zeta|x|}{\sqrt{2}}\right) \\
&\quad = \frac{\exp\left(\frac{i\zeta|x|}{\sqrt{2}}\right) + \exp\left(\frac{-i\zeta|x|}{\sqrt{2}}\right)}{2} \cdot \exp\left(-\frac{\zeta|x|}{\sqrt{2}}\right) \\
&\quad = \frac{1}{2}\left[\exp\left(\frac{(-1+i)\zeta|x|}{\sqrt{2}}\right) + \exp\left(\frac{(-1-i)\zeta|x|}{\sqrt{2}}\right) \right] \\
&\quad = \frac{1}{2}\Bigg[\exp\left(-\frac{|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} + \sin\frac{\theta}{4}\right)\right) \cdot \exp\left(\frac{i|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} - \sin\frac{\theta}{4}\right)\right) \\
&\qquad - \exp\left(-\frac{|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} - \sin\frac{\theta}{4}\right)\right) \cdot \exp\left(-\frac{i|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} + \sin\frac{\theta}{4}\right)\right) \Bigg].
\end{aligned} \tag{2.13}$$

Note that $\cos(\theta/4) - \sin(\theta/4) > 0$ because of $0 < \theta < \pi$. Thus, from (2.11)–(2.13)

$$\int_{\mathbf{R}} |K_\theta(x)|\, dx \leqslant \frac{\pi}{2} \int_{\mathbf{R}} \left[\exp\left(-\frac{|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} + \sin\frac{\theta}{4}\right)\right) + \exp\left(-\frac{|x|}{\sqrt{2}}\left(\cos\frac{\theta}{4} - \sin\frac{\theta}{4}\right)\right)\right] dx. \quad (2.14)$$

Since the right-hand side of (2.14) is bounded by a constant C_{θ_0} for $\theta \in (0, \theta_0]$ with $\theta_0 \in (0, \pi)$, we observe that

$$\|K_\theta\|_{L^1(\mathbf{R})} \leqslant C_{\theta_0} \qquad \text{for } \theta \in (0, \theta_0]. \quad (2.15)$$

Next, we calculate $K^\lambda(x)$ based on the estimate for $K_\theta(x)$. Take $\Sigma_{\theta_0} \ni \lambda (= re^{i\theta})$, then by changing the variable $\xi = r^{1/4}\eta$ and recalling the definition of $K^\lambda(x)$ in (2.1), we have

$$\begin{aligned} K^\lambda(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{\lambda + \xi^4}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{re^{i\theta} + \xi^4}\, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{r^{3/4}} \int_{-\infty}^{\infty} \frac{e^{ir^{1/4}x\eta}}{e^{i\theta} + \eta^4}\, d\eta = \frac{1}{r^{3/4}} K_\theta(r^{1/4}x). \end{aligned} \quad (2.16)$$

Thus from (2.16), we have

$$\begin{aligned} \|K^\lambda\|_{L^1} &= \int_{-\infty}^{\infty} |K^\lambda(x)|\, dx = \int_{-\infty}^{\infty} \frac{1}{r^{3/4}} |K_\theta(r^{1/4}x)|\, dx \\ &= \frac{1}{r} \int_{-\infty}^{\infty} |K_\theta(y)|\, dy = \frac{1}{r} \|K_\theta\|_{L^1}. \end{aligned} \quad (2.17)$$

From (2.15), (2.17) and Young's inequality, we have

$$\begin{aligned} \|(K^\lambda * f)\|_{L^\infty} &\leqslant \|K^\lambda\|_{L^1} \cdot \|f\|_{L^\infty} \\ &\leqslant C_{\theta_0} |\lambda|^{-1} \|f\|_{L^\infty}, \qquad \text{for } \lambda \in \Sigma_{\theta_0}. \end{aligned}$$

□

We set

$$(\mathbf{K}^\lambda f)(x) = (K^\lambda * f)(x) \quad \text{for } f \in L^\infty(\mathbf{R}).$$

By Lemma 2.5 this operator $\mathbf{K}^\lambda$ is a bounded operator in $L^\infty(\mathbf{R})$.

LEMMA 2.6 Let $\lambda \in \mathbb{C} \setminus \{0\}$ satisfy $|\arg \lambda| < \pi$.

(i) The range

$$R(\mathbf{K}^\lambda) = W^{4,\infty}(\mathbf{R}),$$

where

$$W^{4,\infty}(\mathbf{R}) = \{\varphi \in L^\infty(\mathbf{R}); \partial_x^4 \varphi \in L^\infty(\mathbf{R})\},$$

$\partial_x^4 \varphi$ is the fourth order derivative of φ in the sense of distribution.

(ii) The operator $\mathbf{K}^\lambda$ is injective.

Proof. (i) By definition we have

$$(\lambda + \partial_x^4)(\mathbf{K}^\lambda f) = f,$$

for $f \in L^\infty(\mathbf{R})$ in the sense of distribution. This implies that

$$\partial_x^4(\mathbf{K}^\lambda f) = f - \lambda \mathbf{K}^\lambda f \in L^\infty(\mathbf{R}).$$

Thus $R(\mathbf{K}^\lambda) \subset W^{4,\infty}(\mathbf{R})$. If $h \in W^{4,\infty}(\mathbf{R})$, then take $f = (\lambda + \partial_x^4)h \in L^\infty(\mathbf{R})$ to get $h = \mathbf{K}^\lambda f$. This implies the converse inclusion. We conclude $R(\mathbf{K}^\lambda) = W^{4,\infty}(\mathbf{R})$.

(ii) We consider $K^\lambda * f = 0$ for $f \in L^\infty(\mathbf{R})$. Taking the Fourier transform we observe that $\hat{f} = 0$ in the sense of distribution, since $\lambda + |\xi|^4 \neq 0$ for all $\xi \in \mathbf{R}$ so that $s/(\lambda + |\xi|^4) \in \mathcal{S}$ for $s \in \mathcal{S}$, where $\mathcal{S}$ is the space of all rapidly decreasing functions (see [24, Chapter VII, Section 3]). This implies $f = 0$ (this calculation is justified for any Schwartz' tempered distribution f). □

REMARK 2.7 We warn the reader that in higher dimensional problem it is difficult to characterize the range.

We define the closed linear operator A by

$$A(\lambda) := -(\mathbf{K}^\lambda)^{-1} + \lambda I,$$

where I denotes the identity operator.

LEMMA 2.8 Let λ be as in Lemma 2.6. Then $A = A(\lambda)$ is independent of λ. Moreover

$$D(A) = W^{4,\infty}(\mathbf{R}) \text{ and } Au = -\partial_x^4 u \text{ (in the sense of distribution) for } u \in W^{4,\infty}(\mathbf{R}).$$

Proof. By definition $D(A) = R(\mathbf{K}^\lambda) = W^{4,\infty}(\mathbf{R})$. Formally, for $u \in W^{4,\infty}(\mathbf{R})$

$$\widehat{Au} = -(\lambda + |\xi|^4)\hat{u} + \lambda\hat{u} = -\widehat{\partial_x^4 u}.$$

This identity is justified in the sense of tempered distribution so that $Au = -\partial_x^4 u$ and A is independent of the choice of λ. □

REMARK 2.9 (i) For higher dimension case, the domain of the corresponding operator to $-\Delta^2$ is

$$D(-\Delta^2) = \{u \in \bigcap_{p \geqslant 1} W^{4,p}_{\mathrm{loc}}(\mathbf{R}); -\Delta^2 u \in L^\infty(\mathbf{R})\},$$

see Lunardi [20, Theorem 3.2.4]. In one dimensional case this space is $W^{4,\infty}(\mathbf{R})$.

(ii) The independence of A with respect to λ is usually proved by the resolvent identity

$$\mathbf{K}^\lambda - \mathbf{K}^\mu = (\mu - \lambda)\mathbf{K}^\mu \mathbf{K}^\lambda = (\mu - \lambda)\mathbf{K}^\lambda \mathbf{K}^\mu.$$

Here we are able to use the explicit representation.

THEOREM 2.10 (i) The operator A generates a non-C_0 bounded and bounded analytic semigroup e^{tA} in $L^\infty(\mathbf{R})$. In particular

$$\|e^{tA} f\|_\infty \leqslant C_1 \|f\|_\infty \quad \text{for all } t > 0,$$
$$\left\| \frac{d}{dt} e^{tA} f \right\|_\infty \leqslant \frac{C_2}{t} \|f\|_\infty \quad \text{for all } t > 0,$$

with some constants C_1 and C_2 independent of t and $f \in L^\infty(\mathbf{R})$. Moreover, for $k = 1, 2, 3$,

$$\|\partial_x^k e^{tA} f\|_\infty \leqslant \frac{C_3}{t^{k/4}} \|f\|_\infty \quad \text{for all } t > 0$$

with C_3 independent of t and $f \in L^\infty(\mathbf{R})$.

(ii) The closure $\overline{D(A)}$ of $D(A)$ in $L^\infty(\mathbf{R})$ equals $BUC(\mathbf{R})$.

(iii) The operator A generates a C_0-bounded, bounded analytic semigroup e^{tA} in $BUC(\mathbf{R})$.

(iv) The assertions (i)–(iii) still hold if one replaces $L^\infty(\mathbf{R})$ by $L^\infty_{\text{even}}(\mathbf{R})$, $BUC(\mathbf{R})$ by $BUC_{\text{even}}(\mathbf{R})$ and $W^{4,\infty}(\mathbf{R})$ by $W^{4,\infty}_{\text{even}}(\mathbf{R})$.

Proof. (i) This is standard once we have the resolvent estimate in Lemma 2.5 (see, e.g., [20, Proposition 2.1.1]) for the resolvent $\mathbf{K}^\lambda = (\lambda - A)^{-1}$. We give the proof for the global boundedness for the reader's convenience.

By definition

$$e^{tA} = \frac{1}{2\pi i} \int_L e^{t\lambda} (\lambda - A)^{-1} \, d\lambda.$$

One is allowed to take L as $L = L_t^+ \cup L_t^- \cup S_t$ with

$$L_t^\pm = \{\lambda \in \mathbb{C}; |\lambda| \geqslant 1/t, |\arg \lambda| = \pm\theta\}, \quad S_t = \{\lambda \in \mathbb{C}; |\lambda| = 1/t, |\arg \lambda| \leqslant \theta\},$$

where $\theta \in (\pi/2, \theta_0)$. On $L_t^\pm$, by changing the variable $\lambda = \rho e^{\pm i\theta}$, the operator norm is estimated as

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{L_t^\pm} e^{t\lambda} (\lambda - A)^{-1} \, d\lambda \right\| &\leqslant \frac{C_{\theta_0}}{2\pi} \int_{1/t}^\infty \frac{e^{t\rho \cos\theta}}{\rho} \, d\rho \\ &= \frac{C_{\theta_0}}{2\pi} \int_1^\infty \frac{e^{r \cos\theta}}{r} \, dr =: M_\pm, \end{aligned} \tag{2.18}$$

On S_t, by changing the variable $\lambda = e^{i\phi}/t$ ($\phi \in (-\theta, \theta)$), we observe that

$$\left\| \frac{1}{2\pi i} \int_{S_t} e^{t\lambda} (\lambda - A)^{-1} \, d\lambda \right\| \leqslant \frac{C_{\theta_0}}{2\pi} \int_{-\theta}^\theta e^{cos\phi} \, d\phi =: M_0. \tag{2.19}$$

Since the rightest-hand sides of (2.18) and (2.19) are finite and independent of t we observe that

$$\|e^{tA}\| \leqslant M_0 + M_+ + M_- =: C_1,$$

which is the boundedness of the semigroup. The bounded analyticity

$$t \left\| \frac{d}{dt} e^{tA} \right\| \leqslant C_2 \tag{2.20}$$

can be proved similarly. First we observe

$$\frac{d}{dt}e^{tA} = \frac{1}{2\pi i}\int_L \lambda e^{t\lambda}(\lambda - A)^{-1}\,d\lambda. \tag{2.21}$$

For λ on the path $L_t^{\pm}$, we have the estimate

$$\left\|\lambda e^{t\lambda}(\lambda - A)^{-1}\right\| \leqslant C_{\theta_0}\frac{|\lambda|}{|\lambda|}e^{t|\lambda|\cos\theta}. \tag{2.22}$$

We deduce from (2.21) and (2.22) that

$$\begin{aligned}\left\|\frac{d}{dt}e^{tA}\right\| &\leqslant \frac{1}{2\pi}\cdot 2\cdot\int_0^\infty C_{\theta_0}e^{t\rho\cos\theta}\,d\rho\\ &= \frac{C_{\theta_0}}{\pi}\left(-\frac{1}{t\cos\theta}\right) =: \frac{C_2}{t}.\end{aligned}$$

For estimates of derivatives we may assume $k = 1, 2, 3$ since other cases are reduced to this case by the boundedness of the operator $tAe^{tA} = t\partial_t e^{tA}$ which is just proved. We first note that Lemma 2.5 is extended to

$$\|\partial_x^k K^\lambda * f\| \leqslant C_{\theta_0}|\lambda|^{-1+k/4}\|f\|_\infty.$$

Since

$$\partial_x^k e^{tA} = \frac{1}{2\pi i}\int_L e^{t\lambda}\partial_x^k \mathbf{K}_\lambda\,d\lambda,$$

the above estimate for $\partial_x^k\mathbf{K}_\lambda = \partial_x^k K^\lambda *$ yields the desired estimates.

(ii) For a given $f \in BUC(\mathbf{R})$ it is well-known that $f_\varepsilon := \rho_\varepsilon * f \to f$ in $L^\infty(\mathbf{R})$ as $\varepsilon \to 0$ where ρ_ε is a mollifier, i.e.,

$$\rho_\varepsilon(x) = \varepsilon^{-1}\rho(\frac{x}{\varepsilon}) \quad \text{with} \quad \rho \in C^\infty(\mathbf{R}), 0 \leqslant \rho \leqslant 1, \operatorname{supp}\rho \subset (-1, 1) \quad \text{and} \quad \int_{\mathbf{R}}\rho(x)\,dx = 1.$$

It is easy to see that $\rho_\varepsilon * f \in C^\infty(\mathbf{R})$, $f_\varepsilon \in C^\infty(\mathbf{R})$ and $\|\partial_x^m f_\varepsilon\|_\infty < \infty$ for all $m = 1, 2, \ldots$. In particular $f_\varepsilon \in W^{4,\infty}(\mathbf{R})$. Since $\partial_x^m f_\varepsilon = \partial_x^m\rho_\varepsilon * f$,

$$\|\partial_x^m f_\varepsilon\|_\infty \leqslant \|\partial_x^m\rho_\varepsilon\|_{L^1}\|f\|_\infty.$$

Since $W^{4,\infty}(\mathbf{R})$ is contained in $BUC(\mathbf{R})$, this implies $\overline{W^{4,\infty}(\mathbf{R})} = BUC(\mathbf{R})$.

(iii) Since e^{tA} maps from $BUC(\mathbf{R})$ to $W^{4,\infty}(\mathbf{R}) \subset BUC(\mathbf{R})$ for $t > 0$, one may interpret e^{tA} as a semigroup in $BUC(\mathbf{R})$ and its generator A has a dense domain. Thus e^{tA} is a C_0-semigroup.

(iv) This is trivial since e^{tA} preserves evenness. □

REMARK 2.11 Since $\|K^\lambda\|_{L^1} \leqslant C_{\theta_0}|\lambda|^{-1}$ for $\lambda \in \Sigma_{\theta_0}$, an argument similar to the proof of Theorem 2.10 yields

$$\|e^{tA}f\|_{L^1} \leqslant C_0\|f\|_{L^1} \quad \text{for all } t > 0$$

with C_0 independent of $f \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$.

2.4 *Hölder seminorm represented by an analytic semigroup*

In this subsection we shall prove the equivalence of the Hölder seminorm $[f]_\gamma$ and an interpolation seminorm.

LEMMA 2.12 Let $\theta = \gamma/4$ and $0 < \gamma < 1$. Let

$$[f]_{F_\theta} := \sup_{r>0} r^{1-\theta}|Ae^{rA}f|_\infty$$

for $f \in L^\infty(\mathbf{R})$, where e^{tA} is the semigroup generated by the bi-Laplacian in $L^\infty(\mathbf{R})$ as defined in Theorem 2.10. Then there are constants M_1 and M_2 independent of $f \in L^\infty(\mathbf{R})$ such that

$$[f]_\gamma \leqslant M_1[f]_{F_\theta}, \tag{2.23}$$

$$[f]_{F_\theta} \leqslant M_2[f]_\gamma. \tag{2.24}$$

Proof. We set

$$F_0 := L^\infty(\mathbf{R}), \quad F_1 := W^{4,\infty}(\mathbf{R}).$$

The real interpolation space

$$F_\theta = (F_0, F_1)_{\theta,\infty} \tag{2.25}$$

is characterized by a Besov space $B^\gamma_{\infty,\infty}$; see [5, Theorem 6.2.4]. By a characterization of the Besov space we know

$$B^\gamma_{\infty,\infty} = C^\gamma(\mathbf{R})$$

for $0 < \gamma < 1$, see for instance [25, Section 2.5.7]. Thus

$$F_\theta = C^\gamma(\mathbf{R}).$$

In the meanwhile there is a characterization of a real interpolation space by an analytic semigroup such as $D_A(\theta,\infty) = (F_0, F_1)_{\theta,\infty}$ where $F_1 = D(A)$ and e^{tA} is an analytic semigroup in F_0, see e.g., [20, Proposition 2.2.2]. It yields that

$$\|f : (F_0, F_1)_{\theta,\infty}\|$$

is equivalent to

$$\|f\|_{F_0} + \sup_{0<r<1} r^{1-\theta}\|Ae^{rA}f\|_{F_0}.$$

For choice of F_0, F_1 and A the second norm is equivalent to

$$|f|_\infty + [f]_{F_\theta} \tag{2.26}$$

since $\sup_{1<r<\infty} r^{1-\theta}\|Ae^{rA}f\|_\infty \leqslant C|f|_\infty$. The characterization of a Hölder space by semigroup norm is of course well-known; see [20, Theorem 3.1.12] where $F_0 = C(\mathbf{R})$, the space of bounded continuous function and $A = \partial_x^2$. However, we have given here an outline for the reader's convenience since $A = -\partial_x^4$ and $F_0 = L^\infty(\mathbf{R})$.

The characterization of the Hölder norm (2.25) implies that the norm (2.26) is equivalent to $|f|_\infty + [f]_\gamma$. We shall prove (2.23) since the other inequality (2.24) can be proved similarly. By the above characterization there is a constant M_1 such that

$$[f]_\gamma \leqslant M_1\{|f|_\infty + [f]_{F_\theta}\} \tag{2.27}$$

for all $f \in L^\infty(\mathbf{R})$. We plug f_λ into f of this inequality with

$$f_\lambda(x) = \frac{f(\lambda x)}{\lambda^\gamma}$$

for $\lambda > 0$. Note that $[f]_\gamma$ seminorm and $[f]_{F_\theta}$ seminorm is invariant under this scaling. However,

$$|f_\lambda|_\infty = \frac{|f|_\infty}{\lambda^\gamma}.$$

Thus (2.27) yields

$$[f]_\gamma \leqslant M_1 \left\{ \frac{|f|_\infty}{\lambda^\gamma} + [f]_{F_\theta} \right\}.$$

Since $\lambda > 0$ is arbitrary, we conclude (2.23). □

3. Quasilinear equation with linear boundary conditions

In this section, we study the nonlinear problem (1.1)–(1.4). In order to solve (1.1)–(1.4), we decompose v into $v = u + U^L$, where U^L is the solution of the linear equation (1.6)–(1.9) given in Lemma 2.1. We rewrite the equation (1.1) of u into

$$\frac{\partial u}{\partial t} + \frac{\partial U^L}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{1}{(1+(u_x+U_x^L)^2)^{1/2}}\frac{\partial}{\partial x}\left(\frac{u_{xx}+U_{xx}^L}{(1+(u_x+U_x^L)^2)^{3/2}}\right)\right]. \tag{3.1}$$

Recalling that $\partial U^L/\partial t = -\partial^4 U^L/\partial x^4$, we observe that (3.1) becomes

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[\frac{1}{(1+(u_x+U_x^L)^2)^{1/2}}\left(\frac{u_{xxx}+U_{xxx}^L}{(1+(u_x+U_x^L)^2)^{3/2}} - \frac{3(u_{xx}+U_{xx}^L)^2(u_x+U_x^L)}{(1+(u_x+U_x^L)^2)^{5/2}}\right)\right] \\
&\qquad + \frac{\partial^4 U^L}{\partial x^4} \\
&= -\frac{\partial}{\partial x}\left(\frac{1}{(1+(u_x+U_x^L)^2)^2}(u_{xxx}+U_{xxx}^L) - \frac{3(u_x+U_x^L)(u_{xx}+U_{xx}^L)^2}{(1+(u_x+U_x^L)^2)^3}\right) + \frac{\partial^4 U^L}{\partial x^4} \\
&= -\partial_x^4 u - \partial_x\left[\left(\frac{1}{(1+(u_x+U_x^L)^2)^2} - 1\right)(u_{xxx}+U_{xxx}^L)\right. \\
&\qquad \left. - \frac{3(u_x+U_x^L)(u_{xx}+U_{xx}^L)^2}{(1+(u_x+U_x^L)^2)^3}\right].
\end{aligned} \tag{3.2}$$

We define

$$h(p) = \frac{1}{(1+p^2)^2} - 1, \quad g(p,q) = \frac{3pq^2}{(1+p^2)^3},$$

and

$$\Xi(p,q,r) = h(p)r - g(p,q).$$

Then (3.2) becomes

$$\frac{\partial u}{\partial t} = -\partial_x^4 u - \partial_x\Big(\Xi(u_x+U_x^L, u_{xx}+U_{xx}^L, u_{xxx}+U_{xxx}^L)\Big).$$

3.1 *Hölder estimate for the perturbed term*

We shall estimate the term $\Xi(v_x, v_{xx}, v_{xxx})$. We estimate the Hölder norm of products and composite functions.

LEMMA 3.1 (i) Let $\gamma \in (0,1)$ and let $F \in C^1(\mathbf{R})$, i.e., F is a function of C^1-class defined on $\mathbf{R}$. Then

$$[\varphi\psi]_\gamma \leqslant |\varphi|_\infty[\psi]_\gamma + [\varphi]_\gamma|\psi|_\infty,$$
$$[F(\varphi)]_\gamma \leqslant \sup\{|F'(p)|; |p| \leqslant |\varphi|_\infty\}[\varphi]_\gamma,$$

for all $\varphi, \psi \in C^\gamma(\mathbf{R})$.

(ii) If $\varphi, \psi \in h^\gamma(\mathbf{R})$, then $\varphi\psi \in h^\gamma(\mathbf{R})$ and $F(\varphi) \in h^\gamma(\mathbf{R})$.

Proof. We observe that

$$\begin{aligned}[\varphi\psi]_{\gamma,x,y} &= \frac{|\varphi(x)(\psi(x)-\psi(y)) + \psi(y)(\varphi(x)-\varphi(y))|}{|x-y|^\gamma}\\ &\leqslant |\varphi(x)|[\psi]_{\gamma,x,y} + |\psi(y)|[\varphi]_{\gamma,x,y}.\end{aligned}$$

This yields the first inequality of Lemma 3.1 by taking the supremum over $x, y \in \mathbf{R}, x \neq y$. By the characterization of little Hölder space in Definition 2.2 (iii) the above estimate for $[\varphi\psi]_{\gamma,x,y}$ implies $\varphi\psi \in h^\gamma(\mathbf{R})$ if $\varphi, \psi \in h^\gamma(\mathbf{R})$.

Similarly, we have

$$\begin{aligned}[F(\varphi)]_{\gamma,x,y} &= \frac{|F(\varphi(x)) - F(\varphi(y))|}{|x-y|^\gamma}\\ &= \frac{\left|\left(\int_0^1 F'(\theta\varphi(x) + (1-\theta)\varphi(y))\,d\theta\right)(\varphi(x)-\varphi(y))\right|}{|x-y|^\gamma}\\ &\leqslant \int_0^1 |F'(\theta\varphi(x) + (1-\theta)\varphi(y))|\,d\theta[\varphi]_{\gamma,x,y}\\ &\leqslant \sup\{|F'(p)|; |p| \leqslant |\varphi|_\infty\}[\varphi]_{\gamma,x,y}.\end{aligned}$$

This yields the second inequality of Lemma 3.1 by taking the supremum over $x, y \in \mathbf{R}, x \neq y$. This estimate implies that $F(\varphi) \in h^\gamma(\mathbf{R})$ for $\varphi \in h^\gamma(\mathbf{R})$. □

PROPOSITION 3.2 There are constants $\widetilde{C}_1$ and $\widehat{C}_1$ such that

$$[h(v_x)v_{xxx}]_\gamma \leqslant \widetilde{C}_1\{|v_x|_\infty[v_{xxx}]_\gamma + [v_x]_\gamma|v_{xxx}|_\infty\},$$
$$[g(v_x, v_{xx})]_\gamma \leqslant \widehat{C}_1\{|v_x|_\infty[v_{xx}^2]_\gamma + [v_x]_\gamma|v_{xx}^2|_\infty\}.$$

holds for all $v \in C^{3+\gamma}(\mathbf{R})$. If $v \in h^{3+\gamma}(\mathbf{R})$, then $h(v_x)v_{xxx} \in h^\gamma(\mathbf{R})$ and $g(v_x, v_{xx}) \in h^\gamma(\mathbf{R})$.

Proof. By Lemma 3.1 (i) we have

$$\begin{aligned}[h(v_x)v_{xxx}]_\gamma &\leqslant |h(v_x)|_\infty[v_{xxx}]_\gamma + [h(v_x)]_\gamma|v_{xxx}|_\infty\\ &\leqslant \sup|h'|\cdot|v_x|_\infty\cdot[v_{xxx}]_\gamma + \sup|h'|\cdot[v_x]_\gamma\cdot|v_{xxx}|_\infty\\ &\leqslant \widetilde{C}_1\{|v_x|_\infty[v_{xxx}]_\gamma + [v_x]_\gamma|v_{xxx}|_\infty\}.\end{aligned}$$

Similarly by setting $g_1(v_x) := 3v_x/(1+v_x^2)^3$, we have

$$\begin{aligned}[g(v_x, v_{xx})]_\gamma &= [g_1(v_x)v_{xx}^2]_\gamma \\ &\leqslant |g_1(v_x)|_\infty [v_{xx}^2]_\gamma + [g_1(v_x)]_\gamma |v_{xx}^2|_\infty \\ &\leqslant \sup|g_1'| \cdot |v_x|_\infty \cdot [v_{xx}^2]_\gamma + \sup|g_1'| \cdot [v_x]_\gamma \cdot |v_{xx}^2|_\infty \\ &\leqslant \widehat{C}_1(|v_x|_\infty [v_{xx}^2]_\gamma + [v_x]_\gamma |v_{xx}^2|_\infty).\end{aligned}$$

The statement for $h(v_x)v_{xxx} \in h^\gamma(\mathbf{R})$ and $g(v_x, v_{xx}) \in h^\gamma(\mathbf{R})$ for $v \in h^{3+\gamma}(\mathbf{R})$ follows from Lemma 3.1 (ii). □

3.2 *Existence of a mild solution*

We shall construct a solution of an integral equation corresponding to (1.10)–(1.12). Let A be a closed operator corresponding to $-\partial_x^4$ in $BUC_{\text{even}}(\mathbf{R})$ so that e^{tA} is a C_0-analytic semigroup in $BUC_{\text{even}}(\mathbf{R})$ (Theorem 2.10). Let Ξ be as in the beginning of Section 3. Unfortunately, the term $\Xi(v_x, v_{xx}, v_{xxx})$ for $v = u + U^L$ may not attain zero at $x = 0$ because of the second order derivative of v_{xx} even for an even smooth function v. We introduce a modified odd extension operator $\wp$ as

$$\wp : h^\gamma(\overline{\mathbf{R}^+}) \to h^\gamma_{\text{odd}}(\mathbf{R}); \varphi \mapsto \begin{cases} \varphi(x) - \varphi(0), & \text{if } x \geqslant 0, \\ -(\varphi(-x) - \varphi(0)), & \text{if } x < 0, \end{cases}$$

so that $\partial_x(\wp\Xi)$ is an even function. This enables us to define $e^{tA}(\partial_x(\wp\Xi))$ as an even function. We are in position to state our main result.

THEOREM 3.3 Let U^L be the solution of the linear equation given in Lemma 2.1 depending on β. Let γ be in $(0, 1)$. Then there exist $\delta_0 > 0$ and $\beta_0 > 0$ independent of $T > 0$ such that if $\beta \in (0, \beta_0)$ there exists a unique

$$u \in BUC_{1-\alpha}(J, h^{3+\gamma}_{\text{even}}(\mathbf{R})) \cap L^\infty(J, BUC^1_{\text{even}}(\mathbf{R})),$$

with $\alpha = 1/2 - \gamma/4$ and $J = [0, T]$ which solves

$$u(t) = e^{tA}a - \int_0^t e^{(t-s)A}\partial_x\Big(\wp\Xi(u_x + U_x^L, u_{xx} + U_{xx}^L, u_{xxx} + U_{xxx}^L)\Big)(s)\,ds, \quad t \in J$$

for any $a \in BUC^1(\mathbf{R})$ with $\|a\|_{1,\infty} < \delta_0$. The solution u exists for all time interval. Moreover, there exists a constant $C = C(\gamma, \delta_0, \beta_0)$ such that

$$t^{1-\alpha}[u]_{3+\gamma}(t) + |u_x|_\infty(t) \leqslant C \quad \text{for all } t > 0.$$

If $a \equiv 0$, then u is self-similar in the sense that $u_\lambda = u$ for all $\lambda > 0$, where $u_\lambda(x,t) = \lambda^{-1}u(\lambda x, \lambda^4 t)$.

Proof. We first recall a characterization of the little Hölder space by a real interpolation space. We set

$$F_0 := BUC(\mathbf{R}), \quad F_1 := BUC^4(\mathbf{R}).$$

As discussed in the proof of Lemma 2.12 we observe that

$$C^{\gamma}(\mathbf{R}) = (F_0, F_1)_{\gamma/4,\infty}, \quad C^{3+\gamma}(\mathbf{R}) = (F_0, F_1)_{(3+\gamma)/4,\infty}$$

for $\gamma \in (0, 1)$, see [20, Proposition 2.2.2, Theorem 3.1.12]. A little Hölder space is characterized as a continuous interpolation space as

$$h^{\gamma}(\mathbf{R}) = (F_0, F_1)^0_{\gamma/4,\infty}, \quad h^{3+\gamma}(\mathbf{R}) = (F_0, F_1)^0_{(3+\gamma)/4,\infty}$$

for the definition of continuous interpolation spaces, see [8, Section 2] and [20, Definition 1.2.2, Definition 1.2.8]. This can be proved by a semigroup characterization of $(F_0, F_1)^0_{\theta,\infty}$, [20, Proposition 2.2.2].

We shall use the space of even functions, i.e., functions invariant under the transformation $f(z) \mapsto f(-z)$. Since interpolation commutes with this transformation, we observe that

$$(F_{0,\text{even}}, F_{1,\text{even}})_{\theta,\infty} = (F_0, F_1)_{\theta,\infty,\text{even}}.$$

In particular

$$h^{\gamma}_{\text{even}}(\mathbf{R}) = (F_{0,\text{even}}, F_{1,\text{even}})_{\gamma/4,\infty} = (F_0, F_1)_{\gamma/4,\infty,\text{even}}.$$

We next prepare a family of space-time functions. For positive constants $M, M_\infty > 0$ we set

$$\begin{aligned} \mathcal{Z}_{M,M_\infty}(J) := \{u \in BUC_{1-\alpha}(J, h^{3+\gamma}_{\text{even}}(\mathbf{R})) \cap L^\infty(J, BUC^1_{\text{even}}(\mathbf{R}));\\ u(0) = 0, \|u\|_{BUC_{1-\alpha}(J,h^{3+\gamma}_{\text{even}}(\mathbf{R}))} \leqslant M, \|u\|_{L^\infty(J,BUC^1_{\text{even}}(\mathbf{R}))} \leqslant M_\infty\}, \end{aligned} \tag{3.3}$$

equipped with the norm $\|u\|_{\mathcal{Z}_{M,M_\infty}(J)} := \max(\|u\|_{BUC_{1-\alpha}(J,h^{3+\gamma}_{\text{even}}(\mathbf{R}))}, \|u\|_{L^\infty(J,BUC^1_{\text{even}}(\mathbf{R}))})$. We then define a mapping Γ for $u \in \mathcal{Z}_{M,M_\infty}(J)$ as

$$\Gamma u := -\int_0^t e^{(t-s)A}\partial_x(\mathcal{P}\Xi(v_x, v_{xx}, v_{xxx}))(s)\, ds, \quad v = u + U^L. \tag{3.4}$$

Note that $\partial_x(\mathcal{P}\Xi)$ is even because $\mathcal{P}\Xi$ is odd. Apparently, $e^{(t-s)A}\partial_x(\mathcal{P}\Xi)$ is not well-defined for $v \in h^{3+\gamma}(\mathbf{R})$. We have to extend $e^{tA}\partial_x$ to the operator in $BUC(\mathbf{R})$. This can be done as follows. We first note that $e^{tA}\partial_x f = \partial_x e^{tA} f$ holds for $f \in BUC^1(\mathbf{R})$. Since

$$|\partial_x e^{tA} f|_\infty \leqslant \frac{C}{t^{1/4}}|f|_\infty$$

by Theorem 2.10 (i), this commutation formula can be extended for $f \in BUC(\mathbf{R})$ and $e^{tA}\partial_x f$ is well-defined for $f \in BUC(\mathbf{R})$ and $t > 0$.

We first prove Theorem 3.3 by assuming that the initial data a equals zero. We shall show that if M and M_∞ is chosen small, Γ maps $\mathcal{Z}_{M,M_\infty}(J)$ into itself and has a fixed point in $\mathcal{Z}_{M,M_\infty}(J)$ which implies the existence of a unique solution of (1.10)–(1.12).

Step 1 (Γ maps $\mathcal{Z}_{M,M_\infty}(J)$ into itself). For a fixed $T > 0$ we introduce the equivalent norms for the little Hölder spaces h^γ, $h^{3+\gamma}$ and BUC^1 by

$$\|f\|_{h^\gamma(\mathbf{R})} := \frac{|f|_\infty}{T^\theta} + [f]_\gamma, \tag{3.5}$$

$$\|f\|_{h^{3+\gamma}(\mathbf{R})} := \frac{|f|_\infty}{T^{3/4+\theta}} + \frac{|f'|_\infty}{T^{1/2+\theta}} + \frac{|f''|_\infty}{T^{1/4+\theta}} + \frac{|f'''|_\infty}{T^\theta} + [f''']_\gamma, \tag{3.6}$$

$$\|f\|_{BUC^1(\mathbf{R})} := \frac{|f|_\infty}{T^{1/4}} + |f'|_\infty, \tag{3.7}$$

where $|\cdot|$ is the L^∞-norm and $[\cdot]$ is the Hölder seminorm in Subsection 2.2. Our motivation to introduce the equivalent norms (3.5)–(3.7) (for the little Hölder space) is that we construct a global-in-time solution. In particular, we intend to have estimates with constant independent of T for solution in $(0,T)$. To do so, we would like to arrange so that the power of the time t (which shall appear in estimating the norm of the solution) is cancelled out. Thus, the definition of the equivalent norms (3.5)–(3.7) are quite reasonable. In fact, if one defines

$$m := \|U^L\|_{BUC_{1-\alpha}(J,h^{3+\gamma}(\overline{\mathbf{R}^+}))}, \quad m_\infty := \|U^L\|_{L^\infty(J,BUC^1(\overline{\mathbf{R}^+}))},$$

by using norms defined in (3.5)–(3.7) (with $\mathbf{R}$ replaced by $\overline{\mathbf{R}^+}$), by self-similarity of U^L, the constants m and m_∞ are independent of the choice of T. Moreover, from the explicit formula of U^L in Lemma 2.1, one can choose m and m_∞ sufficiently small by taking the contact angle β sufficiently small. We begin with

$$\begin{aligned}\|\Gamma u\|_{BUC_{1-\alpha}(J,h^{3+\gamma}_{\text{even}}(\mathbf{R}))} &= \sup_{t\in \dot{J}} t^{1-\alpha}\|\Gamma u\|_{h^{3+\gamma}(\mathbf{R})}\\ &= \frac{1}{T^{3/4+\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\Gamma u|_\infty + \frac{1}{T^{1/2+\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\partial_x\Gamma u|_\infty\\ &\quad + \frac{1}{T^{1/4+\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\partial_x^2\Gamma u|_\infty + \frac{1}{T^{\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\partial_x^3\Gamma u|_\infty\\ &\quad + \sup_{t\in\dot{J}} t^{1-\alpha}[\partial_x^3\Gamma u]_\gamma. \end{aligned}\tag{3.8}$$

To estimate (3.8) we use a seminorm

$$[w]_{F_\theta} := \sup_{r>0} r^{1-\theta}\|Ae^{rA}w\|_{F_0}, \quad \theta = \gamma/4,$$

which is equivalent to the seminorm $[w]_\gamma$ as proved in Lemma 2.12. First we shall calculate the last term of (3.8). The idea to estimate Γu is that we split the time integral into the two parts near the origin and t, i.e., integral over $(0,t/2)$ and $(t/2,t)$. After splitting the integral, we estimate the former part and the latter part respectively. The way to estimate these terms is similar to Da Prato–Grisvard and Angenent construction. A key step is to estimate $\int_{t/2}^t Ae^{(t-s)A}f\,ds$ (which is a kind of a *singular integral*) by using interpolation spaces. (Da Prato–Grisvard and Angenent have proven the maximal regularity result based on this technique in [9] and [1]).

$$\begin{aligned} &t^{1-\alpha}[\partial_x^3\Gamma u]_\gamma\\ &= t^{1-\alpha}\left[\int_0^t \partial_x^3 e^{(t-s)A}\partial_x(\wp\Xi(v_x,v_{xx},v_{xxx}))(s)\,ds\right]_\gamma\\ &= t^{1-\alpha}\left[\int_0^t \partial_x^4 e^{(t-s)A}\wp\Xi(v_x,v_{xx},v_{xxx})(s)\,ds\right]_\gamma\\ &\lesssim t^{1-\alpha}\left[\int_0^t Ae^{(t-s)A}\wp\Xi(v_x,v_{xx},v_{xxx})(s)\,ds\right]_{F_\theta}\\ &\leq t^{1-\alpha}\left\{\left[\int_0^{t/2} Ae^{(t-s)A}\wp\Xi(s)\,ds\right]_{F_\theta} + \left[\int_{t/2}^t Ae^{(t-s)A}\wp\Xi(s)\,ds\right]_{F_\theta}\right\}\end{aligned}$$

$$\begin{aligned}
&\lesssim t^{1-\alpha}\left\{\int_0^{t/2}\|Ae^{(t-s)A}\|_{\mathfrak{L}(F_\theta,F_\theta)}\cdot[\wp\Xi(s)]_{F_\theta}\,ds+\sup_{r>0}r^{1-\theta}\int_{t/2}^{t}\|A^2e^{(r+t-s)A}\wp\Xi(s)\|_{F_\theta}\,ds\right\}\\
&\lesssim t^{1-\alpha}\left\{\int_0^{t/2}(t-s)^{-1}[\wp\Xi(s)]_{F_\theta}\,ds\right.\\
&\quad\left.+\sup_{r>0}r^{1-\theta}\int_{t/2}^{t}\|A^2e^{(r+t-s)A}\|_{\mathfrak{L}(F_\theta,F_0)}\cdot[\wp\Xi(s)]_{F_\theta}\,ds\right\}\\
&\lesssim t^{1-\alpha}\left\{\int_0^{t/2}(t-s)^{-1}[\wp\Xi(s)]_\gamma\,ds+\sup_{r>0}r^{1-\theta}\int_{t/2}^{t}(r+t-s)^{-2+\theta}[\wp\Xi(s)]_\gamma\,ds\right\}. \qquad (3.9)
\end{aligned}$$

Here we use the symbol $\lesssim$ when we suppress a numerical constant C depending only on exponents. In other words, we simply write $a \lesssim b$ instead of $a \leqslant Cb$.

To estimate $[\wp\Xi(s)]_\gamma = [h(v_x)\partial_x^3 v - g(v_x, v_{xx}, v_{xxx})]_{h^\gamma(\overline{\mathbf{R}^+})}$ we shall use Proposition 3.2 in $\overline{\mathbf{R}^+}$ and the interpolation inequality between BUC^1-norm and $h^{3+\gamma}$-seminorm as

$$\begin{aligned}
[\varphi']_\gamma &\lesssim [\varphi]_{1+\gamma} \lesssim |\varphi|_{1,\infty}^{2/(2+\gamma)}[\varphi]_{3+\gamma}^{\gamma/(2+\gamma)},\\
[(\varphi'')^2]_{h^\gamma} &\leqslant 2|\varphi''|_\infty[\varphi'']_\gamma \lesssim |\varphi|_{2,\infty}\cdot[\varphi]_{2+\gamma}\\
&\lesssim |\varphi|_{1,\infty}^{(1+\gamma)/(2+\gamma)}[\varphi]_{3+\gamma}^{1/(2+\gamma)}\cdot|\varphi|_{1,\infty}^{1/(2+\gamma)}[\varphi]_{3+\gamma}^{(1+\gamma)/(2+\gamma)}\\
&= c|\varphi|_{1,\infty}[\varphi]_{3+\gamma}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
[h(v_x)v_{xxx}]_\gamma &\leqslant \widetilde{C}_1(|v_x|_\infty[v_{xxx}]_\gamma+[v_x]_\gamma|v_{xxx}|_\infty)\\
&\lesssim \widetilde{C}_1(|v|_{1,\infty}[v]_{3+\gamma}+|v|_{1,\infty}^{2/(2+\gamma)}[v]_{3+\gamma}^{\gamma/(2+\gamma)}\cdot|v|_{1,\infty}^{\gamma/(2+\gamma)}[v]_{3+\gamma}^{2/(2+\gamma)})\\
&\leqslant \widetilde{C}_1\left((M_\infty+m_\infty)(s^{1-\alpha}[v]_{3+\gamma})s^{\alpha-1}+|v|_{1,\infty}(s^{1-\alpha}[v]_{3+\gamma})s^{\alpha-1}\right)\\
&\leqslant \widetilde{C}_1\left((M_\infty+m_\infty)(M+m)s^{\alpha-1}+(M_\infty+m_\infty)(M+m)s^{\alpha-1}\right)\\
&= 2\widetilde{C}_1(M_\infty+m_\infty)(M+m)s^{\alpha-1}, \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
[g(v_x, v_{xx})]_\gamma &\leqslant \widehat{C}_1\{|v_x|_\infty[v_{xx}^2]_\gamma+[v_x]_\gamma|v_{xx}|^2\}\\
&\lesssim \widehat{C}_1\{|v|_{1,\infty}\cdot|v|_{1,\infty}[v]_{3+\gamma}+|v|_{1,\infty}^{2/(2+\gamma)}[v]_{3+\gamma}^{\gamma/(2+\gamma)}\cdot|v|_{1,\infty}^{2(1+\gamma)/(2+\gamma)}[v]_{3+\gamma}^{2/(2+\gamma)}\}\\
&= \widehat{C}_1\{|v|_{1,\infty}^2[v]_{3+\gamma}+|v|_{1,\infty}^2[v]_{3+\gamma}\}\\
&= 2\widehat{C}_1\left(|v|_{1,\infty}^2(s^{1-\alpha}[v]_{3+\gamma})s^{\alpha-1}\right)\\
&= 2\widehat{C}_1(M_\infty+m_\infty)^2(M+m)s^{\alpha-1}. \qquad (3.11)
\end{aligned}$$

In (3.10) and (3.11) all norms should be interpreted as a norm over $\overline{\mathbf{R}^+}$ not $\mathbf{R}$ since U^L is defined in $\overline{\mathbf{R}^+}$.

Now we are in position to estimate (3.9). We need to estimate the integral

$$r^{1-\theta}t^{1-\alpha}\int_{t/2}^{t}(r+t-s)^{-2+\theta}s^{\alpha-1}\,ds.$$

Since $s^{\alpha-1} \leqslant (t/2)^{\alpha-1}$ for $s \in [t/2, t]$ we have

$$
\begin{aligned}
r^{1-\theta} t^{1-\alpha} \int_{t/2}^{t} (r+t-s)^{-2+\theta} s^{\alpha-1}\, ds &\leqslant r^{1-\theta} t^{1-\alpha} \left(\frac{t}{2}\right)^{\alpha-1} \int_{t/2}^{t} (r+t-s)^{-2+\theta}\, ds \\
&= \frac{2^{1-\alpha} r^{1-\theta}}{1-\theta} \Big[(r+t-s)^{-1+\theta} \Big]_{s=t/2}^{s=t} \\
&= \frac{2^{1-\alpha}}{1-\theta} r^{1-\theta} \left(r^{-1+\theta} - \left(r + \frac{t}{2}\right)^{-1+\theta} \right) \\
&\leqslant \frac{2^{1-\alpha}}{1-\theta} r^{1-\theta} \cdot r^{-1+\theta} = \frac{2^{1-\alpha}}{1-\theta}. \qquad (3.12)
\end{aligned}
$$

From (3.10)–(3.12), we have

$$
\begin{aligned}
&(3.9) \\
&\lesssim t^{1-\alpha} \Bigg\{ \int_0^{t/2} (t-s)^{-1} [\widetilde{C}_1 (M_\infty + m_\infty)(M+m) s^{\alpha-1} + \widehat{C}_1 (M_\infty + m_\infty)^2 (M+m) s^{\alpha-1}]\, ds \\
&\quad + \sup_{r>0} r^{1-\theta} \int_{t/2}^{t} (r+t-s)^{-2+\theta} [\widetilde{C}_1 (M_\infty + m_\infty)(M+m) s^{\alpha-1} \\
&\qquad\qquad + \widehat{C}_1 (M_\infty + m_\infty)^2 (M+m) s^{\alpha-1}\, ds] \Bigg\} \\
&\lesssim \left(\int_0^{1/2} (1-\sigma)^{-1} \sigma^{\alpha-1}\, d\sigma \right) [\widetilde{C}_1 (M_\infty + m_\infty)(M+m) + \widehat{C}_1 (M_\infty + m_\infty)^2 (M+m)] \\
&\quad + \frac{2^{1-\alpha}}{1-\theta} [\widetilde{C}_1 (M_\infty + m_\infty)(M+m) + \widehat{C}_1 (M_\infty + m_\infty)^2 (M+m)]. \qquad (3.13)
\end{aligned}
$$

Estimating for the other terms of (3.8) proceeds similarly. The L^∞-norm estimate for the integrand $t^{1-\alpha} \int_0^t \partial_x^k e^{(t-s)A} \partial_x (\wp \Xi)\, ds\, (k = 0, 1, 2, 3)$ is different. For example, we estimate the fourth terms of (3.8) as

$$
\begin{aligned}
t^{1-\alpha} \left| \int_0^t \partial_x^3 e^{(t-s)A} \partial_x (\wp \Xi)\, ds \right|_\infty &= t^{1-\alpha} \left| \int_0^t \partial_x^4 e^{(t-s)A} \wp \Xi\, ds \right|_\infty \\
&\leqslant t^{1-\alpha} \int_0^t \| A e^{(t-s)A} \wp \Xi \|_{F_0}\, ds \\
&\leqslant t^{1-\alpha} \int_0^t \| A e^{(t-s)A} \|_{\mathfrak{L}(F_\theta, F_0)} [\wp \Xi]_{F_\theta}\, ds \\
&\leqslant C t^{1-\alpha} \left(\int_0^t (t-s)^{-1+\theta} [\wp \Xi]_{F_\theta}\, ds \right). \qquad (3.14)
\end{aligned}
$$

The term $[\wp \Xi]_{F_\theta}$ is estimated by constants (depending on M and M_∞) times $s^{\alpha-1}$. Thus, taking supremum of (3.14) in $(0, T)$ yields T^θ, which cancels out the term $1/T^\theta$ in (3.8). We also recall that from (3.7)

$$
\| \Gamma u \|_{L^\infty(J, BUC^1(\mathbf{R}))} = \frac{1}{T^{1/4}} \sup_{t \in \dot{J}} |\Gamma u|_\infty + \sup_{t \in \dot{J}} |\partial_x \Gamma u|_\infty. \qquad (3.15)
$$

Estimating (3.15) proceeds similarly as above. If m and m_∞ are taken sufficiently small, then, with a suitable choices of M and M_∞, we can show that

$$\max(\|\Gamma u\|_{BUC_{1-\alpha}(J,h^{3+\gamma}_{\text{even}}(\mathbf{R}))}, \|\Gamma u\|_{L^\infty(J,BUC^1_{\text{even}}(\mathbf{R}))}) \leqslant \max(M, M_\infty).$$

The evenness of Γu is easy since $e^{tA}\partial_x f$ for odd f is even. Thus, we have shown that $\Gamma(\mathcal{Z}_{M,M_\infty}(J)) \subset \mathcal{Z}_{M,M_\infty}(J)$. In particular Γu is well-defined. The smallness of M, M_∞, m, m_∞ are independent of $T > 0$ since all constants appearing in our estimate is independent of $T > 0$.

Step 2 (Γ is a contraction on $\mathcal{Z}_{M,M_\infty}(J)$). We take $u_i \in \mathcal{Z}_{M,M_\infty}(J)$ $(i = 1, 2)$. Let $v_i := u_i + U^L$ $(i = 1, 2)$ and

$$\Xi_i\big((v_i)_x, (v_i)_{xx}, (v_i)_{xxx}\big) := h\big((v_i)_x\big)(v_i)_{xxx} - g\big((v_i)_x, (v_i)_{xx}\big) \ (i = 1, 2).$$

Then we have

$$\Gamma u_1 - \Gamma u_2 = -\int_0^t e^{(t-s)A}\partial_x\big(\mathcal{P}(\Xi_1 - \Xi_2)\big)(s)\, ds,$$

where

$$\begin{aligned}\Xi_1 - \Xi_2 &= h\big((v_1)_x\big)(v_1)_{xxx} - g\big((v_1)_x, (v_1)_{xx}\big)\\ &\qquad - h\big((v_2)_x\big)(v_2)_{xxx} + g\big((v_2)_x, (v_2)_{xx}\big)\\ &= h\big((v_1)_x\big)(u_1 - u_2)_{xxx} + \big[h\big((v_1)_x\big) - h\big((v_2)_x\big)\big](v_2)_{xxx}\\ &\qquad - \big[g\big((v_1)_x, (v_1)_{xx}\big) - g\big((v_2)_x, (v_2)_{xx}\big)\big]\\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.16}$$

Now we estimate the right-hand side of (3.16) with respect to Hölder seminorm respectively. The argument, however, proceeds similarly as in Step 1, therefore we leave the detailed computations to the reader.

$$\begin{aligned}[I_1]_\gamma &= \big[h\big((v_1)_x\big)(u_1 - u_2)_{xxx}\big]_\gamma\\ &\leqslant \widetilde{C}_2\big\{|(v_1)_x|_\infty[(u_1 - u_2)_{xxx}]_\gamma + [(v_1)_x]_\gamma|(u_1 - u_2)_{xxx}|_\infty\big\}. \end{aligned} \tag{3.17}$$

Next, by the fundamental theorem of calculus, we observe that

$$\begin{aligned} I_2 &= \Big(h\big((v_1)_x\big) - h\big((v_2)_x\big)\Big)(v_2)_{xxx}\\ &= \left(\int_0^1 h'(\theta(u_1)_x + (1-\theta)(u_2)_x + U^L_x)\, d\theta\right)(u_1 - u_2)_x(v_2)_{xxx}, \end{aligned} \tag{3.18}$$

where $h'(v) = 6v/(1 + v^2)^4$.

To estimate (3.18) with respect to Hölder seminorm, we use Lemma 3.1 to get

$$[\varphi_1\varphi_2\varphi_3]_\gamma \leqslant [\varphi_1]_\gamma|\varphi_2|_\infty|\varphi_3|_\infty + |\varphi_1|_\infty[\varphi_2]_\gamma|\varphi_3|_\infty + |\varphi_1|_\infty|\varphi_2|_\infty[\varphi_3]_\gamma. \tag{3.19}$$

From (3.18) and (3.19) we have

$$[I_2]_\gamma = [(h((v_1)_x) - h((v_2)_x))(v_2)_{xxx}]_\gamma$$

$$
\begin{aligned}
&\leqslant \widetilde{C}_3\Big\{\Big(\int_0^1 [h'(\theta(u_1)_x + (1-\theta)(u_2)_x + U_x^L)]_\gamma \, d\theta\Big)|(u_1-u_2)_x|_\infty |(v_2)_{xxx}|_\infty \\
&\quad + \Big(\int_0^1 |h'(\theta(u_1)_x + (1-\theta)(u_2)_x + U_x^L)|_\infty \, d\theta\Big)[(u_1-u_2)_x]_\gamma |(v_2)_{xxx}|_\infty \\
&\quad + \Big(\int_0^1 |h'(\theta(u_1)_x + (1-\theta)(u_2)_x + U_x^L)|_\infty \, d\theta\Big)|(u_1-u_2)_x|_\infty [(v_2)_{xxx}]_\gamma\Big\} \\
&\leqslant \widetilde{C}_3\Big\{\big([u_1]_{1+\gamma} + [u_2]_{1+\gamma} + [U^L]_{1+\gamma}\big)\,|u_1-u_2|_{1,\infty}|v_2|_{3,\infty} \\
&\quad + \big(|u_1|_{1,\infty} + |u_2|_{1,\infty} + |U^L|_{1,\infty}\big)\,[u_1-u_2]_{1+\gamma}|v_2|_{3,\infty} \\
&\quad + \big(|u_1|_{1,\infty} + |u_2|_{1,\infty} + |U^L|_{1,\infty}\big)\,|u_1-u_2|_{1,\infty}[v_2]_{3+\gamma}\Big\}. \qquad (3.20)
\end{aligned}
$$

The term $[I_3]_\gamma$ is similarly estimated. In fact, by the fundamental theorem of calculus, we observe that

$$
\begin{aligned}
g(p_1,q_1) - g(p_2,q_2) &= \Big(\int_0^1 D_1 g(\theta p_1 + (1-\theta)p_2, \theta q_1 + (1-\theta)q_2)\, d\theta\Big)(p_1-p_2) \\
&\quad + \Big(\int_0^1 D_2 g(\theta p_1 + (1-\theta)p_2, \theta q_1 + (1-\theta)q_2)\, d\theta\Big)(q_1-q_2), \qquad (3.21)
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 g &= \frac{3(1-5p^2)q^2}{(1+p^2)^4}, \qquad & D_2 g &= \frac{6pq}{(1+p^2)^3}, \\
p_i &= (u_i + U^L)_x, & q_i &= (u_i + U^L)_{xx},\ (i = 1,2).
\end{aligned}
$$

We estimate the right-hand side of (3.21) with respect to Hölder seminorm.

$$
\begin{aligned}
\big[(D_1 g)(p_1-p_2)\big]_\gamma &\leqslant [D_1 g]_\gamma |p_1-p_2|_\infty + |D_1 g|_\infty [p_1-p_2]_\gamma \\
&\leqslant \left|\frac{3(1-5p^2)}{(1+p^2)^4}\right|_\infty [q^2]_\gamma |p_1-p_2|_\infty + \left[\frac{3(1-5p^2)}{(1+p^2)^4}\right]_\gamma |q^2|_\infty |p_1-p_2|_\infty \\
&\qquad + \left|\frac{3(1-5p^2)}{(1+p^2)^4}\right|_\infty |q^2|_\infty [p_1-p_2]_\gamma \\
&\leqslant \widehat{C}_2\{|q|_\infty [q]_\gamma |p_1-p_2|_\infty + [p]_\gamma |q|_\infty^2 |p_1-p_2|_\infty \\
&\qquad + |q|_\infty^2 [p_1-p_2]_\gamma\}, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
[(D_2 g)(q_1-q_2)]_\gamma &\leqslant [D_2 g]_\gamma |q_1-q_2|_\infty + |D_2 g|_\infty [q_1-q_2]_\gamma \\
&\leqslant \left|\frac{6p}{(1+p^2)^3}\right|_\infty [q]_\gamma |q_1-q_2|_\infty + \left[\frac{6p}{(1+p^2)^3}\right]_\gamma |q|_\infty [q_1-q_2]_\gamma \\
&\qquad + \left|\frac{6p}{(1+p^2)^3}\right|_\infty |q|_\infty [q_1-q_2]_\gamma \\
&\leqslant \widehat{C}_3\{[q]_\gamma |q_1-q_2|_\infty + [p]_\gamma |q|_\infty [q_1-q_2]_\gamma + |q|_\infty [q_1-q_2]_\gamma\}. \qquad (3.23)
\end{aligned}
$$

The estimates (3.22) and (3.23) yield that

$$\begin{aligned}
[I_3]_\gamma \leqslant \widehat{C}_2\Big\{&\big(|(u_1)_{xx}|_\infty + |(u_2)_{xx}|_\infty + |U^L_{xx}|_\infty\big)\\
&\times \big([(u_1)_{xx}]_\gamma + [(u_2)_{xx}]_\gamma + [U^L_{xx}]_\gamma\big)\,|u_1 - u_2|_\infty\\
&+ \big([(u_1)_x]_\gamma + [(u_2)_x]_\gamma + [U^L_x]_\gamma\big)\\
&\times \big(|(u_1)_{xx}|^2_\infty + |(u_2)_{xx}|^2_\infty + |U^L_{xx}|^2_\infty\big)\,|u_1 - u_2|_\infty\\
&\big(|(u_1)_{xx}|^2_\infty + |(u_2)_{xx}|^2_\infty + |U^L_{xx}|^2_\infty\big)\,[u_1 - u_2]_\gamma\Big\}\\
&+ \widehat{C}_3\Big\{\big([(u_1)_{xx}]_\gamma + [(u_2)_{xx}]_\gamma + [U^L_{xx}]_\gamma\big)\,|(u_1)_{xx} - (u_2)_{xx}|_\infty\\
&+ \big([(u_1)_x]_\gamma + [(u_2)_x]_\gamma + [U^L_x]_\gamma\big)\big(|(u_1)_{xx}|_\infty + |(u_2)_{xx}|_\infty + |U^L_{xx}|_\infty\big)\\
&\times [(u_1)_{xx} - (u_2)_{xx}]_\gamma\\
&+ \big(|(u_1)_{xx}|_\infty + |(u_2)_{xx}|_\infty + |U^L_{xx}|_\infty\big)\,[(u_1)_{xx} - (u_2)_{xx}]_\gamma\Big\}. \qquad (3.24)
\end{aligned}$$

By (3.17), (3.20) and (3.24) the term $\sup_{t\in \dot{J}} t^{1-\alpha}[\partial_x^3(\Gamma u_1 - \Gamma u_2)]_\gamma$ is estimated by

$$\begin{aligned}
\sup_{t\in\dot{J}} t^{1-\alpha}[\partial_x^3(\Gamma u_1 - \Gamma u_2)]_{h^\gamma} \leqslant &\left(\int_0^{1/2}(1-\sigma)^{-1}\sigma^{\alpha-1}\,d\sigma + \frac{2^{1-\alpha}}{1-\theta}\right)\\
&\times \Big\{\widetilde{C}_2\Big[(M_\infty + m_\infty) + (M_\infty + m_\infty)^{2/(2+\gamma)}(M+m)^{\gamma/(2+\gamma)}\Big]\\
&+ \widetilde{C}_3\big[(2M_\infty^{2/(2+\gamma)}M^{\gamma/(2+\gamma)} + m_\infty^{2/(2+\gamma)}m^{\gamma/(2+\gamma)})(M_\infty + m_\infty)^{\gamma/(2+\gamma)}(M+m)^{2/(2+\gamma)}\\
&+ (2M_\infty + m_\infty)(M_\infty + m_\infty)^{\gamma/(2+\gamma)}(M+m)^{2/(2+\gamma)} + (2M_\infty + m_\infty)(M+m)\big]\\
&+ \widehat{C}_2\big[(2M_\infty^{(1+\gamma)/(2+\gamma)}M^{1/(2+\gamma)} + m_\infty^{(1+\gamma)/(2+\gamma)}m^{1/(2+\gamma)})\\
&\times (2M_\infty^{1/(2+\gamma)}M^{(1+\gamma)/(2+\gamma)} + m_\infty^{1/(2+\gamma)}m^{(1+\gamma)/(2+\gamma)})\\
&+ (2M_\infty^{2/(2+\gamma)}M^{\gamma/(2+\gamma)} + m_\infty^{2/(2+\gamma)}m^{\gamma/(2+\gamma)})\\
&\times (2M_\infty^{2(1+\gamma)/(2+\gamma)}M^{2/(2+\gamma)} + m_\infty^{2(1+\gamma)/(2+\gamma)}m^{2/(2+\gamma)})\\
&+ (2M_\infty^{2(1+\gamma)/(2+\gamma)}M^{2/(2+\gamma)} + m_\infty^{2(1+\gamma)/(2+\gamma)}m^{1/(2+\gamma)})\big]\\
&+ \widehat{C}_3\big[(2M_\infty^{1/(2+\gamma)}M^{(1+\gamma)/(2+\gamma)} + m_\infty^{1/(2+\gamma)}m^{(1+\gamma)/(2+\gamma)})\\
&+ (2M_\infty^{2/(2+\gamma)}M^{\gamma/(2+\gamma)} + m_\infty^{2/(2+\gamma)}m^{\gamma/(2+\gamma)})\\
&\times (2M_\infty^{(1+\gamma)/(2+\gamma)}M^{1/(2+\gamma)} + m_\infty^{(1+\gamma)/(2+\gamma)}m^{1/(2+\gamma)})\\
&+ (2M_\infty^{(1+\gamma)/(2+\gamma)}M^{1/(2+\gamma)} + m_\infty^{(1+\gamma)/(2+\gamma)}m^{1/(2+\gamma)})\big]\Big\}. \qquad (3.25)
\end{aligned}$$

The estimates for the other terms

$$\frac{1}{T^\theta}\sup_{t\in\dot{J}} t^{1-\alpha}|\partial_x^3(\Gamma u_1 - \Gamma u_2)|_\infty, \quad \frac{1}{T^{1/4+\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\partial_x^2(\Gamma u_1 - \Gamma u_2)|_\infty,$$
$$\frac{1}{T^{1/2+\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\partial_x(\Gamma u_1 - \Gamma u_2)|_\infty, \quad \frac{1}{T^{3/4+\theta}}\sup_{t\in\dot{J}} t^{1-\alpha}|\Gamma(u_1 - u_2)|_\infty$$

proceed similarly. Thus, we have actually estimated $\|\Gamma u_1 - \Gamma u_2\|_{BUC_{1-\alpha}(J,h^{3+\gamma}_{\text{even}}(\mathbf{R}))}$. The estimate of $\|\Gamma u_1 - \Gamma u_2\|_{L^\infty(J,BUC^1_{\text{even}}(\mathbf{R}))}$ also proceed similarly as above.

The smallness of the constants m, m_∞ and our suitable choices of the constants M, M_∞ shows that $\|\Gamma u_1 - \Gamma u_2\|_{\mathcal{Z}_{M,M_\infty}(J)}$ is less than or equal to $(1/2)\|u_1 - u_2\|_{\mathcal{Z}_{M,M_\infty}(J)}$, which concludes that Γ is a contraction on $\mathcal{Z}_{M,M_\infty}(J)$. By invoking the Banach contraction mapping principle, there exists a unique solution u of $u = \Gamma u$ in $\mathcal{Z}_{M,M_\infty}(J)$. The rescaled function u_λ also satisfies (1.10)–(1.12). By uniqueness, we conclude that $u = u_\lambda$ which implies u is self-similar.

The proof proceeds similarly in the case of $a \not\equiv 0$. What we have to do for example is to estimate $t^{1-\alpha}[\partial_x^3 e^{tA} a]_\gamma$.

$$
\begin{aligned}
t^{1-\alpha}[\partial_x^3 e^{tA} a]_\gamma &= t^{1-\alpha}[A^{1/2} e^{tA} \partial_x a]_\gamma \\
&\leqslant t^{1-\alpha} \|A^{1/2} e^{tA}\|_{\mathfrak{L}(F_0,F_\theta)} \cdot \|a'\|_{F_0} \\
&\leqslant t^{1-\alpha} \cdot t^{-1/2-\theta} \|a'\|_{F_0} \\
&= \|a'\|_{F_0}.
\end{aligned}
\tag{3.26}
$$

In the last line of (3.26), we use the relation $\alpha = 1/2 - \gamma/4$. Thus, if $\|a'\|_{F_0} = |a'|_\infty$ is sufficiently small, we can arrange the quantity $t^{1-\alpha}[\partial_x^3 e^{tA} a]_\gamma$ is small. In fact, we can prove that

$$
\|e^{tA} a\|_{BUC_{1-\alpha}(J,h^{3+\gamma}_{\text{even}}(\mathbf{R}))} + \|e^{tA} a\|_{L^\infty(J,BUC^1(\mathbf{R}))} \leqslant C\|a\|_{1,\infty}.
$$

Hence, we can show the unique existence of the solution of $u = e^{tA} a + \Gamma u$ as before when $\|a\|_{1,\infty}$ is small. The desired estimate follows by construction. □

It must be noted that u is not differentiable with respect to the time t with values in F_0. For this reason we shall define the term *weak solution* to prove that u solves formally (3.2) at least $\mathbf{R}^+ \times J$ (so that $v = u + U^L$ is the desired solution).

DEFINITION 3.4 (Weak solution) We say that $v \in BUC_{1-\alpha}(J, h^{3+\gamma}(\overline{\mathbf{R}^+})) \cap L^\infty(J, BUC^1(\overline{\mathbf{R}^+}))$ is a weak solution of the problem (1.1)–(1.3) if

$$
\int_0^T \langle v, \varphi_t \rangle \, dt = -\langle \varphi(x,0), a \rangle - \int_0^T \left\langle \partial_x \varphi, \frac{1}{(1+v_x^2)^{1/2}} \frac{\partial}{\partial x} \left(\frac{v_{xx}}{(1+v_x^2)^{3/2}} \right) \right\rangle dt \tag{3.27}
$$

for all $\varphi \in C_c^\infty(\mathbf{R}^+ \times [0,T))$ and satisfies (1.2) and (1.3). Here we denote $\langle \cdot, \cdot \rangle$ by the canonical pair. If $f, g \in L^2(\mathbf{R}^+)$, then $\langle f, g \rangle = \int_0^\infty f g \, dx$.

LEMMA 3.5 Let u be the solution which we constructed in Theorem 3.3. Then $v = u + U^L$ is a weak solution.

Proof. Recall that

$$
u = e^{tA} a - \int_0^t e^{(t-s)A} \partial_x \mathcal{P} \Xi(s) \, ds. \tag{3.28}
$$

Multiplying φ_t to the both sides of (3.28) and integrating over $\overline{\mathbf{R}^+} \times [0,T]$ we have

$$
\int_0^T \langle u, \varphi_t \rangle \, dt = \int_0^T \langle e^{tA}, \varphi_t \rangle \, dt - \int_0^T \left\langle \varphi_t, \int_0^t e^{(t-s)A} \partial_x \mathcal{P} \Xi(s) \, ds \right\rangle dt. \tag{3.29}
$$

We calculate the right-hand side of (3.29) respectively. The first term can be calculated as follows.

$$\begin{aligned}\int_0^T \langle e^{tA}a, \varphi_t\rangle\, dt &= \int_0^T \int_0^\infty e^{tA}a\varphi_t\, dx\, dt\\ &= \int_0^\infty \int_0^T e^{tA}a\varphi_t\, dt\, dx\\ &= \int_0^\infty \Big\langle -\int_0^T \partial_t(e^{tA}a)\varphi\, dt + \Big[e^{tA}a\varphi\Big]_{t=0}^{t=T}\Big\rangle\, dx\\ &= -\int_0^\infty \int_0^T Ae^{tA}a\varphi\, dt\, dx - \int_0^\infty \varphi(x,0)a\, dx\\ &= -\int_0^T \langle A\varphi, e^{tA}a\rangle\, dt - \langle \varphi(x,0), a\rangle. \end{aligned} \tag{3.30}$$

To calculate the second term note that A is self-adjoint, i.e., $A^* = A$. We shall transfer the semigroup e^{rA} in the coupling $\langle\cdot,\cdot\rangle$. Thanks to the self-adjointness we can actually transfer the semigroup e^{rA}.

$$\begin{aligned}&-\int_0^T \Big\langle \varphi_t, \int_0^t e^{(t-s)A}\partial_x \wp \Xi(s)\, ds\Big\rangle\, dt\\ &\quad= -\int_0^T \int_0^t \langle \varphi_t, e^{(t-s)A}\partial_x \wp \Xi(s)\rangle\, ds\, dt\\ &\quad= -\int_0^T \int_0^t \langle e^{(t-s)A^*}\varphi_t, \partial_x \wp \Xi(s)\rangle\, ds\, dt\\ &\quad= -\int_0^T \int_0^t \langle e^{(t-s)A}\varphi_t, \partial_x \wp \Xi(s)\rangle\, ds\, dt\\ &\quad= -\int_0^T \int_s^T \langle e^{(t-s)A}\varphi_t, \partial_x \wp \Xi(s)\rangle\, dt\, ds\\ &\quad= -\int_0^T \Big(-\int_s^T \langle \partial_t(e^{(t-s)A})\varphi, \partial_x \wp \Xi(s)\rangle\, dt + \Big[\langle e^{(t-s)A}\varphi, \partial_x \wp \Xi(s)\rangle\Big]_{t=s}^{t=T}\Big)\, ds\\ &\quad= \int_0^T \int_s^T \langle Ae^{(t-s)A}\varphi, \partial_x \wp \Xi(s)\rangle\, dt\, ds + \int_0^T \langle \varphi(s,x), \partial_x \wp \Xi(s)\rangle\, ds\\ &\quad= \int_0^T \int_0^t \langle A\varphi, e^{(t-s)A}\partial_x \wp \Xi(s)\rangle\, ds\, dt - \int_0^T \langle \partial_x\varphi, \wp \Xi\rangle\, dt\\ &\quad= \int_0^T \langle A\varphi, -u + e^{tA}a\rangle\, dt - \int_0^T \langle \partial_x\varphi, \wp \Xi\rangle\, dt. \end{aligned} \tag{3.31}$$

From (3.30) and (3.31) we get

$$\begin{aligned}\int_0^T \langle u, \varphi_t\rangle\, dt = -\int_0^T \langle A\varphi, e^{tA}a\rangle\, dt - \langle \varphi(x,0), a\rangle + \int_0^T \langle A\varphi, -u + e^{tA}a\rangle\, dt\\ - \int_0^T \langle \partial_x\varphi, \wp \Xi\rangle\, dt\end{aligned}$$

$$= -\int_0^T \langle A\varphi, u\rangle\, dt - \langle \varphi(x,0), a\rangle + \int_0^T \langle \partial_x \varphi, \wp \Xi\rangle\, dt$$

$$= -\int_0^T \langle \partial_x \varphi, \partial_x^3 u\rangle\, dt - \int_0^T \langle \partial_x \varphi, \wp \Xi\rangle\, dt - \langle \varphi(x,0), a\rangle. \tag{3.32}$$

On the other hand we calculate $\int_0^T \langle U^L, \varphi_t\rangle\, dt$ as

$$\int_0^T \langle U^L, \varphi_t\rangle\, dt = -\int_0^T \langle \partial_t U^L, \varphi\rangle\, dt$$

$$= \int_0^T \langle \partial_x^4 U^L, \varphi\rangle\, dt = -\int_0^T \langle \partial_x \varphi, \partial_x^3 U^L\rangle\, dt. \tag{3.33}$$

Summing up (3.32) and (3.33), we have

$$\int_0^T \langle v, \varphi_t\rangle\, dt = -\langle \varphi(x,0), a\rangle - \int_0^T \langle \partial_x \varphi, \partial_x^3 (u + U^L)\rangle\, dt - \int_0^T \langle \partial_x \varphi, \wp \Xi\rangle\, dt$$

$$= -\langle \varphi(x,0), a\rangle$$

$$- \int_0^T \left\langle \partial_x \varphi, \frac{1}{(1 + (u + U^L)_x^2)^2} \partial_x^3 (u + U^L) - \frac{3(u + U^L)_x (u + U^L)_{xx}^2}{(1 + (u + U^L)_x^2)^3} \right\rangle dt$$

$$= -\langle \varphi(x,0), a\rangle - \int_0^T \left\langle \partial_x \varphi, \frac{1}{(1 + v_x^2)^{1/2}} \frac{\partial}{\partial x} \left(\frac{v_{xx}}{(1 + v_x^2)^{3/2}} \right) \right\rangle dt.$$

Here the operator $\wp$ disappears since $\int_0^T \langle \partial_x \varphi, c\rangle\, dt = 0$ for any constant c. □

REMARK 3.6 (i) It is likely that the constructed solution u is smooth by using linear parabolic theory for higher order equation. The fuller study of the regularity of a general solution lies outside the scope of this paper. If $a \equiv 0$, then u is self-similar so that $v = u + U^L$ is self-similar. The self-similar solution v is a solution of ODE so it must be smooth.

(ii) The estimate (3.12) is similar to that in [1, Theorem 2.14] by Angenent. We have given a simple proof.

3.3 *Stability of a self-similar solution*

In this subsection we discuss the stability of a self-similar solution. Let u be the mild solution which we constructed in Theorem 3.3. For $\lambda > 0$, we set

$$v_\lambda(x,t) := \frac{1}{\lambda} v(\lambda x, \lambda^4 t), \quad a_\lambda(x) := \frac{1}{\lambda} a(\lambda x)$$

Since v is the solution of the problem (1.1)–(1.3), v_λ is also the solution of the problem (1.1)–(1.3) with the initial data a_λ.

THEOREM 3.7 The rescaled u_λ uniformly converges to $\tilde{u}$ as $\lambda \to \infty$ on any compact sets in $(0,T] \times \mathbf{R}$, that is, $v_\lambda = u_\lambda + U^L$ uniformly converges to the self-similar solution $\tilde{v} := \tilde{u} + U^L$. In particular, $t^{-1/4} v(t^{1/4} x, t) \to Z(x)$ as $t \to \infty$ locally uniformly in $[0, \infty)$ when Z is the profile function of $\tilde{v}$.

To show Theorem 3.7, we need a preliminary lemma. For the definition of the weighted Hölder space $C^{\alpha}_{\beta}((0,T],E)$, see [20, Chapter 4].

LEMMA 3.8 Let u be the mild solution which we constructed in Theorem 3.3. Then

$$u \in C^{\frac{1+\gamma}{4}}_{\frac{1+\gamma}{2}}\big((0,T], h^{2+\gamma}_{\text{even}}(\mathbf{R})\big)$$

and there exists $C = C(\gamma,\delta_0,\beta_0)$ such that

$$\|u\|_{C^{\frac{1+\gamma}{4}}_{\frac{1+\gamma}{2}}((0,T],h^{2+\gamma}_{\text{even}}(\mathbf{R}))} \leqslant C.$$

Proof. We consider the difference

$$\begin{aligned} u(t)-u(s) &= (e^{tA}-e^{sA})a - \int_s^t e^{(t-\tau)A}\partial_x \wp \Xi(\tau)\,d\tau - \int_0^s (e^{(t-\tau)A}-e^{(s-\tau)A})\partial_x\wp\Xi(\tau)\,d\tau \\ &= -\int_s^t Ae^{rA}a\,dr - \int_s^t e^{(t-\tau)A}\partial_x\wp\Xi(\tau)\,d\tau + \int_0^s\int_{s-\tau}^{t-\tau} Ae^{rA}\partial_x\wp\Xi(\tau)\,dr\,d\tau, \end{aligned}$$

where $\Xi(\tau) = \Xi(v_x,v_{xx},v_{xxx})(\cdot,\tau)$ as before. Recalling the definition of the weighted Hölder space (cf. [20, Chapter 4]), we have

$$[u]_{C^{\frac{1+\gamma}{4}}_{\frac{1+\gamma}{2}}(J,h^{2+\gamma}_{\text{even}}(\mathbf{R}))} = \sup_{0<\varepsilon<T} \varepsilon^{\frac{1+\gamma}{2}}[u]_{C^{\frac{1+\gamma}{4}}([\varepsilon,T],h^{2+\gamma}_{\text{even}}(\mathbf{R}))}.$$

Thus, for $\varepsilon \leqslant s < t \leqslant T$ we calculate

$$\begin{aligned} \varepsilon^{\frac{1+\gamma}{2}}[\partial_x^2 u(t)-\partial_x^2 u(s)]_\gamma &\leqslant \varepsilon^{\frac{1+\gamma}{2}}\left[-\partial_x^2\left(\int_s^t Ae^{rA}a\,dr\right)\right]_\gamma \\ &\quad + \varepsilon^{\frac{1+\gamma}{2}}\left[\partial_x^2\int_s^t e^{(t-\tau)A}\partial_x\wp\Xi(\tau)\,d\tau\right]_\gamma \\ &\quad + \varepsilon^{\frac{1+\gamma}{2}}\left[\partial_x^2\int_0^s\int_{s-\tau}^{t-\tau} Ae^{rA}\partial_x\wp\Xi(\tau)\,dr\,d\tau\right]_\gamma \\ &=: J_1+J_2+J_3. \end{aligned}$$

The term J_1 is estimated by

$$\begin{aligned} J_1 &\leqslant \varepsilon^{\frac{1+\gamma}{2}}\left[\int_s^t \partial_x Ae^{rA}\partial_x a\,dr\right]_\gamma \leqslant \varepsilon^{\frac{1+\gamma}{2}}\int_s^t \|\partial_x Ae^{rA}\|_{\mathfrak{L}(L^\infty(\mathbf{R}),h^\gamma(\mathbf{R}))}\cdot|\partial_x a|_\infty\,dr \\ &\lesssim \varepsilon^{\frac{1+\gamma}{2}}\int_s^t r^{-1-\frac{1}{4}-\frac{\gamma}{4}}\,dr\cdot\|a\|_{BUC^1(\mathbf{R})} \\ &\lesssim \frac{4}{1+\gamma}\varepsilon^{\frac{1+\gamma}{2}}\frac{(t-s)^{\frac{1+\gamma}{4}}}{s^{\frac{1+\gamma}{4}}t^{\frac{1+\gamma}{4}}}\|a\|_{BUC^1(\mathbf{R})} \leqslant \frac{4}{1+\gamma}(t-s)^{\frac{1+\gamma}{4}}\|a\|_{BUC^1(\mathbf{R})}. \end{aligned} \tag{3.34}$$

Now we estimate J_2 by

$$J_2 \leqslant \int_s^t \|\partial_x^3 e^{(t-\tau)A}\|_{\mathfrak{L}(h^\gamma(\mathbf{R}),h^\gamma(\mathbf{R}))} \cdot [\Xi(\tau)]_\gamma \, d\tau \lesssim c_1(\gamma, \delta_0, \beta_0) \int_s^t (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau. \quad (3.35)$$

If $s < t/2$, then we have

$$\varepsilon^{\frac{1+\gamma}{2}} \int_s^t (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau = \varepsilon^{\frac{1+\gamma}{2}} \left(\int_s^{t/2} + \int_{t/2}^t \right) (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau.$$

For $\tau \in [s, t/2]$, note that $\varepsilon \leqslant \tau$ and $\tau \leqslant t - \tau$. Thus, we observe

$$\begin{aligned} \varepsilon^{\frac{1+\gamma}{2}} \int_s^{t/2} (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau &\leqslant \varepsilon^{\frac{1+\gamma}{2}} \int_s^{t/2} (t-\tau)^{-3/4+\gamma/4} (t-\tau)^{-\gamma/4} \tau^{\frac{1+\gamma}{2}} \tau^{\alpha-1} \, d\tau \\ &= \int_s^{t/2} (t-\tau)^{-3/4+\gamma/4} \left(\frac{\tau}{t-\tau} \right)^{\gamma/4} d\tau \\ &\leqslant \int_s^{t/2} (t-\tau)^{-3/4+\gamma/4} \, d\tau \leqslant \frac{4}{1+\gamma} (t-s)^{\frac{1+\gamma}{4}}. \end{aligned} \quad (3.36)$$

For $\tau \in [t/2, t]$, we observe

$$\begin{aligned} \varepsilon^{\frac{1+\gamma}{2}} \int_{t/2}^t (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau &\leqslant \varepsilon^{\frac{1+\gamma}{2}} \left(\frac{t}{2} \right)^{\alpha-1} \int_{t/2}^t (t-\tau)^{-3/4} \, d\tau \\ &= 4\varepsilon^{\frac{1+\gamma}{2}} \left(\frac{t}{2} \right)^{\alpha-1} \left(\frac{t}{2} \right)^{1/4} \\ &\leqslant 4 \cdot 2^{\frac{1+\gamma}{2}} \left(\frac{t}{2} \right)^{\frac{1+\gamma}{2}} \left(\frac{t}{2} \right)^{\alpha-3/4} \leqslant 2^{\frac{5+\gamma}{2}} (t-s)^{\frac{1+\gamma}{4}}. \end{aligned} \quad (3.37)$$

If $s \geqslant t/2$, then we calculate in the same way as (3.37) to get

$$\begin{aligned} \varepsilon^{\frac{1+\gamma}{2}} \int_s^t (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau &\leqslant \varepsilon^{\frac{1+\gamma}{2}} \int_{t/2}^t (t-\tau)^{-3/4} \tau^{\alpha-1} \, d\tau \\ &\leqslant 2^{\frac{5+\gamma}{2}} (t-s)^{\frac{1+\gamma}{4}}. \end{aligned} \quad (3.38)$$

From (3.35)–(3.38) we have

$$J_2 \lesssim c_1 (t-s)^{\frac{1+\gamma}{4}}. \quad (3.39)$$

The term J_3 is estimated by

$$
\begin{aligned}
J_3 &\leqslant \varepsilon^{\frac{1+\gamma}{2}} \int_0^s \int_{s-\tau}^{t-\tau} \|\partial_x^3 A e^{rA}\|_{\mathfrak{L}(h^\gamma(\mathbf{R}),h^\gamma(\mathbf{R}))} [\Xi(\tau)]_\gamma \, d\tau \\
&\lesssim c_1 \varepsilon^{\frac{1+\gamma}{2}} \int_0^s \int_{s-\tau}^{t-\tau} r^{-1-3/4} \tau^{\alpha-1} \, dr \, d\tau \\
&= c_1 \varepsilon^{\frac{1+\gamma}{2}} \int_0^s \int_{s-\tau}^{t-\tau} r^{-3/4+\gamma/4} \cdot r^{-1-\gamma/4} \tau^{\alpha-1} \, dr \, d\tau \\
&\leqslant c_1 \varepsilon^{\frac{1+\gamma}{2}} \int_0^s (s-\tau)^{-1-\gamma/4} \tau^{\alpha-1} \, d\tau \int_{s-\tau}^{t-\tau} r^{-3/4+\gamma/4} \, dr \\
&\leqslant \frac{4}{1+\gamma} c_1 \varepsilon^{\frac{1+\gamma}{2}} \left(\int_0^s (s-\tau)^{-1-\gamma/4} \tau^{\alpha-1} \, d\tau \right) (t-s)^{\frac{1+\gamma}{4}} \\
&\leqslant \frac{4}{1+\gamma} c_1 \left(\int_0^1 (1-\sigma)^{-1-\gamma/4} \sigma^{\alpha-1} \, d\sigma \right) (t-s)^{\frac{1+\gamma}{4}}.
\end{aligned} \tag{3.40}
$$

From (3.34), (3.39) and (3.40) we conclude that $u \in C^{\frac{1+\gamma}{4}}_{\frac{1+\gamma}{2}}(J, h^{2+\gamma}_{\text{even}}(\mathbf{R}))$ and there exists a constant $C := C(\gamma, \delta_0, \beta_0)$ such that

$$
[u]_{C^{\frac{1+\gamma}{4}}_{\frac{1+\gamma}{2}}(J, h^{2+\gamma}_{\text{even}}(\mathbf{R}))} \leqslant C.
$$

$\square$

To show Theorem 3.7 we give a simple sufficient condition so that the product converges in L^∞∗-weak sense.

LEMMA 3.9 Assume that the sequence $\{g_j\}$ converges to g in L^∞∗-weak sense and the sequence $\{f_j\}$ is uniformly bounded and converges to f almost everywhere. Then the product sequence $f_j g_j$ converges to fg in L^∞∗-weak sense.

Proof. Let $h \in L^1$. We denote $\langle \cdot, \cdot \rangle$ by the canonical pair as before. Let us consider the difference $\langle f_j g_j, h \rangle - \langle fg, h \rangle$.

$$
\begin{aligned}
\langle f_j g_j, h \rangle - \langle fg, h \rangle &= \langle g_j (f_j - f), h \rangle + \langle f(g_j - g), h \rangle \\
&= \langle f_j - f, g_j h \rangle + \langle g_j - g, fh \rangle
\end{aligned} \tag{3.41}
$$

The second term converges to zero since $fh \in L^1(\mathbf{R})$ and g_j converges to g in L^∞∗-weak sense. Since $|(f_j - f) g_j h|$ is estimated from above as

$$
|(f_j - f)(x) g_j(x) h(x)| \leqslant (\sup_j |f_j|_\infty + |f|_\infty) \sup_j |g_j|_\infty |h(x)|,
$$

the Lebesgue dominated convergence theorem implies that the first term converges to zero as $j \to \infty$. $\square$

Proof of Theorem 3.7. We first note that

$$\begin{aligned}\|a_\lambda\|_{BUC^1(\mathbf{R})} &= |a_\lambda|_\infty + |\partial_x a_\lambda|_\infty \\ &= \frac{1}{\lambda}\sup_x |a(\lambda x)| + \sup_x |a'(\lambda x)| \\ &\leqslant \frac{|a|_\infty}{\lambda} + |a'|_\infty \\ &\leqslant \|a\|_{BUC^1(\mathbf{R})},\end{aligned}$$

if $\lambda \geqslant 1$. Then Theorem 3.3 yields

$$t^{1-\alpha}[u_\lambda]_{3+\gamma}(t) + |\partial_x u_\lambda|_\infty \leqslant C_1, \quad t > 0.$$

By interpolation we observe that

$$|\partial_x^3 u_\lambda|_\infty(t) \leqslant \frac{C_2}{t^{1/2}}, \quad |\partial_x^2 u_\lambda|_\infty(t) \leqslant \frac{C_3}{t^{1/4}}, \quad t > 0, \tag{3.42}$$

with C_j $(j = 1, 2, 3)$ independent of $\lambda \geqslant 1$. Since

$$\varXi_\lambda := \left(\frac{1}{(1 + (u_\lambda + U^L)_x^2)^2} - 1\right)\partial_x^3 u_\lambda - \frac{3(u_\lambda + U^L)_x (u_\lambda + U^L)_{xx}^2}{(1 + (u_\lambda + U^L)_x^2)^3},$$

applying (3.42) we get

$$|\varXi_\lambda|_\infty(t) \leqslant \frac{2C_2}{t^{1/2}} + \frac{3C_3^2}{t^{1/2}} = \frac{C_4}{t^{1/2}}, \quad t > 0. \tag{3.43}$$

By Lemma 3.8

$$\|u\|_{C^{\frac{1+\gamma}{4}, \frac{1+\gamma}{2}}((0,T], h^{2+\gamma}_{\text{even}}(\mathbf{R}))} \leqslant C,$$

with C independent of $\lambda \geqslant 1$. By this bound we see that u_λ together with its spatially up to second order derivatives is uniformly bounded and equi-continuous in $\overline{\mathbf{R}^+} \times (\delta, T)$ for any $\delta > 0$. By the Ascoli-Arzelà theorem and a diagonal argument, there exists a subsequence $\{u_{\lambda_j}\}_{j=1}^\infty$ such that u_{λ_j} converges to a some continuous function $\tilde{u}$ as $\lambda_j \to \infty$ uniformly on any compact set in $\overline{\mathbf{R}^+} \times (0, T]$ up to the spatial second order derivatives. By (3.42) we may assume that $t^{1/2}\partial_x^3 u_\lambda \to t^{1/2}\partial_x^3 \tilde{u}$ in $*$ weak sense of $L^\infty(\overline{\mathbf{R}^+} \times (0, \infty))$. Thus $t^{1/2}\varXi_{\lambda_j}(t)$ converges to $\varXi = \varXi((\tilde{u} + U^L)_x, (\tilde{u} + U^L)_{xx}, (\tilde{u} + U^L)_{xxx})$ $*$ weakly in $L^\infty(\overline{\mathbf{R}^+} \times (0, \infty))$ by Lemma 3.9 and (3.43).

Since u_{λ_j} solves

$$u_{\lambda_j} = e^{tA} a_{\lambda_j} - \int_0^t e^{(t-s)A}\partial_x \varXi_{\lambda_j}(s)\, ds,$$

letting $j \to \infty$ yields

$$\tilde{u} = 0 - \lim_{j\to\infty}\int_0^t e^{(t-s)A}\partial_x \varXi_{\lambda_j}(s)\, ds = -\int_0^t e^{(t-s)A}\partial_x \varXi(s)\, ds.$$

This is at least true if one interprets the convergence in the sense of distribution in $\mathbf{R}^+$ for a fixed t. Indeed, for $f \in C_c^\infty(\mathbf{R}^+)$

$$\left\langle \int_0^t e^{(t-s)A} \partial_x \wp \Xi_{\lambda_j}(s)\, ds, f \right\rangle = -\int_0^t \langle \wp \Xi_{\lambda_j}(s), \partial_x e^{(t-s)A} f \rangle\, ds$$
$$= -\int_0^t \langle \Xi_{\lambda_j}(s), \partial_x e^{(t-s)A} f \rangle\, ds.$$

Since $\partial_x e^{(t-s)A} f = e^{(t-s)A} \partial_x f$ we see that $\|\partial_x e^{(t-s)A} f\|_{L^1(\mathbf{R})} \leqslant C_0 \|\partial_x f\|_{L^1(\mathbf{R})}$; see Remark 2.11. This is in particular implies that $s^{1/2}\|\partial_x e^{(t-s)A} f\|_{L^1} \in L^1(0,t)$ as a function of s. Since $s^{1/2}\Xi_\lambda \to s^{1/2}\Xi$ in $*$ weak sense in $L^\infty(\mathbf{R}^+ \times (0,t))$, we observe that for any $t > 0$

$$\int_0^t e^{(t-s)A} \partial_x \Xi_{\lambda_j}(s)\, ds \to \int_0^t e^{(t-s)A} \partial_x \Xi(s)\, ds \quad \text{as } j \to \infty$$

in the sense of distribution. (Since

$$\left| \int_0^t e^{(t-s)A} \partial_x \Xi_{\lambda_j}(s)\, ds \right|_\infty \leqslant C_5 \int_0^t \frac{1}{(t-s)^{1/4} s^{1/2}}\, ds = C t^{1/4},$$

the convergence can be actually interpreted in $L^\infty$$*$-weak sense.) We have thus proved that

$$\tilde{u} = -\int_0^t e^{(t-s)A} \partial_x \Xi(s)\, ds.$$

By the uniqueness of solution of integral equations implies that $\tilde{u}$ is self-similar so that $\tilde{v} = \tilde{u} + U^L$ is a self-similar solution. We finally remark that the limit of the sequence is independent of choice of subsequences because the solution of integral equations is unique. We thus conclude that (local uniform) convergence $v_\lambda \to \tilde{v}$ is a full convergence. □

REMARK 3.10 It is important to estimate the depth of the thermal groove, that is, the absolute value of the profile function $Z(\cdot)$ at $x = 0$ with respect to β. However, to discuss the depth of the thermal groove requires further study. We may leave this problem open. As for the second order problem, Hamamuki [16] investigates the depth of thermal groove based on the comparison principle technique.

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